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THE NINE LEMMA AND THE PUSH FORWARD CONSTRUCTION FOR SPECIAL SCHREIER EXTENSIONS OF MONOIDS

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ABSTRACT: We show that the Nine Lemma holds for special Schreier extensions of monoids. This fact is used to obtain a push forward construction for special Schreier extensions with abelian kernel. This construction permits to give a functorial description of the Baer sum of such extensions.

KEYWORDS: monoid, special Schreier extension, Nine Lemma, push forward, Baer sum.
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1. Introduction

Actions of a group B on a group X are classically defined as group homomorphisms from B to the group $\mathrm{Aut}(X)$ of automorphisms of X. There is a well known equivalence between group actions and split extensions, obtained via the semidirect product construction. Monoid actions are defined similarly: an action of a monoid B on a monoid X is a monoid homomorphism from B to the monoid $\mathrm{End}(X)$ of endomorphisms of X. It is not difficult to see that these actions do not correspond to all split epimorphisms of monoids; hence the question of what are the split extensions of monoids that correspond to such actions arises naturally. Such split extensions were identified in [11, 6]: they are the so-called *Schreier split epimorphisms*. A split epimorphism

$$A \underset{f}{\overset{s}{\leftrightarrows}} B$$

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of monoids is a Schreier split epimorphism if every element $a \in A$ can be decomposed uniquely as $a = x \cdot sf(a)$ for some x in the kernel of f.

It turns out that the class of Schreier split epimorphisms has a much better behavior than the class of all split epimorphisms of monoids, in the sense that several homological and algebraic properties of split epimorphisms of groups are still valid for Schreier split epimorphisms, but not for all split epimorphisms of monoids. A paradigmatic example is the Split Short Five Lemma [2, Theorem 4.2]. Another important one is the fact that a Schreier split epimorphism is the cokernel of its kernel [2, Proposition 2.6]. Other important properties of Schreier split epimorphisms have been studied in [1, 3], and extended to other algebraic structures, like semirings and, more generally, monoids with operations [6].

Other interesting properties appear when we consider Schreier relations. An internal reflexive relation (i.e. a reflexive relation which is compatible with the monoid operations) on a monoid A is called a *Schreier reflexive relation* [1, 2] if the split epimorphism given by the first projection and the reflexivity morphism is a Schreier one (see Section 2 below for more details). It happens that every Schreier reflexive relation is transitive, and it is symmetric if and only if the kernel of the first projection is a group [2, Theorem 5.5]. So, Schreier reflexive relations have a property which is typical of all reflexive relations in Mal'tsev varieties [4].

The notions of Schreier reflexive relation and of Schreier congruence allowed to introduce the one of *special Schreier homomorphism* [1]. A monoid homomorphism $f\colon A \to B$ is special Schreier if its kernel congruence is a Schreier one. A special Schreier homomorphism induces a partial division on its domain: the division between two elements of A exists if they have the same image under f (again, see Section 2 below). In particular, the kernel of a special Schreier homomorphism is a group. Moreover, the Short Five Lemma holds for special Schreier extensions, i.e. for special Schreier surjective homomorphisms [1, Proposition 7.2.1].

A special Schreier extension $f\colon A \to B$ with abelian kernel X determines a monoid action of B on X, as it is explained at the beginning of Section 4. Then it is a natural question whether there is an abelian group structure on the set $\mathrm{SExt}(B, X, \varphi)$ of isomorphic classes of special Schreier extensions of a monoid B by an abelian group X inducing the action φ, which generalizes the classical Baer sum of group extensions. The existence of the Baer sum for monoids was deduced in [1] by using categorical arguments. An explicit description of the Baer sum, in terms of factor sets, was then presented in [7]. This gives an interpretation in terms of extensions of the low dimensional cohomology theory for monoids described in [8, 9, 10], which was obtained by generalizing to monoids the classical bar resolution used to compute group cohomology.

In the present paper, we show that the Nine Lemma holds for special Schreier extensions (Section 3). We will use this fact in Section 4 to describe a push forward construction for special Schreier extensions with abelian kernel. More specifically, given a special Schreier extension $f\colon A \to B$ with abelian kernel X, inducing the action φ of B on X, and a morphism $g\colon X \to Y$ of abelian groups which is equivariant with respect to the action φ and to a given action ψ of B on Y, we build a special Schreier extension of B by Y which induces the action ψ and which is universal with respect to all special Schreier extensions of B (in a sense that will be explained in Theorem 4.1 below). This will allow us to give, in Section 5, an alternative, functorial description of the Baer sum of special Schreier extensions with abelian kernel. This new description is important to give a description of cohomology of monoids which is independent from the bar resolution. This cohomological interpretation is material for a future work.

2. Schreier split epimorphisms and special Schreier extensions

The aim of this section is to recall from [6, 2, 1] the notions of Schreier split epimorphism, Schreier congruence and special Schreier extension, that will be used in the rest of the paper.

2.1. Schreier split epimorphisms.

Definition 2.1 ([6], Definition 2.6). *A split epimorphism* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *of monoids is said to be a* Schreier split epimorphism *when, for any $a \in A$, there exists a unique x in the kernel $Ker(f)$ of f such that $a = x \cdot sf(a)$.*

In other terms, a Schreier split epimorphism is a split epimorphism (A, B, f, s) equipped with a unique set-theoretical map $q\colon A \dashrightarrow \mathrm{Ker}(f)$, called the *Schreier retraction* of (A, B, f, s), with the property that, for any $a \in A$, we have:

$$a = q(a) \cdot sf(a).$$

The definition of Schreier split epimorphism of monoids was first implicitly considered in [11], in connection with the notion of Schreier internal category. In [6] it was extended to a wider class of algebraic structures, called *monoids with operations*. Later, in [5], the definition of Schreier split epimorphism was considered in the wider context of Jónsson-Tarski varieties, i.e. varieties (in the sense of universal algebra) whose corresponding theories contain a unique constant 0 and a binary operation + satisfying the equalities $0+x = x+0 = x$ for any x. In the present paper, we restrict our attention only to the case of monoids.

Proposition 2.2 ([2], Proposition 2.4)**.** *A split epimorphism (A, B, f, s) is a Schreier split epimorphism if and only if there exists a set-theoretical map $q\colon A \dashrightarrow Ker(f)$ such that:*

$$\begin{aligned} q(a) \cdot sf(a) &= a \\ q(x \cdot s(b)) &= x \end{aligned}$$

for every $a \in A$, $x \in Ker(f)$ and $b \in B$.

As shown in [6, 1], the Schreier split epimorphisms of monoids correspond to the classical monoid actions: by an action of a monoid B on a monoid X we mean a monoid homomorphism $\varphi\colon B \to \mathrm{End}(X)$, where $\mathrm{End}(X)$ is the monoid of endomorphisms of X. The equivalence is obtained in the following way (we refer to Section 5.2 in [1] for more details). Given a Schreier split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$ with kernel X, the corresponding action $\varphi\colon B \to \mathrm{End}(X)$ is given by

$$\varphi(b)(x) = q(s(b) \cdot x).$$

Conversely, given an action $\varphi\colon B \to \mathrm{End}(X)$, we can build a Schreier split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$ where A is the *semidirect product* of B and X w.r.t. φ. This is the cartesian product $X \times B$, equipped with the monoid operation given by:

$$(x_1, b_1) \cdot (x_2, b_2) = (x_1 \cdot \varphi(b_1)(x_2), b_1 \cdot b_2).$$

Proposition 2.3 ([2], Proposition 3.4)**.** *Every split epimorphism (A, B, f, s) such that B is a group is a Schreier split epimorphism.*

Proof: It suffices to write any $a \in A$ as $a = (a \cdot sf(a)^{-1}) \cdot sf(a)$. ∎

2.2. Schreier internal relations. An *internal relation* on a monoid A is a relation R which is compatible with the monoid operations. It can be described equivalently as a submonoid of the product $A \times A$. By considering the homomorphic inclusion

$$R \rightarrowtail A \times A$$

and by composing it with the two projections of the product, we get two parallel homomorphisms

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A,$$

that are the first and the second projection of the relation. More explicitly, denoting an element of R by a pair (x, y), such that x and y belong to A and are linked by the relation R, we have that $r_1(x, y) = x$ and $r_2(x, y) = y$.

An internal relation is reflexive when r_1 and r_2 have a common section $\sigma\colon A \to R$. In the notation above, we have that $\sigma(a) = (a, a)$ for any $a \in A$.

Definition 2.4 ([2], Definition 5.1)**.** *An internal reflexive relation of monoids*

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & A \arrow[l, "\sigma" description] \end{tikzcd}$$

is a Schreier reflexive relation *if the split epimorphism (R, A, r_1, σ) is a Schreier one.*

It is well known that, in a Malt'sev variety [4], every internal reflexive relation is a congruence. This is false for the variety of monoids. However, a partial version of this result can be recovered for Schreier reflexive relations:

Theorem 2.5 ([2], Theorem 5.5)**.** *Any Schreier reflexive relation is transitive. It is a congruence if and only if $Ker(r_1)$ is a group.*

We will call *Schreier congruence* a Schreier reflexive relation which is a congruence. We will be particularly interested in a specific kind of congruences, the so-called *kernel congruences*: given a monoid homomorphism $f\colon A \to B$, the corresponding kernel congruence $\mathrm{Eq}(f)$ is defined by: $a_1 \mathrm{Eq}(f) a_2$ if and only if $f(a_1) = f(a_2)$. $\mathrm{Eq}(f)$ can be represented by the following diagram:

$$\begin{tikzcd} \mathrm{Eq}(f) \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_2"'] & A, \arrow[l, "\langle 1,1\rangle" description] \end{tikzcd}$$

where $\langle 1,1\rangle$ is the diagonal of A: $\langle 1,1\rangle(a) = (a,a)$. Thanks to the symmetry of the relation, the split epimorphisms $(f_1, \langle 1,1\rangle)$ and $(f_2, \langle 1,1\rangle)$ are isomorphic. Hence, if one of the two is a Schreier split epimorphism, the other is such, too.

2.3. Special Schreier homomorphisms. We recall from [1, 3] the following notion:

Definition 2.6. *A monoid homomorphism $f\colon A \to B$ is a* special Schreier homomorphism *if the kernel congruence Eq(f) is a Schreier congruence.*

In other terms, a monoid homomorphism $f\colon A \to B$ is a special Schreier homomorphism if and only if the split epimorphism

$$\begin{tikzcd} X \arrow[r, "\langle k,0\rangle"] & \mathrm{Eq}(f) \arrow[r, shift right, "f_2"'] & A, \arrow[l, shift right, "\langle 1,1\rangle"'] \end{tikzcd}$$

where $\langle k,0\rangle$ is the morphism sending $x \in X$ to $(x,1)$, is a Schreier split epimorphism.

As a consequence of Theorem 2.5, we have that the kernel of a special Schreier homomorphism, which is isomorphic to the kernel of each of the projections $f_1, f_2\colon \mathrm{Eq}(f) \to A$, is a group. A Schreier split epimorphism is not always a special Schreier homomorphism: it happens if and only if its kernel is a group ([3], Proposition 6.9).

We are interested, in particular, in special Schreier surjective homomorphisms. We recall some relevant facts about them that will be used in the rest of the paper.

Proposition 2.7 ([1], Proposition 7.1.3)**.** *Every special Schreier surjective homomorphism $f\colon A \to B$ is the cokernel of its kernel. In other terms, the following sequence is an extension of B by Ker(f):*

$$Ker(f) \xrightarrow{k} A \xrightarrow{f} B.$$

Thanks to the previous proposition, a special Schreier surjective homomorphism can be called a *special Schreier extension.*

Lemma 2.8 ([7], Corollary 2.8)**.** *Let $f\colon A \to B$ be a special Schreier extension. Denote by X the kernel of f. Then there exists a (unique) map $q\colon Eq(f) \dashrightarrow X$ which satisfies the following conditions, for every $a \in A$, $(a_1, a_2), (a_1', a_2') \in Eq(f)$ and $x \in X$:*

(i) $q(a_1, a_2) \cdot a_2 = a_1$;
(ii) $q(x \cdot a, a) = x$;
(iii) $q(a \cdot x, a) \cdot a = a \cdot x$;
(iv) $q(a_1 \cdot a_1', a_2 \cdot a_2') = q(a_1, a_2) \cdot q(a_2 \cdot q(a_1', a_2'), a_2)$.

We observe that Condition (i) in the previous lemma means that the map q endows the monoid A with a *partial division*: the division between two elements of A exists when they have the same image by the homomorphism f. More precisely:

Corollary 2.9. *A monoid homomorphism $f\colon A \to B$ is a special Schreier homomorphism if and only if, for every $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ there exists a unique element x in the kernel of f such that $a_2 = x \cdot a_1$.*

Proposition 2.10 ([1], Proposition 7.2.1)**.** *The Short Five Lemma holds for special Schreier extensions: given a commutative diagram*

$$\begin{tikzcd} X \arrow[r, tail, "k"] \arrow[d, "u"'] & A \arrow[r, two heads, "f"] \arrow[d, "v"'] & B \arrow[d, "w"] \\ X' \arrow[r, tail, "k'"'] & A' \arrow[r, two heads, "f'"'] & B' \end{tikzcd}$$

whose rows are special Schreier extensions, if u and w are isomorphisms, then also v is.

3. The Nine Lemma

We prove now, separately, the three possible versions of the Nine Lemma for special Schreier extensions (observe that they are independent from each other). We start by recalling the following well-known lemma.

Lemma 3.1. *Given a commutative diagram of the form*

$$\begin{tikzcd} A \arrow[r, tail, "l"] \arrow[d, "t"'] & B \arrow[r, "g"] \arrow[d, "h"'] & C \arrow[d, "m"] \\ X \arrow[r, tail, "k"'] & Y \arrow[r, "f"'] & Z, \end{tikzcd}$$

where $l = Ker(g)$ and $k = Ker(f)$, if m is a monomorphism, then the left hand side square is a pullback.

Theorem 3.2 (the Lower Nine Lemma)**.** *Given a commutative diagram of monoid homomorphisms*

$$\begin{tikzcd} N \arrow[r, tail, "\eta"] \arrow[d, tail, "l"'] & H \arrow[r, two heads, "\lambda"] \arrow[d, tail, "r"'] & K \arrow[d, tail, "s"] \\ X \arrow[r, tail, "\sigma"] \arrow[d, "f"'] & Y \arrow[r, two heads, "\varphi"] \arrow[d, "g"'] & Z \arrow[d, "p"] \\ A \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & C, \end{tikzcd} \tag{1}$$

suppose that the three columns and the first two rows are special Schreier extensions. Then the lower row also is.

Proof: We divide the proof in several steps:

(1) β is a surjective homomorphism, because $\beta g = p\varphi$ is.

(2) $\beta\alpha$ is the zero homomorphism, i.e. $\beta\alpha(a) = 1$ for all $a \in a$. Indeed, $\beta\alpha f = \beta g\sigma = p\varphi\sigma = 0$ and this implies that $\beta\alpha = 0$ since f is surjective.

(3) α is a monomorphism. In order to prove this, let $a_1, a_2 \in A$ be such that $\alpha(a_1) = \alpha(a_2)$. Since f is surjective, there exist $x_i \in X$ such that $f(x_i) = a_i$. Then $g\sigma(x_1) = \alpha f(x_1) = \alpha f(x_2) = g\sigma(x_2)$, which means that $(\sigma(x_1), \sigma(x_2)) \in \mathrm{Eq}(g)$. Since g is a special Schreier homomorphism, there exists a unique $h \in H$ such that $\sigma(x_2) = r(h) \cdot \sigma(x_1)$. Being $\varphi\sigma = 0$, we get that

$$1 = \varphi\sigma(x_2) = \varphi(r(h) \cdot \sigma(x_1)) = \varphi r(h) \cdot \varphi\sigma(x_1) = s\lambda(h) \cdot 1 = s\lambda(h).$$

From the fact that s is a monomorphism we obtain that $\lambda(h) = 1$ or, in other terms, that h belongs to the kernel of λ. Hence there exists a (unique) $n \in N$ such that $\eta(n) = h$. So

$$\sigma(x_2) = r\eta(n) \cdot \sigma(x_1) = \sigma l(n) \cdot \sigma(x_1) = \sigma(l(n) \cdot x_1).$$

σ is a monomorphism, hence $x_2 = l(n) \cdot x_1$. From $fl = 0$ we conclude that

$$a_2 = f(x_2) = fl(n) \cdot f(x_1) = f(x_1) = a_1.$$

(4) A is the kernel of β. From points (2) and (3) we already know that A is contained in the kernel of β. For the other inclusion, suppose that $\beta(b) = 1$. Being g surjective, there exists $y \in Y$ such that $g(y) = b$. Since

$$p\varphi(y) = \beta g(y) = \beta(b) = 1,$$

there exists a (unique) $k \in K$ such that $s(k) = \varphi(y)$. Let $h \in H$ be such that $\lambda(h) = k$. Then

$$\varphi r(h) = s\lambda(h) = s(k) = \varphi(y).$$

But H is a group, because it is the kernel of a special Schreier homomorphism. Hence we get that $\varphi(y \cdot r(h)^{-1}) = 1$. So there is $x \in X$ such that $\sigma(x) = y \cdot r(h)^{-1}$. Call $a = f(x)$. Then

$$\alpha(a) = \alpha f(x) = g\sigma(x) = g(y) \cdot gr(h)^{-1} = g(y) = b$$

and hence $\mathrm{Ker}(\beta) \subseteq A$.

(5) β is a special Schreier homomorphism. We have to prove that, for all $b_1, b_2 \in B$ such that $\beta(b_1) = \beta(b_2)$ there exists a unique $a \in A$ such that $b_2 = \alpha(a) \cdot b_1$. Let us first prove the existence of such an a. Given b_1 and b_2 as above, let $y_i \in Y$ be such that $g(y_i) = b_i$. We have

$$p\varphi(y_1) = \beta g(y_1) = \beta(b_1) = \beta(b_2) = \beta g(y_2) = p\varphi(y_2),$$

hence $(\varphi(y_1), \varphi(y_2)) \in \mathrm{Eq}(p)$. From the fact that p is a special Schreier homomorphism we deduce that there exists a unique $k \in K$ such that $\varphi(y_2) = s(k) \cdot \varphi(y_1)$. Choosing $h \in H$ such that $\lambda(h) = k$, we get

$$\varphi(y_2) = s\lambda(h) \cdot \varphi(y_1) = \varphi r(h) \cdot \varphi(y_1) = \varphi(r(h) \cdot y_1),$$

and so $(r(h) \cdot y_1, y_2) \in \mathrm{Eq}(\varphi)$. φ is a special Schreier homomorphism, hence there exists a unique $x \in X$ such that $y_2 = \sigma(x) \cdot r(h) \cdot y_1$. So we obtain that

$$b_2 = g(y_2) = g(\sigma(x) \cdot r(h) \cdot y_1) = g\sigma(x) \cdot gr(h) \cdot g(y_1) =$$
$$= \alpha f(x) \cdot 1 \cdot g(y_1) = \alpha f(x) \cdot b_1,$$

hence $a = f(x)$ is the element of A we were looking for. To conclude the proof, we need to show that such an a is unique. Suppose that $\bar{a} \in A$ is such that $b_2 = \alpha(\bar{a}) \cdot b_1$. Let $\bar{x} \in X$ be such that $f(\bar{x}) = \bar{a}$; moreover, let $h \in H$ be such that $\varphi(y_2) = \varphi r(h) \cdot \varphi(y_1)$ as above. Then

$$g(\sigma(\bar{x}) \cdot r(h) \cdot y_1) = g\sigma(\bar{x}) \cdot gr(h) \cdot g(y_1) =$$
$$= \alpha f(\bar{x}) \cdot b_1 = \alpha(\bar{a}) \cdot b_1 = b_2 = g(y_2),$$

so that $(\sigma(\bar{x}) \cdot r(h) \cdot y_1, y_2) \in \mathrm{Eq}(g)$. Being g a special Schreier homomorphism, there exists a unique $\bar{h} \in H$ such that $y_2 = r(\bar{h}) \cdot \sigma(\bar{x}) \cdot r(h) \cdot y_1$. Observe now that, since H and X are normal subgroups of the monoid Y, kernels of g and φ, respectively, then the element $r(\bar{h}) \cdot \sigma(\bar{x}) \cdot r(\bar{h})^{-1} \cdot \sigma(\bar{x})^{-1}$ belongs to $H \cap X$. Indeed, it is immediate to see that it belongs both to the kernels of g and φ. But the intersection $H \cap X$ is N, because the upper left square in Diagram (1) is a pullback (thanks to Lemma 3.1). This means that there exists $n \in N$ such that

$$\sigma l(n) = r(\bar{h}) \cdot \sigma(\bar{x}) \cdot r(\bar{h})^{-1} \cdot \sigma(\bar{x})^{-1}$$

or, in other terms,

$$r(\bar{h}) \cdot \sigma(\bar{x}) = \sigma l(n) \cdot \sigma(\bar{x}) \cdot r(\bar{h}).$$

Hence

$$y_2 = \sigma l(n) \cdot \sigma(\bar{x}) \cdot r(\bar{h}) \cdot r(h) \cdot y_1 = \sigma(l(n) \cdot \bar{x}) \cdot r(\bar{h} \cdot h) \cdot y_1.$$

Applying φ to this last equality and using that $\varphi\sigma = 0$ we get

$$\varphi(y_2) = \varphi\sigma(l(n) \cdot \bar{x}) \cdot \varphi r(\bar{h} \cdot h) \cdot \varphi(y_1) =$$
$$= \varphi r(\bar{h} \cdot h) \cdot \varphi(y_1) = s\lambda(\bar{h} \cdot h) \cdot \varphi(y_1).$$

But, being p a special Schreier homomorphism, we know that there exists a unique $k \in K$ such that $\varphi(y_2) = s(k) \cdot \varphi(y_1)$. We proved that both $\lambda(h)$ and $\lambda(\bar{h} \cdot h)$ satisfy this equation, and hence

$$\lambda(\bar{h}) \cdot \lambda(h) = \lambda(\bar{h} \cdot h) = \lambda(h).$$

Since H is a group, this implies that $\lambda(\bar{h}) = 1$. So there exists $\bar{n} \in N$ such that $\eta(\bar{n}) = \bar{h}$. From this we get

$$y_2 = r(\bar{h}) \cdot \sigma(\bar{x}) \cdot r(h) \cdot y_1 =$$
$$= r\eta(\bar{n}) \cdot \sigma(\bar{x}) \cdot r(h) \cdot y_1 = \sigma l(\bar{n} \cdot \bar{x}) \cdot r(h) \cdot y_1.$$

Using now the uniqueness of x as an element of X such that $y_2 = \sigma(x) \cdot r(h) \cdot y_1$, we obtain that $x = l(\bar{n}) \cdot \bar{x}$. Then

$$a = f(x) = fl(\bar{n}) \cdot f(\bar{x}) = 1 \cdot f(\bar{x}) = f(\bar{x}) = \bar{a},$$

and this concludes the proof.

■

Theorem 3.3 (the Upper Nine Lemma)**.** *Given a commutative diagram of monoid homomorphisms*

$$\begin{tikzcd}
N \arrow[r, "\eta"] \arrow[d, tail, "l"'] & H \arrow[r, "\lambda"] \arrow[d, tail, "r"'] & K \arrow[d, tail, "s"] \\
X \arrow[r, tail, "\sigma"] \arrow[d, "f"'] & Y \arrow[r, two heads, "\varphi"] \arrow[d, "g"'] & Z \arrow[d, "p"] \\
A \arrow[r, tail, "\alpha"'] & B \arrow[r, two heads, "\beta"'] & C,
\end{tikzcd} \tag{2}$$

suppose that the three columns and the last two rows are special Schreier extensions. Then the upper row also is.

Proof: (1) η is a monomorphism, because $r\eta = \sigma l$ is.

(2) $\lambda\eta$ is the zero homomorphism. Indeed,

$$s\lambda\eta = \varphi\sigma l = 0$$

and this implies that $\lambda\eta = 0$ since s is a monomorphism.

(3) λ is a surjective homomorphism. Indeed, consider $k \in K$. Since φ is surjective, there exists $y \in Y$ such that $\varphi(y) = s(k)$. Then

$$\beta g(y) = p\varphi(y) = ps(k) = 1,$$

hence there exists $a \in A$ such that $\alpha(a) = g(y)$. Thanks to the surjectivity of f, we find $x \in X$ such that $f(x) = a$. From the equality

$$g\sigma(x) = \alpha f(x) = \alpha(a) = g(y)$$

we obtain that $(\sigma(x), y) \in \mathrm{Eq}(g)$. Being g special Schreier, there exists a unique $h \in H$ such that $y = r(h) \cdot \sigma(x)$. X is a group, so the last equality can be rewritten as $r(h) = y \cdot \sigma(x)^{-1}$. From this we get

$$s\lambda(h) = \varphi r(h) = \varphi(y) \cdot \varphi\sigma(x)^{-1} = \varphi(y) = s(k).$$

s is a monomorphism, so we conclude that $\lambda(h) = k$.

(4) N is the kernel of λ. From points (1) and (2) we already know that N is contained in the kernel of λ. Let then $h \in H$ such that $\lambda(h) = 1$. Then $\varphi r(h) = s\lambda(h) = 1$, hence there exists $x \in X$ such that $\sigma(x) = r(h)$. This means that $\sigma(x) = r(h)$ belongs to the intersection of H and X, which is N thanks to Lemma 3.1. Hence there exists $n \in N$ such that $\eta(n) = h$.

(5) Since N, H and K are groups (which is a consequence of the fact that the three columns are special Schreier extensions), the fact that λ is a surjective homomorphism and η is its kernel immediately implies that the upper row of Diagram (2) is a special Schreier extension. Indeed, it follows immediately from Proposition 2.3 that every extension of groups is special Schreier.

■

We would like to stress the strong asymmetry between the proofs of the Lower and the Upper Nine Lemma: the first is much more complicated than the second. This happens because, since the columns are special Schreier extensions, the upper row lies in the category of groups, in which every surjective homomorphism is a special Schreier extension.

Theorem 3.4 (the Middle Nine Lemma)**.** *Given a commutative diagram of monoid homomorphisms*

$$\begin{tikzcd}
N \arrow[r, tail, "\eta"] \arrow[d, tail, "l"'] & H \arrow[r, two heads, "\lambda"] \arrow[d, tail, "r"'] & K \arrow[d, tail, "s"] \\
X \arrow[r, "\sigma"] \arrow[d, "f"'] & Y \arrow[r, "\varphi"] \arrow[d, "g"'] & Z \arrow[d, "p"] \\
A \arrow[r, tail, "\alpha"'] & B \arrow[r, two heads, "\beta"'] & C,
\end{tikzcd} \tag{3}$$

suppose that the three columns, the upper and the lower row are special Schreier extensions. Suppose moreover that $\varphi\sigma = 0$. Then the middle row is a special Schreier extension, too.

Proof: (1) σ is a monomorphism. In order to prove this, let $x_1, x_2 \in X$ be such that $\sigma(x_1) = \sigma(x_2)$. Then

$$\alpha f(x_1) = g\sigma(x_1) = g\sigma(x_2) = \alpha f(x_2).$$

Since α is a monomorphism, we get that $(x_1, x_2) \in \mathrm{Eq}(f)$. Being f special Schreier, there exists a unique $n \in N$ such that $x_2 = l(n) \cdot x_1$

and hence

$$\sigma(x_1) = \sigma(x_2) = \sigma l(n) \cdot \sigma(x_1) = r\eta(n) \cdot \sigma(x_1).$$

Thanks to the fact that g is a special Schreier homomorphism, the last equality forces $r\eta(n) = 1$, and so $n = 1$. Then $x_1 = x_2$.

(2) φ is surjective. Indeed, consider $z \in Z$. Being β surjective, there exists $b \in B$ such that $\beta(b) = p(z)$. Choosing $y \in Y$ with $g(y) = b$ we get

$$p\varphi(y) = \beta g(y) = \beta(b) = p(z).$$

Using that p is special Schreier, we conclude that there exists a unique $k \in K$ such that $z = s(k) \cdot \varphi(y)$. Choosing $h \in H$ such that $\lambda(h) = k$ we obtain that $s(k) = s\lambda(h) = \varphi r(h)$, hence $z = \varphi r(h) \cdot \varphi(y) = \varphi(r(h) \cdot y)$. So φ is surjective.

(3) X is the kernel of φ. Thanks to point (1) and the hypothesis that $\varphi\sigma = 0$, we already know that X is contained in the kernel of φ. Conversely, let $y \in Y$ be such that $\varphi(y) = 1$. Then $\beta g(y) = p\varphi(y) = 1$ and so there exists $a \in A$ such that $\alpha(a) = g(y)$. Choosing $x \in X$ such that $f(x) = a$ we get

$$g\sigma(x) = \alpha f(x) = \alpha(a) = g(y).$$

Using the fact that g is special Schreier we find a unique $h \in H$ such that $y = r(h) \cdot \sigma(x)$. Then

$$1 = \varphi(y) = \varphi r(h) \cdot \varphi\sigma(x) = \varphi r(h) = s\lambda(h).$$

Being s injective, we obtain that $\lambda(h) = 1$, so there is $n \in N$ such that $h = \eta(n)$. But then

$$y = r(h) \cdot \sigma(x) = r\eta(n) \cdot \sigma(x) = \sigma l(n) \cdot \sigma(x) = \sigma(l(n) \cdot x)$$

belongs to the image of X.

(4) φ is a special Schreier homomorphism. Let $y_1, y_2 \in Y$ be such that $\varphi(y_1) = \varphi(y_2)$. We have to show that there exists a unique $x \in X$ such that $y_2 = \sigma(x) \cdot y_1$. Observe that

$$\beta g(y_1) = p\varphi(y_1) = p\varphi(y_2) = \beta g(y_2),$$

so that $(g(y_1), g(y_2)) \in \mathrm{Eq}(\beta)$. Being β special Schreier, there exists a unique $a \in A$ such that

$$g(y_2) = \alpha(a) \cdot g(y_2).$$

Choosing $x \in X$ such that $f(x) = a$, we get that $\alpha(a) = \alpha f(x) = g\sigma(x)$, and so

$$g(y_2) = g\sigma(x) \cdot g(y_1) = g(\sigma(x) \cdot y_1),$$

which means that $(\sigma(x) \cdot y_1, y_2) \in \mathrm{Eq}(g)$. Then there exists a unique $h \in H$ such that

$$y_2 = r(h) \cdot \sigma(x) \cdot y_1.$$

Applying φ to this equality we obtain

$$\varphi(y_2) = \varphi r(h) \cdot \varphi\sigma(x) \cdot \varphi(y_1) = s\lambda(h) \cdot \varphi(y_1).$$

By assumption $\varphi(y_1) = \varphi(y_2)$, hence we have that

$$\varphi(y_1) = s\lambda(h) \cdot \varphi(y_1).$$

But the fact that p is special Schreier forces $\lambda(h) = 1$. Then there exists $n \in N$ such that $\eta(n) = h$. Hence we have that

$$y_2 = r(h) \cdot \sigma(x) \cdot y_1 = r\eta(n) \cdot \sigma(x) \cdot y_1 =$$

$$= \sigma l(n) \cdot \sigma(x) \cdot y_1 = \sigma(l(n) \cdot x) \cdot y_1,$$

so $l(n) \cdot x$ is the element of X we were looking for. It remains to show its uniqueness. For that, suppose there are $x, x' \in X$ such that

$$y_2 = \sigma(x) \cdot y_1 = \sigma(x') \cdot y_1.$$

Then

$$\alpha f(x) \cdot g(y_1) = g\sigma(x) \cdot g(y_1) = g(y_2) = g\sigma(x') \cdot g(y_1) = \alpha f(x') \cdot g(y_1).$$

But A is a group, hence

$$g(y_1) = \alpha(f(x)^{-1} \cdot f(x')) \cdot g(y_1).$$

From the fact that g is special Schreier we conclude that $f(x)^{-1} \cdot f(x') = 1$, which means that $f(x) = f(x')$. Being f special Schreier, there exists a unique $n' \in N$ such that $x' = l(n') \cdot x$. We must show that $n' = 1$. From the equality

$$\sigma(x) \cdot y_1 = \sigma(x') \cdot y_1 = \sigma l(n') \cdot \sigma(x) \cdot y_1 = r\eta(n') \cdot \sigma(x) \cdot y_1,$$

and from the fact that g is special Schreier, we obtain $r\eta(n') = 1$, which means that $n' = 1$ since r and η are monomorphisms. This concludes the proof.

$\blacksquare$

4. The push forward construction

In this section we develop a push forward construction for special Schreier extensions with abelian kernel. This construction will be used later to give a description of the Baer sum of special Schreier extensions which is alternative to the one we gave in [7].

First we need to recall from [1, 7] how to associate an action with a special Schreier extension. Let

$$X \overset{k}{\rightarrowtail} A \overset{f}{\twoheadrightarrow} B \tag{4}$$

be a special Schreier extension with abelian kernel. This means that the split epimorphism

$$X \xrightarrow{\langle k,0\rangle} \mathrm{Eq}(f) \underset{f_2}{\overset{\langle 1,1\rangle}{\leftrightarrows}} A$$

is a Schreier split epimorphism. As we explained in Section 2, this split epimorphism corresponds to a monoid action $\psi\colon A \to \mathrm{End}(X)$ of A on X. Putting then

$$\varphi(b)(x) = \psi(a)(x) = q((a,a)\cdot(x,1)) = q(a\cdot x, a) \tag{5}$$

for any $a \in A$ such that $f(a) = b$, we obtain a monoid homomorphism $\varphi\colon B \to \mathrm{End}(X)$: it is well defined thanks to the fact that X is an abelian group. Then we get an action of B on X. From now on, we will denote by $b \cdot_\varphi x$ the element $\varphi(b)(x)$.

Theorem 4.1. *Consider the following situation:*

$$\begin{tikzcd} X \arrow[r, tail, "k"] \arrow[d, "g"'] & A \arrow[r, two heads, "f"] & B, \\ Y & & \end{tikzcd} \tag{6}$$

where:

- *f is a special Schreier extension with abelian kernel (with Schreier retraction $q\colon Eq(f) \to X$);*
- *$\varphi\colon B \to End(X)$ is the corresponding action of B on X, defined as in (5);*
- *Y is an abelian group, equipped with an action $\psi\colon B \to End(Y)$ of B on it;*

- *g is a morphism of abelian groups which is equivariant w.r.t. the actions, which means that, for all $b \in B$ and all $x \in X$,*

$$g(b \cdot_\varphi x) = b \cdot_\psi g(x).$$

Then there exists a special Schreier extension f' with kernel Y and codomain B, which induces the action ψ and is universal among all such extensions, meaning that given any diagram of the form

$$\begin{tikzcd}
X \arrow[r, tail, "k"] \arrow[d, "g"] \arrow[dd, bend right, "u"'] & A \arrow[r, two heads, "f"] \arrow[d, "g'"'] \arrow[dd, bend left, "v"] & B \arrow[d, equal] \\
Y \arrow[r, tail, "k'"] \arrow[d, "r"] & C \arrow[r, two heads, "f'"] \arrow[d, dotted, "\alpha"'] & B \arrow[d, equal] \\
Z \arrow[r, tail, "l"'] & E \arrow[r, two heads, "p"'] & B,
\end{tikzcd} \tag{7}$$

where p is a special Schreier extension with abelian kernel Z, r is an equivariant morphism, (u, v) is a morphism of extensions and $u = rg$, then there exists a unique monoid homomorphism α such that $v = \alpha g'$ and (r, α) is a morphism of extensions.

Proof: The morphism f and the action ψ induce an action ζ of A on Y given by $\zeta = \psi f \colon A \to \mathrm{End}(Y)$. In other terms, $a \cdot_\zeta y = f(a) \cdot_\psi y$. We can then build the semidirect product $Y \rtimes_\zeta A$ of Y and A w.r.t. ζ. Since Y is an abelian group, this gives us a special Schreier extension with abelian kernel:

$$Y \overset{\langle 1,0\rangle}{\rightarrowtail} Y \rtimes_\zeta A \underset{\pi_A}{\overset{\langle 0,1\rangle}{\leftrightarrows}} A.$$

Consider now the map $h \colon X \to Y \rtimes_\zeta A$ defined by

$$h(x) = (g(x)^{-1}, k(x)).$$

It is clearly injective, since k is. Moreover, it is a homomorphism, indeed:

$$\begin{aligned} h(x_1) \cdot h(x_2) &= (g(x_1)^{-1}, k(x_1)) \cdot (g(x_2)^{-1}, k(x_2)) = \\ &= (g(x_1)^{-1} \cdot (k(x_1) \cdot_\zeta g(x_2)^{-1}), k(x_1) \cdot k(x_2)) = \\ &= (g(x_1)^{-1} \cdot (fk(x_1) \cdot_\psi g(x_2)^{-1}), k(x_1) \cdot k(x_2)), \end{aligned}$$

and since k is the kernel of f the last expression is equal to

$$(g(x_1)^{-1} \cdot g(x_2)^{-1}, k(x_1) \cdot k(x_2)) = (g(x_1 \cdot x_2)^{-1}, k(x_1 \cdot x_2)) = h(x_1 \cdot x_2),$$

where the first equality holds because Y is an abelian group.

Let $c\colon Y \rtimes_\zeta A \to C$ be the cokernel of h, i.e. the quotient w.r.t. the congruence R_h on $Y \rtimes_\zeta A$ generated by $h(X)$. We first observe that this congruence R_h has a very simple description. Indeed, consider the following relation on $Y \rtimes_\zeta A$:

$$(y_1, a_1)R(y_2, a_2) \quad \text{if} \quad \exists\, x \in X \quad \text{such that} \tag{8}$$
$$(y_2, a_2) = (g(x)^{-1}, k(x)) \cdot (y_1, a_1) = (g(x)^{-1} \cdot y_1, k(x) \cdot a_1),$$

where the last equality holds because the elements in the image of k act trivially on A. This relation R is clearly an equivalence relation (symmetry comes from the fact that X is a group). Let us show that it is a congruence, i.e. that it is compatible with the operation in $Y \rtimes_\zeta A$. Then it will be necessarily the congruence R_h generated by $h(X)$. In order to do this, suppose that

$$(y_1, a_1)R(y_2, a_2) \quad \text{and} \quad (y_1', a_1')R(y_2', a_2'),$$

so that there exist $x, x' \in X$ such that

$$(y_2, a_2) = (g(x)^{-1}, k(x)) \cdot (y_1, a_1) = (g(x)^{-1} \cdot y_1, k(x) \cdot a_1)$$

and

$$(y_2', a_2') = (g(x')^{-1}, k(x')) \cdot (y_1', a_1') = (g(x')^{-1} \cdot y_1', k(x') \cdot a_1').$$

We want to prove that there exists $\bar{x} \in X$ such that

$$(y_2, a_2) \cdot (y_2', a_2') = (g(\bar{x})^{-1}, k(\bar{x})) \cdot (y_1, a_1) \cdot (y_1', a_1'). \tag{9}$$

We have

$$(y_2, a_2) \cdot (y_2', a_2') = (g(x)^{-1} \cdot y_1, k(x) \cdot a_1) \cdot (g(x')^{-1} \cdot y_1', k(x') \cdot a_1') = \tag{10}$$
$$= (g(x)^{-1} \cdot y_1 \cdot (k(x) \cdot a_1) \cdot_\zeta (g(x')^{-1} \cdot y_1'), k(x) \cdot a_1 \cdot k(x') \cdot a_1').$$

Observe that $(a_1 \cdot k(x'), a_1) \in \mathrm{Eq}(f)$; hence, by Lemma 2.8, we have

$$kq(a_1 \cdot k(x'), a_1) \cdot a_1 = a_1 \cdot k(x')$$

and so

$$k(x) \cdot a_1 \cdot k(x') \cdot a_1' = k(x) \cdot kq(a_1 \cdot k(x'), a_1) \cdot a_1 \cdot a_1'.$$

This gives us a candidate for the element $\bar{x}$ we were looking for, namely $\bar{x} = x \cdot q(a_1 \cdot k(x'), a_1)$. Replacing this expression in the right side of (9), we get

$$(g(x \cdot q(a_1 \cdot k(x'), a_1))^{-1}, k(x \cdot q(a_1 \cdot k(x'), a_1))) \cdot (y_1, a_1) \cdot (y_1', a_1') =$$

$$= (g(x)^{-1} \cdot gq(a_1 \cdot k(x'), a_1)^{-1}, k(x) \cdot kq(a_1 \cdot k(x'), a_1)) \cdot (y_1 \cdot f(a_1) \cdot_\psi y_1', a_1 \cdot a_1')$$

and, using the fact that the elements of $k(X)$ act trivially, this is equal to

$$(g(x)^{-1} \cdot gq(a_1 \cdot k(x'), a_1)^{-1} \cdot y_1 \cdot f(a_1) \cdot_\psi y_1', k(x) \cdot kq(a_1 \cdot k(x'), a_1) \cdot a_1 \cdot a_1').$$

We already proved that the second component is the same as in (10). Let us check that this is the case for the first component, too. Using the fact that $q(a_1 \cdot k(x'), a_1) = f(a_1) \cdot_\varphi x'$, the first component is equal to

$$g(x)^{-1} \cdot g(f(a_1) \cdot_\varphi x')^{-1} \cdot y_1 \cdot (f(a_1) \cdot_\psi y_1').$$

Using equivariance of g, this is equal to

$$g(x)^{-1} \cdot (f(a_1) \cdot_\psi g(x'))^{-1} \cdot y_1 \cdot (f(a_1) \cdot_\psi y_1').$$

The first component in (10) is

$$g(x)^{-1} \cdot y_1 \cdot (k(x) \cdot a_1) \cdot_\zeta (g(x')^{-1} \cdot y_1') =$$

$$= g(x)^{-1} \cdot y_1 \cdot (a_1 \cdot_\zeta (g(x')^{-1} \cdot y_1')) =$$

$$= g(x)^{-1} \cdot y_1 \cdot (f(a_1) \cdot_\psi g(x'))^{-1} \cdot (f(a_1) \cdot_\psi y_1'),$$

and the two expressions are the same because Y is an abelian group.

Knowing now that $c\colon Y \rtimes_\zeta A \to C$ is the quotient w.r.t. the congruence (8), it is immediate to see that $h(X)$ is the kernel of c, i.e. the zero-class of the relation R_h. Moreover, c is a special Schreier extension. Indeed, suppose that $c(y_1, a_1) = c(y_2, a_2)$. This means that $(y_1, a_1) R_h (y_2, a_2)$, so that there exists $x \in X$ such that

$$(y_2, a_2) = (g(x)^{-1}, k(x)) \cdot (y_1, a_1) = (g(x)^{-1} \cdot y_1, k(x) \cdot a_1).$$

In particular, this says that $a_2 = k(x) \cdot a_1$. But then $(a_2, a_1) \in \mathrm{Eq}(f)$, and f is a special Schreier extension, so x is necessarily equal to $q(a_2, a_1)$. Hence there is a unique $x \in X$ with the required property.

Consider now the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d, tail, "h"] & X \arrow[d, tail, "k"] \\
Y \arrow[r, tail, "\langle 1,0 \rangle"] \arrow[d, equal] & Y \rtimes_\zeta A \arrow[r, two heads, "\pi_A"] \arrow[d, two heads, "c"] & A \arrow[d, two heads, "f"] \\
Y \arrow[r, "k'"'] & C \arrow[r, "f'"'] & B,
\end{tikzcd}$$

where $k' = c\langle 1, 0\rangle$ and f' is induced by the universal property of the cokernel c. By hypothesis and by what we just proved, the three columns and the first two rows are special Schreier extensions. The Lower Nine Lemma (Theorem 3.2) gives then that the lower row also is. This is the push forward of f along g we were looking for.

We still have to prove that the action of B on Y determined by f' coincides with ψ and that our construction is universal. Let us denote by $[(y, a)]$ an element of C, i.e. an equivalence class of the relation R_h. Then

$$f'([(y, a)]) = f'c(y, a) = f\pi_A(y, a) = f(a).$$

Denoting by q' the unique map $\mathrm{Eq}(f') \to Y$ determined by the fact that f' is special Schreier, we have that the action $\chi\colon B \to \mathrm{End}(Y)$ of B on Y induced by f' is given by

$$b \cdot_\chi y = q'([(\bar{y}, \bar{a})] \cdot [(y, 1)], [(\bar{y}, \bar{a})]) \quad \text{for all } \bar{a} \in A \text{ such that } f(\bar{a}) = b.$$

Hence

$$b \cdot_\chi y = q'([(\bar{y} \cdot (\bar{a} \cdot_\zeta y), \bar{a})], [(\bar{y}, \bar{a})]) = q'([(\bar{y} \cdot (b \cdot_\psi y), \bar{a})], [(\bar{y}, \bar{a})]).$$

By definition, $q'([(\bar{y} \cdot (b \cdot_\psi y), \bar{a})], [(\bar{y}, \bar{a})])$ is the unique element $t \in Y$ such that

$$c\langle 1, 0\rangle(t) \cdot [(\bar{y}, \bar{a})] = [(\bar{y} \cdot (b \cdot_\psi y), \bar{a})].$$

But

$$c\langle 1, 0\rangle(t) \cdot [(\bar{y}, \bar{a})] = [(t, 1)] \cdot [(\bar{y}, \bar{a})] = [(t \cdot \bar{y}, \bar{a})].$$

The commutativity of Y and the uniqueness of t force then $t = b \cdot_\psi y$. Hence χ and ψ coincide.

In order to prove the universality of our construction, consider Diagram (7). Let us denote by $\tau\colon B \to \mathrm{End}(Z)$ the action determined by p. We first define a map $\beta\colon Y \rtimes_\zeta A \to E$ by putting

$$\beta(y,a) = lr(y) \cdot v(a).$$

It is a monoid homomorphism, indeed

$$\beta((y_1,a_1)\cdot(y_2,a_2)) = \beta(y_1 \cdot (a_1 \cdot_\zeta y_2), a_1 \cdot a_2) =$$

$$= lr(y_1) \cdot lr(a_1 \cdot_\zeta y_2) \cdot v(a_1) \cdot v(a_2) =$$

$$= lr(y_1) \cdot lr(f(a_1) \cdot_\psi y_2) \cdot v(a_1) \cdot v(a_2) =$$

$$= lr(y_1) \cdot l(f(a_1) \cdot_\tau r(y_2)) \cdot v(a_1) \cdot v(a_2),$$

where the last equality holds because r is equivariant. Observe that

$$f(a_1) \cdot_\tau r(y_2) = pv(a_1) \cdot_\tau r(y_2) = q_p(v(a_1) \cdot lr(y_2), v(a_1)),$$

where q_p is the Schreier map associated with the special Schreier extension p. Then

$$l(f(a_1) \cdot_\tau r(y_2)) \cdot v(a_1) = lq_p(v(a_1) \cdot lr(y_2), v(a_1)) \cdot v(a_1) = v(a_1) \cdot lr(y_2).$$

Hence

$$\beta((y_1,a_1)\cdot(y_2,a_2)) = lr(y_1) \cdot v(a_1) \cdot lr(y_2) \cdot v(a_2) = \beta(y_1,a_1) \cdot \beta(y_2,a_2).$$

Moreover, we have that

$$\beta h(x) = \beta(g(x)^{-1}, k(x)) = lrg(x)^{-1} \cdot vk(x) = lu(x)^{-1} \cdot lu(x) = 1$$

for all $x \in X$. Being c the cokernel of h, we conclude that there exists a unique morphism $\alpha\colon C \to E$ such that $\alpha c = \beta$, and so

$$\alpha g' = \alpha c\langle 0,1\rangle = \beta\langle 0,1\rangle = v.$$

Moreover, (r,α) is a morphism of extensions, indeed:

$$\alpha k'(y) = \alpha c(y,1) = \beta(y,1) = lr(y)$$

and

$$p\alpha([(y,a)]) = p\beta(y,a) = plr(y) \cdot pv(a) = 1 \cdot pv(a) = f(a) = f'([(y,a)]).$$

■

We conclude this section by mentioning that a similar push forward construction has been obtained independently in [12] for a wider class of extensions, whose kernels are commutative monoids but not necessarily abelian groups. However, in [12] a weaker universality of the construction is proved: the existence of a morphism α as in Diagram (7) was obtained only when r is an identity.

5. The Baer sum of special Schreier extensions with abelian kernel

We now show that the push forward construction described in the previous section allows to define the Baer sum of special Schreier extensions with abelian kernel. A construction of the Baer sum was already given in [7], using factor sets as in the case of classical group extensions. We remark that a similar construction was announced in [8]. We will show that the two approaches give the same result. The advantage of the approach via the push forward is that it is functorial, and this can be useful to give an interpretation of cohomology of monoids in terms of special Schreier extensions. We start by recalling briefly the construction given in [7].

Definition 5.1 ([7], Definition 3.1)**.** *Given a monoid B, an abelian group X and an action $\varphi\colon B \to End(X)$ of B on X, a* factor set *is a map $g\colon B \times B \to X$ which satisfies, for all $b, b_1, b_2, b_3 \in B$, the following conditions:*

(i) $g(b, 1) = g(1, b) = 1$;
(ii) $g(b_1, b_2) \cdot g(b_1 \cdot b_2, b_3) = \varphi(b_1)(g(b_2, b_3)) \cdot g(b_1, b_2 \cdot b_3)$.

Given a special Schreier extension with abelian kernel

$$X \overset{k}{\rightarrowtriangle} A \overset{f}{\twoheadrightarrow} B, \tag{11}$$

we can associate with it a factor set in the following way: let $s\colon B \to A$ be a set-theoretical section of f (it exists, since f is surjective). Let us choose s such that $s(1) = 1$. Then, for any $b_1, b_2 \in B$:

$$f(s(b_1) \cdot s(b_2)) = b_1 \cdot b_2 = f(s(b_1 \cdot b_2)).$$

Hence the pair $(s(b_1) \cdot s(b_2), s(b_1 \cdot b_2))$ belongs to $\mathrm{Eq}(f)$. We define a map $g\colon B \times B \to X$ by putting:

$$g(b_1, b_2) = q(s(b_1) \cdot s(b_2), s(b_1 \cdot b_2)),$$

where q is the Schreier map associated with f. Such a map g is a factor set ([7], Proposition 3.3). Moreover, thanks to Proposition 3.4 in [7], the extension (11) is isomorphic to an extension of the form

$$X \overset{\langle 1,0\rangle}{\rightarrowtail} X \times B \overset{\pi_B}{\twoheadrightarrow} B,$$

where the monoid operation on $X \times B$ is defined by:

$$(x_1, b_1) \cdot (x_2, b_2) = (x_1 \cdot \varphi(b_1)(x_2) \cdot g(b_1, b_2), b_1 \cdot b_2).$$

Choosing two different sections for f, the corresponding factor sets differ by an *inner factor set*:

Definition 5.2. *A factor set g is called* inner factor set *if it is of the form*

$$g(b_1, b_2) = h(b_1) \cdot \varphi(b_1)(h(b_2)) \cdot h(b_1 \cdot b_2)^{-1}$$

for some map $h\colon B \to X$ such that $h(1) = 1$.

The set $\mathcal{F}(B, X, \varphi)$ of all the factor sets corresponding to a given action $\varphi\colon B \to \mathrm{End}(X)$ is a subgroup of the abelian group $X^{B\times B}$, where the group operation is the pointwise multiplication. Its subset $\mathcal{IF}(B, X, \varphi)$ of inner factor sets is a normal subgroup of $\mathcal{F}(B, X, \varphi)$. Let us denote by $\mathrm{SExt}(B, X, \varphi)$ the set of isomorphic classes of special Schreier extensions of a monoid B by an abelian group X inducing the action $\varphi\colon B \to \mathrm{End}(X)$. Since the Short Five Lemma holds for special Schreier extensions ([1], Proposition 7.2.1), two special Schreier extensions of B by X are isomorphic as soon as there exists a morphism of extensions between them. We have the following

Theorem 5.3 ([7], Theorem 3.7)**.** *The set $SExt(B, X, \varphi)$ of isomorphic classes of special Schreier extensions of a monoid B by an abelian group X inducing the action $\varphi\colon B \to End(X)$ is in bijection with the factor abelian group*

$$\frac{\mathcal{F}(B, X, \varphi)}{\mathcal{IF}(B, X, \varphi)}.$$

By means of this bijection, we can endow $\mathrm{SExt}(B, X, \varphi)$ with an abelian group structure, which we call *the Baer sum.* The unit of this abelian group is the isomorphic class of the split extension obtained by taking the semidirect product of X and B with respect to the action φ.

We propose now an alternative description of the Baer sum. Given two special Schreier extensions

$$X \overset{k_1}{\rightarrowtail} A_1 \overset{f_1}{\twoheadrightarrow} B \quad \text{and} \quad X \overset{k_2}{\rightarrowtail} A_2 \overset{f_2}{\twoheadrightarrow} B$$

with abelian kernel X which induce the same action $\varphi\colon B \to \mathrm{End}(X)$, let us first consider their direct product:

$$X \times X \overset{k_1 \times k_2}{\rightarrowtail} A_1 \times A_2 \overset{f_1 \times f_2}{\twoheadrightarrow} B \times B$$

and pull it back along the diagonal morphism $\Delta_B\colon B \to B \times B$ defined by $\Delta_B(b) = (b, b)$:

$$\begin{tikzcd}
X \times X \arrow[r, tail, "\langle k_1, k_2 \rangle"] \arrow[d, equal] & P \arrow[r, two heads, "\bar{f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\Delta_B"] \\
X \times X \arrow[r, tail, "k_1 \times k_2"'] & A_1 \times A_2 \arrow[r, two heads, "f_1 \times f_2"'] & B \times B.
\end{tikzcd}$$

Special Schreier extensions are stable under pullback along any morphism ([1], Proposition 7.1.4), hence $\bar{f}$ is a special Schreier extension. Moreover, it is easy to check that the corresponding action $\bar{\varphi}\colon B \to \mathrm{End}(X \times X)$ is given by

$$b \cdot_{\bar{\varphi}} (x_1, x_2) = (b \cdot_{\varphi} x_1, b \cdot_{\varphi} x_2).$$

Since X is an abelian group, its multiplication $m\colon X \times X \to X$ is a homomorphism, and it is equivariant w.r.t. the actions $\bar{\varphi}$ and φ, since

$$(b \cdot_{\varphi} x_1) \cdot (b \cdot_{\varphi} x_2) = b \cdot_{\bar{\varphi}} (x_1, x_2).$$

We can then take the push forward of $\bar{f}$ along m:

$$\begin{tikzcd}
X \times X \arrow[r, tail, "\langle k_1, k_2 \rangle"] \arrow[d, "m"'] & P \arrow[r, two heads, "\bar{f}"] \arrow[d, "c"] & B \arrow[d, equal] \\
X \arrow[r, tail, "k'"'] & C \arrow[r, two heads, "f'"'] & B,
\end{tikzcd}$$

thus obtaining a special Schreier extension f' which induces the same action φ. We now show that such an extension is the same that we would obtain by taking the Baer sum of f_1 and f_2 defined by means of factor sets.

Let us choose two sections s_1 and s_2 of f_1 and f_2, respectively, with the property that $s_i(1) = 1$. The corresponding factor sets are then given by

$$g_1(b, b') = q_1(s_1(b) \cdot s_1(b'), s_1(b \cdot b')),$$

$$g_2(b, b') = q_2(s_2(b) \cdot s_2(b'), s_2(b \cdot b')),$$

where q_1 and q_2 are the Schreier maps associated with f_1 and f_2. We observe that the pullback P is the set

$$P = \{\ (a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2)\ \}.$$

The monoid C is then a quotient of the semidirect product $X \rtimes P$. We can then consider the section s' of f' defined by

$$s'(b) = [(1, s_1(b), s_2(b))].$$

The corresponding factor set is

$$g'(b, b') = q'(s'(b) \cdot s'(b'), s'(b \cdot b')),$$

where q' is the Schreier retraction associated with f'. We want to prove that

$$g'(b, b') = g_1(b, b') \cdot g_2(b, b').$$

Thanks to the properties of q', it suffices to prove that the element $g_1(b, b') \cdot g_2(b, b')$ of X is such that

$$k'(g_1(b, b') \cdot g_2(b, b')) \cdot [(1, s_1(b \cdot b'), s_2(b \cdot b'))] = [(1, s_1(b) \cdot s_1(b'), s_2(b) \cdot s_2(b'))].$$

But

$$k'(g_1(b, b') \cdot g_2(b, b')) = [(g_1(b, b') \cdot g_2(b, b'), 1, 1)],$$

so we have to show that

$$[(g_1(b, b') \cdot g_2(b, b'), 1, 1)] \cdot [(1, s_1(b \cdot b'), s_2(b \cdot b'))] = [(1, s_1(b) \cdot s_1(b'), s_2(b) \cdot s_2(b'))]$$

or, in other terms,

$$[(g_1(b, b') \cdot g_2(b, b'), s_1(b \cdot b'), s_2(b \cdot b'))] = [(1, s_1(b) \cdot s_1(b'), s_2(b) \cdot s_2(b'))].$$

The two equivalence classes coincide if and only if there is a pair $(x_1, x_2) \in X \times X$ such that

$$(1, s_1(b) \cdot s_1(b'), s_2(b) \cdot s_2(b')) = h(x_1, x_2) \cdot (g_1(b, b') \cdot g_2(b, b'), s_1(b \cdot b'), s_2(b \cdot b')),$$

where $h \colon X \times X \to X \rtimes P$ is the monomorphism given by

$$h(x_1, x_2) = ((x_1 \cdot x_2)^{-1}, k_1(x_1), k_2(x_2)).$$

If we choose $x_i = g_i(b, b')$, we get:

$$h(x_1, x_2) \cdot (g_1(b, b') \cdot g_2(b, b'), s_1(b \cdot b'), s_2(b \cdot b')) =$$

$$= ((g_1(b,b') \cdot g_2(b,b'))^{-1}, k_1 g_1(b,b'), k_2 g_2(b,b')) \cdot (g_1(b,b') \cdot g_2(b,b'), s_1(b \cdot b'), s_2(b \cdot b')).$$

Since the elements of P of the form $(k_1(x_1), k_2(x_2))$ act trivially on X (because $\bar{f}(k_1(x_1), k_2(x_2)) = 1$), the last expression is equal to

$$((g_1(b,b') \cdot g_2(b,b'))^{-1} \cdot g_1(b,b') \cdot g_2(b,b'), k_1 g_1(b,b') \cdot s_1(b,b'), k_2 g_2(b,b') \cdot s_2(b,b')) =$$

$$= (1, s_1(b) \cdot s_1(b'), s_2(b) \cdot s_2(b'))$$

and the proof is completed.

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