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# DETERMINANTAL INEQUALITIES FOR $J$-ACCRETIVE DISSIPATIVE MATRICES

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Abstract: In this note we determine bounds for the determinant of the sum of two $J$-accretive dissipative matrices with prescribed spectra.

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## 1. Results

Consider the complex $n$-dimensional space $\mathbb{C}^n$ endowed with the indefinite inner product

$$[x, y]_J = y^* J x, \quad x, y \in \mathbb{C}^n,$$

where $J = I_r \oplus -I_{n-r}$, and corresponding $J$-norm

$$[x, x]_J = |x_1|^2 + \cdots + |x_r|^2 - |x_{r+1}|^2 - \cdots - |x_n|^2.$$

In the sequel we shall assume that $0 < r < n$, except where otherwise stated. The $J$-*adjoint* of $A \in \mathbb{C}^{n\times n}$ is defined and denoted as

$$[A^{\#}x, x] = [x, Ax]$$

or, equivalently, $A^{\#} := JA^*J$. The matrix $A$ is said to be $J$-*Hermitian* if $A^{\#} = A$, and is $J$-*positive definite* (*semi-definite*) if $JA$ is positive definite (semi-definite). This kind of matrices appears on Quantum Physics and in Symplectic Geometry [15] and deserved some investigation (e.g. see [3] and [16] and references therein). An arbitrary matrix $A \in \mathbb{C}^{n\times n}$ may be uniquely written in the form

$$A = \mathrm{Re}^J A + i\mathrm{Im}^J A,$$

where

$$\mathrm{Re}^J A = (A + A^{\#})/2, \ \mathrm{Im}^J A = (A - A^{\#})/(2i)$$

are $J$-Hermitian. This is the so-called $J$-Cartesian decomposition of $A$. $J$-Hermitian matrices share properties with Hermitian matrices, but they also

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have important differences. For instance, they have real and complex eigenvalues, these occurring in conjugate pairs. Nevertheless, the eigenvalues of a $J$-positive matrix are all real, being $r$ positive and $n-r$ negative, according to the $J$-norm of the associated eigenvectors being positive or negative. A matrix $A$ is said to be $J$-*accretive* (resp. $J$-*dissipative*) if $J\mathrm{Re}^J A$ (resp. $J\mathrm{Im}^J A$) is positive definite. If both matrices $J\mathrm{Re}^J A$ and $J\mathrm{Im}^J A$ are positive definite the matrix is said to be $J$-*accretive dissipative.*

We are interested in obtaining determinantal inequalities for $J$-accretive dissipative matrices. There have been published some papers extending determinantal inequalities for positive semi-definite matrices to the setting of accretive-dissipative matrices, or, more generally, to sector matrices. The reader may consult [7, 10, 11, 13, 17] on this topic. Our main goal is to present some related new results. Some of the used techniques in this note may be applied to the problems treated in the above mentioned papers.

Throughout, we shall be concerned with the set

$$D^J(A,C)=\{\det(A+VCV^{\#}) : V\in \mathrm{U}(r,n-r)\},$$

where $A, C\in\mathbb{C}^{n\times n}$ are $J$-unitarily diagonalizable with prescribed eigenvalues and $\mathrm{U}(r,n-r)$ is the group of $J$-unitary transformations in $\mathbb{C}^n$ ($V$ is $J$-*unitary* if $VV^{\#}=I$). The so-called $J$-unitary group is connected, nevertheless it is not compact. As a consequence, $D^J(A,C)$ is connected. This set is invariant under the transformation $C\to UCU^{\#}$ for every $J$-unitary matrix $U$, and, for short, $D^J(A,C)$ is said to be $J$-unitarily invariant.

In the sequel we use the following notation. By $S_n$ we denote the symmetric group of degree $n$, and we shall also consider

$$S_n^r=\{\sigma\in S_n : \sigma(j)=j,\ j=r+1,\dots,n\}, \tag{1.1}$$

$$\hat{S}_n^r=\{\sigma\in S_n : \sigma(j)=j,\ j=1,\dots,r\}. \tag{1.2}$$

Let $\alpha_j,\ \gamma_j\in\mathbb{C}$, $j=1,\dots,n$ denote the eigenvalues of $A$ and $C$, respectively. The $r!(n-r)!$ points

$$z_\sigma=z_{\xi\tau}=\prod_{j=1}^{r}(\alpha_j+\gamma_{\xi(j)})\prod_{j=r+1}^{n}(\alpha_j+\gamma_{\tau(j)}),\ \xi\in S_n^r,\ \tau\in\hat{S}_n^r. \tag{1.3}$$

belong to $D^J(A,C)$.

The purpose of this note, which is in the continuation of [1], is to establish the following results.

**Theorem 1.1.** *Let $J = I_r \oplus -I_{n-r}$, and $A$ and $C$ be $J$- positive matrices with prescribed eigenvalues*

$$\alpha_1 \geq \ldots \geq \alpha_r > 0 > \alpha_{r+1} \geq \ldots \geq \alpha_n \tag{1.4}$$

*and*

$$\gamma_1 \geq \ldots \geq \gamma_r > 0 > \gamma_{r+1} \geq \ldots \geq \gamma_n, \tag{1.5}$$

*respectively. Then*

$$|\det(A + iC)| \geq |\alpha_1 + i\gamma_1| \ldots |\alpha_n + i\gamma_n|.$$

**Remark 1.2.** For $A$, $C$, $J$-positive matrices, let $\beta_j$ and $\lambda_j$, ordered so that $\beta_1 \geq \ldots \geq \beta_n$, $\lambda_1 \geq \ldots \geq \lambda_n$, be the eigenvalues of $JA$ and $JC$, respectively. Then, (cf. [4, pp.182-183]),

$$\prod_{j=1}^{n} |\beta_j + i\lambda_j| \leq |\det(A + iC)| \leq \prod_{j=1}^{n} |\beta_j + i\lambda_{n-j+1}|.$$

**Corollary 1.3.** *Let $J = I_r \oplus -I_{n-r}$, and $B$ be a $J$-accretive dissipative matrix. Assume that the eigenvalues of* $\mathrm{Re}^J B$ *and* $\mathrm{Im}^J B$ *satisfy (1.4) and (1.5), respectively. Then,*

$$|\det(B)| \geq |\alpha_1 + i\gamma_1| \ldots |\alpha_n + i\gamma_n|.$$

**Remark 1.4.** The lower bounds for the determinant obtained in [4] and the new one presented in Theorem 1.1 should be compared. Indeed, let

$$J_3 = \mathrm{diag}(1, 1, -1), \quad A_3 = \mathrm{diag}(3, 1, -2),$$
$$C_3 = \frac{1}{4} \begin{bmatrix} 13 & -\sqrt{3} & 6\sqrt{2} \\ -\sqrt{3} & 15 & -6\sqrt{6} \\ -6\sqrt{2} & 6\sqrt{6} & -20 \end{bmatrix}.$$

The eigenvalues of $C_3$ are $3,\ 1,\ -2$, while the eigenvalues of $J_3C_3$ are $(9 + \sqrt{73})/2,\ 3,\ (9 - \sqrt{73})/2$. Thus, $10 < |\det(A_3 + iC_3)|$ according to the bound in [4] and $6\sqrt{2} < |\det(A_3 + iC_3)|$ according to Theorem 1.1. So, for $n \geq 3$, the new bound may be better.

**Example 1.5.** In order to illustrate the necessity of $A$ and $C$ to be $J$-positive matrices in Theorem 1.1, let $A = \mathrm{diag}(\alpha_1, \alpha_2)$, $C = \mathrm{diag}(\gamma_1, \gamma_2)$, with $\alpha_1 = \gamma_1 = 1$, $\alpha_2 = 3/2$, $\gamma_2 = -2$, and $J = \mathrm{diag}(1, -1)$. We find $(\alpha_1^2 + \gamma_1^2)(\alpha_2^2 + \gamma_2^2) = 27/2$. However, the minimum of $|\det(A + iVBV^{\#}|^2$, for $V$ ranging over the $J$-unitary group , is $49/4$. So, if one of $A, C$ is not $J$-positive, Theorem 1.1 does not hold.

**Theorem 1.6.** *Let $J = I_r \oplus -I_{n-r}$, and $A$ and $C$ be $J$-unitary matrices with prescribed eigenvalues*

$$\alpha_1, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_n$$

*and*

$$\gamma_1 \dots, \gamma_r, \gamma_{r+1}, \dots . \gamma_n,$$

*respectively. Assume moreover that*

$$\frac{\Im\alpha_1}{2(1+\Re\alpha_1)} \leq \dots \leq \frac{\Im\alpha_r}{2(1+\Re\alpha_r)} < 0 < \frac{\Im\alpha_{r+1}}{2(1+\Re\alpha_{r+1})} \leq \dots \leq \frac{\Im\alpha_n}{2(1+\Re\alpha_n)} \tag{1.6}$$

*and*

$$\frac{\Im\gamma_1}{2(1-\Re\gamma_1)} \leq \dots \leq \frac{\Im\gamma_r}{2(1-\Re\gamma_r)} < 0 < \frac{\Im\gamma_{r+1}}{2(1-\Re\gamma_{r+1})} \leq \dots \leq \frac{\Im\gamma_n}{2(1-\Re\gamma_n)}. \tag{1.7}$$

*Then*

$$D^J(A, C) = \{(\alpha_1 + \gamma_1) \dots (\alpha_n + \gamma_n)t : t \geq 1\}.$$

We shall present the proofs of the above results in the next section.

## 2. Proofs

**Lemma 2.1.** *Let $g : \mathrm{U}(r, n-r) \to \mathbb{R}$ be the real valued function defined by*

$$g(U) = \det(I + A_0^{-1}UC_0JU^*JA_0^{-1}UC_0JU^*J),$$

*where $A_0 = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$, $C_0 = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$ and $\alpha_i, \gamma_j$ satisfy (1.4) and (1.5). Then the set*

$$\{U \in \mathrm{U}(r, n-r) : g(U) \leq a\},$$

*where*

$$a > \prod_{j=1}^{n} \left(1 + \frac{\gamma_j^2}{\alpha_j^2}\right),$$

*is compact.*

*Proof*: Notice that $JA_0 > 0, \ JC_0 > 0$, so we may write

$$g(U) = \det(I + WW^*WW^*),$$

where

$$W = (JA_0)^{-1/2}U(JC_0)^{1/2}.$$

The condition $g(U) \leq a$ implies that $W$ is bounded, and is satisfied if we require that $WW^* \leq \kappa I$, for $\kappa > 0$ such that $(1+\kappa^2)^n \leq a$. Thus, also $U$ is bounded. The result follows by Heine-Borel Theorem. ∎

**Proof of Theorem 1.1.**

Under the hypothesis, $A$ is nonsingular. Since the determinant is $J$-unitarily invariant and $C$ is $J$-unitarily diagonalizable, we may consider

$$C = \operatorname{diag}(\gamma_1, \ldots, \gamma_n).$$

We observe that

$$\begin{aligned} |\det(A+iC)|^2 &= \det\left((A+iC)(A-iC)\right) \\ &= \left(\prod_{i=1}^n \alpha_i\right)^2 \det\big((I+iA^{-1}C)(I-iA^{-1}C)\big) \end{aligned}$$

Clearly,

$$\det\big((I+iA^{-1}C)(I-iA^{-1}C)\big) = \det(I + A^{-1}CA^{-1}C).$$

The set of values attained by $|\det(A+iC)|^2$ is an unbounded connected subset of the positive real line. In order to prove the unboundedness, let us consider the $J$-unitary matrix $V$ obtained from the identity matrix $I$ through the replacement of the entries $(r,r)$, $(r+1,r+1)$ by $\cosh\ u$, and the replacement of the entries $(r,r+1)$, $(r+1,r)$ by $\sinh\ u$, $u \in R$. We may assume that $A_0 = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$. A simple computation shows that

$$\begin{aligned} |\det(A_0 + iVCV^{\#})|^2 &= \prod_{j=1}^n (\alpha_j^2 + \gamma_j^2) \\ &-2(\alpha_r - \alpha_{r+1})(\gamma_r - \gamma_{r+1})(\alpha_{r+1}\gamma_r + \alpha_r\gamma_{r+1})(\sinh\ u)^2 \\ &+(\alpha_r - \alpha_{r+1})^2(\gamma_r - \gamma_{r+1})^2(\sinh\ u)^4. \end{aligned}$$

Thus, the set of values attained by $|\det(A_0 + iVCV^{\#})|$ is given by

$$[(\alpha_1^2+\gamma_1^2)^{1/2} \ldots (\alpha_n^2+\gamma_n^2)^{1/2}, +\infty[\ .$$

As a consequence of Lemma 2.1, the set of values attained by $|\det(A+iC)|^2$ is closed and a half-ray in the positive real line. So, there exist matrices $A, C$ such that the endpoint of the half-ray is given by $|\det(A+iC)|^2$. Let us assume that the endpoint of this half-ray is attained at $|\det(A+iC)|^2$.

We prove that $A$ commutes with $C$. Indeed, for $\epsilon \in \mathbb{R}$ and an arbitrary $J$-Hermitian $X$, let us consider the $J$-unitary matrix given as

$$\mathrm{e}^{iX} = i + i\epsilon X - \frac{\epsilon^2}{2}X^2 + \dots .$$

We obtain by some computations

$$\begin{aligned}
&f(\epsilon) := \det(I + A^{-1}\mathrm{e}^{-i\epsilon X}C\mathrm{e}^{i\epsilon X}A^{-1}\mathrm{e}^{-i\epsilon X}C\mathrm{e}^{i\epsilon X}) \\
&= \det(I + A^{-1}CA^{-1}C - i\epsilon(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C]) + \mathrm{O}(\epsilon^2) \\
&= \det(I + A^{-1}CA^{-1}C) \\
&\times \det\big(I - i\epsilon(I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C])\big) \\
&+\mathrm{O}(\epsilon^2) \\
&= \det(I + A^{-1}CA^{-1}C) \\
&\times \exp\big(-i\epsilon \mathrm{tr}((I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C]))\big) \\
&+\mathrm{O}(\epsilon^2),
\end{aligned}$$

where $[X,Y] = XY - YX$ denotes the commutator of the matrices $X$ and $Y$. The function $f(\epsilon)$ attains its minimum at $\det(I + A^{-1}CA^{-1}C)$, if

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0.$$

Then we must have

$$\mathrm{tr}\big((I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C])\big) = 0,$$

for every $J$-Hermitian $X$. That is

$$[C, (A^{-1}C(I + A^{-1}CA^{-1}C)^{-1}A^{-1} + (I + A^{-1}CA^{-1}C)^{-1}A^{-1}CA^{-1})] = 0,$$

and so, performing some computations, we find

$$\begin{aligned}
&[C, (A^{-1}C(I + A^{-1}CA^{-1}C)^{-1}A^{-1}C + (I + A^{-1}CA^{-1}C)^{-1}A^{-1}CA^{-1}C)] \\
&= 2\left[C,\ \frac{A^{-1}CA^{-1}C}{I + A^{-1}CA^{-1}C}\right] = 2\left[C,\ I - \frac{I}{I + A^{-1}CA^{-1}C}\right] \\
&= -2\left[C,\ \frac{I}{I + A^{-1}CA^{-1}C}\right] \\
&= \frac{2I}{I + (A^{-1}C)^2}\left[C,\ (A^{-1}C)^2\right]\frac{I}{I + (A^{-1}C)^2} = 0.
\end{aligned}$$

Thus

$$[C,(A^{-1}C)^2]=0.$$

Assume that $C$, which is in diagonal form, has distinct eigenvalues. Then $(A^{-1}C)^2$ is a diagonal matrix as well as $((JA)^{-1}JC)^2$. Furthermore,

$$((JC)^{1/2}(JA)^{-1}(JC)^{1/2})^2$$

is diagonal. Since $(JC)^{1/2}(JA)^{-1}(JC)^{1/2}$ is positive definite, it is also diagonal, and so are $(JA)^{-1}JC$ and $A^{-1}C$ . Henceforth, $A$ is also a diagonal matrix and commutes with $C$. (If $C$ has multiple eigenvalues we can apply a perturbative technique and use a continuity argument).

For $\sigma\in S_n$, such that $\sigma(1),\dots,\sigma(r)\le r$, we have

$$(\alpha_1^2+\gamma_{\sigma(1)}^2)\dots(\alpha_n^2+\gamma_{\sigma(n)}^2)\ge(\alpha_1^2+\gamma_1^2)\dots(\alpha_n^2+\gamma_n^2).$$

Thus, the result follows. ■

In the proof of Theorem 1.6, the following lemma is used (cf. [1, Theorem 1.1]).

**Lemma 2.2.** *Let $B,D$ be $J$-positive matrices with eigenvalues satisfying*

$$\beta_1\ge\ldots\ge\beta_r>0>\beta_{r+1}\ge\ldots>\beta_n,$$

*and*

$$\delta_1\ge\ldots\ge\delta_r>0>\delta_{r+1}\ge\ldots>\delta_n.$$

*Then*

$$D^J(B,D)=\{(\beta_1+\delta_1)\dots(\beta_n+\delta_n)\ t:t\ge 1\}.$$

**Proof of Theorem 1.6.**

Since, by hypothesis, $A,C$, are $J$-unitary matrices, considering convenient Möbius transformations, it follows that

$$B=\frac{i}{2}\frac{A-I}{A+I},\ D=-\frac{i}{2}\frac{C+I}{C-I} \tag{2.1}$$

are $J$-Hermitian matrices. Since

$$B+D=-i(A+I)^{-1}(C+A)(C-I)^{-1},$$

we obtain

$$\det(B+D)=i^n\frac{\det(A+C)}{\prod_{j=1}^n(1+\alpha_j)(1-\gamma_j)}.$$

Assume that the eigenvalues of $B$ and $D$ are

$$\sigma(B) = \{\beta_1, \ldots, \beta_n\}, \ \sigma(D) = \{\delta_1, \ldots, \delta_n\},$$

respectively. From (2.1) we get,

$$\beta_j = -\frac{\Im\alpha_j}{2(1+\Re\alpha_j)}, \ \delta_j = -\frac{\Im\gamma_j}{2(1-\Re\gamma_j)}.$$

From (1.6) and (1.7) we conclude that

$$\beta_1 \geq \ldots \geq \beta_r > 0 > \beta_{r+1} \geq \ldots > \beta_n,$$

and

$$\delta_1 \geq \ldots \geq \delta_r > 0 > \delta_{r+1} \geq \ldots > \delta_n,$$

so that the matrices $B$ and $D$ are $J$-positive. From Lemma 2.2 it follows that

$$D^J(B, D) = \{(\beta_1 + \delta_1) \ldots (\beta_n + \delta_n)t : t \geq 1\}.$$

Thus, $D^J(A, C)$ is a half-line with endpoint at

$$(\alpha_1 + \gamma_1) \ldots (\alpha_n + \gamma_n),$$

or, more precisely,

$$D^J(A, C) = \{(\alpha_1 + \gamma_1) \ldots (\alpha_n + \gamma_n)\ t : t \geq 1\}. \quad \blacksquare$$

As a final comment, we recall that, related with the problems here considered, there is a longstanding conjecture of Marcus-de Oliveira [14] on the determinant of the sum of normal matrices with prescribed spectra, which still remains open (cf. [2, 6, 8, 12]).

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