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PSEUDO-KAN EXTENSIONS AND DESCENT THEORY

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Abstract: There are two main constructions in classical descent theory: the category of algebras and the descent category, which are known to be examples of weighted bilimits. We give a formal approach to descent theory, employing formal consequences of commuting properties of bilimits to prove classical and new theorems in the context of Janelidze-Tholen "Facets of Descent II", such as Bénabou-Roubaud Theorems, a Galois Theorem, embedding results and formal ways of getting effective descent morphisms. In order to do this, we develop the formal part of the theory on commuting bilimits via pseudomonad theory, studying idempotent pseudomonads and proving a 2-dimensional version of a well known adjoint triangle theorem. Also, we work out the concept of pointwise pseudo-Kan extension, used as a framework to talk about bilimits, commutativity and the descent object. As a subproduct, this formal approach can be an alternative perspective/ guiding template for the development of higher descent theory.

Keywords: adjoint triangles, descent objects, Kan extensions, pseudomonads, (effective) descent morphism, galois theory, bilimits.
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Introduction

Descent theory is a generalization of a solution given by Grothendieck to a problem related to modules over rings [12]. There is a pseudofunctor $\mathsf{Mod} : \mathsf{Ring} \to \mathsf{CAT}$ which associates each ring $\mathcal{R}$ with the category $\mathsf{Mod}(\mathcal{R})$ of right $\mathcal{R}$-modules. The original problem of descent is the following: given a morphism $f : \mathcal{R} \to \mathcal{S}$ of rings, we wish to understand what is the image of $\mathsf{Mod}(f) : \mathsf{Mod}(\mathcal{R}) \to \mathsf{Mod}(\mathcal{S})$. The usual approach to this problem in descent theory is somewhat indirect: firstly, we characterize the morphisms f in Ring such that $\mathsf{Mod}(f)$ is a functor that forgets some "extra structure". Then, we would get an easier problem: verifying which objects of $\mathsf{Mod}(\mathcal{S})$ could be endowed with such extra structure (see, for instance, [20]).

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Given a category $\mathbb{C}$ with pullbacks and a pseudofunctor $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathsf{CAT}$, for each morphism $p : E \to B$ of $\mathbb{C}$, the *descent data* plays the role of such "extra structure" in the basic problem (see [18, 19, 34]). More precisely, in this context, there is a natural construction of a category $\mathcal{D}esc_{\mathcal{A}}(p)$, called descent category, such that the objects of $\mathcal{D}esc_{\mathcal{A}}(p)$ are objects of $\mathcal{A}(E)$ endowed with descent data, which encompasse the 2-dimensional analogue for equality/ 1-dimensional descent: one invertible 2-cell plus coherence. This construction comes with a comparison functor and a factorization; that is to say, we have the commutative diagram below, in which $\mathcal{D}esc_{\mathcal{A}}(p) \to \mathcal{A}(E)$ is the functor which forgets the descent data (see [19]).

$$\begin{tikzcd}
\mathcal{A}(B) \arrow[r, "\phi_p"] \arrow[dr, "\mathcal{A}(p)"'] & \mathcal{D}esc_{\mathcal{A}}(p) \arrow[d] \\
& \mathcal{A}(E)
\end{tikzcd}$$

Therefore the problem is reduced to investigating whether the *comparison* functor ϕ_p is an equivalence. If it is so, p is is said to be of *effective $\mathcal{A}$-descent* and the image of $\mathcal{A}(p)$ are the objects of $\mathcal{A}(E)$ that can be endowed with descent data. Pursuing this strategy, it is also usual to study cases in which ϕ_p is fully faithful or faithful: in these cases, p is said to be, respectively, of *$\mathcal{A}$-descent* or of almost *$\mathcal{A}$-descent*.

Furthermore, we may consider that the descent problem (in dimension 2) is, in a broad context, the characterization of the image (up to isomorphism) of a given functor $F : \mathbb{C} \to \mathbb{D}$. In this case, using the strategy described above, we investigate if $\mathbb{C}$ can be viewed as a category of objects in $\mathbb{D}$ with some extra structure (plus coherence). Thereby, taking into account the original basic problem, we can ask, hence, if F is (co)monadic. Again, we would get a factorization, the Eilenberg-Moore factorization:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\phi"] \arrow[dr, "F"'] & (Co)\mathsf{Alg} \arrow[d] \\
& \mathbb{D}
\end{tikzcd}$$

And this approach leads to what is called "monadic descent theory". Bénabou and Roubaud proved that, if the functor F is induced by a pseudofunctor $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathsf{CAT}$ such that every $\mathcal{A}(p)$ has a left adjoint and $\mathcal{A}$ satisfies the Beck-Chevalley condition, then "monadic descent theory" coincides with

"Grothendieck $\mathcal{A}$-descent theory". More precisely, assuming the hypotheses above, the morphism that induces F is of effective descent if and only if F is monadic [2].

Thereby, in the core of classical descent theory, there are two constructions: the category of algebras and the descent category. These constructions are known to be examples of 2-categorical limits (see [34, 35]). Also, in a 2-categorical perspective, we can say that the general idea of category of objects with "extra structure (plus coherence)" is, indeed, captured by the notion of 2-dimensional limits.

Not contradicting such point of view, Street considered that (higher) descent theory is about the higher categorical notion of limit [34]. Following this posture, we investigate whether pure formal methods and commuting properties of bilimits are useful to prove classical and new theorems in the classical context of descent theory [18, 19, 20, 9].

Willing to give such formal approach, we employ the concept of Kan extension. However, since we only deal with bilimits (see [36]) and we need some good properties w.r.t. pointwise equivalences, we use a weaker notion: pseudo-Kan extension [28], which is stronger than the notion of lax-Kan extension, already studied by Gray in [10].

In this direction, the fundamental standpoint on "classical descent theory" of this paper is the following: the "descent object" of a cosimplicial object in a given context is the image of the initial object of the appropriate notion of Kan extension of such cosimplicial object. More precisely, in our context of dimension 2 (which is the same context of [19]), we get the following result: *The descent category of a pseudocosimplicial object* $\mathcal{A} : \Delta \to \mathsf{CAT}$ *is equivalent to* $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{A}(\mathsf{0})$, in which $\mathrm{j} : \Delta \to \dot{\Delta}$ is the full inclusion of the category of finite nonempty ordinals into the category of finite ordinals and order preserving functions, and $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{A}$ denotes the right pseudo-Kan extension of $\mathcal{A}$ along j. In particular, we develop some of the abstract features of the "classical theory of descent", including Bénabou-Roubaud theorem, as a theory (of pseudo-Kan extensions) of pseudocosimplicial objects or pseudofunctors $\dot{\Delta} \to \mathsf{CAT}$.

This work was motivated by three main aims. Firstly, to get formal proofs of classical results of descent theory. Secondly, to prove new results in the classical context – for instance, formal ways of getting sufficient conditions for a morphism to be effective descent. Thirdly, to get proofs of descent theorems that could be recovered in other contexts, such as in the development of higher descent theory (see, for instance, the work of Hermida [14] and Street [34] in this direction).

In Section 1, we give an idea of our scope: we revisit the context of [18, 19], we show the main results classically used to deal with the problem of characterization of effective descent morphisms and we present classical results that we prove in Sections 10 and 11: that is to say, the embedding results (Theorems 1.1 and 1.2) and the Bénabou-Roubaud Theorem (Theorem 1.3). At the end of this section, we establish a theorem on pseudopullbacks of categories (Theorem 1.4) which is proved in Section 11.

Section 2 explains why we do not use the usual enriched Kan extensions to study commutativity of the 2-dimensional limits related to descent theory: the main point is that we like to have results which works for bilimits in general (not only flexible ones). In Section 3, we establish our main setting: the tricategory of 2-categories, pseudofunctors and pseudonatural transformations. In this setting, we define pseudo-Kan extensions. We give further background material in Section 4, studying weighted bilimits and proving that, similarly to the enriched case, the appropriate notion of pointwise pseudo-Kan extension is actually a pseudo-Kan extension in the presence of weighted bilimits.

Section 5 is the largest one: it contains most of the abstract results needed to get our formal approach to descent theory. In Subsection 5.1, we define and study idempotent pseudomonads. Then, in Subsection 5.2, we study pseudoalgebra structures of idempotent pseudomonads, proving a Biadjoint Triangle Theorem (Theorem 5.11) and giving a result related to the study of pseudoalgebra structures in commutative squares (Corollary 5.12).

In Subsection 5.3, we deal with the technical situation of considering objects that cannot be endowed with pseudoalgebra structures but have comparison morphisms belonging to a special class of morphisms. We finish Section 5 with Subsections 5.4, 5.5 and 5.6 which apply the results of the section to the special case of weighted bilimits and pseudo-Kan extensions: we get, then, results on factorizations, commutativity of weighted bilimits/ pseudo-Kan extensions and exactness/ *(almost/ effective) descent diagrams.*

Section 6 studies *descent objects.* They are given by conical bilimits of pseudocosimplicial objects: in our context, a descent object is given by the pseudo-Kan extension of a pseudocosimplicial object (as explained above). But we finish Section 6 presenting also the strict version of a descent object, which is given by a Kan extension of a special type of 2-diagram. We get, then, the strict factorization of descent theory.

Section 7 gives elementary examples of our context of *effective descent diagrams*. Every weighted bilimit can be seen as an example, but we focus in examples that we use in applications. As mentioned above, the most important examples of bilimits in descent theory are descent objects and Eilenberg-Moore objects: thereby, Section 8 is dedicated to explain how Eilenberg-Moore objects fit in our context, via the free adjunction category of [33].

In Section 9, we study the Beck-Chevalley condition: by doctrinal adjunction [22], this is a condition to guarantee that a pointwise adjunction between pseudoalgebras can be, actually, extended to an adjunction between such pseudoalgebras. We show how it is related to commutativity of weighted bilimits, giving our first version of a Bénabou-Roubaud Theorem (Theorem 9.4).

We apply our results to the usual context [18, 19] of descent theory in Section 10: we prove a general version (Theorem 10.2) of the embedding results (Theorem 1.1), we prove the Bénabou-Roubaud Theorem (Theorem 10.4) and, finally, we give a weak version of Theorem 1.4.

We finish the paper in Section 11: there, we give a stronger result on commutativity (Theorem 11.2) and we apply our results to descent theory, proving Theorem 1.4 and the Galois result of [16] (Theorem 11.6). We also apply Theorem 1.4 to get effective descent morphisms of the category of enriched categories V-Cat, provided that V satisfies some hypotheses. For instance, we apply this result to Top-Cat and Cat-Cat.

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1. Basic Problem

In the context of [15, 18, 19, 20, 26, 32, 7], the very basic problem of descent is the characterization of effective descent morphisms w.r.t. the basic fibration. Although this problem is trivial for some categories (for instance, for locally cartesian closed categories), that is not true in general. The topological case, solved by Tholen and Reiterman [32] and reformulated by Clementino and Hofmann [6, 8], is an important example of such nontrivial problems.

Below, we present some theorems classically used as a framework to deal with this basic problem. In this paper, we show that most of these theorems are consequences of a formal theorem presented in Section 2, while others are consequences of theorems about bilimits. We give such proofs in Section 10.

Firstly, the most fundamental features of descent theory are the descent category and its related factorization. Assuming that $\mathbb{C}$ is a category with pullbacks, if $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathsf{CAT}$ is a pseudofunctor, this factorization is described by Janelidze and Tholen in [19].

$$\begin{tikzcd} & \mathcal{D}esc_{\mathcal{A}}(p) \arrow[d] \\ \mathcal{A}(B) \arrow[ur, "\phi_p"] \arrow[r, "\mathcal{A}(p)"'] & \mathcal{A}(E) \end{tikzcd}$$

We show in Subsection 6.1 that the concept of Kan extension encompasses these features. In fact, the comparison functor and the factorization described above come from the unit and the triangular identity of the adjunction $[\mathrm{t}, \mathsf{CAT}] \dashv \mathcal{R}an_{\mathrm{t}}$.

Secondly, for the nontrivial problems, the usual approach to study (basic/ universal) effective/ almost descent morphisms is the embedding in well behaved categories, in which "well behaved category" means just that we know which are the effective descent morphisms of this category. For this matter, there are some theorems in [18] and [26]. That is to say, the embedding results:

Theorem 1.1 ([18]). *Let $\mathbb{C}$ and $\mathbb{D}$ be categories with pullbacks and $U : \mathbb{C} \to \mathbb{D}$ be a pullback preserving functor.*

(1) *If U is faithful, then U reflects almost descent morphisms;*
(2) *If U is fully faithful, then U reflects descent morphisms.*

Theorem 1.2 ([18])**.** *Let $\mathbb{C}$ and $\mathbb{D}$ be categories with pullbacks. If $U : \mathbb{C} \to \mathbb{D}$ is a fully faithful pullback preserving functor and $U(p)$ is of effective descent in $\mathbb{D}$, then p is of effective descent if and only if it satisfies the following property: whenever the diagram below is a pullback in $\mathbb{D}$, there is an object C in $\mathbb{C}$ such that $U(C) \cong A$.*

$$\begin{tikzcd} U(P) \arrow[r] \arrow[d] & A \arrow[d] \\ U(E) \arrow[r, "U(p)"'] & U(B) \end{tikzcd}$$

We also show in Section 10 that Theorem 1.1 is a very easy consequence of formal and commuting properties of pseudo-Kan extensions (Corollary 5.20 and Corollary 5.23), while we show in Section 11 that Theorem 1.2 is a consequence of a theorem on bilimits (Theorem 11.3) which also implies the generalized Galois Theorem of [16]. It is interesting to note that, since Theorems 1.1 and 1.2 are just formal properties, they can be applied in other contexts - for instance, for morphisms between pseudofunctors $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathsf{CAT}$ and $\mathcal{B} : \mathbb{D}^{\mathrm{op}} \to \mathsf{CAT}$, as it is explained in Section 10.

Furthermore, Bénabou-Roubaud Theorem [2, 18] is a well known result of Descent Theory. It allows us to understand some problems via monadicity, since it says that monadic descent theory is equivalent to Grothendieck $\mathcal{A}$-descent theory in suitable cases, such as the basic fibration. We demonstrate in Section 10 that it is a corollary of commutativity of bilimits.

Theorem 1.3 (Bénabou-Roubaud [2, 18])**.** *Let $\mathbb{C}$ be a category with pullbacks. If $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathsf{CAT}$ is a pseudofunctor such that, for every morphism $p : E \to B$ of $\mathbb{C}$, $\mathcal{A}(p)$ has left adjoint $\mathcal{A}(p)!$ and the invertible 2-cell induced by $\mathcal{A}$ below satisfies the Beck-Chevalley condition, then the factorization described above is pseudonaturally equivalent to the Eilenberg-Moore factorization. In other words, assuming the hypotheses above, Grothendieck $\mathcal{A}$-descent theory is equivalent to monadic descent theory.*

$$\begin{tikzcd} \mathcal{A}(B) \arrow[r, "\mathcal{A}(p)"] \arrow[d, "\mathcal{A}(p)"'] \arrow[dr, phantom, "\cong"] & \mathcal{A}(E) \arrow[d] \\ \mathcal{A}(E) \arrow[r] & \mathcal{A}(E \times_p E) \end{tikzcd}$$

1.1. Open problems. Clementino and Hofmann [7] studied the problem of characterization of effective descent morphisms for (T, V)-categories provided that V is a lattice. To deal with this problem, they used the embedding (T, V)-$\mathsf{Cat} \to (T, V)$-Grph and Theorems 1.1 and 1.2. However, for more general monoidal categories V, such inclusion is not fully faithful and the characterization of effective descent morphisms still is an open problem even for the simpler case of the category of enriched categories V-Cat.

As an application of the perspective given in this paper, we give some results about effective descent morphisms of V-Cat. They are consequences of formal results given in this paper on effective descent morphisms of categories constructed from other categories: *i.e.* limits of categories. For instance, we prove Theorem 1.4 in Section 11.

Theorem 1.4. *Let $\mathbb{B}, \mathbb{C}, \mathbb{D}$ and $\mathbb{E}$ be categories with pullbacks. Assume that the diagram*

$$\begin{tikzcd} \mathbb{B} \arrow[r, "S"] \arrow[d, "Z"'] \arrow[dr, phantom, "\cong"] & \mathbb{C} \arrow[d, "F"] \\ \mathbb{D} \arrow[r, "G"'] & \mathbb{E} \end{tikzcd}$$

is a pseudopullback such that S, G, F and Z are pullback preserving functors. If p is a morphism in $\mathbb{B}$ such that $S(p), Z(p)$ are of effective descent and $FS(p)$ is a descent morphism, then p is of effective descent.

We can apply the theorem above in some cases of categories of enriched categories: if V is a cartesian closed category satisfying suitable hypotheses, there is a full inclusion V-$\mathsf{Cat} \to \mathsf{Cat}(V)$, in which $\mathsf{Cat}(V)$ is the category of internal categories. When this happens, as a consequence of the formal theorem above, we conclude that the inclusion reflects effective descent morphisms. Since the characterization of effective descent morphisms for internal categories in this setting was already done by Le Creurer [26], we get effective descent morphisms for enriched categories (provided that V satisfies some properties).

2. Kan Extensions

Our perspective is that, in broad terms, descent theory is about reducing a problem of understanding the image of a functor to a problem of understanding the algebras of a (fully) property-like (pseudo)monad [21]. It is easier to understand these pseudoalgebras: they are just the objects that can be endowed with a unique pseudoalgebra structure (up to isomorphism), or, more appropriately, the effective descent points/ objects. More precisely, in dimension 2, we restrict herein our attention to idempotent pseudomonads: every such idempotent pseudomonad comes with a unit η which gives the comparisons $\eta_X : X \to TX$. In this case, an object X can be endowed with a pseudoalgebra structure if and only if η_X is an equivalence.

This setting is precisely sufficient to deal with the classical descent problem [18, 19]. It is known that the descent category and the category of algebras are 2-categorical limits (see, for instance, [35, 36, 17]). Thereby, our standpoint is to deal with descent theory strictly guided by bilimits results.

For the sake of this aim, we focus our study on the pseudomonads coming from a weak notion of right Kan extensions. Actually, since the concept of "right Kan extension" plays the leading role in this work, "Kan extension" means always right Kan extension, while we always make the word "left" explicit when we refer to the dual notion.

We explain below why we need to use a weak notion of Kan extension, instead of employing the fully developed theory of enriched Kan extensions: the natural place of (classical) descent theory is the tricategory of 2-categories, pseudofunctors, pseudonatural transformations and modifications, denoted by 2-CAT. Although we can construct the bilimits related to descent theory as (enriched/ strict) Kan extensions of 2-functors in the 3-category of 2-categories, 2-functors, 2-natural transformations and modifications (see [35, 36]), the necessary replacements [24] do not make computations and formal manipulations any easier.

Further, most of the transformations between 2-functors that are necessary in the development of the theory are pseudonatural. Thus, to work within the "strict world" without employing repeatedly coherence theorems (such as the general coherence result of [24]), we would need a result saying that usual Kan extensions of pseudonaturally equivalent diagrams are pseudonaturally equivalent. This is not true in most of the cases: it is easy to construct examples of pseudonaturally isomorphic diagrams such that their usual Kan extensions are not pseudonaturally equivalent. For instance, consider the 2-category $\mathfrak{A}$ below.

$$d \underset{\beta}{\overset{\alpha}{\rightrightarrows}} c$$

The 2-category $\mathfrak{A}$ has no nontrivial 2-cells. Assume that $\mathfrak{B}$ is the 2-category obtained from $\mathfrak{A}$ adding an initial object s, with inclusion $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$. Now, if $*$ is the terminal category and $\nabla 2$ is the category with two objects and one isomorphism between them (*i.e.* $\nabla 2$ is the localization of the preorder 2 w.r.t. all morphisms), then there are two 2-natural isomorphism classes of diagrams $\mathfrak{A} \to \mathsf{CAT}$ of the type below, while all such diagrams are pseudonaturally isomorphic.

$$* \rightrightarrows \nabla 2$$

And these 2-natural isomorphism classes give pseudonaturally nonequivalent Kan extensions along t. More precisely, if $\mathcal{X}, \mathcal{Y} : \mathfrak{A} \to \mathsf{CAT}$ are such that $\mathcal{X}(d) = \mathcal{Y}(d) = *$, $\mathcal{X}(c) = \mathcal{Y}(c) = \nabla 2$, $\mathcal{X}(\alpha) \neq \mathcal{X}(\beta)$ and $\mathcal{Y}(\alpha) = \mathcal{Y}(\beta)$; then $\mathcal{R}an_{\mathrm{t}}\mathcal{X}(s) = \emptyset$, while $\mathcal{R}an_{\mathrm{t}}\mathcal{Y}(s) = *$. Therefore $\mathcal{R}an_{\mathrm{t}}\mathcal{X}$ and $\mathcal{R}an_{\mathrm{t}}\mathcal{Y}$ are not pseudonaturally equivalent, while $\mathcal{X}$ is pseudonaturally isomorphic to $\mathcal{Y}$.

The usual Kan extensions behave well if we add extra hypotheses related to flexible diagrams (see [3, 4, 24, 28]). However, we do not give such restrictions and technicalities. Thereby we deal with the problems natively in the tricategory 2-CAT, without employing further coherence results. The first step is, hence, to understand the appropriate notion of Kan extension in this tricategory.

3. Pseudo-Kan Extensions

In a given tricategory, if $\mathrm{t} : a \to b$, $f : a \to c$ are 1-cells, we might consider that the right Kan extension of f along t is the right 2-reflection of f along the 2-functor $[\mathrm{t}, c] : [b, c] \to [a, c]$. That is to say, if it exists for all $f : a \to c$, the global Kan extension would be a 2-functor $\mathcal{R}an_{\mathrm{t}} : [a, c] \to [b, c]$ such that $[\mathrm{t}, c] \dashv \mathcal{R}an_j$ is a 2-adjunction. But it happens that, in very important cases, such concept is very restrictive, because it does not take into account the bicategorical structure of the hom-2-categories of the tricategory. Hence, it is possible to consider weaker notions of Kan extension, corresponding to the two other important notions of adjunction between 2-categories [10], that is to say, lax adjunction and biadjunction. For instance, Gray [10] studied the notion of lax-Kan extension.

For the reasons already explained in Section 2, we also consider a weak notion of Kan extension in our tricategory 2-CAT, that is to say, the notion of pseudo-Kan extension. In our case, the need of this concept comes from the fact that, even with many assumptions, the (usual) Kan extension of a pseudofunctor may not exist. Furthermore, we prove in Section 6 that the descent object (descent category) and the Eilenberg-Moore object (Eilenberg-Moore category) can be easily described using our language. In particular, the pseudo-Kan extension of a pseudocosimplicial object gives the descent object (descent category), which agrees with our viewpoint that the appropriate notion of Kan extension of a cosimplicial object gives the descent object for this context.

Before defining pseudo-Kan extension, we need to recall some results of bicategory theory. Most of them can be found in [36, 37]. Firstly, to fix notation, we give the definitions of pseudofunctors, pseudonatural transformations and modifications.

Henceforth, in a given 2-category, we always denote by $\cdot$ the vertical composition of 2-cells and by $*$ their horizontal composition.

Definition 3.1. [Pseudofunctor] Let $\mathfrak{A}, \mathfrak{B}$ be 2-categories. A *pseudofunctor* $\mathcal{A} : \mathfrak{A} \to \mathfrak{B}$ is a pair $(\mathcal{A}, \mathfrak{a})$ with the following data:

- Function $\mathcal{A} : \mathrm{obj}(\mathfrak{A}) \to \mathrm{obj}(\mathfrak{B})$;
- Functors $\mathcal{A}_{XY} : \mathfrak{A}(X, Y) \to \mathfrak{B}(\mathcal{A}(X), \mathcal{A}(Y))$;
- For each pair $g : X \to Y, h : Y \to Z$ of 1-cells in $\mathfrak{A}$, an invertible 2-cell in $\mathfrak{B}$: $\mathfrak{a}_{hg} : \mathcal{A}(h)\mathcal{A}(g) \Rightarrow \mathcal{A}(hg)$;
- For each object X of $\mathfrak{A}$, an invertible 2-cell $\mathfrak{a}_X : \mathrm{Id}_{\mathcal{A}X} \Rightarrow \mathcal{A}(\mathrm{Id}_X)$ in $\mathfrak{B}$;

subject to *associativity*, *identity* and *naturality* axioms [28].

If $\mathcal{A} = (\mathcal{A}, \mathfrak{a}) : \mathfrak{A} \to \mathfrak{B}$ and $(\mathcal{B}, \mathfrak{b}) : \mathfrak{B} \to \mathfrak{C}$ are pseudofunctors, we define the composition as follows: $\mathcal{B} \circ \mathcal{A} := (\mathcal{B}\mathcal{A}, (\mathfrak{b}\mathfrak{a}))$, in which $(\mathfrak{b}\mathfrak{a})_{hg} := \mathcal{B}(\mathfrak{a}_{hg}) \cdot \mathfrak{b}_{\mathcal{A}(h)\mathcal{A}(g)}$ and $(\mathfrak{b}\mathfrak{a})_X := \mathcal{B}(\mathfrak{a}_X) \cdot \mathfrak{b}_{\mathcal{A}(X)}$. This composition is associative and it has trivial identities. A pseudonatural transformation between pseudofunctors $\mathcal{A} \longrightarrow \mathcal{B}$ is a natural transformation in which the usual (natural) commutative squares are replaced by invertible 2-cells plus coherence.

Definition 3.2. [Pseudonatural transformation] If $\mathcal{A}, \mathcal{B} : \mathfrak{A} \to \mathfrak{B}$ are pseudofunctors, a *pseudonatural transformation* $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ is defined by:

- For each object X of $\mathfrak{A}$, a 1-cell $\alpha_X : \mathcal{A}(X) \to \mathcal{B}(X)$ of $\mathfrak{B}$;
- For each 1-cell $g : X \to Y$ of $\mathfrak{A}$, an invertible 2-cell $\alpha_g : \mathcal{B}(g)\alpha_X \Rightarrow \alpha_Y\mathcal{A}(g)$ of $\mathfrak{B}$;

such that axioms of associativity, identity and naturality hold [28].

Firstly, the vertical composition, denoted by $\beta\alpha$, of two pseudonatural transformations $\alpha : \mathcal{A} \Rightarrow \mathcal{B}$, $\beta : \mathcal{B} \Rightarrow \mathcal{C}$ is defined by

$$(\beta\alpha)_W := \beta_W\alpha_W$$

$$\begin{tikzcd} \mathcal{A}(W) \arrow[r, "\beta_W\alpha_W"] \arrow[d, "\mathcal{A}(f)"'] \arrow[dr, phantom, "\overset{(\beta\alpha)_f}{\Longleftarrow}"] & \mathcal{C}(W) \arrow[d, "\mathcal{C}(f)"] \\ \mathcal{A}(X) \arrow[r, "\beta_X\alpha_X"'] & \mathcal{C}(X) \end{tikzcd} \quad := \quad \begin{tikzcd} \mathcal{A}(W) \arrow[r, "\alpha_W"] \arrow[d, "\mathcal{A}(f)"] \arrow[dr, phantom, "\overset{\alpha_f}{\Leftarrow}"] & \mathcal{B}(W) \arrow[r, "\beta_W"] \arrow[d, "\mathcal{B}(f)"'] \arrow[dr, phantom, "\overset{\beta_f}{\Leftarrow}"] & \mathcal{C}(W) \arrow[d, "\mathcal{C}(f)"] \\ \mathcal{A}(X) \arrow[r, "\alpha_X"'] & \mathcal{B}(X) \arrow[r, "\beta_X"'] & \mathcal{C}(X) \end{tikzcd}$$

Secondly, let $(\mathcal{U}, \mathfrak{u}), (\mathcal{L}, \mathfrak{l}) : \mathfrak{B} \to \mathfrak{C}$ and $\mathcal{A}, \mathcal{B} : \mathfrak{A} \to \mathfrak{B}$ be pseudofunctors. If $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$, $\lambda : \mathcal{U} \longrightarrow \mathcal{L}$ are pseudonatural transformations, then the horizontal composition of $\mathcal{U}$ with α, denoted by $\mathcal{U}\alpha$, is defined by: $(\mathcal{U}\alpha)_W :=$ $\mathcal{U}(\alpha_W)$ and $(\mathcal{U}\alpha)_f := \left(\mathfrak{u}_{\alpha_X \mathcal{A}(f)}\right)^{-1} \cdot \mathcal{U}(\alpha_f) \cdot \mathfrak{u}_{\mathcal{B}(f)\alpha_W}$, while the composition $\lambda\mathcal{A}$ is defined trivially. Thereby, we get the (usual) definition of the horizontal composition,

$$(\lambda * \alpha) := (\lambda\mathcal{B})(\mathcal{U}\alpha) \cong (\mathcal{L}\alpha)(\lambda\mathcal{A})$$

Similarly, we get the three types of compositions of modifications.

Definition 3.3. [Modification] Let $\mathcal{A}, \mathcal{B} : \mathfrak{A} \to \mathfrak{B}$ be pseudofunctors. If $\alpha, \beta : \mathcal{A} \Rightarrow \mathcal{B}$ are pseudonatural transformations, a *modification* $\Gamma : \alpha \Longrightarrow \beta$ is defined by the following data:

- For each object X of $\mathfrak{A}$, a 2-cell $\Gamma_X : \alpha_X \Rightarrow \beta_X$ of $\mathfrak{B}$ satisfying one axiom of naturality [28].

It is straightforward to verify that 2-CAT is a tricategory which is locally a 2-category. In particular, we denote by $[\mathfrak{A}, \mathfrak{B}]_{PS}$ the 2-category of pseudofunctors $\mathfrak{A} \to \mathfrak{B}$, pseudonatural transformations and modifications. Also, we have the *bicategorical Yoneda lemma* [36], that is to say, the usual Yoneda embedding $\mathcal{Y} : \mathfrak{A} \to [\mathfrak{A}^{\mathrm{op}}, \mathsf{CAT}]_{PS}$ is locally an equivalence (*i.e.* it induces equivalences between the hom-categories).

A pseudofunctor $\mathcal{A} : \mathfrak{A} \to \mathsf{CAT}$ is said to be *birepresentable* if there is an object W of $\mathfrak{A}$ such that $\mathcal{A}$ is pseudonaturally equivalent to $\mathfrak{A}(W, -) : \mathfrak{A} \to$ CAT. In this case, W is called the birepresentation of $\mathcal{A}$. By the bicategorical Yoneda lemma, birepresentations are unique up to equivalence.

If $\mathcal{L} : \mathfrak{A} \to \mathfrak{B}$ is a pseudofunctor and X is an object of $\mathfrak{B}$, a right bireflection of X along $\mathcal{L}$ is, if it exists, a birepresentation of the pseudofunctor $\mathfrak{B}(\mathcal{L}-, X) : \mathfrak{A}^{\mathrm{op}} \to \mathsf{CAT}$. We say that $\mathcal{L}$ is left biadjoint to $\mathcal{U} : \mathfrak{B} \to \mathfrak{A}$ if, for every object X of $\mathfrak{B}$, $\mathcal{U}(X)$ is the right bireflection of X along $\mathcal{L}$. In this case, we say that $\mathcal{U}$ is right biadjoint to $\mathcal{L}$. This definition of biadjunction is equivalent to Definition 3.4.

Definition 3.4. Let $\mathfrak{U} : \mathfrak{B} \to \mathfrak{A}$ and $\mathcal{L} : \mathfrak{A} \to \mathfrak{B}$ be pseudofunctors between 2-categories. We say that $\mathcal{L}$ is *left biadjoint* to $\mathfrak{U}$, denoted by $\mathcal{L} \dashv \mathfrak{U}$, if there exist

(1) pseudonatural transformations $\eta : \mathrm{Id}_{\mathfrak{A}} \longrightarrow \mathfrak{U}\mathcal{L}$ and $\varepsilon : \mathcal{L}\mathfrak{U} \longrightarrow \mathrm{Id}_{\mathfrak{B}}$
(2) invertible modifications $s : \mathrm{Id}_{\mathcal{L}} \Longrightarrow (\varepsilon\mathcal{L}) \cdot (\mathcal{L}\eta)$ and $t : (\mathfrak{U}\varepsilon) \cdot (\eta\mathfrak{U}) \Longrightarrow \mathrm{Id}_{\mathfrak{U}}$

satisfying coherence equations [28].

In this case, we say that $(\mathcal{L} \dashv \mathfrak{U}, \eta, \varepsilon, s, t)$ is a biadjunction. Sometimes we omit the invertible modifications, denoting a biadjunction by $(\mathcal{L} \dashv \mathfrak{U}, \eta, \varepsilon)$.

By the (bicategorical) Yoneda lemma, if $\mathcal{L} : \mathfrak{A} \to \mathfrak{B}$ is left biadjoint, its right biadjoint is unique up to pseudonatural equivalence. Further, if $\mathcal{L}$ is left 2-adjoint, then it is left biadjoint.

Assume that $\mathfrak{A}, \mathfrak{B}$ are small 2-categories and $\mathfrak{H}$ is a 2-category. If $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ and $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}$ are pseudonfunctors, the (right) *pseudo-Kan extension* of $\mathcal{A}$ along t, denoted by $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}\mathcal{A}$, is, if it exists, a right bireflection of $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}$ along the pseudofunctor

$$[\mathrm{t}, \mathfrak{H}]_{PS} : [\mathfrak{B}, \mathfrak{H}]_{PS} \to [\mathfrak{A}, \mathfrak{H}]_{PS} .$$

A *global pseudo-Kan extension* along $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ is, hence, a right biadjoint of $[\mathrm{t}, \mathfrak{H}]_{PS}$, provided that it exists. That is to say, a pseudofunctor $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}} : [\mathfrak{A}, \mathfrak{H}]_{PS} \to [\mathfrak{B}, \mathfrak{H}]_{PS}$ such that $[\mathrm{t}, \mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}$. Of course, right pseudo-Kan extensions are unique up to pseudonatural equivalence. In this paper, we always assume that the considered global pseudo-Kan extensions exist. In Section 4, we prove that this assumption may be replaced by a stronger (but suitable) one: that is to say, a (bicategorical) completeness condition.

Herein, the expression *Kan extension* refers to the usual notion of Kan extension in CAT-enriched category theory. That is to say, if $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ and $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}$ are 2-functors, the (right) Kan extension of $\mathcal{A}$ along t, denoted by $\mathcal{R}an_{\mathrm{t}}\mathcal{A} : \mathfrak{B} \to \mathfrak{H}$, is (if it exists) the right 2-reflection of $\mathcal{A}$ along the 2-functor $[\mathrm{t}, \mathfrak{H}]$. And the global Kan extension is a right 2-adjoint of $[\mathrm{t}, \mathfrak{H}] : [\mathfrak{B}, \mathfrak{H}] \to [\mathfrak{A}, \mathfrak{H}]$, in which $[\mathfrak{B}, \mathfrak{H}]$ denotes the 2-category of 2-functors $\mathfrak{B} \to \mathfrak{H}$, CAT-natural transformations and modifications.

If $\mathcal{R}an_{\mathrm{t}}\mathcal{A}$ exists, it is not generally true that $\mathcal{R}an_{\mathrm{t}}\mathcal{A}$ is pseudonaturally equivalent to $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}\mathcal{A}$. This is a coherence problem, related to flexible diagrams [4, 24, 3] and to the construction of bilimits via strict 2-limits [35, 36]. For instance, in particular, using the results of [28], we can easily prove, as a corollary of coherence results [4, 24, 28], that, for a given pseudofunctor $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}$ and a 2-functor $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$, we can replace $\mathcal{A}$ by a pseudonaturally equivalent 2-functor $\mathcal{A}' : \mathfrak{A} \to \mathfrak{H}$ such that $\mathcal{R}an_{\mathrm{t}}\mathcal{A}'$ is equivalent to $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}\mathcal{A}' \simeq \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}\mathcal{A}$, provided that $\mathfrak{H}$ satisfies some completeness conditions (for instance, if $\mathfrak{H}$ is CAT-complete).

In Section 6 we show that the descent object of a pseudocosimplicial object $D : \Delta \to \mathsf{CAT}$ is $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}D(0)$, in which $\mathrm{j} : \Delta \to \dot{\Delta}$ is the inclusion of the category of nonempty finite ordinals into the category of finite ordinals. Observe that the Kan extension of a cosimplicial object does not give the descent object: it gives an equalizer (which is the notion of descent for dimension 1), although we might give the descent object via a Kan extension after replacing the (pseudo)cosimplicial objects by suitable strict versions of pseudocosimplicial objects as it is done at Subsection 6.1.

4. Bilimits and pseudo-Kan extensions

Similarly to Kelly's approach for (enriched) Kan extensions, we define what should be called *pointwise (right) pseudo-Kan extension*. Then, we prove that, whenever such pointwise pseudo-Kan extensions exist, they are (equivalent to) the pseudo-Kan extensions [28].

Pointwise right pseudo-Kan extensions are defined via weighted bilimits, the bicategorical analogue for (enriched) weighted limits [36, 37]. Thereby, in the first part of this section, we list some needed results on weighted bilimits.

Definition 4.1. [Weighted bilimit] Assume that $\mathfrak{A}$ is a small 2-category. Let $\mathcal{W} : \mathfrak{A} \to \mathsf{CAT}$, $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}$ be pseudofunctors, the *(weighted) bilimit* of $\mathcal{A}$ with weight $\mathcal{W}$, denoted by $\{\mathcal{W}, \mathcal{A}\}_{\mathrm{bi}}$, if it exists, is the birepresentation of the pseudofunctor

$$\mathfrak{A}^{\mathrm{op}} \to \mathsf{CAT} : \qquad X \mapsto [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathcal{W}, \mathfrak{H}(X, \mathcal{A}-))$$

That is to say, if it exists, a bilimit is an object $\{\mathcal{W}, \mathcal{A}\}_{\mathrm{bi}}$ of $\mathfrak{H}$ endowed with a pseudonatural equivalence (in X) $\mathfrak{A}(X, \{\mathcal{W}, \mathcal{A}\}_{\mathrm{bi}}) \simeq [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathcal{W}, \mathfrak{H}(X, \mathcal{A}-))$. Since, by the (bicategorical) Yoneda lemma, $\{\mathcal{W}, \mathcal{A}\}_{\mathrm{bi}}$ is unique up to equivalence, we refer to it as *the* bilimit.

Firstly, it is easy to see that CAT is bicategorically complete, that is to say, it has all (small) bilimits. More precisely, if $\mathfrak{A}$ is a small 2-category and $\mathcal{W}, \mathcal{A} : \mathfrak{A} \to \mathsf{CAT}$ are pseudofunctors, we have that $\{\mathcal{W}, \mathcal{A}\}_{\mathrm{bi}} \simeq [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathcal{W}, \mathcal{A})$. Moreover, from the bicategorical Yoneda lemma of [36], we get the strong bicategorical Yoneda lemma.

Lemma 4.2 (Yoneda Lemma). *Let $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}$ be a pseudofunctor between 2-categories. There is a pseudonatural equivalence (in X) $\{\mathfrak{A}(X,-), \mathcal{A}\}_{\mathrm{bi}} \simeq \mathcal{A}(X)$.*

There is one important notion remaining: if $\mathfrak{A}$ is a small 2-category, we can give an *ad hoc* definition of the end of a pseudofunctor $T : \mathfrak{A} \times \mathfrak{A}^{\mathrm{op}} \to \mathsf{CAT}$.

Definition 4.3. [End] Let $\mathfrak{A}$ be a small 2-category, and assume that $T : \mathfrak{A}^{\mathrm{op}} \times \mathfrak{A} \to \mathsf{CAT}$ is a pseudofunctor. We define the *(pseudo)end* of T by

$$\int_{\mathfrak{A}} T := [\mathfrak{A}, \mathrm{CAT}]_{PS}(\mathfrak{A}(-,-), T)$$

From the definition above, we get some expected results: they are all analogous to the results of the enriched context of [23]. For instance, it is important to note that, if $\mathfrak{A}$ is a small 2-category and $\mathcal{A}, \mathcal{B} : \mathfrak{A} \to \mathfrak{H}$ are pseudofunctors, we get:

Proposition 4.4. *Let $\mathfrak{A}$ be a small 2-category and $\mathcal{A}, \mathcal{B} : \mathfrak{A} \to \mathfrak{H}$ be pseudofunctors. There is a pseudonatural equivalence*

$$\int_{\mathfrak{A}} \mathfrak{H}(\mathcal{A}-, \mathcal{B}-) \simeq [\mathfrak{A}, \mathfrak{H}]_{PS}(\mathcal{A}, \mathcal{B})$$

Proof: Firstly, observe that a pseudonatural transformation

$$\alpha : \mathfrak{A}(-,-) \longrightarrow \mathfrak{H}(\mathcal{A}-, \mathcal{B}-)$$

corresponds to a collection of 1-cells $\alpha_{(W,X)} : \mathfrak{A}(W,X) \to \mathfrak{H}(\mathcal{A}(W), \mathcal{B}(X))$ and collections of invertible 2-cells

$$\begin{array}{rcl} \alpha_{(Y,f)} : \mathfrak{H}(\mathcal{A}(Y), \mathcal{B}(f))\alpha_{(Y,W)} & \cong & \alpha_{(Y,X)}\mathfrak{A}(Y,f) \\ \alpha_{(f,Y)} : \mathfrak{H}(\mathcal{A}(f), \mathcal{B}(Y))\alpha_{(X,Y)} & \cong & \alpha_{(W,Y)}\mathfrak{A}(f,Y) \end{array}$$

such that, for each object Y of $\mathfrak{A}$, $\alpha_{(Y,-)}$ and $\alpha_{(-,Y)}$ (with the invertible 2-cells above) are pseudonatural transformations. In other words, pseudonatural transformations are transformations which are pseudonatural in each variable.

By the bicategorical Yoneda lemma, we get what we want: more precisely, such a pseudonatural transformation corresponds (up to isomorphism) to a collection of 1-cells

$$\gamma_W := \alpha_{W,W}(\mathrm{Id}_W) : \mathcal{A}(W) \to \mathcal{B}(W)$$

with (coherent) invertible 2-cells $\mathcal{B}(f) \circ \gamma_W \cong \gamma_W \circ \mathcal{A}(f)$. ■

Hence, the original bicategorical Yoneda lemma may be reinterpreted: assume that $\mathcal{A} : \mathfrak{A} \to \mathsf{CAT}$ is a pseudofunctor, then we have the pseudonatural equivalence (in X):

$$\int_{\mathfrak{A}} \mathsf{CAT}(\mathfrak{A}(X,-), \mathcal{A}-) \simeq \mathcal{A}(X).$$

We also need Theorem 4.6 to prove that the "pointwise" pseudo-Kan extension is, indeed, a pseudo-Kan extension. This theorem is the bicategorical analogue to the Fubini theorem in the enriched context.

Lemma 4.5. *Let $\mathfrak{A}, \mathfrak{B}$ be small 2-categories. Assume that*

$$T : \mathfrak{A}^{\mathrm{op}} \times \mathfrak{B}^{\mathrm{op}} \times \mathfrak{B} \times \mathfrak{A} \to \mathsf{CAT}$$

is a pseudofunctor. Then we have a pseudofunctor $T^{^{\mathfrak{B}}} : \mathfrak{A}^{\mathrm{op}} \times \mathfrak{A} \to \mathsf{CAT}$ defined below.

$$T^{^{\mathfrak{B}}}(A,B) := \int_{\mathfrak{B}} T(A,X,X,B)$$

And, clearly, we have $T^{^{\mathfrak{A}}} : \mathfrak{B}^{\mathrm{op}} \times \mathfrak{B} \to \mathsf{CAT}$ such that $T^{^{\mathfrak{A}}}(X,Y)$ is the end analogously defined.

Theorem 4.6 (Fubini's Theorem)**.** *Let $\mathfrak{A}, \mathfrak{B}$ be small 2-categories. Assume that*

$$T : \mathfrak{A}^{\mathrm{op}} \times \mathfrak{B}^{\mathrm{op}} \times \mathfrak{B} \times \mathfrak{A} \to \mathsf{CAT}$$

is a pseudofunctor. Then there are pseudonatural equivalences

$$\int_{\mathfrak{A}\times\mathfrak{B}} T \simeq \int_{\mathfrak{A}} T^{\mathfrak{B}} \simeq \int_{\mathfrak{B}} T^{^{\mathfrak{A}}}.$$

For short, we denote it by

$$\int_{\mathfrak{A}\times\mathfrak{B}} T \simeq \int_{\mathfrak{A}}\int_{\mathfrak{B}} T \simeq \int_{\mathfrak{B}}\int_{\mathfrak{A}} T.$$

Before defining pointwise pseudo-Kan extension, the following, which is mainly used in Section 6, already gives a glimpse of the relation between weighted bilimits and pseudo-Kan extensions.

Theorem 4.7. *Let $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$, $\mathcal{W} : \mathfrak{A} \to \mathsf{CAT}$ be pseudofunctors. If the left pseudo-Kan extension $\mathrm{Ps}\mathcal{L}an_{\mathrm{t}}\mathcal{W}$ exists and $\mathcal{A} : \mathfrak{B} \to \mathfrak{H}$ is a pseudofunctor, then there is an equivalence*

$$\{\mathcal{W}, \mathcal{A}\circ \mathrm{t}\}_{\mathrm{bi}} \simeq \{\mathrm{Ps}\mathcal{L}an_{\mathrm{t}}\mathcal{W}, \mathcal{A}\}_{\mathrm{bi}}$$

whenever one of the weighted bilimits exists.

Proof: Let X be an object of $\mathfrak{H}$. Assuming the existence of $\{\mathcal{W}, \mathcal{A}\circ \mathrm{t}\}_{\mathrm{bi}}$,

$$\mathfrak{H}(X, \{\mathcal{W}, \mathcal{A}\circ \mathrm{t}\}_{\mathrm{bi}}) \simeq [\mathfrak{B}, \mathfrak{H}]_{PS}(\mathcal{W}, \mathfrak{H}(X, \mathcal{A}\circ\mathrm{t}-)) \simeq [\mathfrak{A}, \mathfrak{H}]_{PS}(\mathrm{Ps}\mathcal{L}an_{\mathrm{t}}\mathcal{W}, \mathfrak{H}(X, \mathcal{A}-))$$

are pseudonatural equivalences (in X). Thereby

$$\{\mathcal{W}, \mathcal{A}\circ \mathrm{t}\}_{\mathrm{bi}} \simeq \{\mathrm{Ps}\mathcal{L}an_{\mathrm{t}}\mathcal{W}, \mathcal{A}\}_{\mathrm{bi}}.$$

The proof of the converse is analogous. ■

Let $\mathfrak{H}$ be a 2-category. If we consider the full 2-subcategory $\mathfrak{H}_{\mathcal{Y}}$ of $[\mathfrak{B}^{\mathrm{op}}, \mathsf{CAT}]_{PS}$ such that the objects of $\mathfrak{H}_{\mathcal{Y}}$ are the birepresentable pseudofunctors, the Yoneda embedding $\mathcal{Y} : \mathfrak{H} \to \mathfrak{H}_{\mathcal{Y}}$ is a biequivalence: that is to say, we can choose a pseudofunctor $I : \mathfrak{H}_{\mathcal{Y}} \to \mathfrak{H}$ and pseudonatural equivalences $\mathcal{Y}I \simeq \mathrm{Id}$ and $I\mathcal{Y} \simeq \mathrm{Id}$.

Therefore if $\mathfrak{A}$ is a small 2-category and $\mathfrak{H}$ is a bicategorically complete 2-category, given a pseudofunctor $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}$, there is a pseudofunctor $\{-, \mathcal{A}\}_{\mathrm{bi}} : [\mathfrak{A}, \mathrm{CAT}]^{\mathrm{op}}_{PS} \to \mathfrak{H}$ which is unique up to pseudonatural equivalence and which gives the bilimits of $\mathcal{A}$ [35, 28]. More precisely, since we assume that $\mathfrak{H}$ has all bilimits of $\mathcal{A}$, we are assuming that the pseudofunctor $L : [\mathfrak{A}, \mathrm{CAT}]^{\mathrm{op}}_{PS} \to [\mathfrak{H}^{\mathrm{op}}, \mathrm{CAT}]_{PS}$, in which

$$L(\mathcal{W}) : \mathfrak{B}^{\mathrm{op}} \to \mathsf{CAT} : \qquad X \mapsto [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathcal{W}, \mathfrak{H}(X, \mathcal{A}-))$$

is such that $L(\mathcal{W})$ has a birepresentation for every weight $\mathcal{W} : \mathfrak{A} \to \mathsf{CAT}$. Therefore L can be seen as a pseudofunctor $L : [\mathfrak{A}, \mathsf{CAT}]^{\mathrm{op}}_{PS} \to \mathfrak{H}_{\mathcal{Y}}$. Hence we can take $\{-, \mathcal{A}\}_{\mathrm{bi}} := IL$.

Definition 4.8. [Pointwise pseudo-Kan extension] Let $\mathfrak{A}, \mathfrak{B}$ be small 2-categories. Assume that $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ is a pseudofunctor. The pointwise pseudo-Kan extension is defined by

$$\begin{aligned} \mathcal{RAN}_{\mathrm{t}}\mathcal{A} \; : \mathfrak{B} \; &\to \mathfrak{H} \\ X \; &\mapsto \{\mathfrak{B}(X, \mathrm{t}(-)), \mathcal{A}\}_{\mathrm{bi}}, \end{aligned}$$

provided that the weighted bilimit $\{\mathfrak{B}(X, \mathrm{t}(-)), \mathcal{A}\}_{\mathrm{bi}}$ exists in $\mathfrak{H}$ for every object X of $\mathfrak{B}$.

We prove below that the pointwise pseudo-Kan extension is, actually, a pseudo-Kan extension; that is to say, we have a pseudonatural equivalence

$$[\mathfrak{A}, \mathfrak{H}]_{PS}(- \circ \mathrm{t}, \mathcal{A}) \simeq [\mathfrak{B}, \mathfrak{H}]_{PS}(-, \mathcal{RAN}_{\mathrm{t}}\mathcal{A}).$$

Theorem 4.9. *Assume that* $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}, \mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ *are pseudofunctors. If the pointwise right pseudo-Kan extension* $\mathcal{R}AN_{\mathrm{t}}\mathcal{A}$ *is well defined, then* $\mathcal{R}AN_{\mathrm{t}}\mathcal{A} \simeq \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}\mathcal{A}$.

Proof: By the propositions presented in this section and by the definition of a pointwise Kan extension, we have the following pseudonatural equivalences (in S):

$$\begin{aligned}
[\mathfrak{B}, \mathfrak{H}]_{PS}(S, \mathcal{R}AN_{\mathrm{t}}\mathcal{A}) &\simeq \int_{\mathfrak{B}} \mathfrak{H}(S(b), \mathcal{R}AN_{\mathrm{t}}\mathcal{A}(b)) \\
&\simeq \int_{\mathfrak{B}} \mathfrak{H}(S(b), \{\mathfrak{B}(b, \mathrm{t}(-)), \mathcal{A}\}_{\mathrm{bi}}) \\
&\simeq \int_{\mathfrak{B}} [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathfrak{B}(b, \mathrm{t}(-)), \mathfrak{H}(S(b), \mathcal{A}-)) \\
&\simeq \int_{\mathfrak{B}} \int_{\mathfrak{A}} \mathsf{CAT}(\mathfrak{B}(b, \mathrm{t}(a)), \mathfrak{H}(S(b), \mathcal{A}(a))) \\
&\simeq \int_{\mathfrak{A}} \int_{\mathfrak{B}} \mathsf{CAT}(\mathfrak{B}(b, \mathrm{t}(a)), \mathfrak{H}(S(b), \mathcal{A}(a))) \\
&\simeq \int_{\mathcal{A}} \mathfrak{H}(S \circ \mathrm{t}(a), \mathcal{A}(a)) \\
&\simeq [\mathcal{A}, \mathfrak{H}]_{PS}(S \circ \mathrm{t}, \mathcal{A})
\end{aligned}$$

More precisely, the first, fourth, sixth and seventh pseudonatural equivalences come from the fundamental equivalence of ends, while the second and third are, respectively, the definitions of the pointwise pseudo-Kan extension and the definition of bilimit. The remaining pseudonatural equivalence follows from Fubini's theorem. ∎

Moreover, let $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ be fully faithful and $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}$ a pseudofunctor. By the (bicategorical) Yoneda lemma, if the pseudo-Kan extension $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}\mathcal{A}$ exists, it is actually a pseudoextension.

Theorem 4.10. *Let* $\mathfrak{A}, \mathfrak{B}$ *be small* 2*-categories. If* $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ *is a local equivalence (i.e. it induces equivalences between hom-categories) and there is a biadjunction* $[\mathrm{t}, \mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}$*, its counit is a pseudonatural equivalence.*

Proof: It follows from the (bicategorical) Yoneda lemma. More precisely, by Lemma 4.2, if X is an object of $\mathfrak{A}$, $\{\mathfrak{B}(\mathrm{t}(X), \mathrm{t}-), \mathcal{A}\}_{\mathrm{bi}} \simeq \{\mathfrak{A}(X, -), \mathcal{A}\}_{\mathrm{bi}} \simeq \mathcal{A}(X)$. ∎

Henceforth, for simplicity, we always assume that $\mathfrak{H}$ is a bicategorically complete 2-category, or at least $\mathfrak{H}$ has enough bilimits to construct the considered pseudo-Kan extensions as pointwise pseudo-Kan extensions.

Remark 4.11. The pointwise pseudo-Kan extension was studied originally in [28] using the Biadjoint Triangle Theorem proved therein, while the approach presented above was more similar to the usual approach of the enriched case [23].

5. Formal Results: pseudomonads

Although we focus on pseudomonads coming from pseudo-Kan extensions, recall that our broad context of descent is about understanding the image of a "(pseudo)monadic" (pseudo)functor. In our case, if $\mathfrak{H}$ is a bicategorically complete 2-category and $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ is a pseudofunctor between small 2-categories, by the results of Section 4.9 we have a biadjunction $[\mathrm{t}, \mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}$. We know that every biadjunction induces a pseudomonad [25, 28]. In this case, it induces a pseudomonad $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(- \circ \mathrm{t})$. Our interest is to study the objects of $[\mathfrak{B}, \mathfrak{H}]_{PS}$ that can be endowed with pseudoalgebra structure [28], that is to say, the image of the forgetful Eilenberg-Moore 2-functor

$$\mathsf{Ps}\text{-}\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(- \circ \mathrm{t})\text{-}\mathsf{Alg} \to [\mathfrak{B}, \mathfrak{H}]_{PS}\,.$$

Definition 5.1. [Effective Diagrams] Let $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ be a pseudofunctor between small 2-categories and $\mathfrak{H}$ be a bicategorically complete 2-category. Herein, a pseudofunctor $\mathcal{A} : \mathfrak{B} \to \mathfrak{H}$ is said to be of *effective* t*-descent* if $\mathcal{A}$ can be endowed with a $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(- \circ \mathrm{t})$-pseudoalgebra structure.

We study more closely pseudomonads coming from pseudo-Kan extensions along local equivalences, that is to say pseudomonads $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(- \circ \mathrm{t})$ in which $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ induces equivalences between the hom-categories. Actually, most of the important examples we deal with are about such pseudomonads $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(- \circ \mathrm{t})$ in which $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ is the full inclusion of a small 2-category $\mathfrak{A}$ into a small 2-category $\dot{\mathfrak{A}}$ with only one extra object a.

We show in Subsection 5.4 that these right pseudo-Kan extensions induce a special kind of pseudomonads: idempotent pseudomonads. Thereby this section is dedicated to the study of this type of pseudomonads.

5.1. Idempotent Pseudomonads. Instead of considering the broad context, since we deal only with idempotent pseudomonads, we give an elementary approach focusing on them. The main benefit of this approach is that idempotent pseudomonads have only free pseudoalgebras. Recall that a *pseudomonad* $\mathcal{T}$ on a 2-category $\mathfrak{H}$ consists of a sextuple $(\mathcal{T}, \mu, \eta, \Lambda, \rho, \Gamma)$, in which $\mathcal{T} : \mathfrak{H} \to \mathfrak{H}$ is a pseudofunctor, $\mu : \mathcal{T}^2 \longrightarrow \mathcal{T}, \eta : \mathrm{Id}_{\mathfrak{H}} \longrightarrow \mathcal{T}$ are pseudonatural transformations and

$$
\begin{tikzcd}
\mathcal{T} \arrow[r, "\eta_{\mathcal{T}}"] \arrow[dr, equal, bend right=40] \arrow[r, phantom, shift right=6, "\overset{\Lambda}{\Longleftarrow}"] & \mathcal{T}^2 \arrow[d, "\mu"] & \mathcal{T} \arrow[l, "\mathcal{T}\eta"'] \arrow[dl, equal, bend left=40] \arrow[l, phantom, shift left=6, "\overset{\rho}{\Longleftarrow}"] \\
& \mathcal{T} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{T}^3 \arrow[r, "\mathcal{T}\mu"] \arrow[d, "\mu_{\mathcal{T}}"'] \arrow[dr, phantom, "\overset{\Gamma}{\Longleftarrow}"] & \mathcal{T}^2 \arrow[d, "\mu"] \\
\mathcal{T}^2 \arrow[r, "\mu"'] & \mathcal{T}
\end{tikzcd}
$$

are invertible modifications satisfying the following coherence equations [30, 28]:

- Identity:

$$
\begin{tikzcd}
& \mathcal{T}^2 \arrow[dl, "\mathcal{T}\eta\mathcal{T}"'] \arrow[dr, "\mathcal{T}\eta\mathcal{T}"] \arrow[dd, "\mathrm{Id}_{\mathcal{T}^2}" description] & \\
\mathcal{T}^3 \arrow[dr, "\mu\mathcal{T}"'] \arrow[r, phantom, "\overset{\rho\mathcal{T}}{\Longleftarrow}"] & & \mathcal{T}^3 \arrow[dl, "\mathcal{T}\mu"] \arrow[l, phantom, "\overset{\widehat{\mathcal{T}\Lambda}}{\Longleftarrow}"] \\
& \mathcal{T}^2 \arrow[d, "\mu"] & \\
& \mathcal{T} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathcal{T}^2 \arrow[d, "\mathcal{T}\eta\mathcal{T}"] & \\
& \mathcal{T}^3 \arrow[dl, "\mu\mathcal{T}"'] \arrow[dr, "\mathcal{T}\mu"] & \\
\mathcal{T}^2 \arrow[dr, "\mu"'] \arrow[rr, phantom, "\overset{\Gamma}{\Longleftarrow}"] & & \mathcal{T}^2 \arrow[dl, "\mu"] \\
& \mathcal{T} &
\end{tikzcd}
$$

- Associativity:

$$
\begin{tikzcd}
\mathcal{T}^4 \arrow[r, "\mathcal{T}^2\mu"] \arrow[d, "\mu\mathcal{T}^2"'] \arrow[dr, "\mathcal{T}\mu\mathcal{T}"] & \mathcal{T}^3 \arrow[dr, "\mathcal{T}\mu"] \arrow[d, phantom, "\overset{\widehat{\mathcal{T}\Gamma}}{\Longleftarrow}"] & \\
\mathcal{T}^3 \arrow[dr, "\mu\mathcal{T}"'] \arrow[r, phantom, "\overset{\Gamma\mathcal{T}}{\Longleftarrow}"] & \mathcal{T}^3 \arrow[r, "\mathcal{T}\mu"] \arrow[d, "\mu\mathcal{T}"] \arrow[dr, phantom, "\overset{\Gamma}{\Longleftarrow}"] & \mathcal{T}^2 \arrow[d, "\mu"] \\
& \mathcal{T}^2 \arrow[r, "\mu"'] & \mathcal{T}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{T}^4 \arrow[r, "\mathcal{T}^2\mu"] \arrow[d, "\mu\mathcal{T}^2"'] \arrow[dr, phantom, "\overset{\mu_{\mu}^{-1}}{\Longleftarrow}"] & \mathcal{T}^3 \arrow[d, "\mu\mathcal{T}"] \arrow[dr, "\mathcal{T}\mu"] & \\
\mathcal{T}^3 \arrow[r, "\mathcal{T}\mu"] \arrow[dr, "\mu\mathcal{T}"'] & \mathcal{T}^2 \arrow[dr] \arrow[r, phantom, "\overset{\Gamma}{\Longleftarrow}"] \arrow[d, phantom, "\overset{\Gamma}{\Longleftarrow}"] & \mathcal{T}^2 \arrow[d, "\mu"] \\
& \mathcal{T}^2 \arrow[r, "\mu"'] & \mathcal{T}
\end{tikzcd}
$$

in which

$$\widehat{\mathcal{T}\Lambda} := \left(\mathsf{t}_{\mathcal{T}}\right)^{-1} (\mathcal{T}\Lambda) \left(\mathsf{t}_{(\mu)(\eta\mathcal{T})}\right) \qquad \widehat{\mathcal{T}\Gamma} := \left(\mathsf{t}_{(\mu)(\mu\mathcal{T})}\right)^{-1} (\mathcal{T}\Gamma) \left(\mathsf{t}_{(\mu)(\mathcal{T}\mu)}\right)$$

Definition 5.2. [Idempotent pseudomonad] A pseudomonad $(\mathcal{T}, \mu, \eta, \Lambda, \rho, \Gamma)$ is said to be *idempotent* if there is an invertible modification $\eta\mathcal{T} \cong \mathcal{T}\eta$.

Similarly to 1-dimensional monad theory, the name *idempotent pseudomonad* is justified by Lemma 5.3, which says that multiplications of idempotent pseudomonads are pseudonatural equivalences.

Lemma 5.3. *A pseudomonad $(\mathcal{T}, \mu, \eta, \Lambda, \rho, \Gamma)$ is idempotent if and only if its multiplication μ is a pseudonatural equivalence. Furthermore, if it is so, $\eta\mathcal{T}$ is a pseudonatural equivalence inverse of μ.*

Proof: Since $\mu(\eta\mathcal{T}) \cong \mathrm{Id}_{\mathcal{T}} \cong \mu(\mathcal{T}\eta)$, it is obvious that, if μ is a pseudonatural equivalence, then $\eta\mathcal{T} \cong \mathcal{T}\eta$. Therefore $\mathcal{T}$ is idempotent and $\eta\mathcal{T}$ is an equivalence inverse of μ.

Reciprocally, assume that $\mathcal{T}$ is idempotent. By the definition of pseudomonads, there is an invertible modification $\mu(\eta\mathcal{T}) \cong \mathrm{Id}_{\mathcal{T}}$. And, since $\eta\mathcal{T} \cong \mathcal{T}\eta$, we get the invertible modifications

$$(\eta\mathcal{T})\mu \cong (\mathcal{T}\mu)(\eta\mathcal{T}^2) \cong (\mathcal{T}\mu)(\mathcal{T}\eta\mathcal{T}) \cong \mathcal{T}(\mu(\eta\mathcal{T})) \cong \mathrm{Id}_{\mathcal{T}^2}$$

which prove that μ is a pseudonatural equivalence and $\eta\mathcal{T}$ is a pseudonatural equivalence inverse. ■

The reader familiar to lax-idempotent/ KZ-pseudomonads will notice that an idempotent pseudomonad is just a KZ-pseudomonad whose adjunction $\mu \dashv \eta\mathcal{T}$ is actually an adjoint equivalence. Hence, idempotent pseudomonads are fully property-like pseudomonads [21].

Every biadjunction induces a pseudomonad [25, 28]. In fact, we get the multiplication μ from the counit, and the invertible modifications Λ, ρ, Γ come from the invertible modifications of Definition 3.4. Of course, a biadjunction $\mathcal{L} \dashv \mathcal{U}$ induces an idempotent pseudomonad if and only if its unit η is such that $\eta\mathcal{U}\mathcal{L} \cong \mathcal{U}\mathcal{L}\eta$.

As a consequence of this characterization, we have Lemma 5.4 which is necessary to give the Eilenberg-Moore factorization for idempotent pseudomonads.

Lemma 5.4. *If a biadjunction $(\mathcal{L} \dashv \mathcal{U}, \eta, \varepsilon)$ induces an idempotent pseudomonad, then $\eta\mathcal{U} : \mathcal{U} \longrightarrow \mathcal{U}\mathcal{L}\mathcal{U}$ is a pseudonatural equivalence.*

Proof: By the triangular invertible modifications of Definition 3.4, if ε is the counit of the biadjunction $\mathcal{L} \dashv \mathcal{U}$, $(\mathcal{U}\varepsilon)(\eta\mathcal{U}) \cong \mathrm{Id}_{\mathcal{U}}$. Also, since $\mathcal{U}\mathcal{L}\eta \cong \eta\mathcal{U}\mathcal{L}$, we have the following invertible modifications

$$(\eta\mathcal{U})\cdot(\mathcal{U}\varepsilon) \cong (\mathcal{U}\mathcal{L}\mathcal{U}\varepsilon)(\eta\mathcal{U}\mathcal{L}\mathcal{U}) \cong (\mathcal{U}\mathcal{L}\mathcal{U}\varepsilon)(\mathcal{U}\mathcal{L}\eta\mathcal{U}) \cong \mathcal{U}\mathcal{L}(\mathrm{Id}_{\mathcal{U}}) \cong \mathrm{Id}_{\mathcal{U}\mathcal{L}\mathcal{U}}$$

Therefore $\eta\mathcal{U}$ is a pseudonatural equivalence. ■

As mentioned in the beginning of this subsection, besides being fully property-like, one of the main benefits of restricting our attention to idempotent pseudomonads comes from the fact that all their pseudoalgebras are free. In particular, we can avoid the coherence equations [30, 25, 28] used to define the 2-category of pseudoalgebras of a pseudomonad $\mathcal{T}$ when assuming that $\mathcal{T}$ is idempotent.

Definition 5.5. [Pseudoalgebras] Let $(\mathcal{T}, \mu, \eta, \Lambda, \rho, \Gamma)$ be an idempotent pseudomonad on a 2-category $\mathfrak{H}$. We define the 2-category of $\mathcal{T}$-*pseudoalgebras* $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$ as following:

- Objects: the objects of $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$ are the objects X of $\mathfrak{H}$ such that
$$\eta_X : X \to \mathcal{T}(X)$$
is an equivalence;
- The inclusion $\mathrm{obj}(\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}) \to \mathrm{obj}(\mathfrak{H})$ extends to a full inclusion 2-functor
$$\mathcal{J} : \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \to \mathfrak{H}$$

In other words, the inclusion $\mathcal{J} : \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \to \mathfrak{H}$ is defined to be final among the full inclusions $\widehat{\mathcal{J}} : \mathfrak{A} \to \mathfrak{H}$ such that $\eta\widehat{\mathcal{J}}$ is a pseudonatural equivalence.

If $\eta_X : X \to \mathcal{T}(X)$ is an equivalence, then we say that X can be endowed with a pseudoalgebra structure and the left adjoint $a : \mathcal{T}(X) \to X$ to $\eta_X : X \to \mathcal{T}(X)$ is called a pseudoalgebra structure to X. Because we could describe $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$ by means of pseudoalgebras/ pseudoalgebra structures, we often denote the objects of $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$ by small letters a, b.

Theorem 5.6 (Eilenberg-Moore biadjunction)**.** *Let $(\mathcal{T}, \mu, \eta, \Lambda, \rho, \Gamma)$ be an idempotent pseudomonad on a 2-category $\mathfrak{H}$. There is a unique pseudofunctor $\mathcal{L}^{\mathcal{T}}$ such that*

$$\begin{tikzcd} \mathfrak{H} \arrow[rr, "\mathcal{T}"] \arrow[dr, "\mathcal{L}^{\mathcal{T}}"'] & & \mathfrak{H} \\ & \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \arrow[ur, "\mathcal{J}"'] & \end{tikzcd}$$

is a commutative diagram. Furthermore, $\mathcal{L}^{\mathcal{T}}$ is left biadjoint to $\mathcal{J}$.

Proof: Firstly, we define $\mathcal{L}^{\mathcal{T}}(X) := \mathcal{T}(X)$. On one hand, it is well defined, since, by Lemma 5.3,

$$\eta\mathcal{T} : \mathcal{T} \longrightarrow \mathcal{T}^2$$

is a pseudonatural equivalence. On the other hand, the uniqueness of $\mathcal{L}^{\mathcal{T}}$ is a consequence of the fact that $\mathcal{I}$ is a monomorphism.

Now, it remains to show that $\mathcal{L}^{\mathcal{T}}$ is left biadjoint to $\mathcal{I}$. By abuse of language, if a is an object of $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$, we denote by a its pseudoalgebra structure (of Definition 5.5). Then we define the equivalences inverses below

$$\begin{array}{rlrl} \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}(\mathcal{T}(X), b) \to \mathfrak{H}(X, \mathcal{I}(b)) & & \mathfrak{H}(X, \mathcal{I}(b)) \to \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}(\mathcal{T}(X), b) & \\ f \mapsto f\eta_X & & g \mapsto bT(g) & \\ \alpha \mapsto \alpha * \mathrm{Id}_{\eta_X} & & \beta \mapsto \mathrm{Id}_b * T(\beta) & \end{array}$$

It completes the proof that $\mathcal{L}^{\mathcal{T}} \dashv \mathcal{I}$. ■

Theorem 5.7 shows that this biadjunction $\mathcal{L}^{\mathcal{T}} \dashv \mathcal{I}$ satisfies the expected universal property [25] of the 2-category of pseudoalgebras, which is the Eilenberg-Moore factorization. In other words, we prove that our definition of $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$ for idempotent pseudomonads $\mathcal{T}$ agrees with the usual definition [27, 25, 30, 36] of pseudoalgebras for a pseudomonad.

Theorem 5.7 (Eilenberg-Moore). *Let $\mathcal{L} : \mathfrak{A} \to \mathfrak{B}$ be a pseudofunctor. If $\mathcal{L} \dashv \mathcal{U}$ is a biadjunction which induces an idempotent pseudomonad $(\mathcal{T}, \mu, \eta, \Lambda, \rho, \Gamma)$, then we have a unique comparison pseudofunctor $\mathcal{K} : \mathfrak{B} \to \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$ such that*

$$\begin{tikzcd} \mathfrak{B} \arrow[r, "\mathcal{K}"] \arrow[dr, "\mathcal{U}"'] & \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \arrow[d, "\mathcal{I}"] \\ & \mathfrak{A} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{A} \arrow[r, "\mathcal{L}^{\mathcal{T}}"] \arrow[dr, "\mathcal{L}"'] & \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \\ & \mathfrak{B} \arrow[u, "\mathcal{K}"'] \end{tikzcd}$$

commute.

Proof: It is enough to define $\mathcal{K}(X) = \mathcal{U}(X)$ and $\mathcal{K}(f) = \mathcal{U}(f)$. This is well defined, since, by Lemma 5.4, $\eta\mathcal{U} : \mathcal{U} \longrightarrow \mathcal{T}\mathcal{U}$ is a pseudonatural equivalence. ■

Actually, in 2-CAT, every biadjunction $\mathcal{L} \dashv \mathcal{U}$ induces a comparison pseudofunctor and an Eilenberg-Moore factorization [27] as above, in which $\mathcal{T} = \mathcal{U}\mathcal{L}$ denotes the induced pseudomonad. When the comparison pseudofunctor $\mathcal{K}$ is a biequivalence, we say that $\mathcal{U}$ is pseudomonadic. Although there is the

Beck's theorem for pseudomonads [27, 14, 28], the situation is simpler in the setting of idempotent pseudomonads.

Theorem 5.8. *Let $\mathfrak{L} : \mathfrak{A} \to \mathfrak{B}$ be a pseudofunctor and $\mathfrak{L} \dashv \mathfrak{U}$ be a biadjunction. The pseudofunctor $\mathfrak{U}$ is a local equivalence if and only if $\mathfrak{U}$ is pseudomonadic and the induced pseudomonad $(\mathcal{T}, \mu, \eta, \Lambda, \rho, \Gamma)$ is idempotent. Or, equivalently, the counit of $\mathfrak{L} \dashv \mathfrak{U}$ is a pseudonatural equivalence if and only if $\mathfrak{U}$ is pseudomonadic and the induced pseudomonad is idempotent.*

Proof: Firstly, if the counit ε of the biadjunction of $\mathfrak{L} \dashv \mathfrak{U}$ is a pseudonatural equivalence, then $\mu := \mathfrak{U}\varepsilon\mathfrak{L}$ is a pseudonatural equivalence as well. And, thereby, the induced pseudomonad is idempotent. Now, if $a : \mathcal{T}(X) \to X$ is a pseudoalgebra structure to X, we have that

$$\mathcal{K}(\mathfrak{L}(X)) = \mathcal{T}(X) \xrightarrow[\simeq]{a} X.$$

Thereby $\mathfrak{U}$ is pseudomonadic.

Reciprocally, if $\mathfrak{L} \dashv \mathfrak{U}$ induces an idempotent pseudomonad and $\mathfrak{U}$ is pseudomonadic, then we have that $\mathcal{J} \circ \mathcal{K} = \mathfrak{U}$, $\mathcal{K}$ is a biequivalence and $\mathcal{J}$ is a local equivalence. Thereby $\mathfrak{U}$ is a local equivalence and ε is a pseudonatural equivalence. ■

In descent theory, one needs conditions to decide if a given object can be endowed with a pseudoalgebra structure. Idempotent pseudomonads provide the following simplification.

Theorem 5.9. *Let $\mathcal{T} = (\mathcal{T}, \mu, \eta, \Lambda, \rho, \Gamma)$ be an idempotent pseudomonad on $\mathfrak{H}$. Given an object X of $\mathfrak{H}$, the following conditions are equivalent:*

1. *The object X can be endowed with a $\mathcal{T}$-pseudoalgebra structure;*
2. *$\eta_X : X \to \mathcal{T}(X)$ is a pseudosection, i.e. there is $a : \mathcal{T}(X) \to X$ such that $a\eta_X \cong \mathrm{Id}_X$;*
3. *$\eta_X : X \to \mathcal{T}(X)$ is an equivalence.*

Proof: Assume that $\eta_X : X \to \mathcal{T}(X)$ is a pseudosection. By hypothesis, there is $a : \mathcal{T}(X) \to X$ such that $a\eta_X \cong \mathrm{Id}_X$. Thereby

$$\eta_X a \cong \mathcal{T}(a)\eta_{\mathcal{T}(X)} \cong \mathcal{T}(a)\mathcal{T}(\eta_X) \cong \mathcal{T}(a\eta_X) \cong \mathrm{Id}_{\mathcal{T}(X)}.$$

Hence η_X is an equivalence. ■

5.2. Biadjoint Triangle Theorem. The main formal result used in this paper is somehow related to distributive laws of pseudomonads [30, 31]. Roughly, let $\mathcal{T}$ be a pseudomonad on $\mathfrak{H}$ compatible with the pseudomonads $\mathcal{T}_2, \mathcal{T}_1$ and pseudomonads $\mathcal{T}_4, \mathcal{T}_3$. We denote by $\widehat{\mathcal{T}_2}$ the lifting of $\mathcal{T}_2$ to $\mathsf{Ps}\text{-}\mathcal{T}_1\text{-}\mathsf{Alg}$. If $a : \mathcal{T}_1(X) \to X$ is a $\mathcal{T}_1$-pseudoalgebra structure such that a can be endowed with a $\widehat{\mathcal{T}_2}$-pseudoalgebra structure, then X can be endowed with a $\mathcal{T}_3$-pseudoalgebra structure. This is described by the diagram below.

$$\begin{tikzcd}
\mathsf{Ps}\text{-}\widehat{\mathcal{T}_2}\text{-}\mathsf{Alg} \arrow[r] \arrow[d, "\approx"] & \mathsf{Ps}\text{-}\mathcal{T}_1\text{-}\mathsf{Alg} \arrow[dd] \\
\mathsf{Ps}\text{-}\widehat{\mathcal{T}_4}\text{-}\mathsf{Alg} \arrow[d] \arrow[r, phantom, "\simeq"] & \\
\mathsf{Ps}\text{-}\mathcal{T}_3\text{-}\mathsf{Alg} \arrow[r] & \mathfrak{H}
\end{tikzcd}$$

However, we choose a more direct approach, avoiding some technicalities of distributive laws unnecessary to our setting. To give such direct approach, we use the Biadjoint Triangle Theorem 5.11.

Precisely, we give a bicategorical version (for idempotent pseudomonads) of a well known adjoint triangle theorem [1]. It is important to note that this bicategorical version holds for pseudomonads in general [29], so that our restriction to the idempotent version is due to our scope.

Lemma 5.10. *Let $(\mathcal{L} \dashv \mathcal{U}, \eta, \varepsilon)$ and $(\widehat{\mathcal{L}} \dashv \widehat{\mathcal{U}}, \widehat{\eta}, \widehat{\varepsilon})$ be biadjunctions. Assume that $\widehat{\mathcal{L}} \dashv \widehat{\mathcal{U}}$ induces an idempotent pseudomonad and that there is a pseudonatural equivalence*

$$\begin{tikzcd}
\mathfrak{A} & & \mathfrak{B} \arrow[ll, "\mathrm{E}"'] \\
& \mathfrak{C} \arrow[ul, "\mathcal{L}"] \arrow[ur, "\widehat{\mathcal{L}}"'] \arrow[u, phantom, "\simeq"] &
\end{tikzcd}$$

If η_X is a pseudosection, then $\widehat{\eta}_X$ is an equivalence.

Proof: Let X be an object of $\mathfrak{C}$ such that $\eta_X : X \to \mathcal{U}\mathcal{L}(X)$ is pseudosection. By Theorem 5.9, it is enough to prove that $\widehat{\eta}_X$ is a pseudosection, because the pseudomonad induced by $\widehat{\mathcal{L}} \dashv \widehat{\mathcal{U}}$ is idempotent.

To prove that $\widehat{\eta}_X$ is a pseudosection, we construct a pseudonatural transformation $\alpha : \widehat{\mathcal{U}}\widehat{\mathcal{L}} \longrightarrow \mathcal{U}\mathcal{L}$ such that there is an invertible modification

$$\begin{tikzcd}
 & \mathrm{Id}_{\mathfrak{C}} \arrow[dl, "\widehat{\eta}"'] \arrow[dr, "\eta"] \arrow[d, phantom, "\cong"] & \\
\widehat{\mathcal{U}}\widehat{\mathcal{L}} \arrow[rr, "\alpha"] & {} & \mathcal{U}\mathcal{L}
\end{tikzcd}$$

Without loosing generality, we assume that $\mathtt{E} \circ \widehat{\mathcal{L}} = \mathcal{L}$. Then we define $\alpha := (\mathcal{U}\mathtt{E}\widehat{\varepsilon}\widehat{\mathcal{L}})(\eta\widehat{\mathcal{U}}\widehat{\mathcal{L}})$. Indeed,

$$\alpha\widehat{\eta} = (\mathcal{U}\mathtt{E}\widehat{\varepsilon}\widehat{\mathcal{L}})\left(\eta\widehat{\mathcal{U}}\widehat{\mathcal{L}}\right)(\widehat{\eta}) \cong (\mathcal{U}\mathtt{E}\widehat{\varepsilon}\widehat{\mathcal{L}})\,(\mathcal{U}\mathcal{L}\widehat{\eta})\,(\eta) \cong (\mathcal{U}\mathtt{E}\widehat{\varepsilon}\widehat{\mathcal{L}})\left(\mathcal{U}\mathtt{E}\widehat{\mathcal{L}}\widehat{\eta}\right)(\eta) \cong \eta$$

Therefore, if η_X is a pseudosection, so is $\widehat{\eta}_X$. And, as mentioned, by Theorem 5.9, if $\widehat{\eta}_X$ is a pseudosection, it is an equivalence. ∎

Let $\widehat{\mathcal{T}}$ be the idempotent pseudomonad induced by $\widehat{\mathcal{L}} \dashv \widehat{\mathcal{U}}$ and $\mathcal{T}$ the pseudomonad induced by $\mathcal{L} \dashv \mathcal{U}$. Then Lemma 5.10 could be written as following:

If X is an object of $\mathfrak{C}$ that can be endowed with a $\mathcal{T}$-pseudoalgebra structure, then X can be endowed with a $\widehat{\mathcal{T}}$-pseudoalgebra structure, provided that there is a pseudonatural equivalence $\mathtt{E}\widehat{\mathcal{L}} \simeq \mathcal{L}$.

Theorem 5.11. *Let $(\mathcal{L} \dashv \mathcal{U}, \eta, \varepsilon)$ and $(\widehat{\mathcal{L}} \dashv \widehat{\mathcal{U}}, \widehat{\eta}, \widehat{\varepsilon})$ be biadjunctions such that their right biadjoints are local equivalences. If there is a pseudonatural equivalence*

$$\begin{tikzcd}
\mathfrak{A} \arrow[d, phantom, ""] & {} \arrow[d, phantom, "\simeq"] & \mathfrak{B} \arrow[ll, "\mathtt{E}"'] \\
 & \mathfrak{C} \arrow[ul, "\mathcal{L}"] \arrow[ur, "\widehat{\mathcal{L}}"'] &
\end{tikzcd}$$

then $\mathtt{E}$ is left biadjoint to a pseudofunctor $\mathtt{R}$ which is a local equivalence.

Proof: It is enough to define $\mathtt{R} := \widehat{\mathcal{L}}\mathcal{U}$. By Lemma 5.10, $(\widehat{\eta}\mathcal{U}) : \mathcal{U} \longrightarrow \widehat{\mathcal{U}}\widehat{\mathcal{L}}\mathcal{U} = \widehat{\mathcal{U}}\mathtt{R}$ is a pseudonatural equivalence. Thereby we get

$$\begin{aligned}
\mathfrak{A}(\mathtt{E}(b), a) &\simeq \mathfrak{A}(\mathtt{E}\widehat{\mathcal{L}}\widehat{\mathcal{U}}(b), a)\\
&\simeq \mathfrak{A}(\mathcal{L}\widehat{\mathcal{U}}(b), a)\\
&\simeq \mathfrak{C}(\widehat{\mathcal{U}}(b), \mathcal{U}(a))\\
&\simeq \mathfrak{C}(\widehat{\mathcal{U}}(b), \widehat{\mathcal{U}}\mathtt{R}(a))\\
&\simeq \mathfrak{B}(b, \mathtt{R}(a)).
\end{aligned}$$

This completes the proof that $\mathtt{R}$ is right biadjoint to $\mathtt{E}$. ∎

Assume that $\mathcal{A} : \mathfrak{A} \to \mathfrak{B}$ and $\mathcal{B} : \mathfrak{B} \to \mathfrak{C}$ are pseudomonadic pseudofunctors, and their induced pseudomonads are idempotent. Then it is obvious that $\mathcal{B} \circ \mathcal{A} : \mathfrak{A} \to \mathfrak{C}$ is also pseudomonadic and induces an idempotent pseudomonad. Indeed, by Theorem 5.8, this statement is equivalent to: *compositions of right biadjoint local equivalences are right biadjoint local equivalences as well.*

The main results of this section are related to the study of pseudoalgebra structures, knowing other pseudoalgebras structures: a kind of commutativity property. This is related to Theorem 5.11 and established in Corollary 5.12.

Corollary 5.12. *Assume that there is a pseudonatural equivalence*

$$\begin{tikzcd} \mathfrak{A} & \mathfrak{H} \arrow[l, "\mathtt{E}"'] \\ \mathfrak{B} \arrow[u, "\mathcal{L}_{\mathcal{A}}"] \arrow[ur, phantom, "\simeq"] & \mathfrak{C} \arrow[l, "\mathcal{L}_{\mathcal{B}}"] \arrow[u, "\mathcal{L}_{\mathcal{C}}"'] \end{tikzcd}$$

such that $\mathcal{L}_{\mathcal{A}} \dashv \mathcal{A}$, $\mathcal{L}_{\mathcal{B}} \dashv \mathcal{B}$ *and* $\mathcal{L}_{\mathcal{C}} \dashv \mathcal{C}$ *are pseudomonadic biadjunctions inducing idempotent pseudomonads* $\mathcal{T}_{\mathcal{A}}, \mathcal{T}_{\mathcal{B}}, \mathcal{T}_{\mathcal{C}}$. *Then* $\mathtt{E} \dashv \mathtt{R}$ *and* $\mathtt{R}$ *is a local equivalence.*

In particular, if (X, a) *is a* $\mathcal{T}_{\mathcal{B}}$*-pseudoalgebra that can be endowed with a* $\mathcal{T}_{\mathcal{A}}$*-pseudoalgebra structure, then* X *can be endowed with a* $\mathcal{T}_{\mathcal{C}}$*-pseudoalgebra structure as well.*

Lemma 5.10 and Corollary 5.12 can be seen as results on descent theory in our broad context, *i.e.* they give conditions to decide whether a given object can be endowed with a pseudoalgebra structure. In fact, most of the theorems proved in this paper are consequences of successive applications of these results, including Bénabou-Roubaud theorem and other theorems that essentially are consequences of commutativity properties of bilimits.

However it does not deal with the technical "almost descent" aspects. These are taken care in next subsection.

5.3. $\mathfrak{F}$-comparisons. In the classical context of descent [18, 19], instead of restricting attention to effective descent morphisms, we often are interested in almost descent and descent morphisms as well. In the context of idempotent pseudomonads, these are objects that possibly do not have pseudoalgebra structure but have comparison 1-cells belonging to special classes of morphisms.

In this subsection, every 2-category $\mathfrak{H}$ is assumed to be endowed with a special subclass of morphisms $\mathfrak{F}_{\mathfrak{H}}$ satisfying the following properties:

- Every equivalence of $\mathfrak{H}$ belongs to $\mathfrak{F}_{\mathfrak{H}}$;
- $\mathfrak{F}_{\mathfrak{H}}$ is closed under compositions and under isomorphisms;
- If fg and f belongs to $\mathfrak{F}_{\mathfrak{H}}$, g is also in $\mathfrak{F}_{\mathfrak{H}}$.

If f is a morphism of $\mathfrak{H}$ that belongs to $\mathfrak{F}_{\mathfrak{H}}$, we say that f is an $\mathfrak{F}_{\mathfrak{H}}$*-morphism.*

Definition 5.13. Let $(\mathcal{T}, \mu, \eta, \Lambda, \rho, \Gamma)$ be an idempotent pseudomonad on a 2-category $\mathfrak{H}$. An object X is an $(\mathfrak{F}_{\mathfrak{H}}, \mathcal{T})$*-object* if the comparison $\eta_X : X \to \mathcal{T}(X)$ is an $\mathfrak{F}_{\mathfrak{H}}$-morphism.

We say that a pseudofunctor $\mathcal{E} : \mathfrak{H} \to \mathfrak{H}$ *preserves* $(\mathfrak{F}_{\mathfrak{H}}, \mathcal{T})$*-objects* if it takes $(\mathfrak{F}_{\mathfrak{H}}, \mathcal{T})$-objects to $(\mathfrak{F}_{\mathfrak{H}}, \mathcal{T})$-objects.

Theorem 5.14 is a commutativity result for $(\mathfrak{F}_{\mathfrak{H}}, \mathcal{T})$-objects. Similarly to Corollary 5.12, it follows from the construction given in the proof of Lemma 5.10, although it requires some extra hypotheses.

Theorem 5.14. *Let*

$$\begin{tikzcd}
\mathfrak{A} & \mathfrak{H} \arrow[l, "\mathrm{E}"'] \\
\mathfrak{B} \arrow[u, "\mathcal{L}_{\mathcal{A}}"] \arrow[ur, phantom, "\simeq"] & \mathfrak{C} \arrow[l, "\mathcal{L}_{\mathcal{B}}"] \arrow[u, "\mathcal{L}_{\mathcal{C}}"']
\end{tikzcd}$$

be a pseudonatural equivalence such that $\mathcal{L}_{\mathcal{A}} \dashv \mathcal{A}$, $\mathcal{L}_{\mathcal{B}} \dashv \mathcal{B}$ and $\mathcal{L}_{\mathcal{C}} \dashv \mathcal{C}$ are biadjunctions inducing pseudomonads $\mathcal{T}_{\mathcal{A}}, \mathcal{T}_{\mathcal{B}}, \mathcal{T}_{\mathcal{C}}$. Also, we denote by $\mathcal{T}$ the pseudomonad induced by the biadjunction $\mathcal{L}_{\mathcal{A}}\mathcal{L}_{\mathcal{B}} \dashv \mathcal{B}\mathcal{A}$.

Assume that all the right biadjoints are local equivalences, $\mathcal{B}$ takes $\mathfrak{F}_{\mathfrak{B}}$-morphisms to $\mathfrak{F}_{\mathfrak{C}}$-morphisms and $\mathcal{T}_{\mathcal{C}}$ preserves $(\mathfrak{F}_{\mathfrak{C}}, \mathcal{T})$-objects. If X is a $(\mathfrak{F}_{\mathfrak{C}}, \mathcal{T}_{\mathcal{B}})$-object of $\mathfrak{C}$ and $\mathcal{L}_{\mathcal{B}}(X)$ is a $(\mathfrak{F}_{\mathfrak{B}}, \mathcal{T}_{\mathcal{A}})$-object, then X is a $(\mathfrak{F}_{\mathfrak{C}}, \mathcal{T}_{\mathcal{C}})$-object as well.

Proof: By the proof of Lemma 5.10, there is a pseudonatural transformation $\alpha : \mathcal{T}_{\mathcal{C}} \longrightarrow \mathcal{T}$ such that there is an invertible modification

$$\begin{tikzcd}
 & X \arrow[dl, "\eta^{\mathcal{C}}"'] \arrow[dr, "\eta"] \arrow[d, phantom, "\cong"] & \\
\mathcal{T}_{\mathcal{C}} \arrow[rr, "\alpha"] & {} & \mathcal{T}
\end{tikzcd}$$

In particular, if X is an object of $\mathfrak{C}$ satisfying the hypotheses of the theorem, we get an isomorphism

$$\begin{tikzcd} & X \arrow[dl, "\eta^{\mathcal{C}}_X"'] \arrow[dr, "\eta_X"] \arrow[d, phantom, "\cong"] & \\ \mathcal{T}_{\mathcal{C}}(X) \arrow[rr, "\alpha_X"] & & \mathcal{T}(X) \end{tikzcd}$$

in which, by the hypotheses, we conclude that $\eta_X \cong \left(\mathcal{B}\eta^{\mathcal{A}}\mathcal{L}_{\mathcal{B}}\right)_X \cdot \eta^{\mathcal{B}}_X$ is an $\mathfrak{F}_{\mathfrak{C}}$-morphism.

By the properties of the subclass $\mathfrak{F}_{\mathfrak{C}}$, it remains to prove that α_X is an $\mathfrak{F}_{\mathfrak{C}}$-morphism. Recall that α_X is defined by $\alpha_X := (\mathcal{B}\mathcal{A}\varepsilon^{\mathcal{C}}\mathcal{L}_{\mathcal{C}})_X \cdot (\eta\mathcal{T}_{\mathcal{C}})_X$, in which $\varepsilon^{\mathcal{C}}$ is the counit of the biadjunction $\mathcal{L}_{\mathcal{C}} \dashv \mathcal{C}$.

Since $(\mathcal{B}\mathcal{A}\varepsilon^{\mathcal{C}}\mathcal{L}_{\mathcal{C}})_X$ is an equivalence and, by hypothesis, $(\eta\mathcal{T}_{\mathcal{C}})_X$ is a $\mathfrak{F}_{\mathfrak{C}}$-morphism, it follows that α_X is a $\mathfrak{F}_{\mathfrak{C}}$-morphism.

This completes the proof that $\eta^{\mathcal{C}}_X$ is also a $\mathfrak{F}_{\mathfrak{C}}$-morphism. ∎

5.4. Pseudo-Kan extension. As mentioned before, we deal mainly with pseudo-Kan extensions along local equivalences. Actually, our setting reduces to the study of right pseudo-Kan extensions of pseudofunctors $\mathcal{A} : \mathfrak{A} \to \mathfrak{H}$ along t, in which $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ is the full inclusion of a small 2-category $\mathfrak{A}$ into a small 2-category $\dot{\mathfrak{A}}$ which has only one extra object a.

Theorem 5.15 (Factorization). *Let $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ be an inclusion of a small 2-category $\mathfrak{A}$ into a small 2-category $\dot{\mathfrak{A}}$ in which*

$$\mathrm{obj}(\dot{\mathfrak{A}}) = \mathrm{obj}(\mathfrak{A}) \cup \{\mathsf{a}\}\,.$$

If $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is a pseudofunctor, $\mathsf{a} \neq b$ and $f : b \to \mathsf{a}$, $g : \mathsf{a} \to b$ are morphisms of $\dot{\mathfrak{A}}$, we get induced "factorizations" (actually, invertible 2-cells):

$$\begin{tikzcd} \mathcal{A}(b) \arrow[rr, "\mathcal{A}(f)"] \arrow[dr, "f_{\mathcal{A}}"'] \arrow[rr, phantom, bend right=20, "\cong"] & & \mathcal{A}(\mathsf{a}) \\ & \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ\mathrm{t})(\mathsf{a}) \arrow[ur, "\eta^{\mathsf{a}}_{\mathcal{A}}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}(\mathsf{a}) \arrow[rr, "\mathcal{A}(g)"] \arrow[dr, "\eta^{\mathsf{a}}_{\mathcal{A}}"'] \arrow[rr, phantom, bend right=20, "\cong"] & & \mathcal{A}(b) \\ & \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ\mathrm{t})(\mathsf{a}) \arrow[ur, "g_{\mathcal{A}}"'] & \end{tikzcd}$$

in which

$$f_{\mathcal{A}} := \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ\mathrm{t})(f) \circ \varepsilon^{b}_{(\mathcal{A}\circ\mathrm{t})} \qquad\qquad g_{\mathcal{A}} := \varepsilon^{b}_{(\mathcal{A}\circ\mathrm{t})} \circ \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ\mathrm{t})(g)$$

and $\eta^{\mathsf{a}}_{\mathcal{A}}$, $\varepsilon^{b}_{(\mathcal{A}\circ \mathrm{t})}$ *are the* 1*-cells induced by the components of the unit* η *and counit* ε *of the biadjunction* $[\mathrm{t}, \mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}$*, that is to say*

$$\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ \mathrm{t}) \qquad\qquad \varepsilon_{(\mathcal{A}\circ \mathrm{t})} : \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ \mathrm{t})\circ \mathrm{t} \longrightarrow \mathcal{A}\circ \mathrm{t}$$

Proof: By the (triangular) invertible modifications of Definition 3.4,

$$g_{\mathcal{A}} \circ \eta^{\mathsf{a}}_{\mathcal{A}} = \varepsilon^{b}_{(\mathcal{A}\circ \mathrm{t})} \circ \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ \mathrm{t})(g) \circ \eta^{\mathsf{a}}_{\mathcal{A}} \cong \varepsilon^{b}_{(\mathcal{A}\circ \mathrm{t})} \circ \eta^{b}_{\mathcal{A}} \circ \mathcal{A}(g) \cong \mathcal{A}(g)$$

The proof of the factorization of $\mathcal{A}(f)$ is analogous. ∎

Furthermore, when t is a local equivalence, $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}} : [\mathfrak{A}, \mathfrak{H}]_{PS} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ induces an idempotent pseudomonad $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(-\circ \mathrm{t})$. This is a consequence of Theorem 4.10 and Theorem 5.8.

Theorem 5.16. *Let* $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ *be a full inclusion of* $\mathfrak{A}$ *into a small* 2*-category* $\dot{\mathfrak{A}}$ *with only one extra object* a*. We have the following*

- $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}} : [\mathfrak{A}, \mathfrak{H}]_{PS} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ *is pseudomonadic;*
- $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}$ *induces an idempotent pseudomonad* $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(-\circ \mathrm{t})$.

Thereby, by Theorem 5.9, we can easily study the $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(-\circ\mathrm{t})$-pseudoalgebra structures on diagrams, using the unit of the biadjunction $[\mathrm{t}, \mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}$. More precisely:

Theorem 5.17. *Let* $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ *be a local equivalence between small* 2*-categories and* $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ *be a pseudofunctor. The following conditions are equivalent:*

- $\mathcal{A}$ *is of effective* t*-descent;*
- *The component of the unit on* $\mathcal{A}$*/ comparison* $\eta_{\mathcal{A}} : \mathcal{A} \to \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ \mathrm{t})$ *is a pseudonatural equivalence;*
- *The comparison* $\eta_{\mathcal{A}} : \mathcal{A} \to \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ \mathrm{t})$ *is a pseudonatural pseudo-section.*

Moreover, the component of the unit $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A}\circ \mathrm{t})$ is a pseudonatural equivalence if and only if all components of $\eta_{\mathcal{A}}$ are equivalences. But, by Theorem 4.10, assuming that $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ is a full inclusion of a (small) 2-category $\mathfrak{A}$ into a small 2-category $\dot{\mathfrak{A}}$ with only one extra object a, $\eta^{b}_{\mathcal{A}}$ is always an equivalence for all b in $\mathfrak{A}$. Thereby we get:

Theorem 5.18. *Let* $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ *be an inclusion of a small* 2*-category* $\mathfrak{A}$ *into a small* 2*-category* $\dot{\mathfrak{A}}$ *in which* $\mathrm{obj}(\dot{\mathfrak{A}}) := \mathrm{obj}(\mathfrak{A}) \cup \{\mathsf{a}\}$ *is a disjoint union. If* $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ *is a pseudofunctor,* $\mathcal{A}$ *is of effective* t*-descent if and only if* $\eta^{\mathsf{a}}_{\mathcal{A}} : \mathcal{A}(\mathsf{a}) \to \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A} \circ \mathrm{t})(\mathsf{a})$ *is an equivalence.*

5.5. Commutativity. We can apply the main results of this section (Theorem 5.11 and Corollary 5.12) to our context of pseudo-Kan extensions: we get, then, theorems on commutativity.

Let $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$, $\mathrm{h} : \mathfrak{B} \to \dot{\mathfrak{B}}$ be full inclusions of small 2-categories, such that

$$\mathrm{obj}(\dot{\mathfrak{A}}) = \mathrm{obj}(\mathfrak{A}) \cup \{\mathsf{a}\} \text{ and } \mathrm{obj}(\dot{\mathfrak{B}}) = \mathrm{obj}(\mathfrak{B}) \cup \{\mathsf{b}\}$$

are disjoint unions. Unless we explicit otherwise, henceforth we always consider right pseudo-Kan extensions along such type of inclusions. Recall that we are always assuming that $\mathfrak{H}$ is bicategorically complete (or at least, $\mathfrak{H}$ have enough bilimits to define the considered global pointwise pseudo-Kan extensions). In general, we have that (see [37]):

$$\left[\dot{\mathfrak{A}} \times \dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS} \approx \left[\dot{\mathfrak{A}}, \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS}\right]_{PS} \cong \left[\dot{\mathfrak{B}}, \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}\right]_{PS}$$

Thereby every pseudofunctor $\mathcal{A} : \dot{\mathfrak{A}} \times \dot{\mathfrak{B}} \to \mathfrak{H}$ can be seen (up to pseudonatural equivalence) as a pseudofunctor $\mathcal{A} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS}$. Also, $\mathcal{A} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS}$ can be seen as a pseudofunctor $\mathcal{A} : \dot{\mathfrak{B}} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$.

Theorem 5.19. *Let* $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ *be a effective* t*-descent pseudofunctor. Assume that* $\mathfrak{U} : \widehat{\mathfrak{H}} \to \mathfrak{H}$ *is a pseudomonadic pseudofunctor such that its induced pseudomonad* $\mathfrak{T}$ *is idempotent and* $\mathfrak{L} \dashv \mathfrak{U}$*. If* $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ *is a effective* t*-descent pseudofunctor and* $\mathfrak{T}$ *is an idempotent pseudomonad on* $\mathfrak{H}$ *such that the objects of the image of* $\mathcal{A} \circ \mathrm{t}$ *can be endowed with a* $\mathfrak{T}$*-pseudoalgebra structure, then* $\mathcal{A}(\mathsf{a})$ *can be endowed with a* $\mathfrak{T}$*-pseudoalgebra structure.*

Proof: Observe that the pseudonatural equivalence

$$\begin{tikzcd}
\left[\mathfrak{A}, \widehat{\mathfrak{H}}\right]_{PS} & \left[\dot{\mathfrak{A}}, \widehat{\mathfrak{H}}\right]_{PS} \arrow[l, "{[\mathrm{t}, \widehat{\mathfrak{H}}]_{PS}}"'] \\
\left[\mathfrak{A}, \mathfrak{H}\right]_{PS} \arrow[u, "{[\mathfrak{A}, \mathcal{L}]_{PS}}"] \arrow[ur, phantom, "\simeq"] & \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS} \arrow[l, "{[\mathrm{t}, \mathfrak{H}]_{PS}}"'] \arrow[u, "{[\dot{\mathfrak{A}}, \mathcal{L}]_{PS}}"']
\end{tikzcd}$$

satisfies the hypotheses of Corollary 5.12.

If $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is a effective t-descent pseudofunctor such that all the objects of the image of $\mathcal{A} \circ \mathrm{t}$ has $\mathfrak{T}$-pseudoalgebra structure, it means that $\mathcal{A}$ satisfies the hypotheses of Corollary 5.12. *I.e.* $\mathcal{A}$ is a $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(- \circ \mathrm{t})$-pseudoalgebra that can be endowed with a $[\mathfrak{A}, \mathfrak{T}]_{PS}$-pseudoalgebra structure. Thereby, by Corollary 5.12, $\mathcal{A}$ can be endowed with a $\left[\dot{\mathfrak{A}}, \mathfrak{T}\right]_{PS}$-pseudoalgebra structure. ∎

Corollary 5.20. *Let* $\mathcal{A} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS}$ *be a effective* t*-descent pseudofunctor such that all the diagrams in the image of* $\mathcal{A} \circ \mathrm{t}$ *are of effective* h*-descent, then* $\mathcal{A}(\mathsf{a})$ *is also of effective* h*-descent.*

Corollary 5.21. *Let* $\mathcal{A} : \dot{\mathfrak{A}} \times \dot{\mathfrak{B}} \to \mathfrak{H}$ *be a pseudofunctor. Assume that its mates*

$$\widehat{\mathcal{A}} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS} \qquad \bar{\mathcal{A}} : \dot{\mathfrak{B}} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$$

are such that all the diagrams in the image of $\widehat{\mathcal{A}} \circ \mathrm{t}$ *are of effective* h*-descent and all the diagrams in the image of* $\bar{\mathcal{A}} \circ \mathrm{h}$ *are of* t*-effective descent. We have that* $\widehat{\mathcal{A}}(\mathsf{a})$ *is of effective* h*-descent if and only if* $\bar{\mathcal{A}}(\mathsf{b})$ *is of effective* t*-descent.*

Corollary 5.21 is enough to prove Bénabou-Roubaud Theorem and other abstract theorems of Descent Theory that depend only on basic commutativity properties. Next subsection deals with the technical issues of almost descent pseudofunctors.

5.6. Almost descent pseudofunctors. Recall that a 1-cell in a 2-category $\mathfrak{H}$ is called faithful/ fully faithful if its images by the (covariant) representable 2-functors are faithful/ fully faithful.

Definition 5.22. Let $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ be an inclusion of a small 2-category $\mathfrak{A}$ into a small 2-category $\dot{\mathfrak{A}}$ in which $\mathrm{obj}(\dot{\mathfrak{A}}) = \mathrm{obj}(\mathfrak{A}) \cup \{\mathsf{a}\}$ is a disjoint union. If $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is a pseudofunctor, $\mathcal{A}$ is of almost t-descent if $\eta^{\mathsf{a}}_{\mathcal{A}} : \mathcal{A}(\mathsf{a}) \to \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A} \circ \mathrm{t})(\mathsf{a})$ is faithful. If, furthermore, $\eta^{\mathsf{a}}_{\mathcal{A}}$ is fully faithful, we say that $\mathcal{A}$ is of t-descent.

Consider the class $\mathfrak{F}_{[\dot{\mathfrak{A}},\mathfrak{H}]_{PS}}$ of pseudonatural transformations/ morphisms in $\left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ whose components are faithful. This class satisfies the properties described in Subsection 5.3. Also, a pseudofunctor $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of almost descent if and only if $\mathcal{A}$ is a $(\mathfrak{F}_{[\dot{\mathfrak{A}},\mathfrak{H}]_{PS}}, \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A} \circ \mathrm{t}))$-object.

Analogously, if we take the class $\mathfrak{F}'_{[\dot{\mathfrak{A}},\mathfrak{H}]_{PS}}$ of objectwise fully faithful pseudonatural transformations, $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of descent if and only if $\mathcal{A}$ is a $(\mathfrak{F}'_{[\dot{\mathfrak{A}},\mathfrak{H}]_{PS}}, \mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(\mathcal{A} \circ \mathrm{t}))$-object.

Since in our context of right pseudo-Kan extensions along local equivalences the hypotheses of Theorem 5.14 hold, we get the corollaries below. Again, we are considering full inclusions $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$, $\mathrm{h} : \mathfrak{B} \to \dot{\mathfrak{B}}$ as in Subsection 5.5.

Corollary 5.23. *Let $\mathcal{A} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS}$ be a almost t-descent pseudofunctor such that all the pseudofunctors in the image of $\mathcal{A} \circ \mathrm{t}$ are of almost h-descent. In this case, $\mathcal{A}(\mathsf{a})$ is also of almost h-descent.*

Similarly, if $\mathcal{A}$ is of t-descent and all the pseudofunctors of the image of $\mathcal{A} \circ \mathrm{t}$ are of h-descent, then $\mathcal{A}(\mathsf{a})$ is also of h-descent as well.

Corollary 5.24. *Let $\mathcal{A} : \dot{\mathfrak{A}} \times \dot{\mathfrak{B}} \to \mathfrak{H}$ be a pseudofunctor. Assume that its mates*

$$\widehat{\mathcal{A}} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS} \qquad \bar{\mathcal{A}} : \dot{\mathfrak{B}} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$$

are such that all the diagrams in the image of $\widehat{\mathcal{A}} \circ \mathrm{t}$ are of almost h-descent and all the diagrams in the image of $\bar{\mathcal{A}} \circ \mathrm{h}$ are of almost t-descent. In this case,

$$\widehat{\mathcal{A}}(\mathsf{a}) \textit{ is of almost } \mathrm{h}\textit{-descent if and only if } \bar{\mathcal{A}}(\mathsf{b}) \textit{ is of almost } \mathrm{t}\textit{-descent.}$$

If, furthermore, all the pseudofunctors in the image of $\widehat{\mathcal{A}} \circ \mathrm{t}$ are of h-descent and all the pseudofunctors in the image of $\bar{\mathcal{A}} \circ \mathrm{h}$ are of t-descent. Then:

$$\widehat{\mathcal{A}}(\mathsf{a}) \textit{ is of } \mathrm{h}\textit{-descent if and only if } \bar{\mathcal{A}}(\mathsf{b}) \textit{ is of } \mathrm{t}\textit{-descent.}$$

6. Descent Objects

In this section, we give a description of the descent objects in our setting. Let $\mathrm{j} : \Delta \to \dot{\Delta}$ be the full inclusion of the category of finite nonempty ordinals into the category of finite ordinals and order preserving functions. Recall that $\dot{\Delta}$ is generated by its degeneracy and face maps. That is to say, $\dot{\Delta}$ is generated by the diagram

$$\begin{tikzcd}
\mathsf{0} \arrow[r, "d=d^0"] & \mathsf{1} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & \mathsf{2} \arrow[l, "s^0" description] \arrow[r, shift left=2, "d^0"] \arrow[r, "d^1" description] \arrow[r, shift right=2, "d^2"'] & \mathsf{3} \arrow[l, bend right=60, "s^0"'] \arrow[l, bend left=60, "s^1"] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \arrow[l, bend right=50] \arrow[l, bend left=30] \arrow[l, bend left=60]
\end{tikzcd}$$

with the following relations:

$$\begin{aligned}
d^k d^i &= d^i d^{k-1}, \text{if } i < k & \qquad d^0 d &= d^1 d \\
s^k s^i &= s^i s^{k+1}, \text{if } i \leqslant k & \qquad s^k d^i &= \mathrm{id}, \text{ if } i = k \text{ and } i = k+1 \\
s^k d^i &= d^i s^{k-1}, \text{if } i < k & \qquad s^k d^i &= d^{i-1} s^k, \text{ if } i > k+1
\end{aligned}$$

A pseudofunctor $\mathcal{A} : \Delta \to \mathfrak{H}$ is called a *pseudocosimplicial object* of $\mathfrak{H}$. The *descent object of a pseudocosimplicial object* $\mathcal{A} : \Delta \to \mathfrak{H}$ *is* $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{A}(\mathsf{0})$. Theorem 6.3 shows that this definition agrees with Definition 6.2, which is the usual definition of the descent object/ bilimit [37, 18, 17].

Firstly, we need a suitable domain for the weight defined in [37]. Originally, the weight is defined in a "strict version/ free version" of the category Δ_3 defined below. We choose to not do so, postponing any further comment on strictness to Subsection 6.1.

Definition 6.1. The category $\dot{\Delta}_3$ is generated by the diagram:

$$\begin{tikzcd}
\mathsf{0} \arrow[r, "d"] & \mathsf{1} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & \mathsf{2} \arrow[l, "s^0" description] \arrow[r, shift left=2, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=2, "\partial^2"'] & \mathsf{3}
\end{tikzcd}$$

such that:

$$d^1 d = d^0 d \qquad \partial^k d^i = \partial^i d^{k-1}, \text{ if } i < k \qquad s^0 d^0 = s^0 d^1 = \mathrm{id}$$

We denote by $\mathrm{j}_3 : \Delta_3 \to \dot{\Delta}_3$ the full inclusion of the subcategory Δ_3 in which $\mathrm{obj}(\Delta_3) = \{\mathsf{1}, \mathsf{2}, \mathsf{3}\}$. Still, there are obvious inclusions: $\dot{\mathrm{t}}_3 : \dot{\Delta}_3 \to \dot{\Delta}$ and $\mathrm{t}_3 : \Delta_3 \to \Delta$.

Definition 6.2. We denote by $\mathbb{W} : \Delta_3 \to \mathsf{CAT}$ the weight below (defined in [37]), in which $\nabla\mathsf{n}$ denotes the localization of the category/ finite ordinal n w.r.t all the morphisms.

$$\begin{tikzcd}
\nabla 1 \arrow[r, shift left=3] \arrow[r, shift right=3] & \nabla 2 \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \nabla 3
\end{tikzcd}$$

Following [37], if $\mathcal{A} : \Delta \to \mathfrak{H}$ is a pseudofunctor, the descent object of $\mathcal{A}$ is defined to be

$$\mathcal{D}esc(\mathcal{A}) := \{\mathbb{W}, \mathcal{A} \circ \mathrm{t}_3\}_{\mathrm{bi}}\,.$$

Theorem 6.3 (Descent Objects)**.** *Let $\mathcal{A} : \Delta \to \mathfrak{H}$ be a pseudofunctor. The descent object of Definition 6.2 is equivalent to* $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{A}(\mathsf{0})$.

Proof: Recall that we are assuming that $\mathfrak{H}$ has the pointwise right pseudo-Kan extension of $\mathcal{A}$. By Theorem 4.9, $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{A}(\mathsf{0}) = \left\{\dot{\Delta}(\mathsf{0}, \mathrm{j}-), \mathcal{A}\right\}_{\mathrm{bi}}$. However, since $\mathsf{0}$ is the initial object of $\dot{\Delta}$, the 2-functor $\dot{\Delta}(\mathsf{0}, \mathrm{j}-)$ is constant and equal to the terminal category, denoted by $*$. We denote this weight by $*$.

Therefore we just need to prove that the descent object of Definition 6.2 can be seen (up to equivalence) as the weighted bilimit $\{*, \mathcal{A}\}_{\mathrm{bi}}$. Also, by Theorem 4.7, $\{\mathbb{W}, \mathcal{A} \circ \mathrm{t}_3\}_{\mathrm{bi}} \simeq \left\{\mathrm{Ps}\mathcal{L}an_{\mathrm{t}_3}\mathbb{W}, \mathcal{A}\right\}_{\mathrm{bi}}$. Thereby, it remains to prove that $\mathrm{Ps}\mathcal{L}an_{\mathrm{t}_3}\mathbb{W} \simeq *$. But it is easy to verify that, by the definitions, the left pseudo-Kan extention of $\mathbb{W}$ is (equivalent to)

$$\begin{tikzcd}
\nabla 1 \arrow[r, shift left=3, "d^0"] \arrow[r, shift right=3, "d^1"'] & \nabla 2 \arrow[l, "s^0" description] \arrow[r, shift left=3, "d^0"] \arrow[r, "d^1" description] \arrow[r, shift right=3, "d^2"'] & \nabla 3 \arrow[l, bend right=50, "s^0"'] \arrow[l, bend left=50, "s^1"] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \nabla 4 \cdots \arrow[l, bend right=50] \arrow[l, bend left=30] \arrow[l, bend left=50]
\end{tikzcd}$$

in which $\mathrm{Ps}\mathcal{L}an_{\mathrm{t}_3}\mathbb{W}(\mathsf{n}) = \nabla\mathsf{n}$, defined in the obvious way.

At last, there is a unique 2-natural transformation $\alpha : \mathrm{Ps}\mathcal{L}an_{\mathrm{t}_3}\mathbb{W} \longrightarrow *$. Since α is objectwise an equivalence (that is to say, $\alpha_{\mathsf{n}} : \nabla\mathsf{n} \to *$ is an equivalence for each n), α is a pseudonatural equivalence $\mathrm{Ps}\mathcal{L}an_{\mathrm{t}_3}\mathbb{W} \simeq *$. ■

Observe that, by Theorem 6.3, if $\mathcal{A} : \dot{\Delta} \to \mathfrak{H}$ is a pseudofunctor, then $\mathcal{A}$ is of effective j-descent if and only if $\mathcal{A} \circ \dot{\mathrm{t}}_3$ is of effective j_3-descent.

6.1. Strict Descent Objects. In this subsection, we show how we can see descent objects via (strict/ enriched) Kan extensions of 2-diagrams. Although this construction is important to giving a few strict features of descent theory (such as the strict factorization), we barely use the results of this subsection in the rest of the paper (since, as explained in Section 2, we avoid coherence technicalities).

Clearly, unlike the general viewpoint of this paper, in this subsection we have to deal closely with coherence theorems. Most of the coherence replacements used here follow from the 2-monadic approach to general coherence results [24, 4, 28]. Also, to formalize some observations of free 2-categories, we use the concept of computad, defined in [35].

The first step is actually older than the general coherence results: the strictification of a bicategory described in page 27 of [13]. We take the stritification of the 2-category $\dot{\Delta}_3$ and denote it by $\dot{\Delta}_{\mathrm{Str}}$. More precisely, this is defined herein as follows:

Definition 6.4. We denote by $\dot{\Delta}_{\mathrm{Str}}$ the locally preordered 2-category freely generated by the diagram

$$\begin{tikzcd} \mathsf{0} \arrow[r, "d"] & \mathsf{1} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & \mathsf{2} \arrow[l, "s^0" description] \arrow[r, shift left=2, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=2, "\partial^2"'] & \mathsf{3} \end{tikzcd}$$

with the 2-cells:

$$\begin{aligned} \sigma_{01} &: \ \partial^1 d^0 \cong \partial^0 d^0 & \qquad n_0 &: \ s^0 d^0 \cong \mathrm{Id}_{\mathsf{1}} \\ \sigma_{02} &: \ \partial^2 d^0 \cong \partial^0 d^1 & \qquad n_1 &: \ \mathrm{Id}_{\mathsf{1}} \cong s^0 d^1 \\ \sigma_{12} &: \ \partial^2 d^1 \cong \partial^1 d^1 & \qquad \vartheta &: \ d^1 d \cong d^0 d \end{aligned}$$

We consider the full inclusion $\mathrm{j}_{\mathrm{Str}} : \Delta_{\mathrm{Str}} \to \dot{\Delta}_{\mathrm{Str}}$ in which $\mathrm{obj}(\Delta_{\mathrm{Str}}) = \{\mathsf{1}, \mathsf{2}, \mathsf{3}\}$.

Remark 6.5. Observe that the diagram and 2-cells described above define a computad [35], which we denote by $\mathfrak{d}$. Thereby Definition 6.4 is precise in the following sense: there is a forgetful functor between the category of locally preordered 2-categories and 2-functors and the category of computads. This forgetful functor has a left adjoint which gives the locally preordered 2-categories freely generated by each computad.

The (locally preordered) 2-category $\dot{\Delta}_{\mathrm{Str}}$ is, by definition, the image of the computad $\mathfrak{d}$ by this left adjoint functor.

Remark 6.6. Δ_{Str} is the 2-category freely generated by the corresponding diagram and 2-cells σ_{01}, σ_{02}, σ_{12}, n_0, n_1, since there are no equations involving just these 2-cells.

Indeed, $\dot{\Delta}_{\mathrm{Str}}$ and Δ_{Str} are strict replacements of our 2-categories $\dot{\Delta}_3$ and Δ_3 respectively. Actually, $\mathrm{j}_{\mathrm{Str}}$ is the strictification of j_3. By the construction of $\dot{\Delta}_{\mathrm{Str}}$, we get the desired main coherence result of this subsection:

Proposition 6.7. *There are obvious biequivalences $\Delta_{\mathrm{Str}} \approx \Delta_3$ and $\dot{\Delta}_{\mathrm{Str}} \approx \dot{\Delta}_3$ which are bijective on objects. Also, if $\mathfrak{H}$ is any 2-category, $[\Delta_{\mathrm{Str}}, \mathfrak{H}] \to [\Delta_{\mathrm{Str}}, \mathfrak{H}]_{PS}$ is essentially surjective.*

Moreover, for any 2-functor $\mathcal{C} : \Delta_{\mathrm{Str}} \to \mathsf{CAT}$, we have an equivalence

$$[\Delta_{\mathrm{Str}}, \mathsf{CAT}]\,(\dot{\Delta}_{\mathrm{Str}}(\mathsf{0}, \mathrm{j}_{\mathrm{Str}}(-)), \mathcal{C}) \simeq [\Delta_{\mathrm{Str}}, \mathsf{CAT}]_{PS}\,(\dot{\Delta}_{\mathrm{Str}}(\mathsf{0}, \mathrm{j}_{\mathrm{Str}}(-)), \mathcal{C}).$$

Corollary 6.8. *If $\mathcal{A} : \Delta_{\mathrm{Str}} \to \mathfrak{H}$ is a 2-functor,*

$$\mathrm{Ps}\mathcal{R}an_{\mathrm{j}_3}\breve{\mathcal{A}} \simeq \mathrm{Ps}\mathcal{R}an_{\mathrm{j}_{\mathrm{Str}}}\mathcal{A} \simeq \mathcal{R}an_{\mathrm{j}_{\mathrm{Str}}}\mathcal{A}$$

provided that the pointwise Kan extension $\mathcal{R}an_{\mathrm{j}_{\mathrm{Str}}}\mathcal{A}$ exists, in which $\breve{\mathcal{A}}$ is the composition of $\mathcal{A}$ with the biequivalence $\Delta_3 \approx \Delta_{\mathrm{Str}}$.

Assuming that the pointwise Kan extension $\mathcal{R}an_{\mathrm{j}_{\mathrm{Str}}}\mathcal{A}$ exists, $\mathcal{R}an_{\mathrm{j}_{\mathrm{Str}}}\mathcal{A}(\mathsf{0})$ is called the *strict descent diagram* of $\mathcal{A}$. By the last result, the descent object of $\mathcal{A}$ is equivalent to its strict descent object provided that $\mathcal{A}$ has a strict descent object. We get a glimpse of the explicit nature of the (strict) descent object at Theorem 6.9 which gives a presentation to $\dot{\Delta}_{\mathrm{Str}}$.

We denote by $\dot{\mathfrak{d}}$ the 2-category freely generated by the diagram and 2-cells described in Definition 6.4. It is important to note that $\dot{\mathfrak{d}}$ is not locally preordered. Moreover, there is an obvious 2-functor $\dot{\mathfrak{d}} \to \dot{\Delta}_{\mathrm{Str}}$, induced by the unit of the adjunction between the category of 2-categories and the category of locally preordered 2-categories.

Theorem 6.9. *Let $\mathfrak{H}$ be a 2-category. There is a bijection between 2-functors $\mathcal{A} : \dot{\Delta}_{\mathrm{Str}} \to \mathfrak{H}$ and 2-functors $\mathsf{A} : \dot{\mathfrak{d}} \to \mathfrak{H}$ satisfying the following equations:*

- *Associativity:*

$$\begin{tikzcd}
\mathsf{A}(0) \arrow[r, "\mathsf{A}(d)"] \arrow[d, "\mathsf{A}(d)"'] \arrow[dr, phantom, "\xRightarrow{\mathsf{A}(\vartheta)}"] & \mathsf{A}(1) \arrow[r, "\mathsf{A}(d^0)"] \arrow[d, "\mathsf{A}(d^0)"] \arrow[dr, phantom, "\xRightarrow{\mathsf{A}(\sigma_{01})}"] & \mathsf{A}(2) \arrow[d, "\mathsf{A}(\partial^0)"] \\
\mathsf{A}(1) \arrow[r, "\mathsf{A}(d^1)"] \arrow[d, "\mathsf{A}(d^1)"'] & \mathsf{A}(2) \arrow[r, "\mathsf{A}(\partial^1)"] \arrow[d, phantom, "\xRightarrow{\mathsf{A}(\sigma_{12})}"] & \mathsf{A}(3) \arrow[d, "\mathsf{A}(\mathrm{id}_3)"] \\
\mathsf{A}(2) \arrow[rr, "\mathsf{A}(\partial^2)"'] & & \mathsf{A}(3)
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{A}(3) \arrow[dr, phantom, "\xRightarrow{\mathsf{A}(\sigma_{02})}"] & \mathsf{A}(2) \arrow[l, "\mathsf{A}(\partial^0)"'] \arrow[r, equal] & \mathsf{A}(2) \\
\mathsf{A}(2) \arrow[u, "\mathsf{A}(\partial^2)"] & \mathsf{A}(1) \arrow[l, "\mathsf{A}(d^0)"] \arrow[u, "\mathsf{A}(d^1)"'] \arrow[r, phantom, "\xRightarrow{\mathsf{A}(\vartheta)}"] & {} \\
\mathsf{A}(1) \arrow[u, "\mathsf{A}(d^1)"] \arrow[ur, phantom, "\xRightarrow{\mathsf{A}(\vartheta)}"] & \mathsf{A}(0) \arrow[l, "\mathsf{A}(d)"] \arrow[u, "\mathsf{A}(d)"'] \arrow[r, "\mathsf{A}(d)"'] & \mathsf{A}(1) \arrow[uu, "\mathsf{A}(d^0)"']
\end{tikzcd}$$

- *Identity:*

$$\begin{tikzcd}
\mathsf{A}(0) \arrow[r, "\mathsf{A}(d)"] \arrow[d, "\mathsf{A}(d)"'] \arrow[dr, phantom, "\xLeftarrow{\mathsf{A}(\vartheta)}"] & \mathsf{A}(1) \arrow[d, "\mathsf{A}(d^1)"] \arrow[ddr, equal, bend left] \arrow[ddr, phantom, "\xLeftarrow{\mathsf{A}(n_1)}"] & \\
\mathsf{A}(1) \arrow[r, "\mathsf{A}(d^0)"'] \arrow[drr, equal, bend right] \arrow[drr, phantom, "\xLeftarrow{\mathsf{A}(n_0)}"] & \mathsf{A}(2) \arrow[dr, "\mathsf{A}(s^0)"] & \\
& & \mathsf{A}(1)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathsf{A}(0) \arrow[d, bend right, "\mathsf{A}(d)"'] \arrow[d, bend left, "\mathsf{A}(d)"] \arrow[d, phantom, "="] \\
\mathsf{A}(1)
\end{tikzcd}$$

Remark 6.10. Using the strict descent object, we can construct the "strict" factorization described in Section 1. If $\mathcal{A} : \dot{\Delta}_{\mathrm{Str}} \to \mathfrak{H}$ is a 2-functor and $\mathfrak{H}$ has strict descent objects, we get the factorization from the universal property of the right Kan extension of $\mathcal{A} \circ \mathrm{j}_{\mathrm{Str}} : \Delta_{\mathrm{Str}} \to \mathfrak{H}$ along $\mathrm{j}_{\mathrm{Str}}$. More precisely, since $\mathrm{j}_{\mathrm{Str}}$ is fully faithful, we can consider that $\mathcal{R}an_{\mathrm{j}_{\mathrm{Str}}} \mathcal{A} \circ \mathrm{j}_{\mathrm{Str}}$ is actually a strict extension of $\mathcal{A} \circ \mathrm{j}_{\mathrm{Str}}$. Thereby we get the factorization

$$\begin{tikzcd}
& \mathcal{R}an_{\mathrm{j}_{\mathrm{Str}}}(\mathcal{A} \circ \mathrm{j}_{\mathrm{Str}})(0) \arrow[d, "\mathcal{R}an_{\mathrm{j}_{\mathrm{Str}}}(\mathcal{A}\circ \mathrm{j}_{\mathrm{Str}})(d)"] \\
\mathcal{A}(0) \arrow[ur, "\eta_{\mathcal{A}}^{0}"] \arrow[r, "\mathcal{A}(d)"'] & \mathcal{A}(1)
\end{tikzcd}$$

in which $\eta_{\mathcal{A}}^{0}$ is the comparison induced by the unit/ comparison $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{R}an_{\mathrm{j}_{\mathrm{Str}}}(\mathcal{A} \circ \mathrm{j}_{\mathrm{Str}})$.

Remark 6.11. As observed in Section 3, the Kan extension of a 2-functor $\mathcal{A} : \Delta \to \mathfrak{H}$ along j gives the equalizer of $\mathcal{A}(d^0)$ and $\mathcal{A}(d^1)$. This is a consequence of the isomorphism $\mathcal{L}an_{\mathrm{t}_2} * \cong *$, in which t_2 denotes the full inclusion of the category Δ_2,

$$\begin{tikzcd}
1 \arrow[r, shift left=3, "d^0"] \arrow[r, shift right=3, "d^1"'] & 2 \arrow[l, "s^0" description]
\end{tikzcd}$$

such that $s^0 d^0 = \mathrm{Id} = s^0 d^1$, into Δ.

Remark 6.12. [[28]] The 2-category CAT is CAT-complete. In particular, CAT has strict descent objects. More precisely, if $\mathcal{A} : \Delta_{\mathrm{Str}} \to \mathsf{CAT}$ is a 2-functor, then

$$\left\{\dot{\Delta}_{\mathrm{Str}}(\mathsf{0}, -), \mathcal{A}\right\} \cong [\Delta_{\mathrm{Str}}, \mathsf{CAT}] \left(\dot{\Delta}(\mathsf{0}, -), \mathcal{A}\right).$$

Thereby, we can describe the category the strict descent object of $\mathcal{A} : \Delta \to \mathsf{CAT}$ explicitly as follows:

(1) Objects are 2-natural transformations $\mathsf{W} : \dot{\Delta}_{\mathrm{Str}}(\mathsf{0}, -) \longrightarrow \mathcal{A}$. We have a bijective correspondence between such 2-natural transformations and pairs $(W, \varrho_{\mathsf{W}})$ in which W is an object of $\mathcal{A}(\mathsf{1})$ and $\varrho_{\mathsf{W}} : \mathcal{A}(d^1)(W) \to \mathcal{A}(d^0)(W)$ is an isomorphism in $\mathcal{A}(\mathsf{2})$ satisfying the following equations:
   - Associativity:

$$\left(\mathcal{A}(\partial^0)(\varrho_{\mathsf{W}})\right)\left(\mathcal{A}(\sigma_{02})_W\right)\left(\mathcal{A}(\partial^2)(\varrho_{\mathsf{W}})\right)\left(\mathcal{A}(\sigma_{12})_W^{-1}\right) = \left(\mathcal{A}(\sigma_{01})_W\right)\left(\mathcal{A}(\partial^1)(\varrho_{\mathsf{W}})\right)$$

   - Identity:

$$\left(\mathcal{A}(n_0)_W\right)\left(\mathcal{A}(s^0)(\varrho_{\mathsf{W}})\right)\left(\mathcal{A}(n_1)_W\right) = \mathrm{id}_W$$

   If $\mathsf{W} : \dot{\Delta}(\mathsf{0}, -) \longrightarrow \mathcal{A}$ is a 2-natural transformation, we get such pair by the correspondence $\mathsf{W} \mapsto (\mathsf{W}_1(d), \mathsf{W}_2(\vartheta))$.

(2) The morphisms are modifications. In other words, a morphism $\mathsf{m} : \mathsf{W} \to \mathsf{X}$ is determined by a morphism $\mathfrak{m} : W \to X$ such that $\mathcal{A}(d^0)(\mathfrak{m})\varrho_{\mathsf{W}} = \varrho_X \mathcal{A}(d^1)(\mathfrak{m})$.

7. Elementary Examples

In this section, we give elementary (though important) examples of inclusions $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ for which we can study the Ps-$\mathcal{R}an_{\mathrm{t}}(- \circ \mathrm{t})$-pseudoalgebras/ effective t-descent diagrams in the setting of Section 5. The examples of this section are such that a is the initial object of $\dot{\mathfrak{A}}$.

Let $\mathfrak{H}$ be a 2-category with enough bilimits to construct our pseudo-Kan extensions as global pointwise pseudo-Kan extensions. The most simple example is taking the final category $\mathsf{1}$ and the inclusion $\mathsf{0} \to \mathsf{1}$ of the empty category/ empty ordinal. In this case, a pseudofunctor $\mathcal{A} : \mathsf{1} \to \mathfrak{H}$ is of effective descent if and only if this pseudofunctor (which corresponds to an object of $\mathfrak{H}$) is equivalent to the pseudofinal object of $\mathfrak{H}$.

If, instead, we take the inclusion $d^0 : \mathsf{1} \to \mathsf{2}$ of the ordinal $\mathsf{1}$ into the ordinal $\mathsf{2}$ such that d^0 is the inclusion of the codomain object, then a pseudofunctor $\mathcal{A} : \mathsf{2} \to \mathfrak{H}$ corresponds to a 1-cell of $\mathfrak{H}$ and $\mathcal{A}$ is of effective d^0-descent if and only if its image is an equivalence 1-cell. Moreover, $\mathcal{A}$ is almost d^0-descent/ d^0-descent if and only if its image is faithful/ fully faithful. Precisely, the comparison morphism would be the image $\mathcal{A}(\mathsf{0} \xrightarrow{d} \mathsf{1})$ of the only nontrivial 1-cell of $\mathsf{2}$.

Furthermore, we may consider the following 2-categories $\dot{\mathfrak{B}}$. The first one corresponds to the bilimit notion of lax-pullback, while the second corresponds to the notion of pseudopullback.

$$\begin{tikzcd} \mathsf{b} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & e \arrow[d] \\ c \arrow[r] & o \end{tikzcd} \qquad \begin{tikzcd} \mathsf{b} \arrow[r] \arrow[d] & e \arrow[d] \\ c \arrow[r] & o \end{tikzcd}$$

Actually, we can study the exactness of any weighted bilimit in our setting. More precisely, if $\mathcal{W} : \mathfrak{A} \to \mathsf{CAT}$ is a weight, we can define $\dot{\mathfrak{A}}$ adding an extra object a and defining

$$\dot{\mathfrak{A}}(\mathsf{a},\mathsf{a}) := * \qquad \dot{\mathfrak{A}}(\mathsf{a}, b) := \mathcal{W}(b) \qquad \dot{\mathfrak{A}}(b,\mathsf{a}) := \varnothing$$

for each object b of $\mathfrak{A}$. Hence, it remains just to define the unique nontrivial composition, that is to say, we define the functor composition $\circ : \dot{\mathfrak{A}}(b, c) \times \dot{\mathfrak{A}}(\mathsf{a}, b) \to \dot{\mathfrak{A}}(\mathsf{a}, c)$ for each pair of objects b, c of $\mathfrak{A}$ to be the "mate" of

$$\mathcal{W}_{_{bc}} : \dot{\mathfrak{A}}(b, c) \to \mathsf{CAT}(\mathcal{W}(b), \mathcal{W}(c)).$$

Thereby, a pseudofunctor $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of effective t-descent/ t-descent/ almost t-descent if the canonical comparison 1-cell $\mathcal{A}(\mathsf{a}) \to \{\mathcal{W}, \mathcal{A}\circ \mathrm{t}\}_{\mathrm{bi}}$ is an equivalence/ fully faithful/ faithful.

8. Eilenberg-Moore Objects

Let $\mathfrak{H}$ be a 2-category as in the last sections. In [33], a 2-category Adj such that an adjunction in a 2-category corresponds to a 2-functor $\mathsf{Adj} \to \mathfrak{H}$ is described. This 2-category has a full sub-2-category with a full inclusion $\mathrm{m} : \mathsf{Mnd} \to \mathsf{Adj}$ such that monads of $\mathfrak{H}$ correspond to 2-functors $\mathsf{Mnd} \to \mathfrak{H}$. We describe this 2-category below, and we show how it (still) works in our setting. The 2-category Adj has two objects: alg and b. The hom-categories are defined as follows:

$$\mathsf{Adj}(\mathsf{b},\mathsf{b}) := \dot{\Delta} \qquad \mathsf{Adj}(\mathsf{alg},\mathsf{b}) := \Delta_{-} \qquad \mathsf{Adj}(\mathsf{alg},\mathsf{alg}) := \Delta_{-}^{+} \qquad \mathsf{Adj}(\mathsf{b},\mathsf{alg}) := \Delta^{+}$$

in which Δ_{-} denotes the subcategory of Δ with the same objects such that its morphisms preserve initial objects and, analogously, Δ_{+} is the subcategory of Δ with the same objects and last-element-preserving arrows. Finally, Δ_-^+ is just the intersection of both Δ_- and Δ^+.

Then the composition of Adj is such that $\mathsf{Adj}(\mathsf{b}, w) \times \mathsf{Adj}(c, \mathsf{b}) \to \mathsf{Adj}(c, w)$ is given by the usual "ordinal sum" + (given by the usual strict monoidal structure of Δ) for every objects c, w of Adj and

$$\begin{array}{ll} \mathsf{Adj}(\mathsf{alg}, w) \times \mathsf{Adj}(c, \mathsf{alg}) & \to \mathsf{Adj}(c, w) \\ (x, y) & \mapsto x + y - 1 \\ (\phi : x \to x', \upsilon : y \to y') & \mapsto \phi \oplus \upsilon \end{array}$$

in which

$$\phi \oplus \upsilon(i) := \begin{cases} \upsilon(i), & \text{if } i < y \\ \phi(i - m) - 1 + y' & \text{otherwise.} \end{cases}$$

It is straightforward to verify that Adj is a 2-category. We denote by u the 1-cell $1 \in \mathsf{Adj}(\mathsf{alg}, \mathsf{b})$ and by l the 1-cell $1 \in \mathsf{Adj}(\mathsf{b}, \mathsf{alg})$. Also, we consider the following 2-cells

$$\dot{\Delta}(0, 1) \ni n : \mathrm{id}_{\mathsf{b}} \Rightarrow ul \qquad\qquad \Delta_-^+(1, 2) \ni e : lu \Rightarrow \mathrm{id}_{\mathsf{alg}}$$

The 2-category Mnd is defined to be the full sub-2-category of Adj with the unique object b. As mentioned above, we denote its full inclusion by $\mathrm{m} : \mathsf{Mnd} \to \mathsf{Adj}$.

Firstly, observe that $(l \dashv u, n, e)$ is an adjunction in Adj, therefore the image of $(l \dashv u, n, e)$ by a 2-functor is an adjunction. Also, if $(L \dashv U, \eta, \varepsilon)$ is an adjunction in $\mathfrak{H}$, then there is a unique 2-functor $\mathcal{A} : \mathsf{Adj} \to \mathfrak{H}$ such that $\mathcal{A}(u) := U$, $\mathcal{A}(l) := L$, $\mathcal{A}(e) := \varepsilon$ and $\mathcal{A}(u) := \eta$. Thereby, it gives a bijection between adjunctions in $\mathfrak{H}$ and 2-functors $\mathsf{Adj} \to \mathfrak{H}$ [33].

Secondly, as observed in [33], there is a similar bijection between 2-functors $\mathsf{Mnd} \to \mathfrak{H}$ and monads in the 2-category $\mathfrak{H}$. Also, if the pointwise (enriched) Kan extension of a 2-functor $\mathsf{Mnd} \to \mathfrak{H}$ along m exists, it gives the usual Eilenberg-Moore adjunction. Moreover, given a 2-functor $\mathcal{A} : \mathsf{Adj} \to \mathfrak{H}$, if the pointwise Kan extension $\mathcal{R}an_{\mathrm{m}} (\mathcal{A} \circ \mathrm{m})$ exists, the usual comparison $\mathcal{A}(\mathsf{alg}) \to \mathcal{R}an_{\mathrm{m}} (\mathcal{A} \circ \mathrm{m}) (\mathsf{alg})$ is the Eilenberg-Moore comparison 1-cell.

If, instead, $\mathcal{A} : \mathsf{Adj} \to \mathfrak{H}$ is a pseudofunctor, we also get that $\mathcal{A}(l) \dashv \mathcal{A}(u)$ and

$$\left(\mathcal{A}(l) \dashv \mathcal{A}(u), \mathfrak{a}_{ul}^{-1} \mathcal{A}(n) \mathfrak{a}_{\mathsf{b}}, \mathfrak{a}_{\mathsf{alg}}^{-1} \mathcal{A}(e) \mathfrak{a}_{lu}\right)$$

is an adjunction in $\mathfrak{H}$. The unique 2-functor $\mathcal{A}'$ corresponding to this adjunction is pseudonaturally isomorphic to $\mathcal{A}$. Furthermore, the Eilenberg-Moore object is a flexible limit as it is shown in [3].

Proposition 8.1 ([3])**.** *If $\mathfrak{H}$ is any 2-category, $[\mathsf{Adj}, \mathfrak{H}] \to [\mathsf{Adj}, \mathfrak{H}]_{PS}$ is essentially surjective.*

Moreover, for any 2-functor $\mathcal{C} : \mathsf{Adj} \to \mathrm{CAT}$, we have an equivalence

$$[\mathsf{Adj}, \mathrm{CAT}]\left(\mathsf{Adj}(\mathsf{alg}, \mathrm{m}(-)), \mathcal{C}\right) \simeq [\mathsf{Adj}, \mathrm{CAT}]_{PS}\left(\mathsf{Adj}(\mathsf{alg}, \mathrm{m}(-)), \mathcal{C}\right).$$

Corollary 8.2. *If $\mathcal{A} : \mathsf{Mnd} \to \mathfrak{H}$ is a pseudofunctor,*

$$\mathrm{Ps}\mathcal{R}an_{\mathrm{j}_3}\mathcal{A} \simeq \mathrm{Ps}\mathcal{R}an_{\mathrm{m}}\check{\mathcal{A}} \simeq \mathcal{R}an_{\mathrm{m}}\check{\mathcal{A}}$$

provided that the pointwise Kan extension $\mathcal{R}an_{\mathrm{m}}\check{\mathcal{A}}$ exists, in which $\check{\mathcal{A}}$ is a 2-functor pseudonaturally isomorphic to $\mathcal{A}$.

Therefore, if $\mathfrak{H}$ has Eilenberg-Moore objects, then a pseudofunctor $\mathcal{A} : \mathsf{Adj} \to \mathfrak{H}$ is of effective m-descent/ m-descent if and only if $\mathcal{A}(u)$ is monadic/ premonadic. Also, the "factorizations"

$$\begin{tikzcd}
\mathcal{A}(\mathsf{b}) \arrow[rr, "\mathcal{A}(l)"] \arrow[dr, "l_{\mathcal{A}}"'] & \cong & \mathcal{A}(\mathsf{alg}) \\
& \mathrm{Ps}\mathcal{R}an_{\mathrm{m}}(\mathcal{A}\circ \mathrm{m})(\mathsf{alg}) \arrow[ur, "\eta^{\mathsf{alg}}_{\mathcal{A}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}(\mathsf{alg}) \arrow[rr, "\mathcal{A}(u)"] \arrow[dr, "\eta^{\mathsf{alg}}_{\mathcal{A}}"'] & \cong & \mathcal{A}(\mathsf{b}) \\
& \mathrm{Ps}\mathcal{R}an_{\mathrm{m}}(\mathcal{A}\circ \mathrm{m})(\mathsf{alg}) \arrow[ur, "u_{\mathcal{A}}"'] &
\end{tikzcd}$$

described in Theorem 5.15 are pseudonaturally equivalent to the usual Eilenberg-Moore factorizations. Henceforth, these factorizations are called Eilenberg-Moore factorizations (even if the 2-category $\mathfrak{H}$ does not have the strict version of it).

9. The Beck-Chevalley Condition

We keep our general setting in which $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ is a full inclusion as in Subsection 5.5 and $\mathfrak{H}$ is a 2-category such that all considered pseudo-Kan extensions can be constructed pointwise.

Let $\mathcal{T}$ be an idempotent pseudomonad over the 2-category $\mathfrak{H}$. The most obvious consequence of the commutativity results of Section 5 is the following: if an object X of $\mathfrak{H}$ can be endowed with a $\mathcal{T}$-pseudoalgebra structure and there is an equivalence $X \to W$, then W can be endowed with a $\mathcal{T}$-pseudoalgebra as well.

In our setting, we have the following: let $\mathcal{A}, \mathcal{B} : \dot{\mathfrak{A}} \to \mathfrak{H}$ be pseudofunctors. A pseudonatural transformation $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ can be seen as a pseudofunctor

$\mathcal{C}_\alpha : 2 \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$. By Corollaries 5.20 and 5.23, we get the following: if $\mathcal{C}_\alpha(1)$ is of effective t-descent/ t-descent/ almost t-descent and the images of the mate $\dot{\mathfrak{A}} \to [2, \mathfrak{H}]_{PS}$ of $\mathcal{C}_\alpha$ are of effective d^0-descent/ d^0-descent/ almost d^0-descent as well, then $\mathcal{C}_\alpha(0)$ is also of effective t-descent/ t-descent/ almost t-descent. In Section 10, we show that Theorem 1.1 is a particular case of:

Proposition 9.1. *Let $\mathcal{A}, \mathcal{B} : \dot{\mathfrak{A}} \to \mathfrak{H}$ be pseudofunctors and $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ be a pseudonatural transformation. If $\mathcal{B}$ is of effective* t*-descent/* t*-descent/ almost* t*-descent and α is a pseudonatural equivalence/ objectwise fully faithful/ objectwise faithful, then $\mathcal{A}$ is of effective* t*-descent/* t*-descent/ almost* t*-descent as well.*

Definition 9.2. [Beck-Chevalley condition] Let $\mathcal{A}, \mathcal{B} : \dot{\mathfrak{A}} \to \mathfrak{H}$ be pseudofunctors and $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ be a pseudonatural transformation. Assume that, for each object w of $\dot{\mathfrak{A}}$, $\alpha_{_w}$ is right adjoint to some 1-cell $\widehat{\alpha}^{^w}$. We say that α *satisfies the Beck-Chevalley condition* if, for each morphism $f : w \to c$ of $\dot{\mathfrak{A}}$, the mate of the invertible 2-cell $\alpha_{_f} : \mathcal{B}(f)\alpha_{_w} \Rightarrow \alpha_{_c}\mathcal{A}(f)$ w.r.t. the adjunction $\widehat{\alpha}^{^w} \dashv \alpha_{_w}$ is invertible.

By doctrinal adjunction [22], $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies the Beck-Chevalley condition if and only if α is itself a right adjoint in the 2-category $\left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$. In other words, we get:

Lemma 9.3 ([22])**.** *Let $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ be a pseudonatural transformation and $\mathcal{C}_\alpha : 2 \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ be its corresponding pseudofunctor. Consider the inclusion $\mathsf{u} : 2 \to \mathsf{Adj}$ of the morphism u. There is a pseudofunctor $\widehat{\mathcal{C}}_\alpha : \mathsf{Adj} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ such that $\widehat{\mathcal{C}}_\alpha \circ \mathsf{u} = \mathcal{C}_\alpha$ if and only if every component of α is right adjoint and α satisfies the Beck-Chevalley condition.*

Thereby, as trivial consequences of Corollaries 5.21 and 5.24, we get what can be called a generalized version of Bénabou-Roubaud Theorem:

Theorem 9.4. *Let $\mathcal{A}, \mathcal{B} : \dot{\mathfrak{A}} \to \mathfrak{H}$ be pseudofunctors. Assume that $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ is a pseudonatural transformation such that every component of α is right adjoint, α satisfies the Beck-Chevalley condition and, for each object w of $\mathfrak{A}$, $\alpha_{_w}$ is monadic.*

- *If $\mathcal{B}$ is of almost* t*-descent, then: $\widehat{\mathcal{C}}_\alpha : \mathsf{Adj} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ is of almost* m*-descent if and only if $\mathcal{A}$ is of almost* t*-descent;*

- *If* $\mathcal{B}$ *is of* t*-descent, then:* α_{a} *is premonadic if and only if* $\mathcal{A}$ *is of* t*-descent;*
- *If* $\mathcal{B}$ *is of effective* t*-descent, then:* α_{a} *is monadic if and only if* $\mathcal{A}$ *is of effective* t*-descent.*

Proof: Indeed, by the hypotheses, for each item, there is a pseudofunctor $\widehat{\mathcal{C}}_\alpha : \mathsf{Adj} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ satisfying the hypotheses of Corollary 5.21 or Corollary 5.24. ∎

10. Descent Theory

In this section, we establish briefly the setting of [19] and prove all the classical results mentioned in Section 1 for pseudocomsimplicial objects, except Theorem 1.2 which is postponed to Section 11.

Henceforth, let $\mathbb{C}, \mathbb{D}$ be categories with pullbacks and $\mathfrak{H}$ be a 2-category as in the last sections. In the context of [19], given a pseudofunctor $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathfrak{H}$, we say that a morphism $p : E \to B$ of $\mathbb{C}$ is of *effective* $\mathcal{A}$*-descent/* $\mathcal{A}$*-descent/ almost* $\mathcal{A}$*-descent* if $\mathcal{A}_p : \dot{\Delta} \to \mathfrak{H}$ is of effective j-descent/ j-descent/ almost j-descent, where $\mathcal{A}_p$ is the composition of the diagram

$$\mathcal{D}_p : \dot{\Delta}^{\mathrm{op}} \to \mathbb{C}$$

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & E \times_B E \times_B E \arrow[l, bend right, shift right=2] \arrow[l, bend left, shift left=2] \arrow[l, bend right=40, shift right=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & E \times_B E \arrow[l, bend right, shift right=2] \arrow[l, bend left, shift left=2] \arrow[r, shift left] \arrow[r, shift right] & E \arrow[l] \arrow[r, "p"] & B \end{tikzcd}$$

with the pseudofunctor $\mathcal{A}$, in which the diagram above is given by the pullbacks of p along itself, its projections and diagonal morphisms.

We get the usual factorizations of (Grothendieck) $\mathcal{A}$-descent theory [19] from Theorem 5.15, although the usual strict factorization comes from Remark 6.10. More precisely, if $p : E \to B$ is a morphism of $\mathbb{C}$, we get:

$$\begin{tikzcd} \mathcal{A}_p(\mathsf{0}) = \mathcal{A}(B) \arrow[rr, "\mathcal{A}(p)"] \arrow[dr, "\eta^{\mathsf{0}}_{\mathcal{A}\circ\mathcal{D}_p}"'] & \cong & \mathcal{A}_p(\mathsf{1}) = \mathcal{A}(E) \\ & \mathcal{D}esc_{\mathcal{A}}(p) \simeq \mathrm{Ps}\mathcal{R}an_{\mathsf{j}}(\mathcal{A}_p \circ \mathsf{j})(\mathsf{0}) \arrow[ur, "d_{\mathcal{A}_p}"'] & \end{tikzcd}$$

In descent theory, a *morphism* (U, α) between pseudofunctors $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathfrak{H}$ and $\mathcal{B} : \mathbb{D}^{\mathrm{op}} \to \mathfrak{H}$ is a pullback preserving functor $U : \mathbb{C} \to \mathbb{D}$ with a pseudonatural transformation $\alpha : \mathcal{A} \longrightarrow \mathcal{B} \circ U$. Such a morphism is called *faithful/ fully faithful* if α is objectwise faithful/ fully faithful.

For each morphism $p : E \to B$ of $\mathbb{C}$, a morphism (U, α) between pseudofunctors $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathfrak{H}$ and $\mathcal{B} : \mathbb{D}^{\mathrm{op}} \to \mathfrak{H}$ induces a pseudonatural transformation $\alpha^{p} : \mathcal{A}_p \longrightarrow \mathcal{B}_{U(p)}$. Of course, α^{p} is objectwise faithful/ fully faithful if (U, α) is faithful/ fully faithful.

We say that such a morphism (U, α) between pseudofunctors $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathfrak{H}$ and $\mathcal{B} : \mathbb{D}^{\mathrm{op}} \to \mathfrak{H}$ reflects almost descent/ descent/ effective descent morphisms if, whenever $U(p)$ is of almost $\mathcal{B}$-descent/ $\mathcal{B}$-descent/ effective $\mathcal{B}$-descent, p is of almost $\mathcal{A}$-descent/ $\mathcal{A}$-descent/ effective $\mathcal{A}$-descent.

Remark 10.1. Consider the pseudofunctor given by the basic fibration $(\ )^* : \mathbb{C}^{\mathrm{op}} \to \mathsf{CAT}$ in which

$$(p)^* : \mathbb{C}/B \to \mathbb{C}/E$$

is the change of base functor, given by the pullback along $p : E \to B$. For short, we say that a morphism $p : E \to B$ is of effective descent if p is of effective $(\ )^*$-descent.

In this case, a pullback preserving functor $U : \mathbb{C} \to \mathbb{D}$ induces a morphism (U, α) between the basic fibrations $(\ )^* : \mathbb{C}^{\mathrm{op}} \to \mathsf{CAT}$ and $(\ )^* : \mathbb{D}^{\mathrm{op}} \to \mathsf{CAT}$ in which, for each object B of $\mathbb{C}$, α_{B} is given by the evaluation of U. Again, in this case, if U is faithful/ fully faithful, so is the induced morphism (U, α) between the basic fibrations.

We study pseudocosimplicial objects $\mathcal{A} : \dot{\Delta} \to \mathfrak{H}$ and verify the obvious implications within the setting described above. We start with the embedding results (which are particular cases of 9.1):

Theorem 10.2 (Embedding Results)**.** *Let $\mathcal{A}, \mathcal{B} : \dot{\Delta} \to \mathfrak{H}$ be pseudofunctors and $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ a pseudonatural transformation. If α is objectwise faithful and $\mathcal{B}$ is of almost* j*-descent, then so is $\mathcal{A}$.*

If, furthermore, $\mathcal{B}$ is of j*-descent and α is objectwise fully faithful, then $\mathcal{A}$ is of* j*-descent.*

Of course, we have that, if $\mathcal{A} \simeq \mathcal{B}$, then $\mathcal{A}$ is of almost j-descent/ j-descent/ effective j-descent if and only if $\mathcal{B}$ is of almost j-descent/ j-descent/ effective j-descent as well.

Corollary 10.3. *Let (U, α) be a morphism between the pseudofunctors $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathfrak{H}$ and $\mathcal{B} : \mathbb{D}^{\mathrm{op}} \to \mathfrak{H}$ (as defined above).*

- *If (U, α) is faithful, it reflects almost descent morphisms;*
- *If (U, α) is fully faithful, it reflects descent morphisms;*

- *If α is a pseudonatural equivalence, (U, α) reflects and preserves effective descent morphisms, descent morphisms and almost descent morphisms.*

Every pseudofunctor $\mathcal{A} : \dot{\Delta} \to \mathfrak{H}$ gives rise to a pseudonatural transformation $\mathcal{C}_{\mathcal{A}} : 2 \to \left[\dot{\Delta}, \mathfrak{H}\right]_{PS}$, given by the mate of $\mathcal{A} \circ \mathrm{n} : 2 \times \dot{\Delta} \to \mathfrak{H}$, in which n is defined by

$$\mathrm{n} : 2 \times \dot{\Delta} \to \dot{\Delta}$$

$$(a, b) \mapsto b + a \qquad\qquad (d, \mathrm{id}_b) \mapsto \left(d^0 : b \to (b+1)\right)$$

$$(\mathrm{id}_a, d^i) \mapsto \begin{cases} d^i : b \to (b+1), & \text{if } a = 0 \\ d^{i+1} : (b+1) \to (b+2), & \text{otherwise} \end{cases}$$

$$(\mathrm{id}_a, s^i) \mapsto \begin{cases} s^i : b \to (b+1), & \text{if } a = 0 \\ s^{i+1} : (b+1) \to (b+2), & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
0 \arrow[r, "d"] \arrow[d, "d"'] & 1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "d^0"'] & 2 \arrow[l] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d, "d^0"'] & 3 \arrow[l, bend right, "s^0"'] \arrow[l, bend right=60, "s^1"'] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d, "d^0"'] & \cdots \arrow[l, bend right=20, "s^0"'] \arrow[l, bend right=45, "s^1"'] \arrow[l, bend right=70, "s^2"'] \arrow[d, "d^0"'] \\
1 \arrow[r, "d^1"'] & 2 \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d^2"'] & 3 \arrow[l, "s^1"] \arrow[r, shift left=3, "d^1"] \arrow[r, shift left=1, "d^2"] \arrow[r, shift right=1, "d^3"'] & 4 \arrow[l, bend left, "s^1"] \arrow[l, bend left=60, "s^2"] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \arrow[l, bend left=20, "s^1"] \arrow[l, bend left=45, "s^2"] \arrow[l, bend left=70, "s^3"]
\end{tikzcd}$$

We say that a pseudofunctor $\mathcal{A} : \dot{\Delta} \to \mathfrak{H}$ satisfies the descent of shift property (or just *shift property* for short) if the induced pseudonatural transformation $\mathcal{C}_{\mathcal{A}} : 2 \to \left[\dot{\Delta}, \mathfrak{H}\right]_{PS}$ is such that $\mathcal{C}_{\mathcal{A}}(1)$ is of effective j-descent. We get, then, a version of Bénabou-Roubaud Theorem for pseudocosimplicial objects:

Theorem 10.4. *Let $\mathcal{A} : \dot{\Delta} \to \mathfrak{H}$ be a pseudofunctor satisfying the shift property such that $\mathcal{A}(d^0 : b \to (b+1))$ has a left adjoint for every b. If the induced pseudonatural tranformation $\mathcal{C}_{\mathcal{A}} : 2 \to \left[\dot{\Delta}, \mathfrak{H}\right]_{PS}$ satisfies the Beck-Chevalley condition, then the Eilenberg-Moore factorization of $\mathcal{A}(d)$ is pseudonaturally equivalent to its usual factorization of* j*-descent theory. In particular,*

- *$\mathcal{A}$ is of effective* j*-descent iff $\mathcal{A}(d)$ is monadic;*
- *$\mathcal{A}$ is of* j*-descent iff $\mathcal{A}(d)$ is premonadic;*

- $\mathcal{A}$ *is of almost* j*-descent iff the Eilenberg-Moore comparison* 1*-cell of* $\mathcal{A}(d)$ *is faithful.*

It is known that in the context of [19], the shift property always holds. More precisely, it is known that [19]:

Proposition 10.5. *Let* $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathfrak{H}$ *be a pseudofunctor, in which* $\mathbb{C}$ *is a category with pullbacks. If* p *is a morphism of* $\mathbb{C}$*,* $\mathcal{A}_p$ *(defined above as* $\mathcal{A}_p := \mathcal{A} \circ \mathcal{D}_p$*) satisfies the shift property.*

Thereby, by Theorem 10.4, the usual Bénabou-Roubaud Theorem (Theorem 1.3) follows. And, finally, the most obvious consequence of the commutativity properties is that bilimits of effective j-descent diagrams are effective j-descent diagrams. For instance, taking into account Remark 10.1 and realizing that pseudopullbacks of functors induce pseudopullback of overcategories we already get a weak version of Theorem 1.4.

Next section, we study stronger results on bilimits and apply them to descent theory.

11. Further on Bilimits and Descent

Henceforth, let $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$, $\mathrm{h} : \mathfrak{B} \to \dot{\mathfrak{B}}$ be full inclusions of small 2-categories as in Subsection 5.5 and let $\mathfrak{H}$ be a bicategorically complete 2-category.

Definition 11.1. [Pure Structure] A morphism $f : \mathsf{a} \to b$ of $\dot{\mathfrak{A}}$ is called a t-*irreducible* morphism if $b \neq \mathsf{a}$ and the 1-cells of $\dot{\mathfrak{A}}(\mathsf{a}, b)$ are not in the image of

$$\circ : \dot{\mathfrak{A}}(c, b) \times \dot{\mathfrak{A}}(\mathsf{a}, c) \to \dot{\mathfrak{A}}(\mathsf{a}, b),$$

for every $b \neq c$ in $\mathfrak{A}$.

An object c of $\mathfrak{A}$ is called a t-*pure structure object* if each 1-cell g of $\dot{\mathfrak{A}}(\mathsf{a}, c)$ can be factorized through some t-irreducible morphism $f : \mathsf{a} \to b$ such that $b \neq c$. That is to say, c is a t-pure structure object, if for all $g \in \dot{\mathfrak{A}}(\mathsf{a}, c)$ there are a morphism g' and a t-irreducible morphism f such that $g'f = g$.

The full subcategory of the t-pure structure objects of $\mathfrak{A}$ is denoted by $\mathfrak{S}_{\mathrm{t}}$, while the full subcategory of $\dot{\mathfrak{A}}$ of the objects that are not in $\mathfrak{S}_{\mathrm{t}}$ (including a) is denoted by $\dot{\mathfrak{I}}_{\mathrm{t}}$. We also denote by $\mathfrak{I}_{\mathrm{t}}$ the full subcategory of $\dot{\mathfrak{I}}_{\mathrm{t}}$ without the object a. We have full inclusions $\dot{\mathrm{i}}_{\mathrm{t}} : \dot{\mathfrak{I}}_{\mathrm{t}} \to \dot{\mathfrak{A}}$ and $\mathrm{i}_{\mathrm{t}} : \mathfrak{I}_{\mathrm{t}} \to \mathfrak{A}$.

Of course, in particular, if $f : \mathsf{a} \to b$ is a t-irreducible morphism of $\dot{\mathfrak{A}}$, then b is an object of $\mathfrak{I}_{\mathrm{t}}$. We denote by $\mathfrak{g}_{\mathrm{t}} : \overline{\mathfrak{I}_{\mathrm{t}} \times 2} \to \mathfrak{I}_{\mathrm{t}} \times 2$ the full inclusion in which

$$\mathrm{obj}\left(\overline{\mathfrak{I}_{\mathrm{t}} \times \mathsf{2}}\right) := \mathrm{obj}\left(\mathfrak{I}_{\mathrm{t}} \times \mathsf{2}\right) - \left\{(\mathsf{a}, 0)\right\}.$$

Theorem 11.2. *Let* $\mathcal{A}, \mathcal{B} : \dot{\mathfrak{A}} \to \mathfrak{H}$ *be pseudofunctors and* $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ *be an objectwise fully faithful pseudonatural transformation. We assume that* $\mathcal{B}$ *is of effective* t*-descent. We consider the mate of* α*, denoted by* $\mathcal{C}_\alpha : \dot{\mathfrak{A}} \times \mathsf{2} \to \mathfrak{H}$*. The pseudofunctor* $\mathcal{A}$ *is of effective* t*-descent if and only if* $\mathcal{C}_\alpha \circ (\mathrm{i}_{\mathrm{t}} \times \mathrm{Id}_{\mathsf{2}}) : \mathfrak{I}_{\mathrm{t}} \times \mathsf{2} \to \mathfrak{H}$ *is of effective* $\mathfrak{g}_{\mathrm{t}}$*-descent.*

Proof: Without loosing generality, we prove it to $\mathfrak{H} = \mathsf{CAT}$ and get the general result via representable 2-functors. We just need to prove that $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}\mathcal{A} \circ \mathrm{t}(\mathsf{a})$ is equivalent to $\mathrm{Ps}\mathcal{R}an_{\mathfrak{g}_{\mathrm{t}}}\left(\mathcal{C}_\alpha \circ (\mathrm{i}_{\mathrm{t}} \times \mathrm{Id}_{\mathsf{2}}) \circ \mathfrak{g}_{\mathrm{t}}\right)(\mathsf{a}, 0)$.

The pseudonatural transformations $\varrho' : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \to \mathcal{A} \circ \mathrm{t}$ are obviously in bijection with the pseudonatural transformations $\varrho : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \longrightarrow \mathcal{B} \circ \mathrm{t}$ that can be factorized through $\alpha\mathrm{t}$, since $\alpha\mathrm{t}$ is objectwise fully faithful. Also, given $\varrho : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \longrightarrow \mathcal{B} \circ \mathrm{t}$, there exists $\varrho' : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \to \mathcal{A} \circ \mathrm{t}$ such that $\varrho \cong (\alpha\mathrm{t})\varrho'$ if and only if the image of $(\alpha\mathrm{t})_b$ is essentially surjective onto the image of ϱ_b for every b of $\mathfrak{A}$. Also, if such ϱ' exists, it is unique up to isomorphism: it is the pseudopullback of ϱ along $(\alpha\mathrm{t})$.

But, actually, we claim that, for the existence of such ϱ', it is (necessary and) sufficient $(\alpha\mathrm{t})_b$ be essentially surjective onto the image of ϱ_b for every object b of $\mathfrak{I}_{\mathrm{t}}$. That is to say, we just need to verify the lifting property for the objects in $\mathfrak{I}_{\mathrm{t}}$.

Indeed, assume that $\varrho\mathrm{i}_{\mathrm{t}}$ can be lifted by $\alpha\mathrm{t}\mathrm{i}_{\mathrm{t}}$. Given an object c of $\mathfrak{S}_{\mathrm{t}}$ and a morphism $g : \mathsf{a} \to c$, we prove that $\varrho_c(g)$ is in the image of $(\alpha\mathrm{t})_c$ up to isomorphism. Actually, there is a t-irreducible morphism $f : \mathsf{a} \to b$ such that $g'f = f$ for some $g' : b \to c$ morphism of $\mathfrak{A}$, and, by hypothesis, there is an object u of $\mathcal{A}(b)$ such that $(\alpha\mathrm{t})_b(u) \cong \varrho_b(f)$, thereby:

$$\varrho_c(g) = \varrho_c \cdot \left(\dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(g'))\right)(f) \cong \mathcal{B}(g')\varrho_b(f) \cong \mathcal{B}(g')\,(\alpha\mathrm{t})_b\,(u) \cong (\alpha\mathrm{t})_c\left(\mathcal{A}(g')(u)\right)$$

This completes the proof that it is enough to test the lifting property for the objects in $\mathfrak{I}_{\mathrm{t}}$. Now, one should observe that, since $\mathcal{B}$ is of effective t-descent, a pseudonatural transformation

$$\mathfrak{I}_{\mathrm{t}} \times \mathsf{2}((\mathsf{a}, 0), \mathfrak{g}_{\mathrm{t}}-) \longrightarrow \mathcal{C}_\alpha \circ (\mathrm{i}_{\mathrm{t}} \times \mathrm{Id}_{\mathsf{2}}) \circ \mathfrak{g}_{\mathrm{t}}$$

is precisely determined (up to isomorphism) by a pseudonatural transformation
$$\varrho : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \longrightarrow \mathcal{B} \circ \mathrm{t}.$$
(*i.e.*, an object of $\mathcal{B}(\mathsf{a})$), such that $\varrho \mathrm{i}_{\mathrm{t}}$ can be lifted by $\alpha \mathrm{t} \mathrm{i}_{\mathrm{t}}$. That is to say, as we proved, this is just a pseudonatural transformation
$$\varrho' : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \to \mathcal{A} \circ \mathrm{t}.$$
■

We return to the context of Subsection 5.1 and Subsection 5.2. Let $\mathcal{T}$ be an idempotent pseudomonad on a 2-category $\mathfrak{H}$ and X be an object of $\mathfrak{H}$. We say that X is of $\mathcal{T}$-*descent* if the comparison $\eta_X : X \to \mathcal{T}(X)$ is fully faithful. It is important to note that, if $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of t-descent (following Definition 5.22), then $\mathcal{A}$ is of $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(- \circ \mathrm{t})$-descent.

Corollary 11.3. *Let $\mathcal{T}$ be an idempotent pseudomonad on a bicategorically complete 2-category $\mathfrak{H}$ and $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ be a pseudofunctor such that $\mathcal{A}(b)$ is a $\mathcal{T}$-descent object for every b in $\mathfrak{A}$ and both $\mathcal{A}, \mathcal{T} \circ \mathcal{A}$ are of effective* t*-descent. We assume that $\mathcal{A}(b)$ can be endowed with a $\mathcal{T}$-pseudoalgebra structure for every object $b \notin \mathfrak{S}_{\mathrm{t}}$ in $\mathfrak{A}$. Then $\mathcal{A}(\mathsf{a})$ can be endowed with a $\mathcal{T}$-pseudoalgebra structure.*

Corollary 11.4. *Let $\mathcal{A} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS}$ be a effective* t*-descent pseudofunctor such that all the pseudofunctors in the image of $\mathcal{A} \circ \mathrm{t}$ are of* h*-descent. Furthermore, we assume that $\mathcal{A}(b)$ is of effective* h*-descent for every $b \notin \mathfrak{S}_{\mathrm{t}}$ in $\mathfrak{A}$. Then $\mathcal{A}(\mathsf{a})$ is of effective* h*-descent.*

Recall the following full inclusion of 2-categories $\mathrm{h} : \mathfrak{B} \to \dot{\mathfrak{B}}$ described in Section 7.

$$\begin{tikzcd} & e \arrow[d] \\ c \arrow[r] & o \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathsf{b} \arrow[r] \arrow[d] & e \arrow[d] \\ c \arrow[r] & o \end{tikzcd} \tag{$\mathfrak{P}$}$$

As explained there, a diagram $\dot{\mathfrak{B}} \to \mathfrak{H}$ is of effective h-descent if and only if it is a pseudopullback. In this case, the unique object in $\mathfrak{S}_{\mathrm{h}}$ is o. Thereby we get:

Corollary 11.5. *Assume that $\mathcal{A} : \dot{\mathfrak{B}} \to [\dot{\mathfrak{A}}, \mathfrak{H}]_{PS}$ is a pseudopullback diagram. If $\mathcal{A}(c), \mathcal{A}(e) : \dot{\mathfrak{A}} \to \mathfrak{H}$ are of effective* t*-descent and $\mathcal{A}(o) : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of* t*-descent, then $\mathcal{A}(\mathsf{b})$ is of effective* t*-descent.*

Taking into account Remark 10.1 and realizing that pseudopullbacks of functors induce pseudopullback of overcategories, we get Theorem 1.4 as a corollary.

11.1. Applications. In this subsection, we finish the paper giving applications of our results and proving the remaining theorems presented in Section 1. Firstly, considering our inclusion $\mathrm{j} : \Delta \to \dot{\Delta}$, it is important to observe that $\mathsf{1} \notin \mathfrak{S}_{\mathrm{j}}$, while all the other objects of Δ belong to $\mathfrak{S}_{\mathrm{j}}$. We start proving Theorem 4.2 of [16], which is presented therein as a generalized Galois Theorem.

Theorem 11.6 (Galois). *Let $\mathcal{A}, \mathcal{B} : \dot{\Delta} \to \mathsf{CAT}$ be pseudofunctors and $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ be an objectwise fully faithful pseudonatural transformation. We assume that $\mathcal{B}$ is of effective* j*-descent. The pseudofunctor $\mathcal{A}$ is also of effective* j*-descent if and only if the diagram below is a pseudopullback.*

$$\begin{tikzcd}
\mathcal{A}(\mathsf{0}) \arrow[r, "\mathcal{A}(d)"] \arrow[d, "\alpha_{\mathsf{0}}"'] \arrow[dr, phantom, "\overset{\alpha_d}{\Longrightarrow}"] & \mathcal{A}(\mathsf{1}) \arrow[d, "\alpha_{\mathsf{1}}"] \\
\mathcal{B}(\mathsf{0}) \arrow[r, "\mathcal{B}(d)"'] & \mathcal{B}(\mathsf{1})
\end{tikzcd}$$

Proof: Since, in this case, $\mathfrak{I}_{\mathrm{j}} = \mathsf{2}$ and the inclusion $\mathfrak{g}_{\mathrm{j}} : \overline{\mathfrak{I}_{\mathrm{j}} \times \mathsf{2}} \to \mathfrak{I}_{\mathrm{j}} \times \mathsf{2}$ is precisely equal to the inclusion described in the diagram $\mathfrak{P}$, by Theorem 11.2, the proof is complete. ∎

As a consequence of Theorem 11.6, we get a generalization of Theorem 1.2. More precisely, in the context of Section 10 and using the definitions presented there, we get:

Corollary 11.7. *Let (U, α) be a fully faithful morphism between pseudofunctors $\mathcal{A} : \mathbb{C}^{\mathrm{op}} \to \mathfrak{H}$ and $\mathcal{B} : \mathbb{D}^{\mathrm{op}} \to \mathfrak{H}$, in which $\mathbb{C}$ and $\mathbb{D}$ are categories with pullbacks. Assume that $U(p)$ is an effective $\mathcal{B}$-descent morphism of $\mathbb{D}$. Then $p : E \to B$ is of effective $\mathcal{A}$-descent if and only if, whenever there are $u \in \mathcal{B}(B), v \in \mathcal{A}(E)$ such that $\alpha^p_{\mathsf{1}}(u) \cong \mathcal{B}_{U(p)}(d)(v)$, there is $w \in \mathcal{A}(B)$ such that $\alpha^p_{\mathsf{0}}(w) \cong u$.*

Proof: Recall the definitions of $\mathcal{A}_p, \mathcal{B}_{U(p)}, \alpha^p$. Since we already know that $\mathcal{A}_p$ is j-descent, the condition described is precisely the condition necessary and sufficient to conclude that the diagram of Theorem 11.6 is a pseudopullback. ∎

Indeed, taking into account Remark 10.1, we conclude that Theorem 1.2 is actually a immediate consequence of last corollary.

Given a category with pullbacks V, we denote by $\mathsf{Cat}(V)$ the category of internal categories in V. If V is a category with products, we denote by V-Cat the category of small categories enriched over V. We give a simple application of the Theorem 1.4 below.

Lemma 11.8. *If $(V, \times, I)$ is an infinitary lextensive category such that*

$$\begin{array}{rcl} J : \mathsf{Set} & \to & V \\ A & \mapsto & \sum\limits_{a\in A} I_a \end{array}$$

is fully faithful, then the pseudopullback of the projection of the object of objects $U_0 : \mathsf{Cat}(V) \to V$ along J is the category V-Cat.

Proof: We denote by *Span*(V) the usual bicategory of objects of V and spans between them and by V-Mat the usual bicategory of sets and V-matrices between them. Let $\mathit{Span}_{\mathsf{Set}}(V)$ be the full sub-bicategory of *Span*(V) in which the objects are in the image of Set.

Assuming our hypotheses, we have that $\mathit{Span}_{\mathsf{Set}}(V)$ is biequivalent to V-Mat. Indeed, we define "identity" on the objects and, if A, B are sets, take a matrix $M : A \times B \to \mathrm{obj}(V)$ to the obvious span given by the coproduct $\sum\limits_{(x,y)\in A\times B} M(x,y)$, that is to say, the morphism $\sum_{(x,y)\in A\times B} M(x,y) \to A$ is induced by the morphisms $M(x,y) \to I_x$ and the morphism $\sum_{(x,y)\in A\times B} M(x,y) \to B$ is analogously defined.

Since V is lextensive, this defines a biequivalence. Thereby this completes our proof. ∎

Corollary 6.2.5 of [26] says in particular that, for lextensive categories, effective descent morphisms of $\mathsf{Cat}(V)$ are preserved by the projection $U_0 : \mathsf{Cat}(V) \to V$ to the objects of objects. Thereby, by Theorem 1.4, we get:

Theorem 11.9. *If $(V, \times, I)$ is an infinitary lextensive category such that each arrow of V can be factorized as a regular epimorphism followed by a monomorphism and*

$$\begin{array}{rcl} J : \mathsf{Set} & \to & V \\ A & \mapsto & \sum\limits_{a\in A} I_a \end{array}$$

is fully faithful, then $I : V\text{-}\mathsf{Cat} \to \mathsf{Cat}(V)$ *reflects effective descent morphisms.*

Proof: We denote by $U : V\text{-}\mathsf{Cat} \to \mathsf{Set}$ the forgetful functor and by $U_0 : \mathsf{Cat}(V) \to V$ the projection defined above. We have that U_0, U, J and I are pullback preserving functors.

If $p : E \to B$ is a morphism of V-Cat such that $I(p)$ is of effective descent, then $U_0 I(p)$ is of descent (by Corollary 5.2.1 of [26]). Therefore $JU(p)$ is of descent.

Since J is fully faithful, by Theorem 10.3, $U(p)$ is of descent. Therefore, since descent morphisms of Set are of effective descent, we conclude that $U(p)$ is of effective descent. This completes the proof. $\blacksquare$

For instance, Theorem 6.2.8 of [26] and Proposition 11.9 can be applied to the cases of $V = \mathsf{Cat}$ or $V = \mathsf{Top}$:

Corollary 11.10. *A* 2*-functor* F *between* Cat*-categories is of effective descent in* Cat-Cat*, if*

- F *is surjective on objects;*
- F *is surjective on composable triples of* 2*-cells;*
- F *induces a functor surjective on composable pairs of* 2*-cells between the categories of composable pairs of* 1*-cells;*
- F *induces a functor surjective on* 2*-cells between the categories of composable triples of* 1*-cells.*

Corollary 11.11. *A* Top*-functor* F *between* Top*-categories is of effective descent in* Top-Cat*, if* F *induces*

- *effective descent morphisms between the discrete spaces of objects and between the spaces of morphisms in* Top*;*
- *a descent continuous map between the spaces of composable pairs of morphisms in* Top*;*
- *an almost descent continuous map between the spaces of composable triples of morphisms in* Top.

Since the characterization of (effective/ almost) descent morphisms in Top is known [32, 8, 6], the result above gives effective descent morphisms of Top-Cat.

Remark 11.12. We can give further formal results on (basic) effective descent morphisms (context of Remark 10.1). The main technique in this case is to understand our overcategory as a bilimit of other overcategories.

For instance, we study below the categories of morphisms of a given category $\mathbb{C}$ with pullbacks. Consider the full inclusion of 2-categories $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$

$$\begin{tikzcd} \mathsf{0} \arrow[d, "d"] \\ \mathsf{1} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathsf{a} \arrow[rr, "pro_1"] \arrow[dr, "pro_0"'] & \arrow[d, phantom, "\overset{\xi}{\Rightarrow}"] & \mathsf{0} \arrow[dl, "d"] \\ & \mathsf{1} & \end{tikzcd}$$

Given a morphism of $\mathbb{C}$, *i.e.* a functor $F : \mathsf{2} \to \mathbb{C}$, we take the overcategory $Fun(\mathsf{2}, \mathbb{C})/F$ and define $\mathcal{A} : \dot{\mathfrak{A}} \to \mathsf{CAT}$ in which

$$\mathcal{A}(\mathsf{a}) := Fun(\mathsf{2}, \mathbb{C})/F \qquad \mathcal{A}(\mathsf{0}) := \mathbb{C}/F(\mathsf{1}) \qquad \mathcal{A}(\mathsf{1}) := \mathbb{C}/F(\mathsf{0}).$$

Finally, $\mathcal{A}(pro_{_0}), \mathcal{A}(pro_{_1})$ are given by the obvious projections, $\mathcal{A}(d) := F(d)^*$ and the component $\mathcal{A}(\xi)$ in a morphism $\varpi : H \to F$ is given by the induced morphism from $H(\mathsf{0})$ to the pullback.

Observe that $\mathcal{A}$ is of effective t-descent, that is to say, we have that the overcategory $Fun(\mathsf{2}, \mathbb{C})/F$ is a bilimit constructed from overcategories $\mathbb{C}/F(\mathsf{0})$ and $\mathbb{C}/F(\mathsf{1})$. Also, given a natural transformation $\varpi : F \to G$ between functors $\mathsf{2} \to \mathbb{C}$, *i.e.* a morphism of $Fun(\mathsf{2}, \mathbb{C})$, taking Remark 10.1, we can extend $\mathcal{A}$ to a 2-functor $\overline{\mathcal{A}} : \dot{\mathfrak{A}} \to [\dot{\Delta}, \mathsf{CAT}]$ in which $\overline{\mathcal{A}}(\mathsf{a}) := (\ )^*_{\varpi}$, $\overline{\mathcal{A}}(\mathsf{0}) := (\ )^*_{\varpi_1}$ and $\overline{\mathcal{A}}(\mathsf{1}) := (\ )^*_{\varpi_0}$.

The 2-functor $\overline{\mathcal{A}}$ is also of effective t-descent. Therefore, by our results, we conclude that, *if the components* ϖ_1, ϖ_0 *are of (basic) effective descent, so is* ϖ. Analogously, considering the category of spans in $\mathbb{C}$, the morphisms between spans which are objectwise of effective descent are of effective descent.

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