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KZ-MONADIC CATEGORIES AND THEIR LOGIC

JIŘÍ ADÁMEK AND LURDES SOUSA

Abstract: Given an order-enriched category, it is known that all its KZ-monadic subcategories can be described by Kan-injectivity with respect to a collection of morphisms. We prove the analogous result for Kan-injectivity with respect to a collection $\mathcal{H}$ of commutative squares. A square is called a Kan-injective consequence of $\mathcal{H}$ if by adding it to $\mathcal{H}$ Kan-injectivity is not changed.

We present a sound logic for Kan-injectivity consequences and prove that in "reasonable" categories (such as Pos or Top_0) it is also complete for every set $\mathcal{H}$ of squares.

1. Introduction

Scott's continuous lattices were characterized by Escardó as precisely those topological T_0-spaces that are Kan-injective with respect to all embeddings, see [8]. The concept of Kan-injectivity of an object X of an order-enriched category with respect to a morphism $h : A \to B$ Escardó defined as follows: for every morphism $f : A \to X$ we have a commutative triangle

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[dl, "f/h"] \\ X & \end{tikzcd}$$

where f/h is the left Kan-extension of f along h. That is, if $g : B \to X$ fulfils $f \leq gh$, then $f/h \leq g$.

Later Carvalho and Sousa [6] extended the above concept from objects to morphisms: a morphism $u : X \to X'$ is Kan-injective with respect to h when X and X' are Kan-injective objects and for every morphism $f : A \to$

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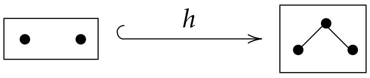
X the following triangle

$$\begin{tikzcd}
B \arrow[r, "f/h"] \arrow[d, "(uf)/h"'] & X \arrow[dl, "u"] \\
X' &
\end{tikzcd}$$

commutes. Example: the morphisms of Top_0 Kan-injective with respect to all embeddings are precisely the continuous maps between continuous lattices preserving all infima. A trivial example: in Pos Kan-injectivity with respect to



defines the subcategory of join-semilattices and their homomorphisms.

Moreover, in locally ranked categories, e.g., in Top_0 or Pos, Kan-injectivity yields a characterization of KZ-monadic categories in the following sense: given a class $\mathcal{H}$ of morphisms, let $\mathsf{LInj}(\mathcal{H})$ be the subcategory of all objects and all morphisms Kan-injective with respect to all members of $\mathcal{H}$. Then

(1) every KZ-monadic subcategory has the form $\mathsf{LInj}(\mathcal{H})$ for some class $\mathcal{H}$ of morphisms, see [6],

and

(2) for every set $\mathcal{H}$ of morphisms the subcategory $\mathsf{LInj}(\mathcal{H})$ is KZ-monadic, see [4].

The topic of our paper is the logic of Kan-injectivity, generalizing the logic of orthogonality studied in [1] and [2]. Observe first that given an ordinary category with its trivial order-enrichment (that is, equality), then $\mathsf{LInj}(\mathcal{H})$ is nothing else than the full subcategory $\mathcal{H}^{\perp}$ on objects X orthogonal to every member $h : A \to B$ of $\mathcal{H}$ (that is, each $f : A \to X$ has a unique factorization through h). The logic of orthogonality aims to characterize those morphisms h for which orthogonality to $\mathcal{H}$ implies that to h, i.e. with $\mathcal{H}^{\perp} = (\mathcal{H} \cup \{h\})^{\perp}$. See Section 4 where the simple logic presented in [1] is recalled.

Analogously, we hoped to present a logic that would characterize, in order-enriched categories, those morphisms for which Kan-injectivity with respect to $\mathcal{H}$ implies that with respect to h. But we have failed so far. What saved our effort was the idea to "enrich" our language by considering, instead of Kan-injectivity with respect to morphisms, Kan-injectivity with respect to commutative squares S:

$$\begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\
A_2 \arrow[r, "h_2"'] & B_2
\end{tikzcd}$$

An object X is *Kan-injective with respect to* S if it is Kan-injective with respect to h_1 and h_2 and for every morphism $f : A_2 \to X$ the following triangle

$$\begin{tikzcd}
B_1 \arrow[d, "b"'] \arrow[dr, "(fa)/h_1"] & \\
B_2 \arrow[r, "f/h_2"'] & X
\end{tikzcd}$$

commutes. And a morphism is Kan-injective with respect to S iff it is Kan-injective with respect to h_1 and h_2.

For every class $\mathcal{H}$ of commutative squares we thus obtain a (non-full) subcategory $\mathsf{LInj}(\mathcal{H})$ analogously to above. These categories characterize again KZ-monadicity: we prove in Section 3 below that the above statements (1) and (2) remain valid. In other words, this richer language does *not* lead to more examples! However, it enables a formulation of a sound logic (see Section 4) which for sets of squares is, under mild size conditions, also complete (see Section 5). It is our present impression that this enrichment of the structure from morphisms to commutative squares is probably necessary: we suspect that no logic for Kan-injectivity with respect to just morphisms is sound and complete.

2.KZ-monadic subcategories

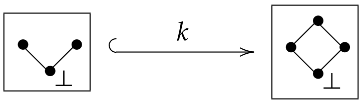
Assumption 2.1. Throughout the paper $\mathcal{X}$ is a category enriched over Pos. All squares in our paper are commutative, so when stating that something is a square, we mean a commutative one.

We introduced Kan-injectivity with respect to a morphism and a square above. Observe that the latter is a generalization of the former: for every morphism $h : A \to B$ let $S(h)$ be the following square

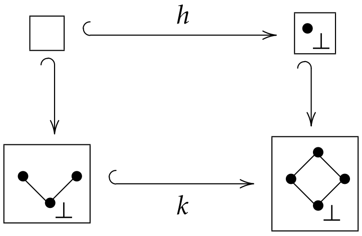
$$S(h) = \begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

Then Kan-injectivity with respect to h and $S(h)$ is the same concept for objects and morphisms.

Example 2.2. In Pos consider Kan-injectivity with respect to $h : \mathbf{0} \hookrightarrow \mathbf{1}$, the empty map into the terminal poset. This means the existence (and preservation) of the least element, $\perp$. Kan-injectivity with respect to the embedding

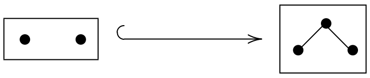


characterizes existence (and preservation) of joins of pairs having a lower bound. Combining those two in a square as follows



yields by Kan-injectivity join-semilattices with $\perp$ and their homomorphisms.

We have already remarked that using squares does not lead to new examples. Indeed, join-semilattices with $\perp$ are also given by Kan-injectivity with respect to h and the following embedding



Example 2.3. Which posets are Kan-injective with respect to all order-embeddings (i.e., regular monomorphisms) in Pos?

As shown in [4], these are precisely the complete lattices, and a monotone map is Kan-injective with respect to order-embeddings iff it preserves joins. We now characterize all squares S that are Kan-injectivity consequences of order-embeddings. That is, such that every complete lattice and every join-preserving map between complete lattices is Kan-injective with respect to S.

Let us denote by Ω_0 the contravariant endofunctor of Pos assigning to every posed X the posed $\Omega_0 X$ of all $\downarrow$-sets of X, and to every monotone function $f : X \to Y$ the function $\Omega_0 f : \Omega_0 Y \to \Omega_0 X$ forming preimages. Observe that $\Omega_0 f$ has a left adjoint

$$(\Omega_0 f)^* : \Omega_0 X \to \Omega_0 Y, \qquad U \mapsto \downarrow f[U] \quad (\text{for all } U \in \Omega_0 X).$$

Proposition 2.4. *For every square*

$$S = \begin{tikzcd} A' \arrow[r, "h"] \arrow[d, "a"'] & B' \arrow[d, "b"] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$

in Pos *the following conditions are equivalent:*

(1) *S is a Kan-injectivity consequence of order-embeddings;*

(2) *h and k are order-embeddings such that whenever $k(z) \le b(y)$, there exists $x \in A'$ with $h(x) \le y$ and $z \le a(x)$;*

and

(3) *h and k are order-embeddings yielding a (commutative) square*

$$\begin{tikzcd} \Omega_0 B' \arrow[r, "\Omega_0 h"] \arrow[d, "(\Omega_0 b)^*"'] & \Omega_0 A' \arrow[d, "(\Omega_0 a)^*"] \\ \Omega_0 B \arrow[r, "\Omega_0 k"'] & \Omega_0 A \end{tikzcd}$$

Proof. The equivalence of (1) and (2) was proved in [14].

$3 \Rightarrow 2$ The inequality $k(z) \le b(y)$ states precisely that

$$z \in \Omega_0 k \cdot (\Omega_0 b)^*(\downarrow y) = (\Omega_0 a)^* \cdot \Omega_0 h(\downarrow y),$$

that is, z lies in $\downarrow a[\Omega_0 h(\downarrow y)]$. This means that an x as in (2) exists.

$2 \Rightarrow 3$ It is easy to verify that the inequality

$$(\Omega_0 a)^* \cdot \Omega_0 h \le (\Omega_0 k) \cdot (\Omega_0 b)^*$$

holds for every square S. Thus we need to show the opposite inequality only. Let V be a $\downarrow$-set of B' and z be an element of $(\Omega_0 k)(\Omega_0 b)^*(V)$, i.e., $z \in k^{-1}[\downarrow b[V]]$. Then for some $y \in V$ we have $k(z) \le b(y)$. Given x as in (2) we conclude $z \in (\Omega_0 a)^* \Omega_0 h(V)$, establishing the required inequality. ■

Example 2.5. The category Top_0 of topological T_0-spaces is order enriched via the opposite of the specialization order. Recall that the specialization order is given by $x \sqsubseteq y$ iff $x \in \overline{\{y\}}$. Thus, for continuous functions $f, g : X \to Y$ we define

$$f \leq g \qquad \text{iff} \qquad g(x) \sqsubseteq f(x) \qquad \text{for all } \ x \in X.$$

Escardó and Flagg proved that Kan-injectivite objects with respect to topological embeddings (= regular monomorphisms) are precisely Scott's continuous lattices, see [9]. Recall that a T_0-space is a continuous lattice iff for the specialization order its topology is the Scott's one, and it is a complete lattice with

$$y = \bigsqcup_{U \in \mathrm{nbh}(y)} (\sqcap U) \qquad \text{for all } \ y \in X.$$

The morphisms Kan-injective with respect to topological embeddings are precisely the continuous functions preserving meets, see [6]. We characterize the squares which are injectivity consequences of topological embeddings. Let $\Omega : \mathsf{Top}_0 \to \mathsf{Pos}^{\mathrm{op}}$ be the functor assigning to a space X the poset ΩX of open sets and to a continuous function $f : X \to Y$ the preimage function $\Omega f : \Omega Y \to \Omega X$. By $(\Omega f)_*$ we denote the right adjoint of Ωf: to an open set U it assigns the union of all open$V \subseteq Y$ with $f^{-1}(V) \subseteq U$.

Proposition 2.6. *A square*

$$S = \begin{tikzcd} A' \arrow[r, "h"] \arrow[d, "a"'] & B' \arrow[d, "b"] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$

in Top_0 *is a Kan-injectivity consequence of topological embeddings iff h and k are topological embeddings yielding the following (commutative) square*

$$\begin{tikzcd} \Omega A \arrow[r, "(\Omega k)_*"] \arrow[d, "\Omega a"'] & \Omega B \arrow[d, "\Omega b"] \\ \Omega A' \arrow[r, "(\Omega h)_*"'] & \Omega B' \end{tikzcd}$$

Proof: (a) Sufficiency. For every continuous lattice X and every embedding $f : A \to X$ the Kan-extension f/k is, as proved in [9], given by

$$f/k(z) = \sqcup\{\sqcap U;\ U \in \Omega X \ \text{ and } \ z \in (\Omega k)_*[\Omega f(U)]\}, \tag{1}$$

and analogously for $(fa)/h$. We are to prove

$$(f/k)\cdot b = (fa)/h.$$

Indeed, given $b \in B'$, then $b(y) \in (\Omega k)_*[\Omega f(U)]$ means that y belongs to $(\Omega b)(\Omega k)_*[\Omega f(U)]$, and, by the commutativity of the last square, this means that $y \in (\Omega h)_*[\Omega(fa)(U)]$. Consequently, taken into account the definitions of $(f/k)(b(y))$ and $((fa)/h)(y)$ given by (1), we conclude the desired equality.

(b) Necessity. Since the Sierpińsky space **S** is Kan-injective with respect to S, k and h are topological embeddings (see 4.3 and 4.4 of [6]). We verify the above square.

Every $V \in \Omega A$ defines the corresponding characteristic function $f_V : A \to$ **S** and, using the formula (1) above, we get (by setting $V = \{1\}$)

$$(f_V/k)(z) = 1 \quad \text{iff} \quad z \in (\Omega k)_*(V).$$

Since $f_V \cdot a = f_{\Omega a(V)}$, the characteristic function of $a^{-1}(V)$, we analogously get

$$(f_V \cdot a)/h(x) = 1 \quad \text{iff} \quad x \in (\Omega h)_*(\Omega a(V)).$$

Thus, our formula $(f_V \cdot a)/h = (f_V/k)\cdot b$ reads:

$$x \in (\Omega h)_*(\Omega a(V)) \quad \text{iff} \quad b(x) \in (\Omega k)_*(V)$$

for all $x \in B'$. That is,

$$(\Omega h)_*(\Omega a(V)) = \Omega b((\Omega k)_*(V)),$$

as desired. ■

Remark 2.7. A slight modification: Scott continuous domains are, as proved by Escardó [8], precisely the T_0-spaces Kan-injective with respect to dense embeddings. And the morphisms Kan-injective with respect to dense embeddings are the continuous functions preserving nonempty meets (see [6]). The above corollary holds analogously, just h and k are required to be dense embeddings.

Example 2.8. Let Loc be the category of locales and localic maps. Thus, the objects of Loc are complete lattices with the infinite distributive law

$$a \wedge \bigvee B = \bigvee \{a \wedge b \,|\, b \in B\},$$

and the morphisms are the monotone maps f which preserve all infima and whose left adjoint f^* preserves finite meets. We recall that a localic map h is an embedding provided that $h^*h = \mathrm{id}$. Johnstone characterized the stably locally compact locales as the locales injective with respect to flat embeddings, i.e. those preserving finite joins, see [10]. Moreover, with convenient morphisms, stably locally compact locales are precisely $\mathsf{LInj}(\mathcal{H})$ for $\mathcal{H}$ = flat embeddings (see [10] and [7]).

Proposition 2.9. *A square*

$$\begin{tikzcd} A' \arrow[r, "h"] \arrow[d, "a"'] & B' \arrow[d, "b"] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$

in Loc *is a Kan-injectivity consequence of flat embeddings iff h and k are flat embeddings yielding the following (commutative) square*

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "a^*"'] & B \arrow[d, "b^*"] \\ A' \arrow[r, "h"'] & B' \end{tikzcd}$$

Proof: Let F_0, F_1 and F_2 be the free frames generated by the empty set, $1 = \{0\}$ and $2 = \{0,1\}$, respectively, and let $f_i : F_i \to F_1$, $i = 0,2$, be the localic maps determined by $f_0(\bot) = 0$, $f_2(0 \vee 1) = 0$ and $f_2(x) = \bot$ for $x \neq \bot, 0 \vee 1$. In [7] flat embeddings were characterized as precisely those morphisms with respect to which both f_0 and f_2 are Kan-injective. Furthermore, it was shown there that for every finitely generated free frame F, in particular for every F_i $(i = 0,1,2)$, given a flat embedding $h : A \to B$ and a morphism $f : A \to F$, the map $(hf^*)_*$ is localic and

$$f/h = (hf^*)_*. \tag{2}$$

(a) Necessity: suppose the given square is a Kan injectivity consequence of flat embeddings. Since we already know that flat embeddings are characterized by means of f_0 and f_2, we only need to prove $ha^* = b^*k$.

Given $x \in A$, we want to show that $b^*k(x) = ha^*(x)$. Let $g : F_1 \to A$ be the frame homomorphism sending 0 to x. By hypothesis, the localic map $g_* : A \to F_1$ satisfies $(g_*/k) \cdot b = (g_*a)/h$, that is, by (2), $(kg)_*b = h(g_*a)^* = ha^*g$. Consequently, by applying the operator $-^*$ to the localic maps $(kg)_*$, b and ha^*g, we obtain $b^*kg(0) = ha^*g(0)$, i.e., $b^*k(x) = ha^*(x)$.

(b) Sufficiency: if the lower square commutes, we prove that, given $f : A \to F_i$, $(fa)/h = (f/k)b$. Indeed, from (2), deduce that

$$(fa)/h = [(h(fa)^*]_* = (ha^* f^*)_* = (b^* k f^*)_* = (kf^*)_* b = (f/k) \cdot b.$$

■

Remark 2.10. (a) Given an order-enriched category $\mathcal{X}$, by a *right adjoint retraction* we mean a morphism $r : X \to Y$ with $r^* : Y \to X$ satisfying $rr^* = \mathrm{id}$ and $r^* r \geq \mathrm{id}$.

(b) Recall that a *KZ-monad* over $\mathcal{X}$ is a monad $\mathbb{T}$ for which

$$T\eta \leq \eta T.$$

Then $\mathbb{T}$-algebras are precisely the right adjoint retractions of $\eta_A : A \to TA$. Thus, $\mathcal{X}^{\mathbb{T}}$ is a (non-full) subcategory of $\mathcal{X}$.

(c) In [4] we proved that KZ-monadic subcategories are precisely the subcategories $\mathcal{K}$ which are

(1) reflective, i.e., the embedding has a left adjoint,
(2) *inserter-ideal*, i.e., contain the inserter $\mathrm{ins}(u,v)$ of every parallel pair (u,v) with u in $\mathrm{Mor}(\mathcal{K})$,

and

(3) closed under right adjoint retractions.

For the last condition recall that a subcategory is *closed under right adjoint retractions* if with every morphism $p : X \to Y$ it contains all morphisms $\overline{p} : \overline{X} \to \overline{Y}$ for which there exists a square with right adjoint retractions x and y as follows

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\ \overline{X} \arrow[r, "\overline{p}"'] & \overline{Y} \end{tikzcd} \tag{3}$$

(d) Every KZ-monadic subcategory $\mathcal{K}$ has the form $\mathcal{K} = \mathsf{LInj}(\mathcal{H})$ for some class $\mathcal{H}$ of morphisms of $\mathcal{X}$ ([6]). And we proved that, conversely, for "reasonable" categories $\mathcal{X}$ (such as Top_0 and Pos), every set $\mathcal{H}$ of morphisms defines a KZ-monadic subcategory $\mathsf{LInj}(\mathcal{H})$. (The latter is not true for proper classes $\mathcal{H}$ in Top_0 as demonstrated by an example in [4]).

Lemma 2.11. *For every class $\mathcal{H}$ of squares the category $\mathsf{LInj}(\mathcal{H})$ is inserter-ideal and closed under right adjoint retractions.*

Proof: A. Given a pair of morphisms $u, v : X \to Y$ and a square

$$\begin{tikzcd} A' \arrow[r, "h'"] \arrow[d, "a"'] & B' \arrow[d, "b"] \\ A \arrow[r, "h"'] & B \end{tikzcd} \tag{4}$$

with respect to which u is Kan-injective, we prove that so is the inserter $i = \mathrm{ins}(u, v) : I \to X$. We have already proved in [4] that the fact that u is Kan-injective with respect to h and h' implies that so is the morphism i. It remains to show that, for every $f : A \to I$, $(f/h)b = (fa)/h'$, or equivalently, since i is mono, $i(f/h)b = i(fa)/h'$. But the last equality follows immediatly from the fact that i is Kan-injective with respect to both h and h' and X is Kan-injective w.r.t. the above square.

B. Given a square (3) with the left adjoints x^* and y^*, respectively, and such that the morphism $p : X \to Y$ is Kan-injective with respect to the square (4), we prove that also $\overline{p}$ is Kan-injective with respect to this square.

(a) $\overline{X}$ is Kan-injective with respect to the above square, and for every morphism $f : A \to \overline{X}$ we have: $f/h = x([x^* f]/h)$.

$$\begin{tikzcd} A' \arrow[r, "h'"] \arrow[d, "a"'] & B' \arrow[d, "b"] \\ A \arrow[r, "h"'] \arrow[d, "f"'] & B \arrow[d, "(x^* f)/h"] \\ \overline{X} \arrow[r, dashed, shift right, "x^*"'] & X \arrow[l, shift right, "x"'] \end{tikzcd}$$

This last formula was proved in [6]. Analogously one proves $(fa)/h' = x([x^* fa]/h')$. Consequently, since the Kan-injectivity of X yields

$$[x^* fa]/h' = ([x^* f]/h)b$$

we conclude $(fa)/h' = (f/h)b$.

(b) $\overline{Y}$ is Kan-injective. This is completely analogous.

(c) $\overline{p}$ is Kan-injective with respect to h and h'. This was also proved in [6].

■

3.The reflection chain

Throughout the rest of the paper $\mathcal{X}$ denotes an order-enriched category with weighted colimits. Given a set $\mathcal{H}$ of squares, we associate with every object X a transfinite chain starting in X. For "reasonable" categories we then prove that there exists a connecting morphism from X in our chain which is the reflection of X in the subcategory $\mathsf{LInj}(\mathcal{H})$.

Definition 3.1. Let $\mathcal{H}$ be a set of squares. For every object X define a transfinite chain X_i ($i \in \mathsf{Ord}$) with connecting morphisms called x_{ij} or simply $X_i \dashrightarrow X_j$ (for $i \le j$). We proceed by transfinite recursion. In the isolated steps, given i we define $X_i \dashrightarrow X_{i+1} \dashrightarrow X_{i+2}$, therefore, we can assume that i is an even ordinal (that is, $i = 2n$ or $i = i_0 + 2n$ for $n < \omega$ and i_0 a limit ordinal).

(a) Initial step. $X_0 = X$.

(b) Limit steps. If i is a limit ordinal then

$$X_i = \underset{j<i}{colim}\, X_j$$

is the colimit of the previous chain with $X_j \dashrightarrow X_i$ forming the colimit cocone.

(c) Isolated step $i \mapsto i+1$ (i even). Consider all pairs (S, f) where

$$S = \begin{tikzcd} A_1 \arrow[r, "h_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\ A_2 \arrow[r, "h_2"'] & B_2 \end{tikzcd} \tag{5}$$

is a member of $\mathcal{H}$ and $f : A_r \to X_i$ a morphism with $r = 1$ or 2. Form the pushout of h_r along f:

$$\begin{tikzcd} A_r \arrow[r, "h_r"] \arrow[d, "f"'] & B_r \arrow[d, "\bar{f}"] \\ X_i \arrow[r, "\overline{h}_r"'] & Q_f \end{tikzcd} \tag{6}$$

Define $X_i \dashrightarrow X_{i+1}$ as the wide pushout of all the morphisms $\overline{h}_r$:

$$\begin{tikzcd}
X_i \arrow[r, "\overline{h}_r"] \arrow[d, dashed] & Q_f \arrow[dl, "q_f"] \\
X_{i+1} &
\end{tikzcd} \tag{7}$$

ranging over all the above pairs (S, f).

Notation: put

$$f /\!\!/ h_r : B_r \xrightarrow{\overline{f}} Q_f \xrightarrow{q_f} X_{i+1} \tag{8}$$

This "approximates" the desired Kan extension f/h_r in the following sense: we get a (commutative) square

$$\begin{tikzcd}
A_r \arrow[r, "h_r"] \arrow[d, "f"'] & B_r \arrow[d, "f /\!\!/ h_r"] \\
X_i \arrow[r, dashed] & X_{i+1}
\end{tikzcd} \tag{9}$$

(d) Isolated step $i+1 \mapsto i+2$ (i even). Consider all triples (S, f, g) consisting of $S \in \mathcal{H}$, as above, and two morphisms forming an inequality as follows:

$$\begin{tikzcd}
A_r \arrow[r, "h_r"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq"] & B_r \arrow[d, "g"] \\
X_i \arrow[r, dashed] & X_{i+1}
\end{tikzcd} \qquad (r = 1 \text{ or } 2) \tag{10}$$

For every decomposition of f as follows

$$f \equiv A_r \xrightarrow{f'} X_j \dashrightarrow X_i \quad (j \leq i \text{ even}) \tag{11}$$

form the coinserter of $A_r \xrightarrow{f' /\!\!/ h_r} X_{j+1} \dashrightarrow X_{i+1}$ and g:

$$c_{f,g} = \mathrm{coins}(x_{j+1,i+1}[f' /\!\!/ h_r], g) : X_{i+1} \to C_{f,g}. \tag{12}$$

And in case $r = 1$ for every decomposition of f as follows

$$f \equiv A_1 \xrightarrow{a} A_2 \xrightarrow{f'} X_i \tag{13}$$

form the coinserter of $B_1 \xrightarrow{b} B_2 \xrightarrow{f' /\!\!/ h_2} X_{i+1}$ and g:

$$c_{f,g} = \mathrm{coins}([f' /\!\!/ h_2].b, g). \tag{14}$$

Define $X_{i+1} \dashrightarrow X_{i+2}$ as the wide pushout of all $c_{f,g}$ above:

$$\begin{tikzcd}
X_{i+1} \arrow[r, "c_{f,g}"] \arrow[d, dashed] & C_{f,g} \arrow[dl, "t_{f,g}"] \\
X_{i+2} &
\end{tikzcd} \tag{15}$$

Lemma 3.2. *Given a morphism $p_0 : X_0 \to P$ where P is Kan-injective with respect to $\mathcal{H}$, there exists a unique cocone $p_i : X_i \to P$ ($i \in \mathsf{Ord}$) of the reflection chain such that for all pairs (S, f) in step $i \mapsto i+1$ the following triangle*

$$\begin{tikzcd}
A_r \arrow[d, "f /\!\!/ h_r"'] \arrow[dr, "(p_i f)/h_r"] & \\
X_{i+1} \arrow[r, "p_{i+1}"'] & P
\end{tikzcd} \tag{16}$$

commutes.

Proof: We only need to prove the isolated step of the transfinite induction. Given p_i, i even, we obtain p_{i+1} as follows. From the following square

$$\begin{tikzcd}
A_r \arrow[r, "h_r"] \arrow[d, "f"'] & B_r \arrow[d, "(p_i f)/h_r"] \\
X_i \arrow[r, "p_i"'] & P
\end{tikzcd}$$

we conclude that the pushout (6) yields a unique factorization morphism $p_f : Q_f \to P$. These morphisms p_f form a cocone of the wide pushout (7). Define p_{i+1} as the unique factorization morphism

$$p_{i+1} \cdot q_f = p_f.$$

It fulfils $p_{i+1} \cdot (f /\!\!/ h_r) = p_{i+1}.q_f \cdot \overline{f} = p_f.\overline{f} = (p_i f) /\!\!/ h_r$. Conversely, whenever the above triangle commutes, then p_{i+1} is a factorization map of the wide pushout (7).

Next we define p_{i+2}: since $X_{i+1} \dashrightarrow X_{i+2}$ is a wide pushout of epimorphisms, p_{i+2} is unique. And for the proof of existence we only need to verify that p_{i+1} factorizes through each $c_{f,g}$. That is:

(a) In case of (11) we want to verifiy the inequality

$$\begin{tikzcd}
B_r \arrow[r, "g"] \arrow[d, "f'/\!\!/h_r"'] & X_{i+1} \arrow[dd, "p_{i+1}"] \\
X_{j+1} \arrow[d, dashed] \arrow[r, phantom, "\leq"] & \\
X_{i+1} \arrow[r, "p_{i+1}"'] & P
\end{tikzcd}$$

The lower passage is, due to $p_{j+1} = p_{i+1} \cdot x_{j+1,i+1}$ and (16), equal to $(p_j.f')/h_r$. By composing (10) with p_{i+1} we get

$$p_j f' \leq p_{i+1} g h_r$$

hence,

$$(p_j f')/h_r \leq p_{i+1} g$$

(b) In case of (13) we want to verify the inequality

$$\begin{tikzcd}
B_1 \arrow[r, "g"] \arrow[d, "b"'] & X_{i+1} \arrow[dd, "p_{i+1}"] \\
B \arrow[d, "f'/\!\!/h_2"'] \arrow[r, phantom, "\leq"] & \\
X_{i+1} \arrow[r, "p_{i+1}"'] & P
\end{tikzcd}$$

Indeed, the lower passage is, due to (16) and injectivity of P with respect to S, equal to

$$[(p_i f')/h_2]b = (p_i f' a)/h_1 = (p_i f)/h_1.$$

By composing (10) with p_{i+1} we get $p_i f \leq p_{i+1}.g.h_1$. This proves $(p_i f)/h_1 \leq p_{i+1}.g$ as desired. ■

Recall that in a category with a factorization system $(\mathcal{E}, \mathcal{M})$ an object is said to have *rank* λ (a regular cardinal) if its hom-functor preserves λ-directed colimits of morphisms in $\mathcal{M}$. We use the following concept introduced in [4]:

Definition 3.3. Let $\mathcal{X}$ be an order-enriched category with a factorization system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{E} \subseteq$ Epi and $\mathcal{M} \subseteq$ Order-Mono (i.e., given $m \in \mathcal{M}$ then $mu \leq mv$ implies $u \leq v$). We call $\mathcal{X}$ *locally ranked* if

(i) it has weighted colimits,
(ii) it is $\mathcal{E}$-cowellpowered,

and

(iii) every object has a rank.

Example 3.4. Pos is locally ranked w.r.t. $\mathcal{E} = \text{Epi}$ and $\mathcal{M} = \text{Order-Embedding}$. Top_0 is locally ranked w.r.t. $\mathcal{E}$ = surjective morphisms and $\mathcal{M}$ = subspace embeddings.

Remark 3.5. Let $\mathcal{X}$ be locally ranked and let (X_i) be a transfinite chain with connecting morphisms $X_i \dashrightarrow X_j (i \leq j)$. By Proposition 4.1 of [11] there exists a chain (Y_i) of monomorphisms in $\mathcal{M}$, a join-preserving function $\varphi : \mathsf{Ord} \to \mathsf{Ord}$ and natural transformations

$$X_i \xrightarrow{\gamma_i} Y_i \xrightarrow{\beta_i} X_{\varphi(i+1)} \ (i \in \mathsf{Ord})$$

such that for all i:

(1) $\beta_i \gamma_i$ is the connecting morphism $X_i \dashrightarrow X_{\varphi(i+1)}$;

(2) the composite $Y_i \xrightarrow{\beta_i} X_{\varphi(i+1)} \dashrightarrow X_j$ lies in $\mathcal{M}$ for all $j \geq \varphi(i+1)$;

and

(3) if i is a limit ordinal, then a colimit of the chain $(Y_j)_{j<i}$ is given by the following cocone

$$Y_j \xrightarrow{\beta_j} X_{\varphi(j+1)} \dashrightarrow X_{\varphi(i)}$$

Moreover, given such a function φ, every join-preserving function $\varphi' \geq \varphi$ works too. Consequently, we can clearly choose φ so that $\varphi(i)$ is even for all ordinals i.

Theorem 3.6. *Let $\mathcal{X}$ be a locally ranked order-enriched category. For every set $\mathcal{H}$ of squares the subcategory* $\mathsf{LInj}(\mathcal{H})$ *is reflective: the reflection of every object X is given by $X \dashrightarrow X_k$ (in Definition 3.1) where k is a suitable cardinal.*

Proof: Apply the above remark to the chain of Definition 3.1, using notation $\hat{i} = \varphi(i+1)$. Since $\mathcal{H}$ is a set, there exists a cardinal λ such that for every square in $\mathcal{H}$ all the four objects involved have rank λ. The cardinal k of our theorem is chosen to be

$$k = \varphi(\lambda).$$

(1) We first prove that the object X_k is Kan-injective with respect to every square in $\mathcal{H}$:

$$S = \begin{tikzcd} A_1 \arrow[r, "h_1"] \arrow[d, "a"'] & A_2 \arrow[d, "b"] \\ B_1 \arrow[r, "h_2"'] & B_2 \end{tikzcd}$$

(a) Kan extensions modulo h_1 and h_2 exist. Indeed, every morphism $f_0 : A_r \to X_k$ $(r = 1, 2)$ has, since A_r has rank λ, a factorization f through some colimit injection of the colimit $X_k = \operatorname*{colim}_{j<\lambda} Y_j$ of Remark 3.5(3):

$$\begin{tikzcd} & & A_r \arrow[dll, "f"'] \arrow[dl, "f'"] \arrow[d, "f_0"] \\ Y_j \arrow[r, "\beta_j"'] & X_{\hat{j}} \arrow[r, dashed] & X_k \end{tikzcd}$$

Put $f' = \beta_j f$. We use Notation (8) and prove that the desired Kan extension is

$$f_0/h_r \equiv A_r \xrightarrow{f'/\!\!/h_r} X_{\hat{j}+1} \dashrightarrow X_k. \tag{17}$$

Indeed, from (9) we get $f_0 = (f_0/h_r) \cdot h_r$ via $f' = (f'/\!\!/h_r) \cdot h_r$:

$$(f_0/h) \cdot h_r \equiv A_r \xrightarrow{f'} X_{\hat{j}+1} \dashrightarrow X_k$$

Next, let $g_0 = B_r \to X_k$ fulfil

$$f_0 \leq g_0 h_r.$$

Since B_r has rank λ, we can find an analogous factorization:

$$\begin{tikzcd} & & B_r \arrow[dll, "g"'] \arrow[dl, "g'"] \arrow[d, "g_0"] \\ Y_i \arrow[r, "\beta_i"'] & X_{\hat{i}} \arrow[r, dashed] & X_k \end{tikzcd}$$

for some ordinal $i < k$. Without loss of generality, $j \leq i$. The above inequality yields the following one:

$$\begin{tikzcd}
A_r \arrow[r, "h_r"] \arrow[dd, "f"'] & B_r \arrow[r, "g"] & Y_i \arrow[d, "\beta_i"] \\
 & \leq & X_{\hat{\imath}} \arrow[d, dashed] \\
Y_j \arrow[r, "\beta_j"'] \arrow[dd, "y_{ji}"'] & X_{\hat{\jmath}} \arrow[r, dashed] \arrow[d, dashed] & X_k \\
 & X_{\hat{\imath}} \arrow[ur, dashed] & \\
Y_i \arrow[ur, "\beta_i"'] & &
\end{tikzcd}$$

Since by Remark 3.5 the colimit morphisms lie in $\mathcal{M}$, (thus, they are order-monomorphisms) this proves

$$y_{ji} \cdot f \leq g \cdot h_r$$

which by composition with β_i yields the inequality

$$\begin{tikzcd}
A_r \arrow[r, "h_r"] \arrow[d, "f'"'] & B_r \arrow[d, "g'"] \\
X_{\hat{\jmath}} \arrow[d, dashed] \arrow[r, phantom, "\leq"] & X_{\hat{\imath}} \arrow[d, dashed] \\
X_{\hat{\imath}} \arrow[r, dashed] & X_{\hat{\imath}+1}
\end{tikzcd}$$

This is an instance of (10) and (11) with respect to $x_{\hat{\imath},\hat{\imath}+1} \cdot g'$. Let $c_{f,g'}$ be the corresponding coinserter (12). Since the map $X_{\hat{\imath}+1} \dashrightarrow X_{\hat{\imath}+2}$ factorizes through $c_{f,g'}$, we obtain the following inequality

$$\begin{tikzcd}
B_r \arrow[r, "g'"] \arrow[d, "f'/\!\!/h_r"'] & X_{\hat{\imath}} \arrow[r, dashed] & X_{\hat{\imath}+1} \arrow[dd, dashed] \\
X_{\hat{\jmath}+1} \arrow[d, dashed] & \leq & \\
X_{\hat{\imath}+1} \arrow[rr, dashed] & & X_{\hat{\imath}+2}
\end{tikzcd}$$

Composed with $X_{\hat{\imath}+2} \dashrightarrow X_k$ this states that

$$x_{\hat{\jmath}+1,k} \cdot (f'/\!\!/h_r) \leq x_{\hat{\imath}k} \cdot g'$$

which is the desired inequality

$$f_0/h_r \leq g_0.$$

(b) It remains to prove that for every $f_0 : A_2 \to X_k$ we have

$$(f_0 a)/h_1 = (f_0/h_2)b.$$

Put $\hat{f}_0 = f_0 a$ and factorize it analogously as f_0 above, assuming (without loss of generality) that the same colimit injection can be applied:

$$\begin{tikzcd}
 & & & A_r \arrow[lll, "\hat{f}"'] \arrow[ll, "\hat{f}'"] \arrow[d, "\hat{f}_0"] \\
Y_j \arrow[r, "\beta_j"'] & X_{\hat{\jmath}} \arrow[rr, dashed] & & X_k
\end{tikzcd}$$

Thus the equality to be proved is, due to (16), the following square

$$\begin{tikzcd}
B_1 \arrow[r, "\hat{f}'/\!\!/h_1"] \arrow[d, "b"'] & X_{\hat{\jmath}+1} \arrow[dd, dashed] \\
B_2 \arrow[d, "f'/\!\!/h_2"'] & \\
X_{\hat{\jmath}+1} \arrow[r, dashed] & X_k
\end{tikzcd} \tag{18}$$

Since $x_{\hat{\jmath}k} \cdot \beta_j$ is monic by Remark 3.5, the equality $\hat{f}_0 = f_0 a$ clearly implies $\hat{f} = f \cdot a$ which multiplied by β_j yields $\hat{f}' = f' \cdot a$. From (9) we thus get the square

$$\begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "a"'] & B_1 \arrow[dd, "\hat{f}'/\!\!/h_1"] \\
A_2 \arrow[d, "f'"'] & \\
X_{\hat{j}} \arrow[r, dashed] & X_{\hat{j}+1}
\end{tikzcd}$$

This is an instance of (10) and (13). Let $c_{f,g}$ be the corresponding coinserter. Since the map $X_{\hat{j}+1} \dashrightarrow X_{\hat{j}+2}$ factorizes through it, we obtain the following inequality

$$\begin{tikzcd}
B_1 \arrow[r, "\hat{f}'/\!\!/h_1"] \arrow[d, "b"'] & X_{\hat{j}+1} \arrow[dd, dashed] \\
B_2 \arrow[d, "f'/\!\!/h_2"'] \arrow[r, phantom, "\leq"] & \\
X_{\hat{j}+1} \arrow[r, dashed] & X_{\hat{j}+2}
\end{tikzcd}$$

which multiplied by $X_{\hat{j}+2} \dashrightarrow X_k$ yields "almost" the desired square (18): indeed, the opposite inequality $(f_0/h_2) \cdot b \geq (f_0 a)/h_1$ is trivial.

(2) For every morphism $p : X_0 \to P$ where P lies in $\mathsf{LInj}(\mathcal{H})$ we prove that the morphism $p_k : X_k \to P$ in Lemma 3.2 is Kan-injective with respect to $\mathcal{H}$. The proof is entirely analogous to that of Part (2) of the proof of Theorem 6.10 of [4]. The proof that the extension of the morphism p via $X \dashrightarrow X_k$ is unique in $\mathsf{LInj}(\mathcal{H})$ is entirely analogous to Part (3) of the proof mentioned above. ■

Remark 3.7. (a) The ordinal k above depends on the choice of the object X. However, given two objects X and $\widetilde{X}$, we can find an ordinal k such that both of the reflection chains for X and $\widetilde{X}$ yield reflections in $\mathsf{LInj}(\mathcal{H})$ after k steps. This follows from the choice $k = \varphi(\lambda)$ made in the above proof, since the same function φ can be used for both chains. (This can be deduced from the fact that we can always use any function $\varphi' \geq \varphi$ preserving joins.)

(b) Denote by

$$R : \mathcal{X} \to \mathsf{LInj}(\mathcal{H})$$

the reflector, i.e., the left adjoint of the embedding. And by $\eta_X : X \to RX$ the reflection morphisms. We have just proved that for all objects X we have

$$\eta_X \equiv X \dashrightarrow X_k$$

How is R characterized on morphisms $u : X \to \widetilde{X}$?

We prove below that

$$Ru = u_k : X_k \to \widetilde{X}_k$$

for the following natural transformation $u_i : X_i \to \widetilde{X}_i$ ($i \in \mathsf{Ord}$). Here $\widetilde{X}_i$ denotes the reflection chain for $\widetilde{X}$ (and we use the obvious notation $\widetilde{Q}_f$, $\widetilde{C}_{f,g}$, etc.).

Notation 3.8. Let $u : X \to \widetilde{X}$ be a morphism of $\mathcal{X}$. We define a natural transformation $u_i : X_i \to \widetilde{X}_i$ ($i \in \mathsf{Ord}$) of the reflection chains of X and $\widetilde{X}$ by the following transfinite induction:

Initial step: $u_0 = u$.

Limit step: this is automatic from naturality.

Isolated step $i \mapsto i+1$ (i even): every pair (S, f) with respect to X_i defines a pair $(S, u_i \cdot f)$ with respect to $\widetilde{X}_i$. For the corresponding pushouts Q_f and $\widetilde{Q}_{u_i f}$, see (6), we get a unique factorization f^* as follows:

$$\begin{tikzcd}
A_r \arrow[r, "h_r"] \arrow[d, "f"'] & B_r \arrow[r, equal] \arrow[d, "\overline{f}"'] & B_r \arrow[dd, "\widetilde{u_i f}"] \\
X_i \arrow[r, "\overline{h}_r"'] \arrow[d, "u_i"'] & Q_f \arrow[dr, "f^*"] & \\
\widetilde{X}_i \arrow[rr, "\widetilde{\overline{h}}_r"'] & & \widetilde{Q}_{u_i f}
\end{tikzcd} \tag{19}$$

Then $(\widetilde{q}_{u_i f} \cdot f^*)\bar{h}_r = \tilde{x}_{i,i+1} \cdot u_i$ is independent of f. Therefore we can define u_{i+1} via the following squares (for all (S, f)):

$$\begin{tikzcd}
Q_f \arrow[r, "f^*"] \arrow[d, "q_f"'] & \widetilde{Q}_{u_i f} \arrow[d, "\widetilde{q}_{u_i f}"] \\
X_{i+1} \arrow[r, "u_{i+1}"'] & \widetilde{X}_{i+1}
\end{tikzcd} \tag{20}$$

Observe also that $(u_i f)/\!\!/h_r$ is the composite of $f/\!\!/h_r$ and u_{i+1}:

$$\begin{tikzcd}
B_r \arrow[r, equal] \arrow[d, "\overline{f}"] \arrow[dd, bend right=60, "f/\!\!/h_r"'] & B_r \arrow[d, "\widetilde{u_i f}"'] \arrow[dd, bend left=60, "(u_i f)/\!\!/h_r"] \\
Q_f \arrow[r, "f^*"] \arrow[d, "q_f"] & \widetilde{Q}_{u_i f} \arrow[d, "\widetilde{q}_{u_i f}"'] \\
X_{i+1} \arrow[r, dashed, "u_{i+1}"'] & \widetilde{X}_{i+1}
\end{tikzcd} \tag{21}$$

Isolated step $i+1 \mapsto i+2$ (i even). Every triple (S,f,g) in the definition of X_{i+2} yields a triple $(S,\widetilde{f},\widetilde{g})$ with respect to $\widetilde{X}_{i+2}$ as follows:

$$\widetilde{f} \equiv A_r \xrightarrow{f} X_i \xrightarrow{u_i} \widetilde{X}_i \text{ and } \widetilde{g} = B_r \xrightarrow{g} X_{i+1} \xrightarrow{u_{i+1}} \widetilde{X}_{i+1}$$

The naturality of (u_j) guarantees that every factorization (11) of f yields the corresponding factorization

$$\widetilde{f} \equiv A_r \xrightarrow{f'} X_j \xrightarrow{u_j} \widetilde{X}_j \dashrightarrow \widetilde{X}_i$$

of $\widetilde{f}$. And we prove below that this leads to a unique morphism $d_{f,g}$ forming the following square

$$\begin{tikzcd}
X_{i+1} \arrow[r, "c_{f,g}"] \arrow[d, "u_{i+1}"'] & C_{f,g} \arrow[d, "d_{f,g}"] \\
\widetilde{X}_{i+1} \arrow[r, "\widetilde{c}_{\widetilde{f},\widetilde{g}}"'] & \widetilde{C}_{\widetilde{f},\widetilde{g}}
\end{tikzcd} \tag{22}$$

Analogously, every factorization (13) of f yields one for $\widetilde{f}$ and we again obtain a square (22). We define u_{i+2} via the following squares (for all (S,f,g)):

$$\begin{tikzcd}
C_{f,g} \arrow[r, "d_{f,g}"] \arrow[d, "t_{f,g}"'] & \widetilde{C}_{f,g} \arrow[d, "\widetilde{t}_{\widetilde{f},\widetilde{g}}"] \\
X_{i+2} \arrow[r, "u_{i+2}"'] & \widetilde{X}_{i+2}
\end{tikzcd} \tag{23}$$

Proposition 3.9. *The above natural transformation $u_i : X_i \to \widetilde{X}_i$ ($i \in \mathsf{Ord}$) is well defined, and $u_k = Ru$ for every morphism $u : X \to \widetilde{X}$.*

Proof: (1) Firstly, given an even ordinal i, the morphisms $q_{u_i f} \cdot f^*$ form a cocone of the wide pushout (7) defining $X_i \dashrightarrow X_{i+1}$, hence, in (21) the morphism u_{i+1} is unique. We need to verify the naturality square. For that observe that the following diagram

$$\begin{tikzcd}
X_i \arrow[rrr, "u_i"] \arrow[ddd, dashed] \arrow[dr, "h_r"] & & & \widetilde{X}_i \arrow[dl, "\widetilde{h}_r"'] \arrow[ddd, dashed] \\
& Q_f \arrow[r, "f^*"] \arrow[ddl, "q_f"] & \widetilde{Q}_{u_i f} \arrow[ddr, "\widetilde{q}_{u_i f}"'] & \\
& & & \\
X_{i+1} \arrow[rrr, "u_{i+1}"'] & & & \widetilde{X}_{i+1}
\end{tikzcd}$$

commutes.

(2) Next we verify the existence of $d_{f,g}$ in (22). In case (11) this is equivalent to proving that $\widetilde{c}_{\widetilde{f},\widetilde{g}} u_{i+1}$ satisfies the following inequality:

$$\begin{tikzcd}
A_r \arrow[rr, "g"] \arrow[d, "f' /\!\!/ h_r"'] & & X_{i+1} \arrow[d, "u_{i+1}"] \\
X_{j+1} \arrow[d, dashed] & \leq & \widetilde{X}_{i+1} \arrow[d, "\widetilde{c}_{\widetilde{f},\widetilde{g}}"] \\
X_{i+1} \arrow[r, "u_{i+1}"'] & \widetilde{X}_{i+1} \arrow[r, "\widetilde{c}_{\widetilde{f},\widetilde{g}}"'] & \widetilde{C}_{\widetilde{f},\widetilde{g}}
\end{tikzcd}$$

Indeed, this follows from the definition of $\widetilde{c}_{\widetilde{f},\widetilde{g}}$ and (21). And in case (13) we need the inequality

$$\begin{tikzcd}
B_1 \arrow[rr, "g"] \arrow[d, "b"'] & & X_{i+1} \arrow[d, "u_{i+1}"] \\
B \arrow[d, "f' /\!\!/ h_2"'] & \leq & \widetilde{X}_{i+1} \arrow[d, "\widetilde{c}_{\widetilde{f},\widetilde{g}}"] \\
X_{i+1} \arrow[r, "u_{i+1}"'] & \widetilde{X}_{i+1} \arrow[r, "\widetilde{c}_{\widetilde{f},\widetilde{g}}"'] & \widetilde{C}_{\widetilde{f},\widetilde{g}}
\end{tikzcd}$$

which also follows from the definition of $\widetilde{c}_{\widetilde{f},\widetilde{g}}$ and (21). The morphisms

$$C_{f,g} \xrightarrow{d_{f,g}} \widetilde{C}_{\widetilde{f},\widetilde{g}} \xrightarrow{\widetilde{t}_{\widetilde{f},\widetilde{g}}} \widetilde{X}_{i+2}$$

form a cocone of the wide pushout (15) due to (22):

$$\widetilde{t}_{\widetilde{f},\widetilde{g}} \cdot d_{f,g} \cdot c_{f,g} = t_{\widetilde{f},\widetilde{g}} \cdot \widetilde{c}_{\widetilde{f},\widetilde{g}} \cdot u_{i+1} = \widetilde{x}_{i+1,i+2} \cdot u_{i+1}.$$

Thus (23) defines u_{i+2} uniquely. Moreover, the naturality square now follows:

$$\begin{tikzcd}
X_{i+1} \arrow[rrr, "u_{i+1}"] \arrow[dd, dashed] \arrow[dr, "c_{f,g}"] & & & \widetilde{X}_{i+1} \arrow[dd, dashed] \arrow[dl, "\widetilde{c}_{\widetilde{f},\widetilde{g}}"'] \\
& C_{f,g} \arrow[r, "d_{f,g}"] \arrow[dl, "t_{f,g}"] & \widetilde{C}_{\widetilde{f},\widetilde{g}} \arrow[dr, "\widetilde{t}_{\widetilde{f},\widetilde{g}}"] & \\
X_{i+1} \arrow[rrr, "u_{i+2}"'] & & & \widetilde{X}_{i+2}
\end{tikzcd}$$

(3) Consequently, for every morphism $u : X \to \widetilde{X}$ the ordinal k of Theorem 3.6 provides a square

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "u"'] & RX \arrow[r, equal] & X_k \arrow[d, "u_k"] \\
\widetilde{X} \arrow[r, "\eta_{\widetilde{X}}"'] & R\widetilde{X} \arrow[r, equal] & \widetilde{X}_k
\end{tikzcd}$$

To prove $u_k = Ru$, we only need to verify that u_k is a morphism of $\mathsf{LInj}(\mathcal{H})$. That is, for every square

$$\begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\
A_2 \arrow[r, "h_2"'] & B_2
\end{tikzcd}$$

in $\mathcal{H}$ and every morphism $f_0 : A_r \to X_k$ the triangle

$$\begin{tikzcd}
B_r \arrow[r, "f_0/h_r"] \arrow[dr, "(u_k f_0)/h_r"'] & X_k \arrow[d, "u_k"] \\
& \widetilde{X}_k
\end{tikzcd}$$

commutes. We use (17) for f_0/h_r, and we assume without loss of generality that the same ordinal j can be used for $(u_k f_0)/h_r$. Then the triangle above commutes due to (21):

$$\begin{tikzcd}
A_r \arrow[r, "f' /\!\!/ h_r"] \arrow[dr, "(u_{\hat{j}} f') /\!\!/ h_r"'] & X_{\hat{j}+1} \arrow[r, dashed] \arrow[d, "u_{\hat{j}+1}"] & X_k \arrow[d, "u_k"] \\
 & \widetilde{X}_{\hat{j}+1} \arrow[r, dashed] & \widetilde{X}_k
\end{tikzcd}$$

■

Remark 3.10. In the following diagram (see (19))

$$\begin{tikzcd}
A_r \arrow[r, "h_r"] \arrow[d, "f"'] & B_r \arrow[d, "\overline{f}"] \\
X_i \arrow[r, "\overline{h}_r"] \arrow[d, "u_i"'] & Q_f \arrow[d, "f^*"] \\
\widetilde{X}_i \arrow[r, "\widetilde{h}_r"'] & \widetilde{Q}_{u_i f}
\end{tikzcd}$$

the lower square is a pushout. This follows from the fact that both the composite and the upper square are pushouts.

4.Kan-injectivity logic

We now introduce a sound logic for deriving a square S from a given class $\mathcal{H}$ of squares of $\mathcal{X}$. (Recall that "square" means a commutative one throughout.) Soundness means that every object and every morphism Kan-injective with respect to members of $\mathcal{H}$ is also Kan-injective with respect to squares derived from $\mathcal{H}$. In the subsequent section we prove that our logic is also complete if $\mathcal{H}$ is small and the base category is locally ranked.

Since for ordinary categories Kan-injectivity w.r.t. a morphism is just the usual orthogonality, it is not surprising that our logic is very close to the orthogonality logic presented in [1]. Let us recall this logic here shortly. Firstly, every isomorphism s has the property that all objects are orthogonal to s. Hence that logic has one axiom

AXIOM $\dfrac{}{s}$ for s an isomorphism.

The deduction rules are such that whenever an object is orthogonal to the assumptions (above the horizontal line), then it is orthogonal also to the conclusion. We have the following deduction rules:

COMPOSITION $\dfrac{h_1 \quad h_2}{h_2 \cdot h_1}$ for morphisms $\xrightarrow{h_1} \xrightarrow{h_2}$

PUSHOUT $\dfrac{h}{k}$ for a pushout

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "k"'] & \cdot \end{tikzcd}$$

WIDE PUSHOUT $\dfrac{h_i \, (i \in I)}{k}$ for a wide pushout

$$\begin{tikzcd} \cdot \arrow[r, "h_i"] \arrow[dr, "k"'] & \cdot \arrow[d, "\bar{h}_i"] \\ & \cdot \end{tikzcd} \quad (i \in I)$$

COEQUALIZER $\dfrac{k}{c}$ for a coequalizer

$$\begin{tikzcd} \cdot \arrow[d, dotted, "k"'] & & \\ \cdot \arrow[r, shift left, "g_2"] \arrow[r, shift right, "g_1"'] & \cdot \arrow[r, "c"] & \cdot \end{tikzcd}$$

and a morphism k with $g_1 k = g_2 k$

WEAK CANCELLATION $\dfrac{h_3 \cdot h_2 \quad h_2 \cdot h_1}{h_1}$ for morphisms $\xrightarrow{h_1} \xrightarrow{h_2} \xrightarrow{h_3}$

This logic is sound in every category $\mathcal{X}$ with colimits. And for small sets of morphisms $\mathcal{H}$ it is complete, i.e., every morphism h such that $\mathcal{H}^{\perp} = (\mathcal{H} \cup \{h\})^{\perp}$ can be derived from $\mathcal{H}$ (and the isomorphisms) provided that $\mathcal{X}$ is *locally ranked,* see [1]. This means that for some factorization system $(\mathcal{E}, \mathcal{M})$

(a) $\mathcal{X}$ is cocomplete and $\mathcal{E}$-cowellpowered

and

(b) every object X has a *rank,* i.e., an infinite cardinal λ such that $\mathcal{X}(X, -)$ preserves λ-directed colimits of $\mathcal{M}$-monics.

The logic presented below deals with collections $\mathcal{H}$ of squares and the following concept:

Definition 4.1. A square S is said to be a *Kan-injectivity consequence* of a class $\mathcal{H}$ of squares provided that every object and every morphism Kan-injective with respect to the members of $\mathcal{H}$ is also Kan-injective with respect to S. Shortly:

$$\mathsf{LInj}(\mathcal{H}) = \mathsf{LInj}(\mathcal{H} \cup \{S\}).$$

Example 4.2. If $\mathcal{H} = \emptyset$, we are speaking about squares w.r.t. everything is Kan-injective. Let us call a square

$$\begin{tikzcd} A_1 \arrow[r, "h_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\ A_2 \arrow[r, "h_2"'] & B_2 \end{tikzcd} \tag{24}$$

split if h_1 and h_2 are left adjoint sections and the Beck-Chevalley condition holds. That is, we have a commutative square

$$\begin{tikzcd} A_1 \arrow[d, "a"'] & B_1 \arrow[l, "h_1^*"'] \arrow[d, "b"] \\ A_2 & B_2 \arrow[l, "h_2^*"] \end{tikzcd} \tag{25}$$

with $h_i^* h_i = \mathrm{id}$ and $h_i h_i^* \leq \mathrm{id}$ for $i = 1, 2$.

Each such square has the property that every object is Kan-injective with respect to it. Indeed, given $f : A_r \to X$ for $r = 1, 2$ the formula for the Kan extension is easily seen to be

$$f/h_r = f \cdot h_r^*$$

from which this fact is obvious. Moreover, this formula implies also that every morphism is Kan-injective w.r.t. S.

This explains why in the following deduction system the split squares can serve as axioms.

The following logic has as formulas (commutative) squares in a given category. Recall that every morphism $h : A \to B$ is represented by the square

$$S(h) = \begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "h"'] & B \end{tikzcd} .$$

Definition 4.3. The *Kan-injectivity Logic* consists of one

AXIOM $\dfrac{}{S}$ for split squares S

and the following deduction rules:

COMPOSITION $\dfrac{S_1,\ S_2}{S}$ for a composite S, horizontal or vertical, of S_1 and S_2

PUSHOUT

$$\dfrac{\begin{tikzcd} {} \arrow[r, "h_1"] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, "h_2"'] & {} \end{tikzcd}}{\begin{tikzcd} {} \arrow[r, "h_r"] \arrow[d, "a"'] & {} \arrow[d] \\ {} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & {} \end{tikzcd}}$$

for a pushout of h_r, $r = 1$ or 2, along an arbitrary morphism a

WIDE PUSHOUT

$$\dfrac{\begin{tikzcd} {} \arrow[r, "h"] \arrow[d, equal] & {} \arrow[d, "b_i\ (i \in I)"] \\ {} \arrow[r, "b_i h"'] & {} \end{tikzcd}}{\begin{tikzcd} {} \arrow[r, "h"] \arrow[d, equal] & {} \arrow[r, "b_j"] & {} \arrow[d, "\bar{b}_j"] \\ {} \arrow[r, "h"'] & {} \arrow[r, "k"'] & {} \end{tikzcd}}$$

for any wide pushout

$$\begin{tikzcd} {} \arrow[d, dotted, "h"'] & \\ {} \arrow[r, "b_i"] \arrow[dr, "k"'] & {} \arrow[d, "\bar{b}_i \quad (i \in I)"] \\ & {} \end{tikzcd}$$

and any $j \in I$

COINSERTER

$$\frac{S\,,\ S_0\,,\ \overline{S}}{\begin{tikzcd} {} \arrow[rr, "k'"] \arrow[d, equal] & & {} \arrow[d, "c"] \\ {} \arrow[r, "k'"'] & {} \arrow[r, "c"'] & {} \end{tikzcd}}$$

for a coinserter
$$\begin{tikzcd} {} \arrow[d, dotted, "k"'] & & \\ {} \arrow[r, shift left, "g_2"] \arrow[r, shift right, "g_1"'] & {} \arrow[r, "c"] & {} \end{tikzcd},$$
a morphism k with $g_1k \leq g_2k$, and a commutative diagram

$$\begin{tikzcd} {} \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "S"] & {} \arrow[r, "g_1"] \arrow[d] & {} \arrow[d, equal] & \\ {} \arrow[r] & {} \arrow[r] & {} \arrow[r, dotted, "c"] & {} \\ {} \arrow[u, "h_0"] \arrow[r, equal] \arrow[ur, phantom, "S_0"] & {} \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\overline{S}"] & {} \arrow[u, "k'"'] & \end{tikzcd}$$

RIGHT CANCELLATION

$$\frac{S\,,\ S(h),\ S(k)}{S_0}$$

for $S = \begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "S_0"] & {} \arrow[r, "h"] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r, "k"'] & {} \end{tikzcd}$

UPPER CANCELLATION

$$\frac{S_i\,,\ \overline{S}_i\ (i \in I)}{S}$$

for $\overline{S}_i = \begin{tikzcd} {} \arrow[r, "h_i"] \arrow[d, "a_i"'] \arrow[dr, phantom, "S_i"] & {} \arrow[d, "b_i"] \\ {} \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "S"] & {} \arrow[d, "b"] \\ {} \arrow[r, "h"'] & {} \end{tikzcd}$

with $(b_i)_{i \in I}$ collectively epic

By a *deduction* of a square S from a collection $\mathcal{H}$ of squares is, as usual, meant a sequence of squares obtained by the application of the above rules where S is the last square and the assumptions are (a) members of $\mathcal{H}$, (b) axiom instances or (c) squares already deduced.

Proposition 4.4. *(Soundness of the Kan-Injectivity Logic) Let $\mathcal{X}$ have weigheted colimits. Every square deduced from a class of squares is a Kan-injectivity consequence of that class.*

Proof: For the soundness of AXIOM see Example 4.2. Therefore, all we need to prove is that for every deduction rule in Definition 4.3 the deduced square S is a Kan-injectivity consequence of the assumptions of that rule. To do so we take an object X Kan-injective with respect to each of the assumptions and verify that X is Kan-injective with respect to S. By doing so we actually give a formula for the Kan extensions needed.

We leave out the verification that also morphisms $u : X \to X'$ Kan-injective with respect to all assumptions are Kan-injective with respect to S. Indeed, due to the formula presented for objects this verification is in each case trivial.

(1) COMPOSITION. (a) Horizontal composition:

$$\begin{tikzcd}
A_1 \arrow[r, "h"] \arrow[d, "a"'] & B_1 \arrow[r, "h'"] \arrow[d, "b"'] & C_1 \arrow[d, "c"] \\
A_2 \arrow[r, "k"'] \arrow[d, "f"'] \arrow[d, "f/k"] & B_2 \arrow[r, "k'"'] \arrow[dl] & C_3 \arrow[dll, "(f/k)/k'"] \\
X & &
\end{tikzcd}$$

Let X be Kan-injective with respect to both of the above squares. We prove for all $f : A_2 \to X$ the formula

$$f/(k'k) = (f/k)/k'.$$

Clearly,

$$[(f/k)/k']k'k = f.$$

Given g with $f \leq gk'k$, we conclude $f/k \leq gk'$, hence $(f/k)/k' \leq g$. Analogously,

$$\overline{f}/(h'h) = (\overline{f}/h)/h' \quad \text{for all } \overline{f} : A_1 \to X.$$

The formula $[fa]/(h'h) = [f/(k'k)] \cdot c$ easily follows.

(b) For a vertical composition:

$$\begin{tikzcd}
A'' \arrow[rr, "h''"] \arrow[d, "a_1"'] & & B'' \arrow[d, "b_1"] \\
A' \arrow[rr, "h'"] \arrow[d, "a_2"'] & S_1 & B' \arrow[d, "b_2"] \\
A \arrow[rr, "h"'] \arrow[dr, "f"'] & S_2 & B \arrow[dl, "f/h"] \\
& X &
\end{tikzcd}$$

we prove that $(fa_2a_1)/h'' = (f/h)b_2b_1$ easily from $(fa_2)/h' = (f/h)b_2$ and $[(fa_2)a_1]/h'' = [(fa_2)/h'] \cdot b_1$.

(2) Pushout. (a) Suppose X is Kan-injective w.r.t. $\begin{tikzcd} {} \arrow[r, "h_1"] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, "h_2"'] & {} \end{tikzcd}$. Let $r = 1$ and $f : \overline{A}_1 \to X$ be given:

$$\begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "\overline{a}"'] & B_1 \arrow[d, "\overline{b}"] \arrow[ddr, "(f\overline{a})/h_1"] & \\
\overline{A}_1 \arrow[r, "\overline{h}_1"'] \arrow[drr, "f"'] & \overline{B}_1 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "f/\overline{h}_1"] & \\
& & X
\end{tikzcd}$$

For the square formed by f and $(f\overline{a})/h_1$ we have a unique factorization, let us call it $f/\overline{h}_1$. This is justified by the lower triangle above together with the implication

$$f \leq g\overline{h}_1 \text{ implies } f/\overline{h}_1 \leq g$$

which we verify easily. The above pushout is conical so to prove $f/\overline{h}_1 \leq g$ we only need a verification when precomposed by $\overline{h}_1$ (this is our assumption) and by $\overline{b}$. To prove $(f/\overline{h}_1)\overline{b} \leq g\overline{b}$, that is, $(f\overline{a})h_1 \leq g\overline{b}$, we just observe that our assumption implies $f\overline{a} \leq g\overline{b}h_1$ due to $\overline{b}h_1 = \overline{h}_1\overline{a}$.

The required rule $(f/h_1)/\overline{b} = (f\overline{a})/h_1$ is the right-hand triangle above.

(b) The proof for $r = 2$ is completely analogous.

(3) Wide pushout. Let X be Kan-injective w.r.t. each of the squares $\begin{tikzcd} {} \arrow[r, "h"] \arrow[d, equal] & {} \arrow[d, "b_i"] \\ {} \arrow[r, "b_i h"'] & {} \end{tikzcd}$. Given a morphism $f : A \to X$

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] & B \arrow[r, "b_i"] & B_i \arrow[d, "\overline{b}_i"] \\
A \arrow[r, "h"'] \arrow[d, "f"'] & B \arrow[r, "k"'] & P \\
X & &
\end{tikzcd}$$

we know that f/h and $f/(b_i h)$ exist and fulfil

$$(f/(b_i h))b_i = f/h.$$

Consequently, for the given wide pushout P there exists a unique morphism

$$\hat{f} : P \to X \ \text{ with } \ \hat{f}\overline{b}_i = f/(b_i h) \ \text{ for } \ i \in I.$$

We prove

$$f/(kh) = \hat{f}.$$

Firstly

$$\hat{f}kh = \hat{f}\overline{b}_i b_i h = [f/(b_i h)]\, b_i h = f.$$

Next if $f \leq gkh$, we prove $\hat{f} \leq g$. It suffices to observe that for all i we have $\hat{f}\overline{b}_i \leq g\overline{b}_i$, and use that our wide pushout is conical. Indeed, we have

$$\hat{f}\overline{b}_i = f/(b_i h) \leq g\overline{b}_i$$

or, equivalently,

$$f \leq g\overline{b}_i b_i h$$

by assumption.

The desired formula

$$(f/(kh)) \cdot \overline{b}_j = f/(b_j h)$$

now follows from $\hat{f}\overline{b}_j = f/(b_j h)$.

(4) COINSERTER. We are given a diagram

$$\begin{tikzcd} A \arrow[r, "k"] & B \arrow[r, shift left, "g_2"] \arrow[r, shift right, "g_1"'] & C \arrow[r, "c"] & D \end{tikzcd}$$

with $g_1 k \leq g_2 k$ and $c = \mathrm{coins}(g_1, g_2)$, and a commutative diagram

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "a"'] \arrow[dr, phantom, "S"] & B \arrow[r, "g_1 = h'b"] \arrow[d, "b"] & C \arrow[d, equal] \\ A_0 \arrow[r, "h"] & B_0 \arrow[r, "h'"] & C \\ R \arrow[u, "h_0"] \arrow[r, equal] \arrow[ur, phantom, "S_0"] & R \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\overline{S}"] & R \arrow[u, "k'"'] \end{tikzcd}$$

Observe that $k' = h'hh_0$. If X is Kan-injective with respect to S, S_0 and $\overline{S}$, we prove Kan-injectivity with respect to the following square:

$$\begin{tikzcd} R \arrow[r, "h_0"] \arrow[d, equal] & A_0 \arrow[rr, "h"] & & B_0 \arrow[rr, "h'"] & & C \arrow[d, "c"] \\ R \arrow[r, "h_0"] & A_0 \arrow[r, "h"] & B_0 \arrow[rr, "h'"] & & C \arrow[r, "c"] & D \end{tikzcd}$$

Kan-injectivity with respect to $\overline{S}$ yields

$$\hat{f} = f/(h'hh_0) \text{ with } \hat{f}h' = f/(hh_0) \tag{26}$$

We prove below that

$$\hat{f}g_1 \leq \hat{f}g_2 \tag{27}$$

which means that $\hat{f}$ factorizes through $c = \mathrm{coins}(g_1, g_2)$. This concludes the proof: the factorization

$$\tilde{f} : C \to X \text{ with } \hat{f} = \tilde{f}c$$

is the desired $f/(ch'hh_0)$. Indeed,

$$\tilde{f}ch'hh_0 = \hat{f}(h'hh_0) = f$$

and given t with $f \leq tch'hh_0$, then $\hat{f} = f/(h'hh_0) \leq tc$. Thus $\tilde{f}c \leq tc$ which implies $\tilde{f} \leq t$. The desired equality

$$[f/(ch'hh_0)] \cdot c = f/(h'hh_0)$$

in other words

$$\hat{f} = \tilde{f}c = f/(h'hh_0)$$

is the definition of $\hat{f}$.

In order to prove (27) first recall that X is Kan-injective with respect to S_0 and S, hence, f/h_0 and $(f/h_0)/h$ exist. From this we easily deduce

$$f/(hh_0) = (f/h_0)/h$$

and then (26) yields

$$\hat{f}h' = (f/h_0)/h. \tag{28}$$

By Kan-injectivity with respect to S this implies

$$\hat{f}g_1 = \hat{f}h'b = [(f/h_0)/h] \cdot b = [(f/h_0) \cdot a]/k. \tag{29}$$

Next recall that COINSERTER also assumes $g_1k \leq g_2k$, thus $\hat{f}g_1k \leq \hat{f}g_2k$, and then, using (29), we obtain (27).

(5) RIGHT CANCELLATION. Let X be Kan-injective with respect to the squares

$$S = \begin{tikzcd} A' \arrow[r, "k'"] \arrow[d, "a"'] & B' \arrow[r, "k"] \arrow[d, "b"] & C' \arrow[d, "c"] \\ A \arrow[r, "h'"'] & B \arrow[r, "h"'] & C \end{tikzcd}, \quad S(h) = \begin{tikzcd} B \arrow[r, "h"] \arrow[d, equal] & C \arrow[d, equal] \\ B \arrow[r, "h"'] & C \end{tikzcd} \quad \text{and} \quad S(k) = \begin{tikzcd} B' \arrow[r, "k"] \arrow[d, equal] & C' \arrow[d, equal] \\ B' \arrow[r, "k"'] & C' \end{tikzcd}.$$

Given a morphism $f : A \to X$, put

$$f/h' = [f/(hh')] \cdot h.$$

Then $(f/h') \cdot h' = f$ is clear. And if $f \leq gh'$, then, recalling Kan-injectivity with respect to $S(h)$, we get $g = (g/h)h$, hence

$$f \leq (g/h)(hh')$$

which implies

$$f/(hh') \leq g/h$$

and yields

$$[f/(h'h)] \cdot h \leq g$$

as desired. Analogously, $(fa)/k' = [(fa)/(kk')] \cdot k$ and this yields

$$(f/h') \cdot b = (fa)/k'$$

as required.

(6) Upper cancellation. Let X be Kan-injective w.r.t. the squares S_i and composites of S_i with S $(i \in I)$. Given a morphism f as follows

$$\begin{tikzcd}
A_i \arrow[rr, "h_i"] \arrow[d, "a_i"'] \arrow[drr, phantom, "S_i"] & & B_i \arrow[d, "b_i"] \\
A \arrow[rr, "k"] \arrow[d, "a"'] \arrow[drr, phantom, "S"] & & B \arrow[d, "b"] \\
A' \arrow[rr, "h"'] \arrow[dr, "f"'] & & B' \arrow[dl, "f/h"] \\
& X &
\end{tikzcd}$$

we have f/h satisfying

$$(f/h)bb_i = (faa_i)/h_i \quad (i \in I).$$

The desired equality

$$(fa)/k = (f/h) \cdot b$$

follows, since (b_i) is collectively epic, from

$$\begin{aligned} [(fa)/k] \cdot b_i &= (faa_i)/h_i && \text{by Kan-injectivity with respect to } S \\ &= (f/h)(bb_i) && \text{by Kan-injectivity with respect to the composite.} \end{aligned}$$

This conclude the proof of soundness. ■

Lemma 4.5. *The following deduction rules are consequences of the Kan-injectivity Logic:*

S-RULE

$$\dfrac{\begin{tikzcd} {} \arrow[r, "h_1"] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, "h_2"'] & {} \end{tikzcd}}{S(h_r)} \qquad \textit{for } r = 1 \textit{ or } 2$$

and

TRANSFER
$$\begin{array}{c}\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "h"'] & \cdot \end{tikzcd} \\ \hline \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "h"'] & \cdot \end{tikzcd}\end{array}$$

Proof: S-RULE is a special case of PUSHOUT: $S(h_r)$ is a pushout square. TRANSFER follows by applying RIGHT CANCELLATION as follows:

$$S(h) = \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, equal] & \cdot \arrow[r, "h"] \arrow[d, "h"'] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "h"'] & \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

■

Lemma 4.6. *The following deduction rule is a consequence of the Kan-Injectivity Logic:*

COEQUALIZER
$$\begin{array}{c}\begin{tikzcd} \cdot \arrow[r, "k"] \arrow[d, "a"'] & \cdot \arrow[d, "b_1"] \\ \cdot \arrow[r, "h"'] & \cdot \end{tikzcd} \quad \begin{tikzcd} \cdot \arrow[r, "k"] \arrow[d, "a"'] & \cdot \arrow[d, "b_2"] \\ \cdot \arrow[r, "h"'] & \cdot \end{tikzcd} \\ \hline \begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, equal] & \cdot \arrow[d, "c"] \\ \cdot \arrow[r, "ch"'] & \cdot \end{tikzcd}\end{array} \qquad \textit{for } c = \mathrm{coeq}(b_1, b_2)$$

Proof: Form coinserters

$$k_1 = \mathrm{coins}(b_1, b_2) \quad \text{and} \quad k_2 = \mathrm{coins}(b_2, b_1).$$

Then we apply COINSERTER to the following diagrams

$$\begin{tikzcd} \cdot \arrow[r, "k"] \arrow[d, "a"'] \arrow[dr, phantom, "S"] & \cdot \arrow[r, "b_i"] \arrow[d, "b_i"] & \cdot \arrow[d, equal] \\ A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "S_0"] & B \arrow[r, equal] \arrow[dr, phantom, "\overline{S}"] & B \\ A \arrow[r, equal] & A \arrow[u, "h"'] \arrow[r, equal] & A \arrow[u, "h"'] \end{tikzcd} \qquad (i = 1, 2)$$

where S_0 follows by TRANSFER and $\overline{S}$ by S-RULE, and we deduce the squares below from the above assumptions:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, "k_1"] \\ A \arrow[r, "k_1 h"'] & K_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] & B \arrow[d, "k_2"] \\ A \arrow[r, "k_2 h"'] & K_2 \end{tikzcd}$$

The coequalizer c is clearly a pushout of k_1 and k_2:

$$\begin{tikzcd} & B \arrow[dl, "k_1"'] \arrow[dr, "k_2"] \arrow[dd, "c"] & \\ K_1 \arrow[dr] & & K_2 \arrow[dl] \\ & C & \end{tikzcd}$$

By applying WIDE PUSHOUT with $s = h$ we derive

$$\begin{tikzcd} {} \arrow[r, "h"] \arrow[d, equal] & {} \arrow[d, "c"] \\ {} \arrow[r, "ch"'] & {} \end{tikzcd}$$

■

Remark 4.7. Let X_i $(i < \lambda)$ be an α-chain with connecting maps x_{ij}. We can construct its colimit using wide pushouts as follows:

First form a wide pushout of all x_{0i}

$$\begin{tikzcd} X_0 \arrow[r, "x_{0i}"] \arrow[dr, "p_0"'] & X_i \arrow[d, "p_i"] & (i < \alpha) \\ & P & \end{tikzcd} \tag{30}$$

For all pairs $i \leq i' < \alpha$ form the following coequalizer

$$\begin{tikzcd} X_i \arrow[r, shift left, "p_i"] \arrow[r, shift right, "p_{i'} x_{ii'}"'] & P \arrow[r, "c_{ii'}"] & C_{ii'} \end{tikzcd} \tag{31}$$

Finally, the wide pushout of all these coequalizers is formed

$$\begin{tikzcd} P \arrow[r, "c_{ii'}"] \arrow[dr, "g"'] & C_{ii'} \arrow[d, "\bar{c}_{ii'}"] \\ & K \end{tikzcd} \tag{32}$$

Then K is the colimit of the given diagram with the following colimit cocone:

$$X_i \xrightarrow{p_i} P \xrightarrow{g} K \quad (i < \alpha).$$

Indeed, since g merges the parallel pair in (31), all gp_i form a cocone.

Let $y_i : X_i \to Y \ \ (i < \alpha)$ be another cocone. Then the unique $y : P \to Y$ with $y_i = yp_i \ \ (i < \alpha)$ clearly merges the parallel pair of (31). Hence, y factorizes through each $c_{ii'}$. Consequently, $y = \overline{y}g$ for a unique $\overline{y} : K \to Y$. This is the desired factorization:

$$y_i = yp_i = \overline{y}(gp_i) \quad \text{for } i < \alpha.$$

The uniqueness of this factorization is easy to verify.

Lemma 4.8. *(Transfinite Composition) Let* $(X_i)_{i\in\alpha}$ *be an* α*-chain with connecting morphisms* $X_i \dashrightarrow X_j$ *and a colimit* $k_i : X_i \to K \ \ (i < \alpha)$*. The following deduction rule follows, for every* $j < \alpha$*, from the Kan-Injectivity Logic.*

TRANSFINITE COMPOSITION

$$\dfrac{\begin{tikzcd} X_0 \arrow[r, dashed] \arrow[d, equal] & X_j \arrow[d, dashed] \\ X_0 \arrow[r, dashed] & X_k \end{tikzcd} \quad (j \le k \le \alpha)}{\begin{tikzcd} X_0 \arrow[r, dashed] \arrow[d, equal] & X_j \arrow[d, "k_j"] \\ X_0 \arrow[r, "k_0"'] & K \end{tikzcd}}$$

Proof: We can assume $k_i = gp_i$ for all i, see the above remark. By applying WIDE PUSHOUT to the premisses of our rule we get, for every $i < \alpha$, the deduction of the square

$$\begin{tikzcd} X_0 \arrow[r, dashed] \arrow[d, equal] & X_i \arrow[d, "p_i"] \\ X_0 \arrow[r, "p_0"'] & P \end{tikzcd} \tag{33}$$

This makes it possible to apply COEQUALIZER to (30) to derive

$$\begin{tikzcd} X_b \arrow[r, "p_0"] \arrow[d, equal] & P \arrow[d, "c_{ii'}"] \\ X_0 \arrow[r, "c_{ii'}p_0"'] & C_{ii'} \end{tikzcd} \tag{34}$$

for all $i \leq i' < \alpha$. Next apply WIDE PUSHOUT to (34) with $h = p_0$ to derive

$$\begin{tikzcd}
X_0 \arrow[r, "p_0"] \arrow[d, equal] & P \arrow[r, "c_{ii'}"] & C_{ii'} \arrow[d, "\overline{c}_{ii'}"] \\
X_0 \arrow[r, "p_0"'] & P \arrow[r, "g"'] & K
\end{tikzcd} .$$

For $i = i' = j$ we can put $c_{jj} = \mathrm{id}$ in (31), thus $g = \overline{c}_{jj}$ in (32). Hence we have derived the following square

$$\begin{tikzcd}
X_0 \arrow[rr, "p_0"] \arrow[d, equal] & & P \arrow[d, "g"] \\
X_0 \arrow[r, "p_0"'] & P \arrow[r, "g"'] & K
\end{tikzcd}$$

Vertical COMPOSITION with (33) (for $i = j$) yields the desired square:

$$\begin{tikzcd}
X_0 \arrow[r, dashed] \arrow[d, equal] & X_j \arrow[d, "h_j = gp_j"] \\
X_0 \arrow[r, "h = gp_0"'] & K
\end{tikzcd}$$

■

5. Proof of Completeness

Throughout this section $\mathcal{H}$ is a set of squares in a locally ranked order-enriched category $\mathcal{X}$. We prove that the Kan-Injectivity Logic is complete.

First a preliminary result. Recall the reflection chain from Section 3.

Lemma 5.1. *For all ordinals $m \leq i$ the squares*

$$\begin{tikzcd}
X_0 \arrow[r, dashed] \arrow[d, equal] & X_m \arrow[d, dashed] \\
X_0 \arrow[r, dashed] & X_i
\end{tikzcd} \qquad (35)$$

can be deduced from $\mathcal{H}$.

Proof: We proceed by transfinite induction in i.

Initial step. Use AXIOM on the split square

$$\begin{tikzcd}
X_0 \arrow[r, equal] \arrow[d, equal] & X_0 \arrow[d, equal] \\
X_0 \arrow[r, equal] & X_0
\end{tikzcd}$$

Limit step. This follows from TRANSFINITE COMPOSITION, see Lemma 4.8. For $m = i$, $S(X_0 \dashrightarrow X_i)$ is obtained by S-RULE (Lemma 4.5).
Isolated step $i \mapsto i+1$ (i even). Given a square

$$\begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "a"'] \arrow[dr, phantom, "S"] & B_1 \arrow[d, "b"] \\
A_2 \arrow[r, "h_2"'] & B_2
\end{tikzcd}$$

in $\mathcal{H}$ and a morphism $f : A_r \to X_i$ $(r = 1 \text{ or } 2)$, we have the following deduction:

$$S$$

PUSHOUT

$$\begin{tikzcd}
{} \arrow[r, "h_r"] \arrow[d, "f"'] & {} \arrow[d, "\overline{f}"] \\
{} \arrow[r, "\overline{h}_r"'] & {}
\end{tikzcd} \tag{36}$$

TRANSFER (Lemma 4.5)

$$\begin{tikzcd}
X_i \arrow[r, equal] \arrow[d, equal] & X_i \arrow[d, "\overline{h}_r"] \\
X_i \arrow[r, "\overline{h}_r"'] & Q_f
\end{tikzcd}$$

WIDE PUSHOUT

$$\begin{tikzcd}
X_i \arrow[r, "\overline{h}_r"] \arrow[d, equal] & Q_f \arrow[d, "q_f"] \\
X_i \arrow[r, dashed] & X_{i+1}
\end{tikzcd} \tag{37}$$

TRANSFER
(Lemma 4.5)

$$\begin{tikzcd}
X_i \arrow[r, equal] \arrow[d, equal] & X_i \arrow[d, dashed] \\
X_i \arrow[r, dashed] & X_{i+1}
\end{tikzcd}$$

(38)

By induction hypothesis, (35) are given. Our task is to verify, for every $m \leq i+1$, the corresponding square. Horizontal COMPOSITION of (38) with (35) for $m = i$ deduces

$$\begin{tikzcd}
X_0 \arrow[r, dashed] \arrow[d, equal] & X_i \arrow[d, dashed] \\
X_0 \arrow[r, dashed] & X_{i+1}
\end{tikzcd}$$

and vertical COMPOSITION of (38) with (35), $m \leq i$, yields the desired square

$$\begin{tikzcd}
X_0 \arrow[r, dashed] \arrow[d, equal] & X_k \arrow[d, dashed] \\
X_0 \arrow[d, equal] & X_i \arrow[d, dashed] \\
X_0 \arrow[r, dashed] & X_{i+1}
\end{tikzcd}$$

(39)

For $m = i+1$, we deduce the square $S(X_0 \dashrightarrow X_{i+1})$ by (39) via S-RULE (Lemma 4.5).

Isolated step $i+1 \mapsto i+2$ (i even). We first observe that the square (9) of Definition 3.1 is deduced from $\mathcal{H}$ by COMPOSITION applied to (36) and (37):

$$\begin{tikzcd}
A_r \arrow[r, "h_r"] \arrow[d, "f"'] & B_r \arrow[d, "\overline{f}"] \arrow[dd, bend left, "f /\!\!/ h_r"] \\
X_i \arrow[r, "\overline{h}_r"'] \arrow[d, equal] & Q_f \arrow[d, "q_f"] \\
X_i \arrow[r, dashed] & X_{i+1}
\end{tikzcd}$$

(40)

For every coinserter (12) we can thus apply COINSERTER to $k = h_r$ and the following diagram

$$\begin{tikzcd}
A_r \arrow[r, "h_r"] \arrow[d, "f'"'] \arrow[dr, phantom, "S"] & B_r \arrow[r] \arrow[d, "f' /\!\!/ h_r"] & {} \arrow[d, equal] & \\
X_j \arrow[r, dashed] \arrow[dr, phantom, "S_0"] & X_{j+1} \arrow[r, dashed] \arrow[dr, phantom, "\overline{S}"] & X_{i+1} \arrow[r, dotted, "c_{f,g}"] & C_{f,g} \\
X_0 \arrow[r, equal] \arrow[u, dashed] & X_0 \arrow[r, equal] \arrow[u, dashed] & X_0 \arrow[u, dashed] &
\end{tikzcd}$$

Indeed, the assumptions are S, by (40), S_0, which is (35) for $m = j$ and $i = j + 1$, and $\overline{S}$, which is (35) with $m = j + 1$ and $i + 1$ in the place of i. Consequently, we deduce the square

$$\begin{tikzcd}
X_0 \arrow[rr, dashed] \arrow[d, equal] & & X_{i+1} \arrow[d, "c_{f,g}"] \\
X_0 \arrow[r, dashed] & X_{i+1} \arrow[r, "c_{f,g}"'] & C_{f,g}
\end{tikzcd} \tag{41}$$

For every coinserter (14), first observe that the square

$$S = \begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "a"'] \arrow[dr, phantom, "S'"] & B_1 \arrow[d, "b"] \\
A_2 \arrow[r, "h_2"'] \arrow[d, "f'"'] & B_2 \arrow[d, "f' /\!\!/ h_2"] \\
X_i \arrow[r, dashed] & X_{i+1}
\end{tikzcd}$$

is deduced from S' and (40) by COMPOSITION. We apply COINSERTER to $k = h_1$ and the following diagram

$$\begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d] \arrow[dr, phantom, "S"] & B_1 \arrow[r] \arrow[d] & X_{i+1} \arrow[d, equal] & \\
X_i \arrow[r, dashed] \arrow[dr, phantom, "S_0"] & X_{i+1} \arrow[r, equal] \arrow[dr, phantom, "\overline{S}"] & X_{i+1} \arrow[r, dotted, "c_{f,g}"] & C_{f,g} \\
X_0 \arrow[r, equal] \arrow[u, dashed] & X_0 \arrow[r, equal] \arrow[u, dashed] & X_0 \arrow[u, dashed] &
\end{tikzcd}$$

where the second and last assumptions of COINSERTER are just instances of (35) which are deduced from $\mathcal{H}$ by induction hypothesis. Thus we again deduce the square (41).

Apply WIDE PUSHOUT to (15) with $h \equiv X_0 \dashrightarrow X_{i+1}$ to deduce all the squares

$$\begin{tikzcd}
X_0 \arrow[r, dashed] \arrow[d, equal] & X_{i+1} \arrow[r, "c_{f,g}"] & C_{f,g} \arrow[d, "t_{f,g}"] \\
X_0 \arrow[r, dashed] & X_{i+1} \arrow[r, dashed] & X_{i+2}
\end{tikzcd} \tag{42}$$

A vertical COMPOSITION with (41) deduces

$$\begin{tikzcd}
X_0 \arrow[r, dashed] \arrow[d, equal] & X_{i+1} \arrow[d, dashed] \\
X_0 \arrow[r, dashed] & X_{i+2}
\end{tikzcd}$$

which vertically composed with (39) yields

$$\begin{tikzcd}
X_0 \arrow[r, dashed] \arrow[d, equal] & X_m \arrow[d, dashed] \\
X_0 \arrow[r, dashed] & X_{i+2}
\end{tikzcd} \qquad \text{for all } m \leq i+1.$$

The remaining case $m = i + 2$ is then deduced by S-RULE. ■

Theorem 5.2. *(Kan-Injectivity Logic is Complete and Sound) A square is a Kan-injectivity consequence of a set of squares iff it can be deduced from that set.*

Proof: For soundness see Proposition 4.4. Let $\mathcal{H}$ be a set of squares and let the following square

$$S \equiv \begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\
A_2 \arrow[r, "h_2"'] & B_2
\end{tikzcd}$$

be a Kan-injectivity consequence of $\mathcal{H}$. We find a deduction of S from $\mathcal{H}$.

Let $R : \mathcal{X} \to \mathsf{LInj}(\mathcal{H})$ denote the reflector, $RX = X_k$ and $Ru = u_k$ (see Proposition 3.9).

(1) We first use the fact that RA_1 and RA_2 are Kan-injective with respect to S, thus, η_{A_1}/h_1 and η_{A_2}/h_2 exist, and we prove that they form a square as follows:

$$\begin{tikzcd}
B_1 \arrow[r, "\eta_{A_1}/h_1"] \arrow[d, "b"'] & RA_1 \arrow[d, "Ra"] \\
B_2 \arrow[r, "\eta_{A_2}/h_2"'] & RA_2
\end{tikzcd}$$

Indeed, the morphism Ra is Kan-injective with respect to h_1, thus

$$Ra \cdot (\eta_{A_1}/h_1) = (Ra \cdot \eta_{A_1})/h_1 = (\eta_{A_2} \cdot a)/h_1$$

and, since RA_2 is Kan-injective with respect to S, we have

$$(\eta_{A_2} \cdot a)/h_1 = (\eta_{A_2}/h_2) \cdot b.$$

(2) The morphism $Rh_1 : RA_1 \to RB_1$ is a left adjoint section. (This was proved in [6] but we include the short proof for he convenience of the reader.) Indeed, since RA_1 is Kan-injective with respect to h_1 (being Kan-injective with respect to S), using the universal property of η_{A_1}, we have $\eta_{A_1}/h_1 = h_1^* \cdot \eta_{B_1}$

$$\begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "\eta_{A_1}"'] & B_1 \arrow[d, "\eta_{B_1}"] \arrow[dl, "\eta_{A_1}/h_1"'] \\
RA_1 \arrow[r, shift left, "Rh_1"] & RB_1 \arrow[l, shift left, "h_1^*"]
\end{tikzcd}$$

for a unique h_1^* in $\mathsf{LInj}(\mathcal{H})$. This is the desired morphism with

$$h_1^* \cdot Rh_1 = id \quad \text{and} \quad Rh_1 \cdot h_1^* \leq id.$$

Indeed, both composites lie in $\mathsf{LInj}(\mathcal{H})$, thus, it is sufficient to verify (a) $h_1^* \cdot Rh_1 \cdot \eta_{A_1} = \eta_{A_1}$ - see the above diagram, and (b) $Rh_1 \cdot h_1^* \cdot \eta_{B_1} \leq \eta_{B_1}$. We use the trivial inequality $(\eta_{B_1} \cdot h_1)/h_1 \leq \eta_{B_1}$ and the above diagram to see that (b) holds.

(3) The square

$$R(S) \equiv \begin{tikzcd} RA_1 \arrow[r, "Rh_1"] \arrow[d, "Ra"'] & RB_1 \arrow[d, "Rb"] \\ RA_2 \arrow[r, "h_2"'] & RB_2 \end{tikzcd}$$

splits. (This is Lemma 3.4 in [14]; here we present a different proof.) Indeed, we have $h_1^* : RB_1 \to RA_1$ in (2) and, analogously, $h_2^* : RB_2 \to RA_2$. It remains to verify the following square

$$\begin{tikzcd} RB_1 \arrow[r, "h_1^*"] \arrow[d, "Rb"'] & RA_1 \arrow[d, "Ra"] \\ RB_2 \arrow[r, "h_2^*"'] & RA_2 \end{tikzcd}$$

It lies in $\mathsf{LInj}(\mathcal{H})$, thus, it is sufficient to prove that it commutes when precomposed by η_{B_1}:

$$\begin{tikzcd} RB_1 \arrow[rr, "h_1^*"] \arrow[ddd, "Rb"'] & & RA_1 \arrow[ddd, "Ra"] \\ & B_1 \arrow[ul, "\eta_{B_1}"] \arrow[ur, "\eta_{A_1}/h_1"'] \arrow[d, "b"'] & \\ & B_2 \arrow[dl, "\eta_{B_2}"'] \arrow[dr, "\eta_{A_2}/h_2"] & \\ RB_2 \arrow[rr, "h_2^*"'] & & RA_2 \end{tikzcd}$$

Indeed, use the square in (1) above.

(4) In part (5) we are going to prove that every naturality square of η, in particular

$$\begin{tikzcd} A_1 \arrow[r, "\eta_{A_1}"] \arrow[d, "a"'] & RA_1 \arrow[d, "Ra"] \\ A_2 \arrow[r, "\eta_{A_2}"'] & RA_2 \end{tikzcd}$$

can be deduced from $\mathcal{H}$. Due to AXIOM, the square $R(S)$ of part (3) is also deducible, and their horizontal composite yields

$$\begin{tikzcd}
A_1 \arrow[r, "\eta_{A_1}"] \arrow[d, "a"'] & RA_1 \arrow[r, "Rh_1"] & RB_1 \arrow[d, "Rb"] \\
A_2 \arrow[r, "\eta_{A_2}"'] & RA_2 \arrow[r, "Rh_2"'] & RB_2
\end{tikzcd}$$

which is the same square as the following composite

$$\begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "a"'] \arrow[dr, phantom, "S"] & B_1 \arrow[r, "\eta_{B_1}"] \arrow[d, "b"] & RB_1 \arrow[d, "Rb"] \\
A_2 \arrow[r, "h_2"'] & B_2 \arrow[r, "\eta_{B_2}"'] & RB_2
\end{tikzcd}$$

Thus, S is deduced via RIGHT CANCELLATION, since the right-hand square is deducible (being, again, a naturality square of η). This concludes the proof.

(5) To prove that naturality squares of η are deducible from $\mathcal{H}$, we consider the squares

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, "u"'] & X_i \arrow[d, "u_i"] \\
\widetilde{X} \arrow[r, dashed] & \widetilde{X}_i
\end{tikzcd} \tag{43}$$

for all ordinals i and prove their deducibility by transfinite induction. By Proposition 3.9 the case $i = k$ is the desired square.

Initial step: the square

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "u"'] & X \arrow[d, "u"] \\
\widetilde{X} \arrow[r, equal] & \widetilde{X}
\end{tikzcd}$$

is split, we can apply AXIOM .

Limit step: Given a limit ordinal i, such that (43) is deducible for every $m < i$ in place of i, compose (43) vertically with (35) to get a deducible (outward) square as follows:

$$\begin{tikzcd}
X_0 \arrow[r, dashed] \arrow[d, equal] & X_m \arrow[d, dashed] \\
X \arrow[r, dashed] \arrow[d, "u"'] & X_i \arrow[d, "u_i"] \\
\widetilde{X} \arrow[r, dashed] & \widetilde{X}_i
\end{tikzcd}$$

The upper square is deducible by Lemma 5.1, and all $X_m \dashrightarrow X_i$ are collectively epic, hence, the desired lower square is deduced by UPPER CANCELLATION .

Isolated step $i \mapsto i+1$ (i even). We are going to derive the square

$$\begin{tikzcd}
X_i \arrow[r, dashed] \arrow[d, "u_i"'] & X_{i+1} \arrow[d, "u_{i+1}"] \\
\widetilde{X}_i \arrow[r, dashed] & X_{i+1}
\end{tikzcd}$$

and compose it horizontally with the square assumed by induction hypothesis.

For that take all pairs (S, f) defining the step $i \mapsto i+1$. Then we apply UPPER CANCELLATION to the following diagram

$$\begin{tikzcd}
X_i \arrow[r, "\overline{h}_r"] \arrow[d, equal] & Q_f \arrow[d, "q_f"] \\
X_i \arrow[r, dashed] \arrow[d, "u_i"'] & X_{i+1} \arrow[d, "u_{i+1}"] \\
\widetilde{X}_i \arrow[r, dashed] & \widetilde{X}_{i+1}
\end{tikzcd}$$

The upper squares are deduced, see (37), and all q_f are collectively epic, thus, we only need a deduction of the composite square. This is, by the definition of u_{i+1} in Remark 3.7 the square

$$\begin{tikzcd}
X_i \arrow[r, "\overline{h}_r"] \arrow[dd, "u_i"'] & Q_f \arrow[d, "f^*"] \\
& \widetilde{Q}_{u_i f} \arrow[d, "\widetilde{q}_{u_i f}"] \\
\widetilde{X}_i \arrow[r, dashed] & \widetilde{X}_{i+1}
\end{tikzcd} \tag{44}$$

Now in order to derive (44), recall the pushout

$$\begin{tikzcd}
X_i \arrow[r, "\overline{h}_r"] \arrow[d, "u_i"'] \arrow[dr, phantom, "S_0"] & Q'_f \arrow[d, "f^*"] \\
\widetilde{X}_i \arrow[r, "\widetilde{h}_r"'] & \widetilde{Q}_{u_i f}
\end{tikzcd}$$

from Notation 3.8 and compose it vertically with the square

$$\begin{tikzcd}
\widetilde{X}_i \arrow[r, "\widetilde{h}_r"] \arrow[d, equal] & \widetilde{Q}_{u_i f} \arrow[d, "\widetilde{q}_{u_i f}"] \\
\widetilde{X}_i \arrow[r, dashed] & \widetilde{X}_{i+1}
\end{tikzcd}$$

The former square can be deduced from (36) via PUSHOUT, for the latter one see (37). Thus (44) is deducible.

Isolated step $i+1 \mapsto i+2$ (i even). Using (42), we can again apply UPPER CANCELLATION :

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, equal] & X_{i+1} \arrow[r, "c_{f,g}"] & C_{f,g} \arrow[d, "\overline{c}_{f,g}"] \\
X \arrow[rr, dashed] \arrow[d, "u"'] & & X_{i+2} \arrow[d, "u_{i+2}"] \\
\widetilde{X} \arrow[rr, dashed] & & \widetilde{X}_{i+2}
\end{tikzcd} \tag{45}$$

We only need to deduce the composite square.

The following composite

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, "u"'] & X_{i+1} \arrow[d, "u_{i+1}"] \\
\widetilde{X} \arrow[r, dashed] \arrow[d, equal] & \widetilde{X}_{i+1} \arrow[d, dashed] \\
\widetilde{X} \arrow[r, dashed] & \widetilde{X}_{i+2}
\end{tikzcd}$$

is deducible due to Lemma 5.1, and the right-hand vertical morphism is

$$X_{i+1} \xrightarrow{c_{f,g}} C_{f,g} \xrightarrow{d_{f,g}} \widetilde{C}_{\widetilde{f},\tilde{g}} \xrightarrow{\widetilde{t}_{\tilde{f},\tilde{g}}} \widetilde{X}_{i+2}$$

by (22). Thus we have deduced the following composite

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[d, equal] & & X_{i+1} \arrow[d, "c_{f,g}"] \\
X \arrow[r, dashed] \arrow[ddd, "u"'] & X_{i+1} \arrow[r, "c_{f,g}"'] & C_{fg} \arrow[d, "d_{f,g}"] \\
& & \widetilde{C}_{\tilde{f},\tilde{g}} \arrow[dd, "\widetilde{t}_{\tilde{f},\tilde{g}}"] \\
& & \\
\widetilde{X} \arrow[rr, dashed] & & \widetilde{X}_{i+2}
\end{tikzcd}$$

The upper square is (41). Moreover, $c_{f,g}$ is an epimorphism (being a coinserter). Thus upper cancellation yields the deduction of the lower square. This is the desired composite square (45): indeed, see (23). ■

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Jiří Adámek
Institute of Theoretical Computer Science, Technical University of Braunschweig, Germany
E-mail address: `adamek@iti.cs.tu-bs.de`

Lurdes Sousa
Polytechnic Institute of Viseu & CMUC, University of Coimbra, Portugal
E-mail address: `sousa@estv.ipv.pt`