Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 16–42

ZERO-TRUNCATED COMPOUND POISSON INTEGER-VALUED GARCH MODELS FOR TIME SERIES

ESMERALDA GONÇALVES AND NAZARÉ MENDES LOPES

Abstract: Starting from the Compound Poisson INGARCH models ([3]), we introduce in this paper a new family of integer-valued models suitable to describe count data without zeros that we name Zero Truncated CP-INGARCH processes. For such class of models, a probabilistic study concerning stationarity, ergodicity and moments existence is developed. The conditional maximum likelihood method is used to consistently estimate the parameters of the conditional Poisson subfamily of models, for which the main asymptotic properties are analyzed. A simulation study illustrating the finite distance behavior of those estimators and a real-data application conclude the paper.

Keywords: Compound Poisson distribution, zero truncated count time series.
AMS Subject Classification (2000): 62M10.

1. Introduction

The usual probability distributions describing the integer-valued models present in literature assume, in general, that the count data to be modeled have zero counts, that is zero is a possible value of their supports.

It may however happen that the expected number of zeros according to the probability distribution of the fitted model is not compatible with those actually occurring. We have in this case an inflation situation, or deflation, of the zero value and in order to correct this phenomenon we have to provide for the possibility to mix such distribution with a point probability. This is for example the case of integer-valued zero inflated models, studied in particular by [10], [6] and [3].

There is yet another type of counting series that structurally exclude the zero value. The number of days of hospitalization in an hospital or the number of days of travel of tourists from a certain country in a period of the year are clear examples of count series without zeros. An interesting and dynamical series of count data without zeros is, for instance, the daily number of occupied beds in a central hospital inpatient service.

Received September 29, 2016.

When the structure of the series is such that it makes no sense the occurrence of zeros, the underlying distribution should not include zero in its support. One possibility to describe such a situation is the truncation of zero in integer-valued models generally compatible with the situation in study.

A class of integer-valued models recently introduced in the literature is the class of integer-valued GARCH processes with compound Poisson conditional distribution (CP-INGARCH). This is a very general class of models that is able to respond to situations compatible with all compound Poisson laws among which stand out the Poisson, Negative Binomial, generalized Poisson and Neyman type A laws. In view of the wideness of the family of distributions associated with these processes it is natural to expect that such distributions truncated in zero may be compatible with count models without null results. This fact led us to introduce a new class of models based on the CP-INGARCH class but without the possibility of zeros. We call these new models CP-INGARCH truncated at zero and denote them briefly as ZTCP-INGARCH.

In Section 2 we recall the definition of the compound Poisson model with values in $\mathbb{N}_0$ with generalized autoregressive conditional heteroskedasticity and introduce the integer-valued compound Poisson truncated at zero model definition (ZTCP-INGARCH). We analyze aspects of its probabilistic structure, namely the existence of moments, and the strict stationarity and ergodicity in a general sub-class. We study then the negative binomial ZTCP-INGARCH model; this model does not belong to that sub-class but, as we shall see, we are able to establish its second order stationarity.

Section 3 includes the estimation of the parameters of the Poisson ZTCP-INGARCH model by conditional maximum likelihood and a simulation study that illustrates the estimation methodology developed. Section 4 concludes with a real-data application.

2. Zero Truncated CP-INGARCH processes

2.1. Zero truncated compound Poisson law.

Let us recall that a real random variable X follows a compound Poisson law with parameter λ, $\lambda > 0$, if its generating function (of probabilities), $g_X(u) = E\left(u^X\right)$, $|u| \leq 1$, is given by

$$g_X(u) = \exp(\lambda(g_Y(u) - 1))$$

with g_Y the generating function of probabilities of a random variable Y. Moreover, in this case $P(X=0)=g_X(0)=\exp(\lambda(g_Y(0)-1))$ and $g_Y(0)=P(Y=0)$.

We say that the real random variable Z follows a compound Poisson law truncated at zero if its generating function is

$$\begin{aligned} g_Z(u) &= E\left(u^Z\right) \\ &= E\left(u^X|X\neq 0\right) \\ &= \sum_{k=1}^{+\infty} u^k P\left(X=k|X\neq 0\right) \\ &= \frac{\exp(\lambda(g_Y(u)-1))-\exp(\lambda(g_Y(0)-1))}{1-\exp(\lambda(g_Y(0)-1))} \end{aligned}$$

2.2. Zero truncated CP-INGARCH model.

Let $X=(X_t,\ t\in\mathbb{Z})$ be a nonnegative integer-valued stochastic process and, for $t\in\mathbb{Z}$, let $\underline{X}_t$ denote the $\sigma-$ field generated by $(X_{t-j}, j\geq 0)$.

Definition. ([3]) The process X is said to follow a compound Poisson GARCH model with values in $\mathbb{N}_0$ with orders p and q ($p,\ q\in\mathbb{N}$) if, for all $t\in\mathbb{Z}$, the characteristic function of X_t conditioned on $\underline{X}_{t-1}$ is given by

$$\Phi_{X_t|\underline{X}_{t-1}}(u)=\exp\left\{i\frac{\lambda_t}{\varphi_t^{'}(0)}\left[\varphi_t(u)-1\right]\right\}, u\in\mathbb{R},$$

with

$$E\left(X_t|\underline{X}_{t-1}\right)=\lambda_t=\alpha_0+\sum_{j=1}^{p}\alpha_j X_{t-j}+\sum_{k=1}^{q}\beta_k\lambda_{t-k}$$

for constants $\alpha_0>0$, $\alpha_j\geq 0$ $(j=1,...,p)$, $\beta_k\geq 0$ $(k=1,...,q)$ and where $(\varphi_t, t\in\mathbb{Z})$ is a family of characteristic functions on $\mathbb{R}$, $\underline{X}_{t-1}-$ measurables, associated to a family of discrete laws with support in $\mathbb{N}_0$ and finite mean (*). i represents the imaginary unit.

In a briefly way, we say that X follows a $CP-INGARCH(p,q)$ model.

If $q=1$ and $\beta_1=0$, the $CP-INGARCH(p,q)$ model is denoted $CP-INARCH(p)$.

*As φ_t is the characteristic function of a discrete distribution with support in $\mathbb{N}_0$ and finite mean, the derivative of $\varphi_t(u)$ at $u=0$, $\varphi_t^{'}(0)$, exists and is nonzero.

As the conditional distribution of X_t on $\underline{X}_{t-1}$ is a discrete compound Poisson law with support in $\mathbb{N}_0$ then for all $t \in \mathbb{Z}$ and conditioned on $\underline{X}_{t-1}$, X_t can be identified in distribution with the random sum

$$X_t \overset{d}{=} X_{t,1} + ... + X_{t,N_t}$$

where N_t is a random variable following a Poisson distribution with parameter $\frac{\lambda_t}{E(X_{t,j})}$ and $X_{t,1}...X_{t,N_t}$ are discrete and independent random variables, with support contained in $\mathbb{N}_0$, independent of N_t and having common characteristic function φ_t, with finite mean. The distribution of $X_{t,j}$ is called compounding distribution and we assume X_t equal to zero if $N_t = 0$.

The previous definition may be rewritten in terms of the generating function of the law of X_t conditioned on $\underline{X}_{t-1}$:

$$G_{X_t|\underline{X}_{t-1}}(u) = \exp\left\{\frac{\lambda_t}{g_t^{'}(1)}\left[g_t\left(u\right) - 1\right]\right\},$$

with $(g_t, t \in \mathbb{Z})$ the family of generating functions associated to the discrete laws of the compounding variables.

We can now introduce the definition of the nonzero integer-valued (or zero truncated) generalized autoregressive conditional heteroscedastic compound Poisson model, briefly $ZTCP - INGARCH(p, q)$.

Definition. The stochastic process $Z = (Z_t, t \in \mathbb{Z})$ follows a $ZTCP - INGARCH(p, q)$ model if, for any $t \in \mathbb{Z}$, the generating function of Z_t conditioned on $\underline{Z}_{t-1}$ is given by

$$G_{Z_t|\underline{Z}_{t-1}}(u) = \frac{\exp\left\{\frac{\lambda_t}{g_t^{'}(1)}\left[g_t\left(u\right) - 1\right]\right\} - \exp\left\{\frac{\lambda_t}{g_t^{'}(1)}\left[g_t\left(0\right) - 1\right]\right\}}{1 - \exp\left\{\frac{\lambda_t}{g_t^{'}(1)}\left[g_t\left(0\right) - 1\right]\right\}}$$

with

$$\lambda_t = \alpha_0 + \sum_{j=1}^{p}\alpha_j Z_{t-j} + \sum_{k=1}^{q}\beta_k\lambda_{t-k}.$$

If $q = 1$ and $\beta_1 = 0$, the $ZTCP - INGARCH(p, q)$ model is denoted $ZTCP - INARCH(p)$.

In order to assure that λ_t is $\underline{Z}_{t-1}-$ measurable we consider, in what follows, $\sum_{k=1}^{q} \beta_k < 1$.

In the following Figures 1 and 2 we present the trajectories and the basic descriptives of a series X following a $CP-INGARCH(1,1)$ model with Poisson conditional law with $\lambda_t = \alpha_0 + \alpha_1 X_{t-1} + \beta_1 \lambda_{t-1}$, and of a Z process following the $ZTCP-INGARCH(1,1)$ model with $\lambda_t = \alpha_0 + \alpha_1 Z_{t-1} + \beta_1 \lambda_{t-1}$, where $\alpha_0 = 0.8, \alpha_1 = 0.5$ and $\beta_1 = 0.3$, which illustrate the probabilistic changes related with the zero truncation.

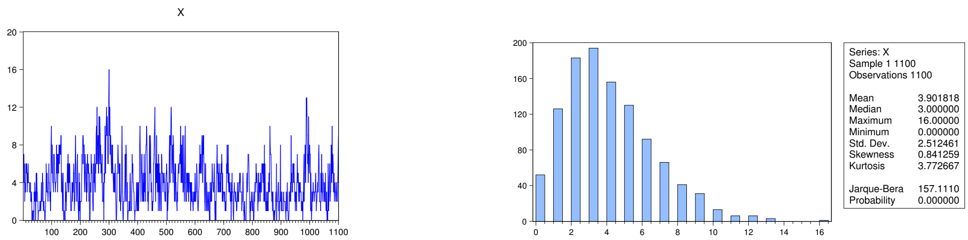

FIGURE 1. X series following a $CP-INGARCH$ model: the time plot and its principal descriptive summaries

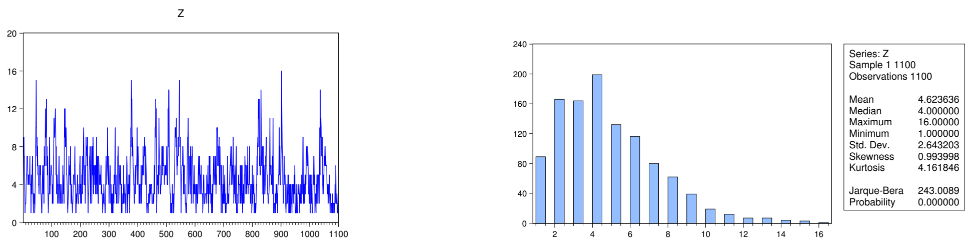

FIGURE 2. Z series following a $ZTCP-INGARCH$ model: the time plot and its principal descriptive summaries

From the relations between the generating function and the moments of the corresponding probability law we deduce

$$E\left(Z_t | \underline{Z}_{t-1}\right) = \frac{\lambda_t}{1 - \exp\left\{ \frac{\lambda_t}{g_t'(1)} \left[g_t(0) - 1 \right] \right\}}$$

$$E\left(Z_t^2|\underline{Z}_{t-1}\right)=\frac{\lambda_t^2+\lambda_t\frac{g_t^{''}(1)}{g_t^{'}(1)}+\lambda_t}{1-\exp\left\{\frac{\lambda_t}{g_t^{'}(1)}\left[g_t\left(0\right)-1\right]\right\}}.$$

2.3. Probabilistic structure.

The general probabilistic study presented in the subsections 2.3.1 and 2.3.2 is developed within the subclass of $ZTCP-INGARCH$ models for which $g_t, t\in\mathbb{Z}$, are deterministic functions. In subsection 2.3.3 a particular $ZTCP-INGARCH$ model with random g_t functions is considered.

2.3.1 Moments

Property. $E\left(Z_t\right)$ exists if and only if $E\left(\lambda_t\right)$ exists.

Proof. Let us assume that $E\left(\lambda_t\right)$ exists. As $g_t\left(0\right)$ is a probability value, $g_t^{'}\left(1\right)>0$ and $\lambda_t\geq\alpha_0$, we deduce

$$E\left(\frac{\lambda_t}{1-\exp\left\{\frac{\lambda_t}{g_t^{'}(1)}[g_t(0)-1]\right\}}\right)\leq E\left(\frac{\lambda_t}{1-\exp\left\{\frac{\alpha_0}{g_t^{'}(1)}[g_t(0)-1]\right\}}\right)=\frac{1}{1-\exp\left\{\frac{\alpha_0}{g_t^{'}(1)}[g_t(0)-1]\right\}}E\left(\lambda_t\right)$$

and so

$$E\left(Z_t\right)=E\left[E\left(Z_t|\underline{Z}_{t-1}\right)\right]=E\left(\frac{\lambda_t}{1-\exp\left\{\frac{\lambda_t}{g_t^{'}(1)}[g_t(0)-1]\right\}}\right)$$

exists.

Otherwise, if $E\left(Z_t\right)$ exists it is enough to take into account that

$$\lambda_t\leq\frac{\lambda_t}{1-\exp\left\{\frac{\lambda_t}{g_t^{'}(1)}[g_t(0)-1]\right\}}$$

to conclude that $E\left(\lambda_t\right)$ exists.

Moreover, it is clear that if the k order moment, $k\in\mathbb{N}$, of one of the processes, Z or λ, exists so does the k order moment of the other. We note that the existence of such moments requires a priori the $k-$ order differentiability of the g_t function.

2.3.2 Strict stationarity and ergodicity

In order to study the stationarity of the truncated process let us assume that the determinist functions g_t are independent of t. In these conditions and if $\sum_{j=1}^{p}\alpha_j + \sum_{k=1}^{q}\beta_k < 1$, the $CP-INGARCH$ model has a strictly stationary and ergodic solution ([3], Theorem 5), $X^* = (X_t^*, t \in \mathbb{Z})$, with generating function of the law of X_t^* conditioned on X_{t-1}^* given by

$$G_{X_t^*|\underline{X}_{t-1}^*}(u) = \exp\left\{\frac{\lambda_t^*}{g_t^{'}(1)}\left[g_t(u)-1\right]\right\}$$

with conditioned expectation $E\left(X_t^*|\underline{X}_{t-1}^*\right) = \lambda_t^* = \alpha_0 + \sum_{j=1}^{p}\alpha_j X_{t-j}^* + \sum_{k=1}^{q}\beta_k\lambda_{t-k}^*$ and we can define the process $Z^* = (Z_t^*, t \in \mathbb{Z})$ such that

$$Z_t^* = X_t^*|X_t^* > 0. \ (^\dagger)$$

This process $Z^* = (Z_t^*, t \in \mathbb{Z})$ is a solution of the $ZTCP-INGARCH$ model with $\lambda_t = \alpha_0 + \sum_{j=1}^{p}\alpha_j Z_{t-j}^* + \sum_{k=1}^{q}\beta_k\lambda_{t-k}$ and, as Z_t^* is a measurable function of a strictly stationary and ergodic process, Z^* is also strictly stationary and ergodic. In fact, we note that $Z^* = (Z_t^*, t \in \mathbb{Z})$ is a stochastic process whose conditional law is the law of $X_t^*|\underline{Z}_{t-1}^*$ conditioned on $X_t^* > 0$. So,

$$\begin{aligned}
E\left(Z_t^*|\underline{Z}_{t-1}^*\right) &= E\left(X_t^*|X_t^*>0 \ \mid \ \underline{Z}_{t-1}^*\right) = E\left[E\left(X_t^*|X_t^*>0 \ \mid \ \underline{X}_{t-1}^*\right) \ \mid \underline{Z}_{t-1}^*\right] \\
&= E\left[\frac{\lambda_t^*}{P^{\,\underline{X}_{t-1}^*}\left(X_t^*>0\right)} \ \mid \underline{Z}_{t-1}^*\right] \\
&= E\left[\frac{\alpha_0 + \sum_{j=1}^{p}\alpha_j X_{t-j}^* + \sum_{k=1}^{q}\beta_k\lambda_{t-k}^*}{1-\exp\left\{\frac{\lambda_t^*}{g_t^{'}(1)}\left[g_t(0)-1\right]\right\}} \mid \underline{Z}_{t-1}^*\right] \\
&= \frac{\alpha_0 + \sum_{j=1}^{p}\alpha_j Z_{t-j}^* + \sum_{k=1}^{q}\beta_k\lambda_{t-k}}{1-\exp\left\{\frac{\lambda_t}{g_t^{'}(1)}\left[g_t(0)-1\right]\right\}}.
\end{aligned}$$

[†] In particular, we know that $E\left(X_t^*\right)$ exists if and only if $E\left(\lambda_t^*\right)$ exists and this happens if and only if $\sum_{j=1}^{p}\alpha_j + \sum_{k=1}^{q}\beta_k < 1$. Moreover, $E\left(\lambda_t^*\right) = E\left(X_t^*\right)$ is independent of t.

Thus $E\left(Z_t^*|\underline{Z}_{t-1}^*\right) = \frac{\lambda_t}{1-\exp\left\{\frac{\lambda_t}{g_t'(1)}[g_t(0)-1]\right\}}$.

From the equality $\lambda_t = \alpha_0 + \sum\limits_{j=1}^{p}\alpha_j Z_{t-j} + \sum\limits_{k=1}^{q}\beta_k\lambda_{t-k}$ we deduce that

$$\left(1-\sum_{k=1}^{q}\beta_k L^k\right)\lambda_t = \alpha_0 + \sum_{j=1}^{p}\alpha_j Z_{t-j}$$

that is,

$$\lambda_t = \frac{\alpha_0}{1-\sum\limits_{k=1}^{q}\beta_k} + \sum_{n=0}^{+\infty}\left(\sum_{k=1}^{q}\beta_k\right)^n\sum_{j=1}^{p}\alpha_j Z_{t-j-kn}.$$

We may now present the following property.

Property. A process Z following a ZTCP-INGARCH model has a strictly stationary and ergodic solution if $\sum\limits_{j=1}^{p}\alpha_j + \sum\limits_{k=1}^{q}\beta_k < 1$. Moreover, if Z is strictly stationary and ergodic then λ is also a strictly stationary and ergodic process.

If Z is stationary in mean the same happens to λ and we have the following relation between the corresponding means:

$$\begin{aligned}\left(1-\sum_{k=1}^{q}\beta_k\right)E(\lambda_t) &= \alpha_0 + \sum_{j=1}^{p}\alpha_j E(Z_t)\\ \Leftrightarrow\ E(\lambda_t) &= \frac{\alpha_0}{1-\sum\limits_{k=1}^{q}\beta_k} + \frac{\sum\limits_{j=1}^{p}\alpha_j}{1-\sum\limits_{k=1}^{q}\beta_k}E(Z_t).\end{aligned}$$

Example. It is clear that every ZT process associated to a strictly stationary and ergodic process is also strictly stationary and ergodic. In particular, the ZT processes associated to the models INGARCH ($g_t(u) = u$), NB-DINARCH (with g_t the generating function of the logarithmic law with parameter $\frac{\alpha-1}{\alpha}, \alpha > 0$), NTA-INGARCH (where g_t is the generating function of the Poisson law with parameter $\phi, \phi > 0$) and GEOMP2-INGARCH (with g_t the generating function of the geometric law with parameter $p, p \in]0,1[$) such that $\sum\limits_{j=1}^{p}\alpha_j + \sum\limits_{k=1}^{q}\beta_k < 1$ have strictly stationary and ergodic solutions as they correspond to models with deterministic and independent of t generating functions g_t ([3]).

2.3.3 Zero truncated NB-INGARCH model

In this subsection we present a zero truncated CP-INGARCH model, not belonging to the general subclass previously considered, for which some probabilistic properties, namely the second order stationarity, may be established. As this model is based on the negative binomial law we begin by recalling that this law is a compound Poisson one.

a) Presentation

The negative binomial law (NB) belongs to the class of compound Poisson laws. In fact, given $r \in \mathbb{N}$ and $p \in \,]0,1[\,$, let $(Y_j, j \geq 1)$ be a sequence of i.i.d. random variables with logarithmic distribution with parameter $1-p$, that is, with probability function given by

$$P\left(Y_j = y\right) = -\tfrac{(1-p)^y}{y \ln p}, y = 1, 2, ...$$

and let N be a random variable following a Poisson distribution with mean $-r \ln p$ and independent of $(Y_j,\ j \geq 1)$. Then the random variable $X = Y_1 + ... + Y_N$ follows a negative binomial law with parameters (r, p), that is,

$$P\left(X = x\right) = \tbinom{x+r-1}{r-1} p^r \left(1-p\right)^x, \ x = 0, 1, ...$$

In these conditions, the generating function of X is

$$g_X(u) = \exp\left\{-r\left[\ln\left(1-(1-p)u\right) - \ln p\right]\right\} = \left(\tfrac{p}{1-(1-p)u}\right)^r$$

taking into account that $\lambda = -r \ln p$ and $g_Y(u) = \frac{\ln(1-(1-p)u)}{\ln p}$. In particular, the geometric law with parameter p appears when $r = 1$.

The NB-INGARCH process ([9]) is obtained considering, conditionally on $\underline{X}_{t-1}$,

$$X_t = Y_{t,1} + ... + Y_{t,N_t}$$

where $Y_{t,1}, Y_{t,2}, ...$ are i.i.d. random variables with logarithmic distribution with parameter $\frac{\lambda_t^*}{1+\lambda_t^*}$, independents of the random variable N_t which follows the Poisson law with parameter $r \ln\left(1 + \lambda_t^*\right)$.

The generating function of the compounding variables is given by

$$g_{Y_t}(u) = \frac{\ln\left(1-\frac{\lambda_t^*}{1+\lambda_t^*}u\right)}{\ln\frac{\lambda_t^*}{1+\lambda_t^*}}.$$

This function is $\underline{X}_{t-1}$- measurable and dependent of t. Nevertheless, as we will see, it is possible to study the stationarity of the corresponding truncated NB-INGARCH model. We note that

$$g_{Y_t}(0) = 0, \qquad g'_{Y_t}(1) = -\frac{\lambda_t^*}{\ln\frac{1}{1+\lambda_t^*}}, \qquad g''_{Y}(1) = -\frac{\lambda_t^{*2}}{\ln\frac{1}{1+\lambda_t^*}}.$$

Let us consider the ZT NB-INGARCH model, $Z = (Z_t, t \in \mathbb{Z})$. The generating function of Z_t conditioned on $\underline{Z}_{t-1}$ is then

$$\begin{aligned} G_{Z_t|\underline{Z}_{t-1}}(u) &= \frac{\exp\left\{\frac{\lambda_t}{g'_t(1)}\left[g_t\left(u\right)-1\right]\right\} - \exp\left\{\frac{\lambda_t}{g'_t(1)}\left[g_t\left(0\right)-1\right]\right\}}{1-\exp\left\{\frac{\lambda_t}{g'_t(1)}\left[g_t\left(0\right)-1\right]\right\}} \\ &= \frac{\exp\left\{-\ln\left(\frac{1}{1+\lambda_t}\right)\left[\frac{\ln\left(1-\frac{\lambda_t}{1+\lambda_t}u\right)}{\ln\left(\frac{1}{1+\lambda_t}\right)}-1\right]\right\} - \exp\left\{\ln\left(\frac{1}{1+\lambda_t}\right)\right\}}{1-\exp\left\{\ln\left(\frac{1}{1+\lambda_t}\right)\right\}} \\ &= \frac{u}{1+\lambda_t-\lambda_t u} \end{aligned}$$

with $\lambda_t = \alpha_0 + \sum\limits_{j=1}^{p}\alpha_j Z_{t-j} + \sum\limits_{k=1}^{q}\beta_k\lambda_{t-k}$.

b) Stationarity

Let us begin by noting that the moments of orders 1 and 2 of the law of Z_t conditioned on $\underline{Z}_{t-1}$ are given by

$$E\left(Z_t|\underline{Z}_{t-1}\right) = \lambda_t + 1$$
$$E\left(Z_t^2|\underline{Z}_{t-1}\right) = \left(2\lambda_t+1\right)\left(\lambda_t+1\right),$$

expressions that coincide with those coming from the general formula of $G_{Z_t|\underline{Z}_{t-1}}$ previously presented.

Taking into account that $E\left(Z_t|\underline{Z}_{t-1}\right) = \lambda_t + 1$, it is easy to establish that the truncated NB-INGARCH model Z is stationary in mean if and only if

$$\sum_{j=1}^{p}\alpha_j + \sum_{k=1}^{q}\beta_k < 1.$$

Under this condition, the processes λ e Z are both stationary in mean and the corresponding (non conditional) means are

$$\begin{aligned} E\left(Z_t\right) &= \mu_Z = \frac{\alpha_0 + 1 - \sum\limits_{k=1}^{q}\beta_k}{1 - \sum\limits_{j=1}^{p}\alpha_j - \sum\limits_{k=1}^{q}\beta_k} \\ E\left(\lambda_t\right) &= \mu_Z - 1 = \frac{\alpha_0 + \sum\limits_{k=1}^{q}\alpha_k}{1 - \sum\limits_{j=1}^{p}\alpha_j - \sum\limits_{k=1}^{q}\beta_k}. \end{aligned}$$

Let us now analyze the second order stationarity of Z. Without lost of generality, we take $p = q$ (the coefficients in excess are considered zero). We have

$$\begin{aligned} E\left(Z_{t-j}\lambda_{t-k}\right) &= E\left[E\left(Z_{t-j}|\underline{Z}_{t-j-1}\right)\lambda_{t-k}\right], \quad \text{if } k \geq j \\ &= E\left(\lambda_{t-j}\lambda_{t-k}\right) + E\left(\lambda_{t-k}\right) \end{aligned}$$

$$\begin{aligned} E\left(Z_{t-j}\lambda_{t-k}\right) &= E\left[Z_{t-j}\left(E\left(Z_{t-k}|\underline{Z}_{t-k-1}\right) - 1\right)\right] \\ &= E\left(Z_{t-j}Z_{t-k}\right) - E\left(Z_{t-j}\right), \quad \text{if } k < j. \end{aligned}$$

By developing $E\left(Z_t Z_{t-h}\right) = E\left[E\left(Z_t|\underline{Z}_{t-1}\right)Z_{t-h}\right] = E\left[\left(\lambda_t + 1\right)Z_{t-h}\right]$ for $h \geq 1$, $E\left(\lambda_t\lambda_{t-h}\right)$ for $h \geq 0$, and using the stationarity in mean of Z, we get

$$E\left(Z_t^2\right) = C + 2[\sum_{j=1}^{p}\alpha_j^2 E\left(Z_{t-j}^2\right) + \sum_{\substack{j,k=1 \\ j\neq k}}^{p}\alpha_j\alpha_k E\left(Z_{t-j}Z_{t-k}\right) +$$
$$+2\sum_{j=1}^{p}\sum_{k=1}^{p}\alpha_j\beta_k E\left(Z_{t-j}\lambda_{t-k}\right) + \sum_{k=1}^{p}\beta_k^2 E\left(\lambda_{t-k}^2\right) + \sum_{\substack{j,k=1 \\ j\neq k}}^{p}\beta_j\beta_k E\left(\lambda_{t-j}\lambda_{t-k}\right)]$$

where $C = 3\mu_\lambda + 1 + 2\alpha_0^2 + 4\alpha_0\left(\mu_\lambda - \alpha_0\right)$. We note that

$$\begin{aligned} E\left(Z_t^2\right) = E\left(2\lambda_t^2 + 3\lambda_t + 1\right) &\Leftrightarrow E\left(Z_t^2\right) = 2E\left(\lambda_t^2\right) + 3\mu_\lambda + 1 \\ &\Leftrightarrow E\left(\lambda_t^2\right) = \frac{E\left(Z_t^2\right) - 3\mu_\lambda - 1}{2} \\ &\Leftrightarrow E\left(\lambda_t^2\right) = \frac{E\left(Z_t^2\right)}{2} - \frac{3\mu_Z - 2}{2} \end{aligned}$$

and from this we deduce that Z is a second order process if and only if the same occurs with λ.

Analogously to what is done in [3], Proposition 1, we obtain

i) $E\left(Z_t^2\right) = b_0 + 2\sum_{j=1}^{p}\left(\alpha_j^2 + \frac{\beta_j^2+2\alpha_j\beta_j}{2}\right)E\left(Z_{t-j}^2\right) +$

$$+4\sum_{j=1}^{p-1}\sum_{k=j+1}^{p}\alpha_k\left(\alpha_j+\beta_j\right)E\left(Z_{t-j}Z_{t-k}\right)+4\sum_{j=1}^{p-1}\sum_{k=j+1}^{p}\beta_k\left(\alpha_j+\beta_j\right)E\left(\lambda_{t-j}\lambda_{t-k}\right)]$$

ii) $E\left(Z_tZ_{t-h}\right) = b_{1,h} + \left(\alpha_h + \frac{\beta_h}{2}\right)E\left(Z_{t-h}^2\right) + \sum_{j=1}^{h-1}\left(\alpha_j+\beta_j\right)E\left(Z_{t-j}Z_{t-h}\right) +$

$$+\sum_{j=h+1}^{p}\alpha_jE\left(Z_{t-j}Z_{t-h}\right) + \sum_{j=h+1}^{p}\beta_jE\left(\lambda_{t-j}\lambda_{t-h}\right), h \geq 1$$

iii) $E\left(\lambda_t\lambda_{t-h}\right) = b_{2,h} + \frac{\alpha_h+\beta_h}{2}E\left(Z_{t-h}^2\right) + \sum_{j=h+1}^{p}\alpha_jE\left(Z_{t-j}Z_{t-h}\right) +$

$$+\sum_{j=1}^{h-1}\left(\alpha_j+\beta_j\right)E\left(\lambda_{t-k}\lambda_{t-h}\right) + \sum_{j=h+1}^{p}\beta_jE\left(\lambda_{t-j}\lambda_{t-h}\right), h \geq 1$$

with

$$b_0 = \left(3+4\alpha_0\right)\left(\mu_Z-1\right)+1-2\alpha_0^2-\left(\frac{3\mu_Z-2}{2}\right)\sum_{k=1}^{p}\left(\beta_k^2+2\alpha_k\beta_k\right)+$$
$$+2\left(\mu_Z-1\right)\sum_{\substack{j,k=1\\ j\leq k}}^{p}\alpha_j\beta_k - 2\mu_Z\sum_{\substack{j,k=1\\ j<k}}^{p}\alpha_k\beta_j$$

$$b_{1,h} = \mu_Z\left(1+\alpha_0-\sum_{k=1}^{h-1}\beta_k\right)+\left(\mu_Z-1\right)\sum_{k=h}^{p}\beta_k - \frac{3\mu_Z-2}{2}\beta_h$$

$$b_{2,h} = \left(\mu_Z-1\right)\left(\alpha_0+\sum_{j=1}^{h}\alpha_j\right)-\mu_Z\sum_{j=h+1}^{p}\alpha_j - \frac{3\mu_Z-2}{2}\left(\alpha_h+\beta_h\right).$$

From these calculations, it is easy to see that the vector W_t with dimension $p+q-1$,

$$W_t = \left[\begin{array}{l} E\left(Z_t^2\right) \\ E\left(Z_t Z_{t-1}\right) \\ ... \\ E\left(Z_t Z_{t-(p-1)}\right) \\ E\left(\lambda_t \lambda_{t-1}\right) \\ ... \\ E\left(\lambda_t \lambda_{t-(q-1)}\right) \end{array}\right]$$

satisfy an autoregressive equation of order $\max(p,q)$

$$W_t = b + \sum_{k=1}^{\max(p,q)} B_k W_{t-k} \qquad (\mathbf{1})$$

with $b = [b_j]_{j=1,\dots,p+q-1}$ a real $p+q-1$ dimensional vector such that

$$b_j = \begin{cases} b_0, & j=1 \\ \left(1+\alpha_0-\sum\limits_{k=1}^{j-2}\beta_k\right)\mu_X + (\mu_X-1)\sum\limits_{k=j-1}^{p}\beta_k - \frac{3\mu_X-2}{2}\beta_{j-1}, & j=2,...,p \\ \left(\alpha_0+\sum\limits_{k=1}^{j-p}\alpha_k\right)(\mu_X-1) - \mu_X\sum\limits_{k=j-p+1}^{p}\alpha_k - \frac{3\mu_X-2}{2}\left(\alpha_{j-p}+\beta_{j-p}\right), & j=p+1,...,p+q-1 \end{cases}$$

and B_k ($k = 1, ..., \max(p,q)$) real square $p+q-1$ dimensional matrices. The coefficientes of these matrices are equal to those of the matrices obtained for the corresponding nontruncated model ([3]).

So, we may write the following property.

Property. A zero truncated NB-INGARCH process, Z, stationary in mean is second order stationary if and only if

$$P(L) = I_{p+q-1} - \sum_{k=1}^{\max(p,q)} B_k L^k$$

is a polinomial matrix such that $\det P(z)$ has all its roots outside the unit circle, where I_{p+q-1} is the identity matrix of $p+q-1$ order and B_k ($k = 1, ...,$ $\max(p,q)$) are the matrices present in the autoregressive equation ($\mathbf{1}$).

c) Particular cases

Let us analyze some particular cases of the zero truncated NB-INGARCH model, namely those with small orders.

c.1) If $p = q = 1$ and $\alpha_1 + \beta_1 < 1$, the matrix B_1 reduce to the scalar $2\left(\alpha_1^2 + \frac{\beta_1^2+2\alpha_1\beta_1}{2}\right)$ and the necessary and sufficient condition of weak stationarity of Z becomes $\alpha_1^2 + (\alpha_1+\beta_1)^2 < 1$.

In this case we have

$$E(Z_t) = \mu_Z = \tfrac{\alpha_0+1-\beta_1}{1-(\alpha_1+\beta_1)}$$

and from $E\left(Z_t^2\right) = b_0 + B_1 E\left(Z_{t-1}^2\right)$ we get

$$E\left(Z_t^2\right) = \tfrac{b_0}{1-B_1}$$

with $b_0 = \left(3 + 4\alpha_0 - \alpha_1\beta_1 - \frac{3}{2}\beta_1^2\right)\mu_Z - 2(\alpha_0+1)^2 + \beta_1^2$ and $B_1 = \alpha_1^2 + (\alpha_1+\beta_1)^2$.

Moreover, for $h \geq 1$,

$$\begin{aligned} E(Z_t Z_{t-h}) &= (1+\alpha_0)\mu_Z + \alpha_1 E(Z_{t-1}Z_{t-h}) + \beta_1\left(E\left(\lambda_t^2\right) + E(\lambda_{t-h})\right) \\ &= (1+\alpha_0)\mu_Z + \beta_1(\mu_Z - 1) + \beta_1\left(\frac{E\left(Z_t^2\right)}{2} - \frac{3\mu_Z - 2}{2}\right) + \alpha_1 E(Z_{t-1}Z_{t-h}) \end{aligned}$$

and we deduce

$$\begin{aligned} E(Z_t Z_{t-h}) &= \left[(1+\alpha_0)\mu_Z + \beta_1(\mu_Z-1) + \frac{\beta_1}{2}E\left(Z_t^2\right) - \frac{\beta_1}{2}(3\mu_Z - 2)\right]\left(\frac{1-\alpha_1^h}{1-\alpha_1}\right) + \alpha_1^h E\left(Z_t^2\right) \\ &= \left[(1+\alpha_0+\beta_1)\mu_Z - \frac{3\beta_1}{2}\mu_Z\right]\left(\frac{1-\alpha_1^h}{1-\alpha_1}\right) + \left[\frac{\beta_1}{2}\left(\frac{1-\alpha_1^h}{1-\alpha_1}\right) + \alpha_1^h\right]E\left(Z_t^2\right) \\ &= \left(1+\alpha_0 - \frac{\beta_1}{2}\right)\left(\frac{1-\alpha_1^h}{1-\alpha_1}\right)\mu_Z + \left[\frac{\beta_1 + (-2\alpha_1 - \beta_1 + 2)\alpha_1^h}{2(1-\alpha_1)}\right]E\left(Z_t^2\right). \end{aligned}$$

The autocovariance function of Z, $\gamma(h) = Cov(Z_t Z_{t-h})$, $h \geq 0$, follow from these equalities.

Remark. We point out that the expressions obtained may be used to estimate the parameters of the model by the moments method in a simple but consistent way. If $\beta_1 = 0$, for example, the estimators of α_0 and α_1 are

$$\widehat{\alpha}_0 = \overline{Z}(1-\widehat{\alpha}_1) - 1, \qquad \widehat{\alpha}_1 = \sqrt{\frac{\overline{Z}\left(1-\overline{Z}\right) + S_Z^2}{2S_Z^2}}$$

where $\overline{Z}$ and S_Z^2 denote, respectively, the empirical mean and variance of a n-sample $(Z_1, ..., Z_n)$ of the process Z.

c.2) If $p=q=2$, B_1 and B_2 are $3-$ order matrices equal to

$$B_1=\begin{bmatrix} 2\alpha_1^2+\beta_1^2+2\alpha_1\beta_1 & 4\alpha_2(\alpha_1+\beta_1) & 4\beta_2(\alpha_1+\beta_1) \\ \alpha_1+\frac{\beta_1}{2} & \alpha_2 & \beta_2 \\ \frac{\alpha_1+\beta_1}{2} & \alpha_2 & \beta_2 \end{bmatrix}$$

$$B_2=\begin{bmatrix} 2\alpha_2^2+\beta_2^2+2\alpha_2\beta_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we get

$$\begin{aligned}\det P(z) &= 1-\left((\alpha_1+\beta_1)^2+\alpha_2+\beta_2+\alpha_1^2\right)z+ \\ &+\left((\alpha_1+\beta_1)^2(\alpha_2+\beta_2)+(\alpha_2+\beta_2)^2+\alpha_2^2-\alpha_1^2\beta_2+\alpha_1^2\alpha_2+2\alpha_1\alpha_2\beta_1\right)z^2+ \\ &+\left((\alpha_2+\beta_2)^3+\alpha_2^2(\alpha_2+\beta_2)\right)z^3.\end{aligned}$$

If, in particular, $\alpha_1=\beta_1=0$ then the roots of $\det P(z)=0$ are

$$z_1=\frac{1}{\alpha_2+\beta_2},\quad z_2=\frac{1}{\sqrt{(\alpha_2+\beta_2)^2+\alpha_2^2}},\quad z_3=-z_2,$$

and the necessary and sufficient condition of weak stationarity of Z becomes $(\alpha_2+\beta_2)^2+\alpha_2^2<1$.

3. Parameter estimation

3.1. Conditional maximum likelihood.

Using the conditional maximum likelihood methodology we estimate in this Section the parameter vector of a stochastic process Z following a ZTCP-INGARCH(p,q) model for which $g(u)=u$, that is, the conditional law is a ZT Poisson one.

The generating function of Z_t conditioned on $\underline{Z}_{t-1}$ is then

$$G_{Z_t|\underline{Z}_{t-1}}(u)=\frac{\exp(\lambda_t(u-1))-\exp(-\lambda_t)}{1-\exp(-\lambda_t)}$$

with $\lambda_t=\alpha_0+\sum\limits_{j=1}^{p}\alpha_j Z_{t-j}+\sum\limits_{k=1}^{q}\beta_k\lambda_{t-k}$. Thus, the probability function of the conditioned law is

$$P\left(Z_t=k\mid\underline{Z}_{t-1}\right)=\frac{G^{(k)}_{Z_t|\underline{Z}_{t-1}}(0)}{k!}=\frac{\exp(-\lambda_t)(\lambda_t)^k}{(1-\exp(-\lambda_t))k!},\ k=1,2,...$$

The conditional likelihood function associated to n observations $Z_1,...,Z_n$ conditionally to the initial values is

$$L(\Theta) = \prod_{t=1}^{n} \frac{\exp(-\lambda_t)(\lambda_t)^{Z_t}}{(1-\exp(-\lambda_t))Z_t!}$$

where $\Theta = (\alpha_0, \alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q)^T = (\theta_0, \theta_1, ..., \theta_p, \theta_{p+1}, ..., \theta_{p+q})^T$

The log-likelihood function is given by

$$\begin{aligned}
\mathcal{L}(\Theta) &= \log L(\Theta) = \sum_{t=1}^{n} \left[Z_t \log(\lambda_t) - \lambda_t - \log(Z_t!) - \log(1-\exp(-\lambda_t))\right] \\
&= \sum_{t=1}^{n} l_t(\Theta)
\end{aligned}$$

with $l_t(\Theta) = Z_t \log(\lambda_t) - \lambda_t - \log(Z_t!) - \log(1-\exp(-\lambda_t))$.

The first derivatives of l_t in order to θ_i, $i = 0, ..., p+q$ are

$$\begin{aligned}
\frac{\partial l_t}{\partial \theta_i} &= Z_t \frac{\partial \lambda_t}{\partial \theta_i} \frac{1}{\lambda_t} - \frac{\partial \lambda_t}{\partial \theta_i} - \frac{-\exp(-\lambda_t)\left(-\frac{\partial \lambda_t}{\partial \theta_i}\right)}{1-\exp(-\lambda_t)} \\
&= \frac{\partial \lambda_t}{\partial \theta_i}\left(\frac{Z_t}{\lambda_t} - 1 - \frac{\exp(-\lambda_t)}{1-\exp(-\lambda_t)}\right) \\
&= \frac{\partial \lambda_t}{\partial \theta_i}\left(\frac{Z_t}{\lambda_t} - \frac{1}{1-\exp(-\lambda_t)}\right) \qquad (2)
\end{aligned}$$

and the second derivatives are

$$\begin{aligned}
\frac{\partial^2 l_t}{\partial \theta_i \partial \theta_j} &= \frac{\partial^2 \lambda_t}{\partial \theta_i \partial \theta_j}\left(\frac{Z_t}{\lambda_t} - \frac{1}{1-\exp(-\lambda_t)}\right) + \frac{\partial \lambda_t}{\partial \theta_i}\left[-\frac{Z_t}{\lambda_t^2}\frac{\partial \lambda_t}{\partial \theta_j} + (1-\exp(-\lambda_t))^{-2}\exp(-\lambda_t)\frac{\partial \lambda_t}{\partial \theta_j}\right] \\
&= \frac{\partial^2 \lambda_t}{\partial \theta_i \partial \theta_j}\left(\frac{Z_t}{\lambda_t} - \frac{1}{1-\exp(-\lambda_t)}\right) + \left[-\frac{Z_t}{\lambda_t^2} + \frac{\exp(-\lambda_t)}{(1-\exp(-\lambda_t))^2}\right]\frac{\partial \lambda_t}{\partial \theta_i}\frac{\partial \lambda_t}{\partial \theta_j} \qquad (3)
\end{aligned}$$

for $0 \leq i, j \leq p+q$. Moreover,

$$\begin{aligned}
\frac{\partial \lambda_t}{\partial \alpha_0} &= 1 + \sum_{k=1}^{q} \beta_k \frac{\partial \lambda_{t-k}}{\partial \alpha_0}; \\
\frac{\partial \lambda_t}{\partial \alpha_i} &= Z_{t-i} + \sum_{k=1}^{q} \beta_k \frac{\partial \lambda_{t-k}}{\partial \alpha_i}, i = 1, ..., p; \\
\frac{\partial \lambda_t}{\partial \beta_j} &= \lambda_{t-j} + \sum_{k=1}^{q} \beta_k \frac{\partial \lambda_{t-k}}{\partial \beta_j}, j = 1, ..., q.
\end{aligned}$$

Following [8] we deduce that the conditional maximum likelihood estimator, $\widehat{\Theta}$, is strongly consistent for Θ_0, the true value of Θ, and asymptotically normal. Namely, if n is large enough, the distribution of $\widehat{\Theta}$ may be approached by the following distribution:

$$\widehat{\Theta} \overset{\bullet}{\sim} N\left(\Theta_0, \left[n\mathcal{I}\left(\Theta_0\right)\right]^{-1}\right)$$

where $\mathcal{I}\left(\Theta_0\right)$ is the information matrix evaluated at Θ_0.

In order to estimate the asymptotic covariance matrix of $\widehat{\Theta}$, let us consider

$$W_t^T = \left(1, Z_{t-1}, ..., Z_{t-p}, \lambda_{t-1}, ..., \lambda_{t-q}\right)$$

and let ∇f denote the gradient of any function f. We have

$$\nabla \lambda_t = W_t + \sum_{k=1}^{q} \beta_k \nabla \lambda_{t-k}.$$

The equation (2) may now be written as

$$\nabla l_t = \left(\tfrac{Z_t}{\lambda_t} - \tfrac{1}{1-\exp(-\lambda_t)}\right) \nabla \lambda_t$$

and the equation (3) becomes

$$\mathbf{H}_t = \left(\tfrac{Z_t}{\lambda_t} - \tfrac{1}{1-\exp(-\lambda_t)}\right) \nabla\left(\nabla^T \lambda_t\right) - \left[\tfrac{Z_t}{\lambda_t^2} - \tfrac{\exp(-\lambda_t)}{\left(1-\exp(-\lambda_t)\right)^2}\right] \nabla \lambda_t \nabla^T \lambda_t.$$

Taking expectations in both sides of the equation (3) we obtain

$$E\left(\tfrac{\partial^2 l_t}{\partial \theta_i \partial \theta_j} | \underline{Z}_{t-1}\right) =$$

$$\begin{aligned} &= E\left[\frac{\partial^2 \lambda_t}{\partial \theta_i \partial \theta_j}\left(\frac{Z_t}{\lambda_t} - \frac{1}{1-\exp\left(-\lambda_t\right)}\right) - \left(\frac{Z_t}{\lambda_t^2} - \frac{\exp\left(-\lambda_t\right)}{\left(1-\exp\left(-\lambda_t\right)\right)^2}\right) \frac{\partial \lambda_t}{\partial \theta_i}\frac{\partial \lambda_t}{\partial \theta_j} | \underline{Z}_{t-1}\right] \\ &= \frac{\partial^2 \lambda_t}{\partial \theta_i \partial \theta_j} E\left(\frac{Z_t}{\lambda_t} - \frac{1}{1-\exp\left(-\lambda_t\right)} | \underline{Z}_{t-1}\right) - E\left(\frac{Z_t}{\lambda_t^2} - \frac{\exp\left(-\lambda_t\right)}{\left(1-\exp\left(-\lambda_t\right)\right)^2} | \underline{Z}_{t-1}\right) \frac{\partial \lambda_t}{\partial \theta_i}\frac{\partial \lambda_t}{\partial \theta_j}. \end{aligned}$$

But from $E\left(Z_t | \underline{Z}_{t-1}\right) = \frac{\lambda_t}{1-\exp(-\lambda_t)}$ we deduce

$$E\left(\frac{Z_t}{\lambda_t} | \underline{Z}_{t-1}\right) = \frac{1}{1-\exp\left(-\lambda_t\right)}$$

and

$$E\left(\frac{Z_t}{\lambda_t^2} | \underline{Z}_{t-1}\right) = \frac{1}{\lambda_t\left(1-\exp\left(-\lambda_t\right)\right)}.$$

So

$$E\left(\tfrac{\partial^2 l_t}{\partial \theta_i \partial \theta_j} | \underline{Z}_{t-1}\right) = -\left(\tfrac{1}{\lambda_t(1-\exp(-\lambda_t))} - \tfrac{\exp(-\lambda_t)}{\left(1-\exp(-\lambda_t)\right)^2}\right) \tfrac{\partial \lambda_t}{\partial \theta_i}\tfrac{\partial \lambda_t}{\partial \theta_j}.$$

Consequently

$$-E\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\right)=E\left[\left(\frac{1}{\lambda_t(1-\exp(-\lambda_t))}-\frac{\exp(-\lambda_t)}{(1-\exp(-\lambda_t))^2}\right)\frac{\partial\lambda_t}{\partial\theta_i}\frac{\partial\lambda_t}{\partial\theta_j}\right].$$

In an analogous way, from (2) we get

$$E\left(\frac{\partial l_t}{\partial\theta_i}\frac{\partial l_t}{\partial\theta_j}|\underline{Z}_{t-1}\right)=E\left(\frac{\partial\lambda_t}{\partial\theta_i}\frac{\partial\lambda_t}{\partial\theta_j}\left(\frac{Z_t}{\lambda_t}-\frac{1}{1-\exp(-\lambda_t)}\right)^2|\underline{Z}_{t-1}\right).$$

Taking into account that $E\left(Z_t^2|\underline{Z}_{t-1}\right)=\frac{\lambda_t^2+\lambda_t}{1-\exp(-\lambda_t)}$ we deduce

$$\begin{aligned}
E\left(\frac{\partial l_t}{\partial\theta_i}\frac{\partial l_t}{\partial\theta_j}|\underline{Z}_{t-1}\right) &= \frac{\partial\lambda_t}{\partial\theta_i}\frac{\partial\lambda_t}{\partial\theta_j}E\left(\left(\frac{Z_t}{\lambda_t}\right)^2-\frac{2Z_t}{\lambda_t\left(1-\exp\left(-\lambda_t\right)\right)}+\left(\frac{1}{1-\exp\left(-\lambda_t\right)}\right)^2|\underline{Z}_{t-1}\right)\\
&= \frac{\partial\lambda_t}{\partial\theta_i}\frac{\partial\lambda_t}{\partial\theta_j}\left[\frac{1+\frac{1}{\lambda_t}}{1-\exp\left(-\lambda_t\right)}-2\left(\frac{1}{1-\exp\left(-\lambda_t\right)}\right)^2+\left(\frac{1}{1-\exp\left(-\lambda_t\right)}\right)^2\right]\\
&= \frac{\partial\lambda_t}{\partial\theta_i}\frac{\partial\lambda_t}{\partial\theta_j}\left[\frac{\lambda_t+1}{\lambda_t\left(1-\exp\left(-\lambda_t\right)\right)}-\left(\frac{1}{1-\exp\left(-\lambda_t\right)}\right)^2\right].
\end{aligned}$$

Consequently

$$\begin{aligned}
E\left(\frac{\partial l_t}{\partial\theta_i}\frac{\partial l_t}{\partial\theta_j}\right) &= E\left[\frac{\partial\lambda_t}{\partial\theta_i}\frac{\partial\lambda_t}{\partial\theta_j}\left(\frac{\lambda_t+1}{\lambda_t\left(1-\exp\left(-\lambda_t\right)\right)}-\frac{1}{\left(1-\exp\left(-\lambda_t\right)\right)^2}\right)\right]\\
&= E\left[\frac{\partial\lambda_t}{\partial\theta_i}\frac{\partial\lambda_t}{\partial\theta_j}\left(\frac{1}{1-\exp\left(-\lambda_t\right)}+\frac{1}{\lambda_t\left(1-\exp\left(-\lambda_t\right)\right)}-\frac{1}{\left(1-\exp\left(-\lambda_t\right)\right)^2}\right)\right]\\
&= E\left[\frac{\partial\lambda_t}{\partial\theta_i}\frac{\partial\lambda_t}{\partial\theta_j}\left(\frac{-\exp\left(-\lambda_t\right)}{\left(1-\exp\left(-\lambda_t\right)\right)^2}+\frac{1}{\lambda_t\left(1-\exp\left(-\lambda_t\right)\right)}\right)\right].
\end{aligned}$$

We deduce that this ZT model satisfy the information matrix equality:

$$-E\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\right)=E\left(\frac{\partial l_t}{\partial\theta_i}\frac{\partial l_t}{\partial\theta_j}\right),$$

and so the matrices

$$\widehat{S}_n=\frac{1}{n}\sum_{t=1}^{n}\nabla l_t\nabla^T l_t$$

and

$$\widehat{D}_n=-\frac{1}{n}\sum_{t=1}^{n}\nabla\left[\nabla^T l_t\right]$$

are consistent estimates for the information matrix. Thus, both can be used to estimate the asymptotic covariance matrix of the conditional maximum likelihood estimator.

3.2. Simulation study.

To implement the estimation methodology and analyze its performance in a concrete situation we consider now a stochastic process Z following a ZTCP-INGARCH(1, 1) model with $g(u) = u$ and $\lambda_t = 0.8+0.5Z_{t-1}+0.3\lambda_{t-1}$.

We generate 1100 observations. In order to minimize the effect of the initial conditions we discard the first 100 observations. The estimates obtained for the parameters using the conditional maximum likelihood method for samples of 200, 600 and 1000 observations are presented in the Table 1. When the number of observations increase we observe an increasing proximity between the estimates and the true values of the model parameter vector, as the standard errors reveal. Moreover this estimation improvement is also assessed by the significant decrease in the mean square error, given by

$$RMS^2 = \frac{1}{n}\sum_{t=1}^{n}\left(Z_t - \frac{\widehat{\lambda}_t}{1-\exp\left(-\widehat{\lambda}_t\right)}\right)^2.$$

Table 1. Maximum likelihood estimates of the parameters of the model ZTCP-INGARCH(1,1) with $\lambda_t = 0.8 + 0.5Z_{t-1} + 0.3\lambda_{t-1}$, the corresponding standard errors and probabilities and the root mean square error ($n = 200, 600$ and 1000).

ZTCP INGARCH(1,1)	Estimates				RMS
		Coefficient	Std. Error	Prob.	
$n = 200$	$\widehat{\alpha}_0$	0.934752	0.450732	0.0381	1.964743
	$\widehat{\alpha}_1$	0.573235	0.071171	0.0000	
	$\widehat{\beta}_1$	0.168460	0.127505	0.1864	
		Coefficient	Std. Error	Prob.	
$n = 600$	$\widehat{\alpha}_0$	0.751109	0.244264	0.0021	1.945211
	$\widehat{\alpha}_1$	0.508055	0.043687	0.0000	
	$\widehat{\beta}_1$	0.294458	0.070838	0.0000	
		Coefficient	Std. Error	Prob.	
$n = 1000$	$\widehat{\alpha}_0$	0.804141	0.188584	0.0000	1.911525
	$\widehat{\alpha}_1$	0.495506	0.035423	0.0000	
	$\widehat{\beta}_1$	0.280772	0.055280	0.0000	

We study also the effect of modeling the same observations ($n = 1000$) by other models of the class ZTCP-INGARCH with orders different from the true ones. For the comparison we use the log-likelihood function and Akaike and Schwarz criteria values.

Table 2. Conditional maximum likelihood estimates of the parameters of the models, with the corresponding standard errors and probabilities, the log-likelihood function and Akaike and Schwarz criteria.

Model	Estimates				Log L	Akaike criterion	Schwarz criterion
ZTCP-INGARCH(1,1)		Coeff.	Std. Error	Prob.	2672.787	-5.339575	-5.324851
	$\widehat{\alpha}_0$	0.729979	0.197370	0.0002			
	$\widehat{\alpha}_1$	0.493813	0.036731	0.0000			
	$\widehat{\beta}_1$	0.337999	0.056966	0.0000			
ZTCP-INARCH(1)		Coeff.	Std. Error	Prob.	2652.876	-5.301753	-5.291937
	$\widehat{\alpha}_0$	1.643538	0.131514	0.0000			
	$\widehat{\alpha}_1$	0.620958	0.028528	0.0000			
ZTCP-INARCH(2)		Coeff.	Std. Error	Prob.	2672.180	-5.338359	-5.323636
	$\widehat{\alpha}_0$	1.241163	0.145437	0.0000			
	$\widehat{\alpha}_1$	0.502453	0.036001	0.0000			
	$\widehat{\alpha}_2$	0.205675	0.034193	0.0000			
ZTCP-INGARCH(1,2)		Coeff.	Std. Error	Prob.	2673.008	-5.338017	-5.318386
	$\widehat{\alpha}_0$	0.773800	0.217414	0.0004			
	$\widehat{\alpha}_1$	0.494038	0.036828	0.0000			
	$\widehat{\beta}_1$	0.371542	0.072992	0.0000			
	$\widehat{\beta}_2$	-0.043666	0.068497	0.5238			
CP-INGARCH(1,1)		Coeff.	Std. Error	Prob.	2646.227	-5.286453	-5.271730
	$\widehat{\alpha}_0$	1.008047	0.204170	0.0000			
	$\widehat{\alpha}_1$	0.469816	0.037518	0.0000			
	$\widehat{\beta}_1$	0.320104	0.058181	0.0000			

Comparing the results with those of the ZTCP-INGARCH(1,1) model used to generate the observations (line 2 of Table 2), we note that in the two ZTCP-INARCH models considered (lines 3 and 4 of Table 2) the log-likelihood function has smaller values. We also point out the expected non significance of the last parameter estimate in the ZTCP-INGARCH(1,2) model (line 5 of Table 2). Moreover the minimum values of the Akaike and Schwarz criteria are obtained for the ZTCP-INGARCH$(1,1)$. However we note that the ZTCP-INARCH(2) model competes well with the true model which is certainly related with the great coefficient of Z_{t-1} in the evolution of λ_t.

We also study the effect of modeling the same observations by a CP-INGARCH(1,1) model and the results, reported in the last line of Table 2, although not very different from the previous ones, present the smallest value of the log-likelihood function and the greatest one for Akaike and Schwarz criteria for the five models considered. The small differences between the standard and the zero-truncated Poisson models may be explained by the value of the mean of the count response variable ([5]), greater than 4, which corresponds to a negligible probability for the occurrence of zero values. In order to see the influence of the mean, we repeat the analysis considering now 1000 observations from a ZTCP-INGARCH$(1, 1)$ model with a smaller mean by considering $\lambda_t = 0.5 + 0.3Z_{t-1} + 0.2\lambda_{t-1}$.

Table 3. Conditional maximum likelihood estimates of the parameters of the models, with the corresponding standard errors and probabilities, the log-likelihood function and Akaike and Schwarz criteria ($\lambda_t = 0.5 + 0.3Z_{t-1} + 0.2\lambda_{t-1}$).

Model	Estimates				Log L	Akaike criterion	Schwarz criterion
ZTCP-INGARCH(1,1)		Coeff.	Std. Error	Prob.	-498.9299	1.003860	1.018583
	$\widehat{\alpha}_0$	0.462356	0.194389	0.0174			
	$\widehat{\alpha}_1$	0.245658	0.050299	0.0000			
	$\widehat{\beta}_1$	0.285840	0.146120	0.0504			
CP-INGARCH(1,1)		Coeff.	Std. Error	Prob.	-751.0212	1.508042	1.522766
	$\widehat{\alpha}_0$	1.187953	0.269543	0.0000			
	$\widehat{\alpha}_1$	0.176817	0.066177	0.0075			
	$\widehat{\beta}_1$	0.207045	0.197160	0.2937			

We note that in this case the differences between the true model and the non truncated one are significant, namely in which concerns the closeness between the parameters estimates and the true values as the standard errors reinforce. Moreover, taking into account all the criteria used, we verify that in this case the CP-INGARCH(1,1) model gives clearly worse results than the ZTCP- INGARCH(1,1) one. We conclude that the standard model is not so adequate as the ZT one introduced in this paper to reproduce the zero truncation characteristic of the initial observations.

4. Real-data example

In this section, we want to assess the improvement provided with real data when using a model ZTCP-INGARCH instead of a standard CP-INGARCH.

In order to do this let us consider the time series of the quarterly counts of poliomyelitis cases in the United States of America starting from January 1970 and ending in December 1983 (56 observations), obtained in the Forecasting Principles site (http:// www.forecastingprinciples.com).

Figure 3 presents the original series and its principal descriptive summaries. In Figure 4 the empirical autocorrelations and partial autocorrelations are displayed.

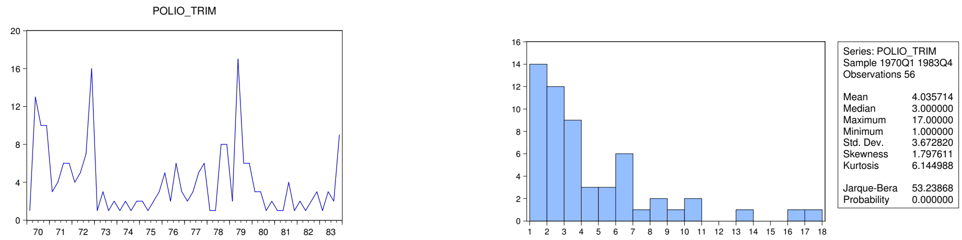

FIGURE 3. Quarterly poliomyelitis series: the time plot and its principal descriptive summaries

We have in fact a strictly positive integer-valued time series which values are relatively low. The empirical mean and variance of the series are 4.0357 and 13.4896. The empirical analysis of autocorrelation functions show a serial dependence at order 2 of this time series.

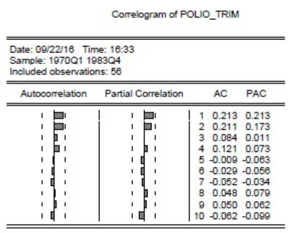

FIGURE 4. Quarterly poliomyelitis series: the sample autocorrelations and partial autocorrelations

The data are thus fitted by the ZTCP-INARCH(2) and CP-INARCH(2) models. Conditional maximum likelihood parameter estimates and their standard errors are summarized in Table 4. The lower values of the standard errors for the ZTCP-INARCH(2) led, in this case, to more accurate estimates. Moreover, the Akaike and Schwarz criteria and the values of the log-likelihood function allow us to conclude that the zero truncated model provide a better fit than the CP-INARCH one to the quarterly poliomyelitis data. This improvement is naturally due to the nature of the data that describes a rare phenomenon which occurs effectively with some regularity.

Table 4. Conditional maximum likelihood estimates of the parameters, with the corresponding standard errors and probabilities, the log-likelihood function and Akaike and Schwarz criteria for the quarterly poliomyelitis cases.

Model	Estimates				Log L	Akaike criterion	Schwarz criterion
ZTCP-INARCH(2)		Coeff.	Std. Error	Prob.	89.59489	-3.207218	-3.096719
	$\widehat{\alpha}_0$	1.778501	0.323187	0.0000			
	$\widehat{\alpha}_1$	0.282240	0.051970	0.0000			
	$\widehat{\alpha}_2$	0.224450	0.054547	0.0000			
CP-INARCH(2)		Coeff.	Std. Error	Prob.	87.53110	-3.130782	-3.088166
	$\widehat{\alpha}_0$	2.043173	0.344851	0.0000			
	$\widehat{\alpha}_1$	0.264551	0.052925	0.0000			
	$\widehat{\alpha}_2$	0.207091	0.056697	0.0003			

5. Conclusion

Compound Poisson INGARCH processes are a wide family of integer-valued models, recently introduced, that are able to describe simultaneously characteristics of count data like different kinds of conditional heteroscedasticity or overdispersion. But, these kind of processes all assume that the count data in analysis have zero counts.

Many times the count systems to be modeled structurally exclude zeros; so, in order to model such data properly, the underlying probability distribution should preclude null outcomes. The potential of compound Poisson probability distributions to describe huge different characteristics of count data justify the introduction of the new class of zero truncated models inspired in the CP-INGARCH ones but amended to exclude zeros.

We point out that the main subfamily of models here studied, namely the class of ZTCP-INGARCH processes with g_t deterministic and independent of t, may accommodate a significant number of models useful in applications. As relevant examples we should refer the ZT Poisson and the ZT Neyman Type A ones, corresponding respectively to $g_t(u) = u$ and $g_t(u) = \exp\left(\phi\left(u-1\right)\right)$ Furthermore, according to the Pseudo-Conditional Maximum Likelihood methodology, the estimator obtained for the parameter vector of a ZT Poisson-INGARCH (1,1) model is also a consistent estimator of the corresponding vector of a general ZTCP-INGARCH (1,1) model.

In conclusion, the probabilistic and statistical study developed, although restricted to a subclass of the wide family considered, is enough consistent for using these models in applications related to rare phenomena but with effective occurrences, as we hope having illustrated with the simulation and real data studies presented.

Acknowledgement. This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

References

[1] R. Ferland, A. Latour, D. Oraichi, *Integer-valued GARCH models*, Journal of Time Series Analysis 27 (2006) 923-942.
[2] Ch.Francq, J.M. Zakoian, *GARCH models, Structure, Statistical inference and Financial applications*, Wiley, 2010.
[3] E. Gonçalves, N. Mendes-Lopes, F. Silva, *Infinitely divisible distributions in integer-valued GARCH models*, Journal of Time Series Analysis 36 (2015) 503-527.
[4] E. Gonçalves, N. Mendes-Lopes, F. Silva, *Zero-inflated compound Poisson distributions in integer-valued GARCH models*, Statistics 50 (2016) 558-578.
[5] J.M. Hilbe, *Modeling Count Data*, Cambridge University Press, 2014.
[6] S. Lee, Y. Lee and C. Chen, (2015) *Parameter change test for zero-inflated generalized Poisson autoregressive models*, Statistics, page DOI:10.1080/02331888.2015.1083020.
[7] K.F. Turkman, M.G. Scotto, P.Z. Bermudez, *Non-linear time series, Extreme Events and Integer Value Problems*, Springer, 2014.
[8] A.A. Weiss, *Asymptotic theory for ARCH models: Estimation and Testing*, Econometric Theory, Vol. 2, 1 (1986) 107-131.
[9] F. Zhu, *A negative binomial integer-valued GARCH model*, Journal of Time Series Analysis, 32 (2011) 54-67.
[10] F. Zhu, *Zero-inflated Poisson and negative binomial integer-valued GARCH models*, J. Stat. Plann. and Inf, 142 (2012) 826-839.

Esmeralda Gonçalves
CMUC, Departamento de Matemática, Universidade de Coimbra, Portugal
E-mail address: esmerald@mat.uc.pt

Nazaré Mendes Lopes
CMUC, Departamento de Matemática, Universidade de Coimbra, Portugal
E-mail address: nazare@mat.uc.pt