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ON LINEAR SPECTRAL TRANSFORMATIONS AND THE LAGUERRE-HAHN CLASS

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ABSTRACT: We study the Christoffel, Geronimus, and Uvarov transformations for Laguerre-Hahn orthogonal polynomials on the real line. It is analysed the modification of the corresponding difference-differential equations that characterize the systems of orthogonal polynomials and the consequences for the three-term recurrence relation coefficients.

KEYWORDS: Orthogonal polynomials; Laguerre-Hahn class; Riccati differential equation; Christoffel transformation; Geronimus transformation; Uvarov transformation.
MATH. SUBJECT CLASSIFICATION (2010): 33C47, 42C05.

1. Introduction

Spectral transformations of orthogonal polynomials is a widely known theme in the literature of special functions and applications (see, for instance, [11, 16, 17, 21] and references therein). The canonical linear spectral transformations are related to the Christoffel, Geronimus, and Uvarov modifications, that is, the modification of the orthogonality measure by a polynomial, a rational function, and the addiction of a mass point, respectively. A basic topic of research concerns the modification of real weights and the formulae expressing the new orthogonal polynomials in terms of the original system. The seminal references on this topic are traced back to [5, 9, 10, 19, 20]. More recent literature can be found, amongst many others, in [3, 16, 17, 21, 22].

The main motivation for the present work relies on studies concerning the Christoffel, Geronimus, and Uvarov transformations related to semi-classical measures. The semi-classical character is preserved under such transformations [22]. Basic structures, such as the recurrence relation for the orthogonal polynomials, the spectral derivatives of the system, and the class of the corresponding linear functional, have been extensively studied (see [3, 16, 21]

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and its references). Generalizations and extensions of semi-classical families of orthogonal polynomials are well-known, their study can be placed within the so-called Laguerre-Hahn families. Similarly to the semi-classical case, the Laguerre-Hahn class is also closed under the Christoffel, Geronimus, and Uvarov transformations [6, 8, 13, 14]. A systematic study of Laguerre-Hahn orthogonal polynomials on the real line was initiated in [7], showing some basic properties such as the so-called structure relations, that is, systems of difference-differential equations involving the orthogonal polynomials, $\{P_n\}_{n\geq 0}$, and the associated polynomials, $\{P_n^{(1)}\}_{n\geq 0}$ (cf. Section 2). In the matrix form, these systems are given by

$$A\Psi_n' = \mathcal{M}_n\Psi_n + \mathcal{N}_n\Psi_{n-1}\,, \quad n \geq 0\,, \quad \Psi_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \end{bmatrix}^T . \tag{1}$$

Here, A is a polynomial and $\mathcal{M}_n, \mathcal{N}_n$ are 2×2 matrices whose entries are polynomials [1]. In such a setting, in the present paper we study the Christoffel, Geronimus, and Uvarov transformations for Laguerre-Hahn orthogonal polynomials within the modifications of systems of type (1). The main goal is to study the derivatives of the system (1) under rational modifications of the orthogonality weight and, as a consequence, to analyse the corresponding modifications on the three-term recurrence relation coefficients of the polynomials.

The reminder of the paper is organized as follows. In Section 2 we give the basic results on Laguerre-Hahn orthogonal polynomials and we introduce notation to be used in the sequel. In Section 3 we deduce formulae in the matrix form, involving the modified polynomials in terms of the original system. The modifications in the Laguerre-Hahn class are studied Section 4. The semi-classical class is analysed in Section 4.2.

2. Preliminary results and notations

Let $\mathbb{P} = \operatorname{span}\{x^k : k \in \mathbb{N}_0\}$ be the linear space of polynomials with complex coefficients, and let $\mathbb{P}^*$ be its algebraic dual space. We will denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}^*$ on $f \in \mathbb{P}$. Given the moments of u, $u_n = \langle u, x^n\rangle$, $n \geq 0$, where we take $u_0 = 1$, the principal minors of the corresponding Hankel matrix are defined by $H_n = \det(u_{i+j})_{i,j=0}^n$, where, by convention, $H_{-1} = 1$. The functional u is said to be quasi-definite (respect., positive-definite) if $H_n \neq 0$ (respect., $H_n > 0$), for all $n \geq 0$.

Let $u \in \mathbb{P}^*$ and let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials such that $\deg(P_n) = n$. The basis $\{P_n\}_{n\geq 0}$ is said to be a sequence of orthogonal polynomials with respect to u if

$$\langle u, P_n P_m \rangle = h_n \delta_{n,m}\,,\ h_n \neq 0\,, \quad n,m \geq 0\,. \tag{2}$$

Throughout the paper we shall take each P_n monic, that is, $P_n(x) = x^n+$ lower degree terms, and we will denote $\{P_n\}_{n\geq 0}$ by SMOP.

The equivalence between the quasi-definiteness of u and the existence of a SMOP with respect to u is well-known in the literature of orthogonal polynomials [4, 18]. Furthermore, if u is positive-definite, then it has an integral representation in terms of a positive Borel measure, μ, supported on an infinite point set, $I \subseteq \mathbb{R}$, such that

$$u_n = \langle u, x^n \rangle = \int_I x^n \, d\mu(x)\,, \ \ n \geq 0\,, \tag{3}$$

and the orthogonality condition (2) becomes

$$\int_I P_n(x) P_m(x) d\mu(x) = h_n \delta_{n,m}\,, \ \ h_n > 0\,, \quad n,m \geq 0\,.$$

If μ is an absolutely continuous measure supported on I, and w denotes its Radon-Nikodym derivative with respect to the Lebesgue measure, then we will also say that $\{P_n\}_{n\geq 0}$ is orthogonal with respect to w.

Associated with any SMOP there exist sequences $\{\gamma_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ of positive real numbers and real numbers, respectively, such that the three term recurrence relation holds [18]:

$$P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n \geq 1\,, \tag{4}$$

with $P_0(x) = 1$ and $P_1(x) = x - \beta_0$.

Given a SMOP with respect to u, the sequence of associated polynomials of the first kind is defined by

$$P_n^{(1)}(x) = \langle u_t, \frac{P_{n+1}(x) - P_{n+1}(t)}{x - t} \rangle\,, \quad n \geq 0\,,$$

where u_t denotes the action of u on the variable t.

The sequence $\{P_n^{(1)}\}_{n\geq 0}$ also satisfies a three-term recurrence relation,

$$P_n^{(1)}(x) = (x - \beta_n) P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x)\,, \quad n \geq 1\,, \tag{5}$$

with $P_{-1}^{(1)}(x) = 0$ and $P_0^{(1)}(x) = 1$.

The Stieltjes function of u is defined by $S(x) = \sum_{n=0}^{\infty} \frac{u_n}{x^{n+1}}$. Note that if u is positive-definite, defined by (3), then S is given by

$$S(x) = \int_I \frac{d\mu(t)}{x-t}, \quad x \in \mathbb{C} \setminus I .$$

The sequence of functions of the second kind corresponding to $\{P_n\}_{n\geq 0}$ is defined as follows:

$$q_{n+1} = P_{n+1}S - P_n^{(1)}, \quad n \geq 0, \; q_0 = S .$$

Whenever u is positive-definite, defined by (3), the q_n's are given in terms of an integral formula,

$$q_n(x) = \int_I \frac{P_n(t)}{x-t} d\mu(t), \quad n \geq 0 .$$

The Stieltjes function S is said to be Laguerre-Hahn if there exist polynomials A, B, C, D, with $A \neq 0$, such that it satisfies a Riccati differential equation [15]

$$AS' = BS^2 + CS + D . \tag{6}$$

The corresponding sequence of orthogonal polynomials is called Laguerre-Hahn. If $B = 0$, then S is said to be Laguerre-Hahn affine or semi-classical.

Note that equation (6) is equivalent to the distributional equation [7, 15]

$$\mathcal{D}(Au) = \psi u + B(x^{-1}u^2), \quad \psi = A' + C .$$

Furthermore, if u is positive-definite, defined in terms of a weight, w, then the semi-classical character of u, $\mathcal{D}(Au) = \psi u$, $\deg(\psi) \geq 1$, is equivalent to $w'/w = C/A$ with w satisfying the boundary conditions [15]

$$x^n A(x) w(x)|_{a,b} = 0, \quad n \geq 0 ,$$

where a, b (eventually a and/or b infinite) are linked with the roots of A. In such a case, w is the weight function on the support $I = [a, b]$.

Throughout the paper we will use the following matrices associated with the SMOP $\{P_n\}_{n\geq 0}$:

$$\Psi_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \end{bmatrix}^T, \quad \mathcal{Y}_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}, \quad n \geq 0 . \tag{7}$$

In the account of (4) and (5), we have the recurrence relations

$$\Psi_n(x) = (x-\beta_n)\Psi_{n-1}(x) - \gamma_n\Psi_{n-2}(x)\,, \quad n \geq 1\,, \tag{8}$$

$$\mathcal{Y}_n = \mathcal{A}_n\mathcal{Y}_{n-1}\,, \;\; \mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1\,, \tag{9}$$

with initial conditions

$$\Psi_{-1} = \begin{bmatrix} P_0 & P_{-1}^{(1)} \end{bmatrix}^T,\; \Psi_0 = \begin{bmatrix} P_1 & P_0^{(1)} \end{bmatrix}^T,\; \mathcal{Y}_0 = \begin{bmatrix} x-\beta_0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As usual, $\mathcal{A}_n$ is called the transfer matrix.

The matrices (7) will play a relevant role in the sequel. Indeed, Laguerre-Hahn orthogonal polynomials are characterized in terms of differential systems for $\{\Psi_n\}_{n\geq 0}$: there holds the equivalence between (6) and

$$A\Psi_n' = \mathcal{M}_n\Psi_n + \mathcal{N}_n\Psi_{n-1}\,, \quad n \geq 0\,. \tag{10}$$

Here, A is the same as in (6) and $\mathcal{M}_n, \mathcal{N}_n$ are 2×2 matrices whose entries are bounded degree polynomials depending on the coefficients of the Riccati equation [1, Theorem 1]. In the semi-classical case, that is, $B \equiv 0$ in (6), when dealing with weights, there holds the equivalence between the semi-classical character of w, $w'/w = C/A$, and the differential system

$$A\mathcal{Y}_n' = \mathcal{B}_n\mathcal{Y}_n\,, \quad n \geq 1\,. \tag{11}$$

Here, $\mathcal{B}_n$ is a 2×2 matrix whose entries are bounded degree polynomials depending on the polynomials A, C (see [2, Theorem 2] and [12]).

The three canonical transformations, the Christoffel, Geronimus and Uvarov transforms, are defined as follows [22].

(a) Christoffel: the weight is modified as

$$\tilde{w}(x) = (x-\alpha)w(x)\,, \tag{12}$$

the polynomial modification is

$$(x-\alpha)\tilde{P}_n(x) = P_{n+1}(x) - a_nP_n(x)\,, \quad a_n = P_{n+1}(\alpha)/P_n(\alpha)\,. \tag{13}$$

(b) Geronimus: the weight is modified as

$$\tilde{w}(x) = \frac{w(x)}{x-\alpha}\,, \tag{14}$$

the polynomial modification is

$$\tilde{P}_n(x) = P_n(x) - b_nP_{n-1}(x)\,, \quad b_n = q_n(\alpha)/q_{n-1}(\alpha)\,. \tag{15}$$

(c) Uvarov: the weight is modified as

$$\tilde{w}(x) = w(x) + \delta_\alpha\,, \tag{16}$$

the polynomial modification is

$$(x-\alpha)\tilde{P}_n(x) = P_{n+1}(x) + c_n P_n(x) + d_n P_{n-1}(x)\,, \tag{17}$$

with

$$c_n = \alpha - \beta_n + \frac{P_n(\alpha)P_{n-1}(\alpha)}{1+K_{n-1}(\alpha,\alpha)}\,,\quad d_n = \gamma_n + \frac{P_n^2(\alpha)}{1+K_{n-1}(\alpha,\alpha)}\,.$$

Here, K_n is defined by $K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{h_k}$, $\;h_k = \prod_{i=1}^{k} \gamma_i\,.$

Throughout the text, $(X)_{(i,j)}$ will denote the (i,j) entry in the matrix X.

3. The Christoffel, Geronimus, and Uvarov transformations: matrix relations

In this section we will write the matrix modifications for $\{\Psi_n\}_{n\geq 0}$ under the Christoffel, Geronimus, and Uvarov transformations.

3.1. Christoffel transformation.

Theorem 1. *Let $\{P_n\}_{n\geq 0}$ be a SMOP related to a weight w and let $\{\tilde{P}_n\}_{n\geq 0}$ be the SMOP related to the modified weight (12). Denote by $\{\Psi_n\}_{n\geq 0}$ and $\{\tilde{\Psi}_n\}_{n\geq 0}$ the corresponding sequences defined through (7). The following relation holds:*

$$(x-\alpha)\widetilde{\Psi}_n(x) = \mathcal{S}_n(x;\alpha)\Psi_{n+1}(x) + \mathcal{T}_n(x;\alpha)\Psi_n(x)\,,\quad n\geq 0\,, \tag{18}$$

where

$$\mathcal{S}_n(x;\alpha) = \begin{bmatrix} 1 & 0 \\ -1 & x-\alpha \end{bmatrix}\,,\quad \mathcal{T}_n(x;\alpha) = -a_{n+1}\mathcal{S}_n(x;\alpha)\,. \tag{19}$$

Proof: Let us first prove the identity

$$\begin{aligned}(x-\alpha)\widetilde{P_n^{(1)}}(x) = a_{n+1}P_{n+1}(x) - P_{n+2}(x)& \\ + (x-\alpha)\left(P_{n+1}^{(1)}(x) - a_{n+1}P_n^{(1)}(x)\right)&\,. \end{aligned}\tag{20}$$

Using the definition of $\widetilde{P_n^{(1)}}$, we get

$$(x-\alpha)\widetilde{P_n^{(1)}}(x) = (x-\alpha)\int_I \frac{\tilde{P}_{n+1}(x)-\tilde{P}_{n+1}(t)}{x-t}(t-\alpha)w(t)dt\,.$$

The use of (13) in the equation above gives us

$$(x-\alpha)\widetilde{P_n^{(1)}}(x) = \int_I \frac{(t-\alpha)A_n(x)-(x-\alpha)A_n(t)}{x-t}w(t)dt\,,$$

with $A_n(x)=P_{n+2}(x)-a_{n+1}P_{n+1}(x)$. Thus, we get

$$(x-\alpha)\widetilde{P_n^{(1)}}(x) = \int_I \frac{tA_n(x)-xA_n(t)}{x-t}w(t)dt - \alpha\int_I \frac{A_n(x)-A_n(t)}{x-t}w(t)dt\,.$$

Note that

$$\begin{aligned}\int_I \frac{tA_n(x)-xA_n(t)}{x-t}w(t)dt &= \int_I \frac{(t-x)A_n(x)-(x-t)A_n(t)}{x-t}w(t)dt \\ &+ \int_I \frac{xA_n(x)-tA_n(t)}{x-t}w(t)dt\,,\end{aligned}$$

thus,

$$\int_I \frac{tA_n(x)-xA_n(t)}{x-t}w(t)dt = -A_n(x)+xP_{n+1}^{(1)}(x)-a_{n+1}xP_n^{(1)}(x)\,, \tag{21}$$

where we have used $\displaystyle\int_I(-A_n(x)-A_n(t))w(t)dt=-A_n(x)$ together with the recurrence relation (4). Also,

$$\int_I \frac{A_n(x)-A_n(t)}{x-t}w(t)dt = P_{n+1}^{(1)}(x)-a_{n+1}P_n^{(1)}(x)\,. \tag{22}$$

Therefore, (21) and (22) yield (20).

Equation (18) is the matrix form of (13) and (20). ■

In what follows we will see how the recurrence coefficients of the transformed orthogonal polynomials can be obtained through the matrix relations (18).

Corollary 1. *Under the notation of (13), the recurrence coefficients of the SMOP $\{\tilde{P}_n\}_{n\geq 0}$ related to (12) are transformed according to the formulas*

$$\tilde{\beta}_n = \beta_{n+1}+a_{n+1}-a_n\,,\quad \tilde{\gamma}_n=\gamma_n\frac{a_n}{a_{n-1}}\,. \tag{23}$$

Proof: Multiply the recurrence relation

$$\widetilde{\Psi}_n = (x-\tilde{\beta}_n)\widetilde{\Psi}_{n-1} - \tilde{\gamma}_n\widetilde{\Psi}_{n-2}\,, \quad n \geq 1\,,$$

by $(x-\alpha)$ and use (18), thus obtaining

$$\begin{aligned}\mathcal{S}_n\Psi_{n+1} + \mathcal{T}_n\Psi_n = (x-\tilde{\beta}_n)\left(\mathcal{S}_{n-1}\Psi_n + \mathcal{T}_{n-1}\Psi_{n-1}\right)& \\ - \tilde{\gamma}_n\left(\mathcal{S}_{n-2}\Psi_{n-1} + \mathcal{T}_{n-2}\Psi_{n-2}\right)&\,.\end{aligned}$$

The use of the recurrence relation (8) in the above equality yields

$$\mathcal{M}_{n,1}\Psi_n = \mathcal{M}_{n,2}\Psi_{n-1}\,, \quad n \geq 0\,, \tag{24}$$

where

$$\mathcal{M}_{n,1} = (x-\beta_{n+1})\mathcal{S}_n + \mathcal{T}_n - (x-\tilde{\beta}_n)\mathcal{S}_{n-1} - \frac{\tilde{\gamma}_n}{\gamma_n}\mathcal{T}_{n-2}\,, \tag{25}$$

$$\mathcal{M}_{n,2} = \gamma_{n+1}\mathcal{S}_n + (x-\tilde{\beta}_n)\mathcal{T}_{n-1} - \tilde{\gamma}_n\mathcal{S}_{n-2} - \frac{\tilde{\gamma}_n}{\gamma_n}(x-\beta_n)\mathcal{T}_{n-2}\,. \tag{26}$$

Taking into account (19), then $(\mathcal{M}_{n,1})_{(1,2)} = (\mathcal{M}_{n,2})_{(1,2)} = 0$. Hence, (24) yields

$$\left(\mathcal{M}_{n,1}\right)_{(1,1)} P_{n+1} = \left(\mathcal{M}_{n,2}\right)_{(1,1)} P_n\,.$$

As P_n and P_{n+1} do not share zeros (see, e.g., [4]) and the degrees of $\left(\mathcal{M}_{n,j}\right)_{(1,1)}$, $j=1,2$, are uniformly bounded, there follows

$$\left(\mathcal{M}_{n,1}\right)_{(1,1)} = \left(\mathcal{M}_{n,2}\right)_{(1,1)} = 0\,.$$

Furthermore, in the account of (19), we have $\left(\mathcal{M}_{n,1}\right)_{(2,1)} = -\left(\mathcal{M}_{n,1}\right)_{(1,1)}$, $\left(\mathcal{M}_{n,1}\right)_{(2,2)} = (x-\alpha)\left(\mathcal{M}_{n,1}\right)_{(1,1)}$. Thus, $\mathcal{M}_{n,1}$ is the null matrix. Therefore, (24) reads

$$\left(\mathcal{M}_{n,2}\right)_{(2,1)} P_n = -\left(\mathcal{M}_{n,2}\right)_{(2,2)} P^{(1)}_{n-1}\,.$$

As P_n and $P^{(1)}_{n-1}$ do not share zeros (see, e.g., [4]) and the degrees of $\left(\mathcal{M}_{n,2}\right)_{(2,j)}$, $j=1,2$, are uniformly bounded, there follows

$$\left(\mathcal{M}_{n,2}\right)_{(2,1)} = \left(\mathcal{M}_{n,2}\right)_{(2,2)} = 0\,,$$

that is, $\mathcal{M}_{n,2}$ is also a null matrix. Therefore, we get

$$(x-\beta_{n+1})\mathcal{S}_n + \mathcal{T}_n - (x-\tilde{\beta}_n)\mathcal{S}_{n-1} - \frac{\tilde{\gamma}_n}{\gamma_n}\mathcal{T}_{n-2} = 0\,, \tag{27}$$

$$\gamma_{n+1}\mathcal{S}_n + (x-\tilde{\beta}_n)\mathcal{T}_{n-1} - \tilde{\gamma}_n\mathcal{S}_{n-2} - \frac{\tilde{\gamma}_n}{\gamma_n}(x-\beta_n)\mathcal{T}_{n-2} = 0\,. \tag{28}$$

Equation (27) yields only one non-trivial equation,

$$\tilde{\beta}_n = \beta_{n+1} + a_{n+1} - \frac{\tilde{\gamma}_n}{\gamma_n} a_{n-1} \,. \tag{29}$$

Equation (28) yields only one non-trivial equation, from which we get

$$a_n = \frac{\tilde{\gamma}_n}{\gamma_n} a_{n-1} \,. \tag{30}$$

Equations (29) and (30) yield (23). ∎

3.2. Geronimus transformation.

Theorem 2. *Let $\{P_n\}_{n\geq 0}$ be a SMOP related to a weight w and let $\{\tilde{P}_n\}_{n\geq 0}$ be the SMOP related to the modified weight (14). Denote by $\{\Psi_n\}_{n\geq 0}$ and $\{\widetilde{\Psi}_n\}_{n\geq 0}$ the corresponding sequences defined through (7). The following relation holds:*

$$(x-\alpha)\widetilde{\Psi}_n(x) = \mathcal{S}_n(x;\alpha)\Psi_{n+1}(x) + \mathcal{T}_n(x;\alpha)\Psi_n(x)\,, \quad n \geq 0\,, \tag{31}$$

where

$$\mathcal{S}_n(x;\alpha) = \begin{bmatrix} 1 - \frac{(\alpha-\tilde{\beta}_{n+1})b_{n+1}}{\gamma_{n+1}} - \frac{\tilde{\gamma}_{n+1}}{\gamma_{n+1}} + \frac{(x-\beta_n)\tilde{\gamma}_{n+1}b_n}{\gamma_n\gamma_{n+1}} & 0 \\ \frac{b_{n+1}}{\gamma_{n+1}} & \frac{b_{n+1}}{\gamma_{n+1}} \end{bmatrix}, \tag{32}$$

$$\mathcal{T}_n(x;\alpha) = \begin{bmatrix} t_n & 0 \\ 1 - \frac{(x-\beta_{n+1})b_{n+1}}{\gamma_{n+1}} & 1 - \frac{(x-\beta_{n+1})b_{n+1}}{\gamma_{n+1}} \end{bmatrix}, \tag{33}$$

where

$$t_n = -b_{n+2} + \tilde{\beta}_{n+1} - \alpha + \frac{\tilde{\gamma}_{n+1}b_n}{\gamma_n} - (x-\beta_{n+1})((\mathcal{S}_n)_{(1,1)} - 1)\,.$$

Proof: Let us first prove the identity

$$(x-\alpha)\widetilde{P_n^{(1)}}(x) = P_{n+1}(x) - b_{n+1}P_n(x) + P_n^{(1)}(x) - b_{n+1}P_{n-1}^{(1)}(x)\,. \tag{34}$$

Using the definition of $\widetilde{P_n^{(1)}}$, we get

$$\begin{aligned}(x-\alpha)\widetilde{P_n^{(1)}}(x) &= (x-\alpha)\int_I \frac{\tilde{P}_{n+1}(x)-\tilde{P}_{n+1}(t)}{x-t}\frac{w(t)}{t-\alpha}dt \\ &= \int_I \frac{\tilde{P}_{n+1}(x)-\tilde{P}_{n+1}(t)}{t-\alpha}w(t)dt + \int_I \frac{\tilde{P}_{n+1}(x)-\tilde{P}_{n+1}(t)}{x-t}w(t)dt\,. \end{aligned} \tag{35}$$

Note the first integral in (35),

$$\begin{aligned}\int_I \frac{\tilde{P}_{n+1}(x)-\tilde{P}_{n+1}(t)}{t-\alpha}w(t)dt &= (\tilde{P}_{n+1}(x)-\tilde{P}_{n+1}(\alpha))\int_I \frac{w(t)}{t-\alpha}dt \\ &\quad - \int_I \frac{\tilde{P}_{n+1}(t)-\tilde{P}_{n+1}(\alpha)}{t-\alpha}w(t)dt \\ &= -\left(P_{n+1}(x)-b_{n+1}P_n(x)-P_{n+1}(\alpha)+b_{n+1}P_n(\alpha)\right)S(\alpha) \\ &\quad - P_n^{(1)}(\alpha)+b_{n+1}P_{n-1}^{(1)}(\alpha)\,, \end{aligned} \tag{36}$$

where we have used (15).

The second integral in (35) gives us

$$\int_I \frac{\tilde{P}_{n+1}(x)-\tilde{P}_{n+1}(t)}{x-t}w(t)dt = P_n^{(1)}(x)-b_{n+1}P_{n-1}^{(1)}(x)\,. \tag{37}$$

Therefore, (36) and (37) yield

$$\begin{aligned}(x-\alpha)\widetilde{P_n^{(1)}}(x) &= -\left(P_{n+1}(x)-b_{n+1}P_n(x)-P_{n+1}(\alpha)+b_{n+1}P_n(\alpha)\right)S(\alpha) \\ &\quad -P_n^{(1)}(\alpha)+b_{n+1}P_{n-1}^{(1)}(\alpha)+P_n^{(1)}(x)-b_{n+1}P_{n-1}^{(1)}(x)\,.\end{aligned}$$

As $\widetilde{P_n^{(1)}}$ is monic, there must hold $-S(\alpha)=1$, thus,

$$(x-\alpha)\widetilde{P_n^{(1)}}(x) = P_{n+1}(x)-b_{n+1}P_n(x)+P_n^{(1)}(x)-b_{n+1}P_{n-1}^{(1)}(x)+A_n(\alpha)\,,$$

with $A_n(\alpha)=-P_{n+1}(\alpha)+b_{n+1}P_n(\alpha)-P_n^{(1)}(\alpha)+b_{n+1}P_{n-1}^{(1)}(\alpha)$.

Note that $A_n(\alpha)=0$, hence, we obtain (34).

Now let us write, in the matrix notation, (15) and (34), that is,

$$(x-\alpha)\widetilde{\Psi}_n = \mathcal{A}\Psi_{n+1}+\mathcal{B}_n\Psi_n+\mathcal{C}_n\Psi_{n-1}+\mathcal{D}_n\Psi_{n-2}\,,$$

with

$$\mathcal{A}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad \mathcal{B}_n=\begin{bmatrix}\tilde{\beta}_{n+1}-b_{n+2}-\alpha & 0\\ 1 & 1\end{bmatrix},$$

$$\mathcal{C}_n=\begin{bmatrix}(\alpha-\tilde{\beta}_{n+1})b_{n+1}+\tilde{\gamma}_{n+1} & 0\\ -b_{n+1} & -\,b_{n+1}\end{bmatrix},\quad \mathcal{D}_n=\begin{bmatrix}-\tilde{\gamma}_{n+1}b_n & 0\\ 0 & 0\end{bmatrix}.$$

The use of the recurrence relation (8) yields

$$(x-\alpha)\widetilde{\Psi}_n = \mathcal{S}_n\Psi_{n+1}+\mathcal{T}_n\Psi_n\,,$$

with

$$\mathcal{S}_n = \mathcal{A} - \frac{1}{\gamma_{n+1}}\mathcal{C}_n - \frac{(x-\beta_n)}{\gamma_n\gamma_{n+1}}\mathcal{D}_n \,,$$
$$\mathcal{T}_n = \mathcal{B}_n + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\mathcal{C}_n + \left(\frac{(x-\beta_n)(x-\beta_{n+1})}{\gamma_n\gamma_{n+1}} - \frac{1}{\gamma_n}\right)\mathcal{D}_n \,.$$

Hence, we obtain (31) with the matrices $\mathcal{S}_n, \mathcal{T}_n$ given in (32) and (33). ∎

Corollary 2. *Under the notation of (15), the recurrence coefficients of the SMOP $\{\tilde{P}_n\}_{n\geq 0}$ related to (14) are transformed according to the formulas*

$$\tilde{\beta}_n = \beta_n + b_{n+1} - b_n \,, \quad \tilde{\gamma}_n = \gamma_{n-1}\frac{b_n}{b_{n-1}} \,. \tag{38}$$

Proof: Proceeding as in the proof of Corollary 1 we obtain (24),

$$\mathcal{M}_{n,1}\Psi_n = \mathcal{M}_{n,2}\Psi_{n-1} \,, \quad n \geq 0 \,,$$

with $\mathcal{M}_{n,1}, \mathcal{M}_{n,2}$ given by (25) and (26), respectively.

Note that $(\mathcal{M}_{n,1})_{(1,2)} = (\mathcal{M}_{n,2})_{(1,2)} = 0$. Hence, following the same arguments as in the proof of Corollary 1, there follows

$$(\mathcal{M}_{n,1})_{(1,1)} = (\mathcal{M}_{n,2})_{(1,1)} = 0 \,.$$

Furthermore, as $(\mathcal{M}_{n,1})_{(2,1)} = (\mathcal{M}_{n,1})_{(2,2)}$ and $(\mathcal{M}_{n,2})_{(2,1)} = (\mathcal{M}_{n,2})_{(2,2)}$, after some computations we obtain that $\mathcal{M}_{n,1}$ is the null matrix and, consequently, $\mathcal{M}_{n,2}$ must also be null. Therefore, we have (27)–(28).

From position $(2,1)$ in (27) we get the equation for the $\tilde{\gamma}_n$'s in (38),

$$\tilde{\gamma}_n = \gamma_{n-1}\frac{b_n}{b_{n-1}} \,, \tag{39}$$

as well as

$$1 + \frac{\tilde{\beta}_n}{\gamma_n}b_n - \frac{\tilde{\gamma}_n}{\gamma_n}\left(1 + \frac{b_{n-1}}{\gamma_{n-1}}b_{n-1}\right) = 0 \,, \tag{40}$$

thus, after some computations following from the use of (39) in (40), we get

$$\tilde{\beta}_n = \beta_{n-1} + \frac{\gamma_{n-1}}{b_{n-1}} - \frac{\gamma_n}{b_n} \,. \tag{41}$$

From position $(2,1)$ in (28) we get

$$\beta_{n-1} + \frac{\gamma_{n-1}}{b_{n-1}} - \frac{\gamma_n}{b_n} = \beta_n + b_{n+1} - b_n \,. \tag{42}$$

Therefore, (41) and (42) yield the equation for the $\tilde{\beta}_n$'s in (38). ∎

3.3. Uvarov transformation.

Theorem 3. *Let* $\{P_n\}_{n\geq 0}$ *be a SMOP related to a weight* w *and let* $\{\tilde{P}_n\}_{n\geq 0}$ *be the SMOP related to the modified weight (16). Denote by* $\{\Psi_n\}_{n\geq 0}$ *and* $\{\widetilde{\Psi}_n\}_{n\geq 0}$ *the corresponding sequences defined through (7). The following relation holds:*

$$(x-\alpha)^2\widetilde{\Psi}_n(x) = \mathcal{S}_n(x;\alpha)\Psi_{n+1}(x) + \mathcal{T}_n(x;\alpha)\Psi_n(x)\,, \quad n\geq 0\,, \tag{43}$$

where

$$\mathcal{S}_n(x;\alpha) = \left(1-\frac{d_{n+1}}{\gamma_{n+1}}\right)\begin{bmatrix} x-\alpha & 0 \\ 1 & x-\alpha\end{bmatrix}, \tag{44}$$

$$\mathcal{T}_n(x;\alpha) = \left(c_{n+1}+\frac{(x-\beta_{n+1})}{\gamma_{n+1}}d_{n+1}\right)\begin{bmatrix} x-\alpha & 0 \\ 1 & x-\alpha\end{bmatrix}. \tag{45}$$

Proof: Let us first prove the identity

$$(x-\alpha)^2\widetilde{P_n^{(1)}}(x) = (x-\alpha)P_{n+1}^{(1)}(x)+(x-\alpha)c_{n+1}P_n^{(1)}(x)+(x-\alpha)d_{n+1}P_{n-1}^{(1)}(x) \\ + P_{n+2}(x)+c_{n+1}P_{n+1}(x)+d_{n+1}P_n(x)\,. \tag{46}$$

Using the definition of $\widetilde{P_n^{(1)}}$, we get

$$\begin{aligned}(x-\alpha)\widetilde{P_n^{(1)}}(x) &= \int_I \frac{(x-\alpha)\tilde{P}_{n+1}(x)-(x-\alpha)\tilde{P}_{n+1}(t)}{x-t}\tilde{w}(t)dt \\ &= \int_I \frac{(x-\alpha)\tilde{P}_{n+1}(x)-(t-\alpha)\tilde{P}_{n+1}(t)}{x-t}\tilde{w}(t)dt \\ &- \int_I \tilde{P}_{n+1}(t)\tilde{w}(t)dt\,.\end{aligned}$$

Note that

$$\int_I \tilde{P}_{n+1}(t)\tilde{w}(t)dt = 0\,. \tag{47}$$

Also,

$$\begin{aligned}\int_I \frac{(x-\alpha)\tilde{P}_{n+1}(x)-(t-\alpha)\tilde{P}_{n+1}(t)}{x-t}\tilde{w(t)}dt \\ = \int_I \frac{(x-\alpha)\tilde{P}_{n+1}(x)-(t-\alpha)\tilde{P}_{n+1}(t)}{x-t}w(t)dt \\ + \int_I \frac{(x-\alpha)\tilde{P}_{n+1}(x)-(t-\alpha)\tilde{P}_{n+1}(t)}{x-t}\delta_\alpha(t)\,.\end{aligned} \tag{48}$$

In account of (17), for the first integral in (48) we have

$$\begin{aligned}\int_I \frac{(x-\alpha)\tilde{P}_{n+1}(x)-(t-\alpha)\tilde{P}_{n+1}(t)}{x-t}w(t)dt \\ = P^{(1)}_{n+1}(x)+c_{n+1}P^{(1)}_n(x)+d_{n+1}P^{(1)}_{n-1}(x)\,.\end{aligned}$$

For the second integral in (48) we have

$$\int_I \frac{(x-\alpha)\tilde{P}_{n+1}(x)-(t-\alpha)\tilde{P}_{n+1}(t)}{x-t}\delta_\alpha(t) = \tilde{P}_{n+1}(x)\,.$$

Hence, we obtain (46).

Now let us write, in the matrix form, (17) and (46). We get

$$(x-\alpha)^2\widetilde{\Psi}_n = \mathcal{A}\Psi_{n+1}+\mathcal{B}_n\Psi_n+\mathcal{C}_n\Psi_{n-1}\,,$$

with

$$\mathcal{A} = \begin{bmatrix} x-\alpha & 0 \\ 1 & x-\alpha \end{bmatrix}, \quad \mathcal{B}_n = c_{n+1}\mathcal{A}, \quad \mathcal{C}_n = d_{n+1}\mathcal{A}\,.$$

The use of the recurrence relation (8) yields

$$(x-\alpha)^2\widetilde{\Psi}_n = \mathcal{S}_n\Psi_{n+1}+\mathcal{T}_n\Psi_n\,,$$

with

$$\mathcal{S}_n = \mathcal{A}-\frac{1}{\gamma_{n+1}}\mathcal{C}_n\,, \quad \mathcal{T}_n = \mathcal{B}_n+\frac{(x-\beta_{n+1})}{\gamma_{n+1}}\mathcal{C}_n\,.$$

Hence, we obtain (43) with the matrices $\mathcal{S}_n, \mathcal{T}_n$ given in (44) and (45). ■

Corollary 3. *Under the notation of (17), the recurrence coefficients of the SMOP* $\{\tilde{P}_n\}_{n\geq 0}$ *related to (16) are transformed according to the formulas*

$$\tilde{\beta}_n = \beta_{n+1}+c_n-c_{n+1}\,, \quad \tilde{\gamma}_n = \gamma_{n-1}\frac{d_n}{d_{n-1}}\,. \tag{49}$$

Proof: Proceeding as in the proof of Corollary 1 and Corollary 2 we obtain equations (27)–(28). From position $(1,1)$ in (27) we get the equation for the $\tilde{\gamma}_n$'s in (49), and after some computations we obtain, from (28), the equation for the $\tilde{\beta}_n$'s. ■

3.4. Iterations - General formulas. We denote the iterated Christoffel transformation by

$$w^{[K]}(x) = \prod_{j=1}^{K}(x-\alpha_j)w(x)\,, \tag{50}$$

and the iterated Geronimus transformation by

$$w_{[L]}(x) = \frac{1}{\prod_{j=1}^{L}(x-\nu_j)}w(x)\,. \tag{51}$$

The monic orthogonal polynomials related to (50) and (51) will be denoted by $P_{n,K,\cdot}$ and $P_{n,\cdot,L}$, respectively. The corresponding vectors defined in (7) related to $w^{[K]}$ will be denoted by $\Psi_{n,K,\cdot}$, and the ones related to $w_{[L]}$ will be denoted by $\Psi_{n,\cdot,L}$. Recall that the vectors related to w are denoted by Ψ_n.

Theorem 4. *For the modification of the weight (50), the following relation holds, for all $n \geq 0$:*

$$\prod_{j=1}^{K}(x-\alpha_j)\Psi_{n,K,\cdot}(x) = \mathcal{S}_n(x;\alpha_k,\ldots,\alpha_1)\Psi_{n+1}(x) + \mathcal{T}_n(x;\alpha_k,\ldots,\alpha_1)\Psi_n(x)\,, \tag{52}$$

where the matrices $\mathcal{S}_n(x;\alpha_k,\ldots,\alpha_1), \mathcal{T}_n(x;\alpha_k,\ldots,\alpha_1)$ are defined recursively through

$$\begin{aligned}
\mathcal{S}_n(x;\alpha_k,\ldots,\alpha_1) &= (x-\beta_{n+2})\mathcal{S}_n(x;\alpha_k)\mathcal{S}_{n+1}(x;\alpha_{k-1},\ldots\alpha_1)\\
&\quad +\mathcal{S}_n(x;\alpha_k)\mathcal{T}_{n+1}(x;\alpha_{k-1},\ldots,\alpha_1) + \mathcal{T}_n(x;\alpha_k)\mathcal{S}_n(x;\alpha_{k-1},\ldots,\alpha_1),\\
\mathcal{T}_n(x;\alpha_k,\ldots,\alpha_1) &= -\gamma_{n+2}\mathcal{S}_n(x;\alpha_k)\mathcal{S}_{n+1}(x;\alpha_{k-1},\ldots,\alpha_1)\\
&\quad +\mathcal{T}_n(x;\alpha_k)\mathcal{T}_n(x;\alpha_{k-1},\ldots,\alpha_1)\,,
\end{aligned}$$

with initial conditions

$$\mathcal{S}_n(x;\alpha_1) = \begin{bmatrix} 1 & 0 \\ -1 & x-\alpha_1 \end{bmatrix}, \quad \mathcal{T}_n(x;\alpha_1) = -a_{n+1}\mathcal{S}_n(x;\alpha_1)\,.$$

For the modification of the weight (51), the following relation holds:

$$\prod_{j=1}^{L}(x-\nu_j)\Psi_{n,\cdot,L}(x)=\mathcal{S}_n(x;\nu_L,\dots,\nu_1)\Psi_{n+1}(x)+\mathcal{T}_n(x;\nu_L,\dots,\nu_1)\Psi_n(x)\,, \tag{53}$$

where the matrices $\mathcal{S}_n(x;\nu_L,\dots,\nu_1),\mathcal{T}_n(x;\nu_L,\dots,\nu_1)$ *are defined recursively through*

$$\begin{aligned}\mathcal{S}_n(x;\nu_L,\dots,\nu_1)&=(x-\beta_{n+2})\mathcal{S}_n(x;\nu_L)\mathcal{S}_{n+1}(x;\nu_{L-1},\dots,\nu_1)\\&\quad+\mathcal{S}_n(x;\nu_L)\mathcal{T}_{n+1}(x;\nu_{L-1},\dots,\nu_1)+\mathcal{T}_n(x;\nu_L)\mathcal{S}_n(x;\nu_{L-1},\dots,\nu_1),\\ \mathcal{T}_n(x;\nu_L,\dots,\nu_1)&=-\gamma_{n+2}\mathcal{S}_n(x;\nu_L)\mathcal{S}_{n+1}(x;\nu_{L-1},\dots,\nu_1)\\&\quad+\mathcal{T}_n(x;\nu_L)\mathcal{T}_n(x;\nu_{L-1},\dots,\nu_1)\,,\end{aligned}$$

with initial conditions

$$\begin{aligned}\mathcal{S}_n(x;\nu_1)&=\begin{bmatrix}1-\frac{(\nu_1-\tilde{\beta}_{n+1})b_{n+1}}{\gamma_{n+1}}-\frac{\tilde{\gamma}_{n+1}}{\gamma_{n+1}}+\frac{(x-\beta_n)\tilde{\gamma}_{n+1}b_n}{\gamma_n\gamma_{n+1}} & 0\\ \frac{b_{n+1}}{\gamma_{n+1}} & \frac{b_{n+1}}{\gamma_{n+1}}\end{bmatrix},\\ \mathcal{T}_n(x;\nu_1)&=\begin{bmatrix}t_n & 0\\ 1-\frac{(x-\beta_{n+1})b_{n+1}}{\gamma_{n+1}} & 1-\frac{(x-\beta_{n+1})b_{n+1}}{\gamma_{n+1}}\end{bmatrix},\end{aligned}$$

where

$$t_n=-b_{n+2}+\tilde{\beta}_{n+1}-\nu_1+\frac{\tilde{\gamma}_{n+1}b_n}{\gamma_n}-(x-\beta_{n+1})((\mathcal{S}_n(x;\nu_1))_{(1,1)}-1)\,.$$

4. Modifications within the Laguerre-Hahn class

4.1. The Laguerre-Hahn class: the Christoffel, Geronimus, and Uvarov transformations. It is well known that the Laguerre-Hahn class is closed under the Christoffel, Geronimus, and Uvarov transformations [13, 14].

Recall that Laguerre-Hahn orthogonal polynomials are characterized in terms of a Riccati equation for the Stieltjes function,

$$AS'=BS^2+CS+D\,,$$

or, equivalently , through a structure relation (10) for the corresponding Ψ_n,

$$A\Psi_n'=\mathcal{M}_n\Psi_n+\mathcal{N}_n\Psi_{n-1}\,,\quad n\geq 0\,,$$

where $\mathcal{M}_n,\mathcal{N}_n$ are matrices whose entries are bounded degree polynomials [1, Theorem 1].

Hence, there holds the equivalence between the Riccati equation for a modification of the Stieltjes function under the christoffel, Geronimus, Uvarov transformations, say, $\tilde{S}$, such that

$$\tilde{A}\tilde{S}' = \tilde{B}\tilde{S}^2 + \tilde{C}\tilde{S} + \tilde{D}\,,$$

and a structure relation for the corresponding $\widetilde{\Psi}_n$,

$$\tilde{A}\widetilde{\Psi}'_n = \tilde{\mathcal{M}}_n\tilde{\Psi}_n + \tilde{\mathcal{N}}_n\widetilde{\Psi}_{n-1}\,, \quad n \geq 0\,. \tag{54}$$

In order to construct the difference-differential equation (54) for the transformed $\widetilde{\Psi}_n$, it is sufficient to know the difference-differential equation (10) for Ψ_n and the matrices $\mathcal{S}_n, \mathcal{T}_n$ which result from the Christoffel-Geronimus-Uvarov transformations (cf. (18), (31), (43)), say,

$$\pi\widetilde{\Psi}_n = \mathcal{S}_n\Psi_{n+1} + \mathcal{T}_n\Psi_n\,.$$

Indeed, we have the following result.

Theorem 5. *Let $\{P_n\}_{n\geq 0}$ be a SMOP related to a Laguerre-Hahn Stieltjes function S satisfying $AS' = BS^2 + CS + D$, equivalently, the corresponding $\{\Psi_n\}_{n\geq 0}$ satisfying the difference-differential equation (10),*

$$A\Psi'_n = \mathcal{M}_n\Psi_n + \mathcal{N}_n\Psi_{n-1}\,, \quad n \geq 0\,.$$

Let $\{\tilde{P}_n\}_{n\geq 0}$ be the SMOP related to the modified Stieltjes function under the Christoffel or Geronimus transformations (12) or (14), say $\tilde{S}$ satisfying

$$\tilde{A}\tilde{S}' = \tilde{B}\tilde{S}^2 + \tilde{C}\tilde{S} + \tilde{D}\,. \tag{55}$$

Set the relations of the type given in (18), (31) written as

$$\pi\widetilde{\Psi}_n = \mathcal{S}_n\Psi_{n+1} + \mathcal{T}_n\Psi_n\,. \tag{56}$$

There holds the equivalence between (55) and the difference-differential equation

$$\tilde{A}\widetilde{\Psi}'_n = \tilde{\mathcal{M}}_n\widetilde{\Psi}_n + \tilde{\mathcal{N}}_n\widetilde{\Psi}_{n-1} \tag{57}$$

with $\tilde{A} = A\pi$, and the matrices $\tilde{\mathcal{M}}_n, \tilde{\mathcal{N}}_n$ given by

$$\tilde{\mathcal{M}}_n = \left(\widehat{\mathcal{S}}_n + \frac{1}{\gamma_{n+1}}\tilde{\mathcal{N}}_n\mathcal{T}_{n-1}\right)\mathcal{S}_n^{-1}\,, \quad \tilde{\mathcal{N}}_n = \left(\widehat{\mathcal{T}}_n\mathcal{T}_n^{-1} - \widehat{\mathcal{S}}_n\mathcal{S}_n^{-1}\right)\mathcal{V}_n^{-1}\,, \tag{58}$$

where

$$\widehat{\mathcal{S}}_n = \pi\left(A\mathcal{S}_n' + \mathcal{S}_n\mathcal{M}_{n+1} - \frac{1}{\gamma_{n+1}}\mathcal{T}_n\mathcal{N}_n\right) - A\pi'\mathcal{S}_n\,, \tag{59}$$

$$\widehat{\mathcal{T}}_n = \pi\left(A\mathcal{T}_n' + \mathcal{S}_n\mathcal{N}_{n+1} + \mathcal{T}_n\left(\mathcal{M}_n + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\mathcal{N}_n\right)\right) - A\pi'\mathcal{T}_n\,, \tag{60}$$

$$\mathcal{V}_n = \frac{1}{\gamma_{n+1}}\mathcal{T}_{n-1}\mathcal{S}_n^{-1} + \left(\mathcal{S}_{n-1} + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\mathcal{T}_{n-1}\right)\mathcal{T}_n^{-1}\,. \tag{61}$$

Proof: Take derivatives in (56), multiply by the polynomial A, and use (10), thus obtaining

$$A\pi'\widetilde{\Psi}_n + A\pi\widetilde{\Psi}_n' = \mathcal{S}_{n,1}\Psi_{n+1} + \mathcal{T}_{n,1}\Psi_n\,, \tag{62}$$

with

$$\mathcal{S}_{n,1} = A\mathcal{S}_n' + \mathcal{S}_n\mathcal{M}_{n+1} - \frac{1}{\gamma_{n+1}}\mathcal{T}_n\mathcal{N}_n,$$

$$\mathcal{T}_{n,1} = A\mathcal{T}_n' + \mathcal{S}_n\mathcal{N}_{n+1} + \mathcal{T}_n\left(\mathcal{M}_n + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\mathcal{N}_n\right)\,.$$

Multiply (62) by π and use again (56), thus obtaining

$$\hat{A}\widetilde{\Psi}_n' = \widehat{\mathcal{S}}_n\Psi_{n+1} + \widehat{\mathcal{T}}_n\Psi_n\,, \tag{63}$$

with

$$\hat{A} = A\pi^2,\ \ \widehat{\mathcal{S}}_n = \pi\mathcal{S}_{n,1} - A\pi'\mathcal{S}_n,\ \ \widehat{\mathcal{T}}_n = \pi\mathcal{T}_{n,1} - A\pi'\mathcal{T}_n\,.$$

Let us now deduce (58). We consider the relation (57) with $\tilde{A} = A\pi$, that is,

$$A\pi\widetilde{\Psi}_n' = \tilde{\mathcal{M}}_n\widetilde{\Psi}_n + \tilde{\mathcal{N}}_n\widetilde{\Psi}_{n-1}\,.$$

Multiplying the above equation by π and using (56) and (63), as well as the recurrence relation (8), we get

$$\mathcal{M}_{n,1}\Psi_{n+1} = \mathcal{M}_{n,2}\Psi_n\,,$$

with

$$\mathcal{M}_{n,1} = \widehat{\mathcal{S}}_n - \tilde{\mathcal{M}}_n\mathcal{S}_n + \frac{1}{\gamma_{n+1}}\tilde{\mathcal{N}}_n\mathcal{T}_{n-1}\,, \tag{64}$$

$$\mathcal{M}_{n,2} = -\widehat{\mathcal{T}}_n + \tilde{\mathcal{M}}_n\mathcal{T}_n + \tilde{\mathcal{N}}_n\mathcal{S}_{n-1} + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\tilde{\mathcal{N}}_n\mathcal{T}_{n-1}\,. \tag{65}$$

Taking into account the structure of the matrices defining $\mathcal{M}_{n,1}$ and $\mathcal{M}_{n,2}$, we conclude that $\mathcal{M}_{n,1}$ and $\mathcal{M}_{n,2}$ are null matrices. Hence, we obtain that $\tilde{\mathcal{M}}_n, \tilde{\mathcal{N}}_n$ are given by (58). ■

4.2. The semi-classical class: the Christoffel and Geronimus transformations. The semi-classical class is closed under the Christoffel and Geronimus transformations [15].

Recall that, whenever dealing with weights, there holds the equivalence between the semi-classical character of w, say $w'/w = C/A$, and the differential system (11) for the corresponding $\mathcal{Y}_n$,

$$A\mathcal{Y}_n' = \mathcal{B}_n\mathcal{Y}_n\,, \quad n \geq 1\,,$$

where $\mathcal{B}_n$ is a matrix of polynomial entries having uniformly bounded degrees [2, Theorem 2].

Let us now consider $\tilde{w}$, a modification of w under the Christoffel or Geronimus transformation. As $\tilde{w}$ is semi-classical, there holds a differential system for the corresponding $\tilde{\mathcal{Y}}_n = \begin{bmatrix} \tilde{P}_{n+1} & \tilde{q}_{n+1}/\tilde{w} \\ \tilde{P}_n & \tilde{q}_n/\tilde{w} \end{bmatrix}$,

$$\tilde{A}\tilde{\mathcal{Y}}_n' = \tilde{\mathcal{B}}_n\tilde{\mathcal{Y}}_n\,, \quad n \geq 1\,. \tag{66}$$

In what follows we see how $\tilde{\mathcal{B}}_n$ in (66) and the transfer matrices of $\{\tilde{\mathcal{Y}}_n\}_{n\geq 0}$, henceforth denoted by $\tilde{\mathcal{A}}_n$, relate to $\mathcal{B}_n$ in (11) and to the transfer matrices $\mathcal{A}_n$ of $\{\mathcal{Y}_n\}_{n\geq 0}$.

Theorem 6. *Let $\{P_n\}_{n\geq 0}$ be a SMOP related to a semi-classical weight w with the corresponding $\{\mathcal{Y}_n\}_{n\geq 0}$ satisfying (11),*

$$A\mathcal{Y}_n' = \mathcal{B}_n\mathcal{Y}_n\,, \quad n \geq 1\,.$$

Let $\mathcal{A}_n$ be the transfer matrix of $\mathcal{Y}_n$.
a) Christoffel case.
a.1) The sequence $\{\tilde{\mathcal{Y}}_n\}_{n\geq 0}$ related to the modified weight (12), $\tilde{w}(x) = (x-\alpha)w(x)$, satisfies

$$(x-\alpha)\tilde{\mathcal{Y}}_n = \mathcal{C}_n\mathcal{Y}_n\,, \quad \mathcal{C}_n = \mathcal{A}_{n+1} - \boldsymbol{A}_n\,, \quad n \geq 1\,, \tag{67}$$

where $\boldsymbol{A}_n = \begin{bmatrix} a_{n+1} & 0 \\ 0 & a_n \end{bmatrix}$, $a_n = P_{n+1}(\alpha)/P_n(\alpha)$;
a.2) $\{\tilde{\mathcal{Y}}_n\}_{n\geq 0}$ satisfies a linear system (66), $\tilde{A}\tilde{\mathcal{Y}}_n' = \tilde{\mathcal{B}}_n\tilde{\mathcal{Y}}_n\,,\ n \geq 1\,,$ where

$$\tilde{A} = (x-\alpha)A\,, \quad \tilde{\mathcal{B}}_n = \left((x-\alpha)(A\mathcal{C}_n' + \mathcal{C}_n\mathcal{B}_n) - A\mathcal{C}_n\right)\mathcal{C}_n^{-1}\,; \tag{68}$$

a.3) The transfer matrices of $\tilde{\mathcal{Y}}_n$ and $\mathcal{Y}_n$ are related through

$$\tilde{\mathcal{A}}_n = \mathcal{C}_n \mathcal{A}_n \mathcal{C}_{n-1}^{-1}\,, \quad n \geq 1\,. \tag{69}$$

b) Geronimus case.

b.1) The sequence $\{\tilde{\mathcal{Y}}_n\}_{n\geq 0}$ related to the modified weight (14), $\tilde{w}(x) = \dfrac{w(x)}{(x-\alpha)}$, satisfies

$$\tilde{\mathcal{Y}}_n = \mathcal{G}_n \mathcal{Y}_n\,, \quad \mathcal{G}_n = \mathcal{A}_{n+1} - \boldsymbol{B}_n\,, \tag{70}$$

where $\boldsymbol{B}_n = \begin{bmatrix} b_{n+1} & 0 \\ 0 & b_n \end{bmatrix}$, $b_n = q_n(\alpha)/q_{n-1}(\alpha)$;

b.2) $\{\tilde{\mathcal{Y}}_n\}_{n\geq 0}$ satisfies a linear system (66), $\tilde{A}\tilde{\mathcal{Y}}_n' = \tilde{\mathcal{B}}_n \tilde{\mathcal{Y}}_n\,, \quad n \geq 1\,,$ where

$$\tilde{A} = (x-\alpha)A\,, \quad \tilde{\mathcal{B}}_n = (x-\alpha)\left(A\mathcal{G}_n' + \mathcal{G}_n\mathcal{B}_n\right)\mathcal{G}_n^{-1}\,; \tag{71}$$

b.3) The transfer matrices of $\tilde{\mathcal{Y}}_n$ and $\mathcal{Y}_n$ are related through

$$\tilde{\mathcal{A}}_n = \mathcal{G}_n \mathcal{A}_n \mathcal{G}_{n-1}^{-1}\,. \tag{72}$$

Proof: Christoffel case.
a.1) The sequence of functions of the second kind related to $\tilde{w}$, $\{\tilde{q}_n\}$, satisfies

$$\tilde{q}_n(x) = q_{n+1}(x) - a_n q_n(x)\,, \quad n \geq 1\,. \tag{73}$$

Thus,

$$(x-\alpha)\tilde{\mathcal{Y}}_n = \mathcal{Y}_{n+1} - \mathbf{A}_n \mathcal{Y}_n\,.$$

The use of (9), $\mathcal{Y}_{n+1} = \mathcal{A}_{n+1}\mathcal{Y}_n$, in the equation above yields (67).

a.2) Take derivatives in (67), then multiply the resulting equation by $(x-\alpha)A$ and use (67) again, thus obtaining

$$(A\mathcal{C}_n + \tilde{\mathcal{B}}_n\mathcal{C}_n)\mathcal{Y}_n = (x-\alpha)\left(A\mathcal{C}_n' + \mathcal{C}_n\mathcal{B}_n\right)\mathcal{Y}_n\,.$$

Hence, we have the linear system (66) with the data (68).

a.3) Relation (69) follows from the use of the recurrence relation for $\mathcal{Y}_n$ as well as for $\tilde{\mathcal{Y}}_n$, into (67).

Geronimus case.
b.1) The sequence of functions of the second kind related to $\tilde{w}$, $\{\tilde{q}_n\}$, satisfies

$$(x-\alpha)\tilde{q}_n(x) = q_n(x) - b_n q_{n-1}(x)\,, \quad n \geq 1\,.$$

Thus,

$$\tilde{\mathcal{Y}}_n = \mathcal{Y}_{n+1} - \mathbf{B}_n \mathcal{Y}_n\,.$$

The use of (9), $\mathcal{Y}_{n+1} = \mathcal{A}_{n+1}\mathcal{Y}_n$, in the equation above yields (70).

b.2) Take derivatives in (70), then multiply the resulting equation by $(x-\alpha)A$ and use (70) again, thus obtaining

$$\tilde{\mathcal{B}}_n \mathcal{G}_n \mathcal{Y}_n = (x-\alpha)\left(A\mathcal{G}_n' + \mathcal{G}_n \mathcal{B}_n\right)\mathcal{Y}_n \,.$$

Hence, we have the linear system (66) with the data (71).

b.3) Relation (72) follows from the use of the recurrence relation for $\mathcal{Y}_n$ as well as for $\tilde{\mathcal{Y}}_n$ into (70).

■

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