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ON SEMICLASSICAL ORTHOGONAL POLYNOMIALS VIA POLYNOMIAL MAPPINGS

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Abstract: Orthogonal polynomials via polynomial mappings in the framework of the semiclassical class are considered. It is proved that this class is stable under polynomial transformations. Several consequences of this fact are deduced. As an application cubic transformations for semiclassical orthogonal polynomials of class at most 2 are analyzed, recovering and extending some results proved recently for class 1, and producing new examples of semiclassical orthogonal polynomials of classes 1 and 2.

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1. Introduction

The semiclassical orthogonal polynomials (OP) are the sole creation of the French mathematician J. Shohat [25]. They arise as a natural extension of the well known classical OP of Hermite, Laguerre, Jacobi, and Bessel, having been a focus of great research activity since the 1980's, specially after the contributions of another French mathematician, P. Maroni [18, 19, 20, 21]. As expected, many aspects of the theory were developed by seeking properties which generalize the properties of the classical OP, e.g., a second order homogeneous linear differential equation with polynomial coefficients, the Pearson's (distributional) differential equation, and the so-called structure relations. Nowadays the subject is a very active research area both on the setting of continuous OP as well as on the discrete $\Delta_\omega-$OP and $q-$OP ones, not only from a theoretical point of view —were the algebraic aspects of the theory still play the central role—, but also because they arise naturally in

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connection with other branches of Mathematics and Mathematical Physics. For instance, the connection of semiclassical OP with Painlevé equations emerged as an important topic of research. On the other hand, they appear as a breeding ground for the construction of explicit examples of orthogonal polynomial sequences (OPS) out of the classical family of OP.

The aim of our work is to study semiclassical OP via polynomial mappings, as a natural extension of our results discussed in [10]. The later is a subject that received special attention along the last decades. Indeed, the theory of OP via polynomial mappings begun with the discovery of the well known quadratic relation involving Hermite and Laguerre OP of parameters $\pm\frac{1}{2}$, and the interest on the subject grew after a question asked by Chihara about the existence of OPS such that one of the OPS is obtained from the other one via a cubic transformation. The answer to this question was given by Barrucand and Dickinson in [2], and after this work several papers appeared in the literature involving OP obtained via general polynomials mappings, the most influential one being the paper [11] by Geronimo and Van Assche, where an important connection with the so-called sieved OP discovered by Al-Salam, Allaway, and Askey [1] was pointed out.

The structure of the paper is as follows. In Section 2 we compile some known results on OP, including a brief review on semiclassical OP as well as the relevant results needed along this work on OP and polynomial mappings. In Section 3 we state our main results. We complement the study contained in [10] by considering here OP via polynomial mappings in the framework of the semiclassical class. We also present relations involving the Stieltjes formal series of the regular functionals with respect to which the sequences of polynomials appearing in the polynomial mapping are orthogonal, and then we use these relations to show that semiclassical families are stable under polynomial transformations. We further deduce several consequences of this fact, e.g. stating precise results about the class of a semiclassical OPS obtained via a polynomial transformation on another semiclassical OPS. In Section 4 we determine all the semiclassical OPS, $\{p_n\}_{n\geq 0}$, of class at most 2 obtained via cubic transformations of the form

$$p_{3n}(x) = q_n\big(x^3 + px + r\big)\ , \quad p, r \in \mathbb{C}\ ,$$

requiring that $\{q_n\}_{n\geq 0}$ be a classical OPS. We give a complete description of such OPS $\{p_n\}_{n\geq 0}$ (of class at most 2) and show that $\{q_n\}_{n\geq 0}$ is necessarily a Jacobi or Laguerre family, improving and extending the results given recently in [28] where the authors require $\{p_n\}_{n\geq 0}$ to be semiclassical of class 1. Finally, we summarize in Tables 3 and 4 new examples of semiclassical OPS of classes 1 and 2.

2. Background

In this section we recall some basic facts concerning the general theory of OP that will be needed along the remaining sections.

2.1. Basic tools. The linear space of all polynomials with complex coefficients will be denoted by $\mathcal{P}$, and its algebraic dual by $\mathcal{P}^*$, i.e., $\mathcal{P}^*$ is the space of all linear functionals $\mathbf{u}:\mathcal{P}\to\mathbb{C}$. We recall that $\mathcal{P}$ may be carried with a topology for which all the linear functionals become continuous—and hence the equality $\mathcal{P}'=\mathcal{P}^*$ holds, where $\mathcal{P}'$ denotes the topological dual of $\mathcal{P}$—, namely the topology of the strict inductive limit of the spaces $\mathcal{P}_n$, being $\mathcal{P}_n$ the linear subspace of $\mathcal{P}$ of all polynomials of degree at most n endowed with any norm. We also define $\mathcal{P}_{-1}:=\{0\}$, the trivial subspace. For details (and rigorous justifications) we refer the reader to the works [18, 19, 20, 21] by Maroni (see also [24]) and [4] by Brezinski and Maroni. All the elements needed concerning strict inductive limit topologies may be found e.g. in the books by Trèves [29], Reed and Simon [26], and Lax [13].

We denote by $\langle\mathbf{u},f\rangle$ the action of a functional $\mathbf{u}\in\mathcal{P}'$ over $f\in\mathcal{P}$, and by

$$u_n:=\langle\mathbf{u},x^n\rangle\quad(n\in\mathbb{N}_0)$$

the moment of order n of $\mathbf{u}$. As usual, given $\mathbf{u}\in\mathcal{P}'$ and $\phi\in\mathcal{P}$, we define the (distributional) derivative of $\mathbf{u}$ and the left-multiplication of $\mathbf{u}$ by ϕ, as the functionals $D\mathbf{u},\phi\mathbf{u}\in\mathcal{P}'$ defined by

$$\langle D\mathbf{u},f\rangle:=-\langle\mathbf{u},f'\rangle\ ,\quad\langle\phi\mathbf{u},f\rangle:=\langle\mathbf{u},\phi f\rangle\ ,\quad f\in\mathcal{P}\ .$$

A sequence $\{f_n\}_{n\geq 0}$ such that $f_n\in\mathcal{P}_n\backslash\mathcal{P}_{n-1}$ for every n will be called a simple set of polynomials. The dual basis associated with a simple set $\{f_n\}_{n\geq 0}$ is the sequence $\{\mathbf{e}_n\}_{n\geq 0}$, with $\mathbf{e}_n\in\mathcal{P}'$, such that

$$\langle\mathbf{e}_n,f_j\rangle:=\delta_{n,j}\quad(n,j\in\mathbb{N}_0)\ ,$$

where $\delta_{n,j}$ is the Kronecker symbol.

In the next we recall the definition of OPS and some of the main facts concerning such sequences. Let $\mathbf{u}\in\mathcal{P}'$ and $\{p_n\}_{n\geq 0}\subset\mathcal{P}$. We say that

$\{p_n\}_{n\geq 0}$ is an OPS with respect to $\mathbf{u}$ if it is a simple set of polynomials and the orthogonality conditions

$$\langle \mathbf{u}, p_n p_m \rangle = k_n \delta_{n,m} \quad (n, m \in \mathbb{N}_0)$$

hold, where $\{k_n\}_{n\geq 0}$ is a sequence of nonzero complex numbers. In such case, $\mathbf{u}$ is called regular or quasi-definite, and $\{p_n\}_{n\geq 0}$ the associated OPS. If the leading coefficient of each p_n is equal to 1, i.e., $p_n(x) = x^n$+lower degree terms, we will refer to $\{p_n\}_{n\geq 0}$ as a monic OPS. It is well know that any monic OPS $\{p_n\}_{n\geq 0}$ is characterized by a three-term recurrence relation of the form

$$p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x) \ ,$$

with initial conditions $p_0(x) := 1$ and $p_1(x) := x - \beta_0$, being $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ sequences of complex numbers such that $\gamma_n \neq 0$ for all n. In particular, if $\beta_n \in \mathbb{R}$ and $\gamma_n > 0$ for all n, then the linear functional $\mathbf{u}$ with respect to which $\{p_n\}_{n\geq 0}$ is an OPS admits an integral representation such as

$$\langle \mathbf{u}, f \rangle = \int_{\mathbb{R}} f \, \mathrm{d}\mu \quad (f \in \mathcal{P}) \ ,$$

where μ is a nontrivial positive Borel measure with finite moments of all orders (i.e., $\int_{\mathbb{R}} |x|^n \, \mathrm{d}\mu < \infty$ for all $n \in \mathbb{N}_0$). In such a case, we say that $\{p_n\}_{n\geq 0}$ is an OPS in the positive-definite sense, or, equivalently, $\mathbf{u}$ is regular in the positive-definite sense. We also recall that if $\{p_n\}_{n\geq 0}$ is a monic OPS with respect to $\mathbf{u} \in \mathcal{P}'$ (not necessarily in the positive-definite sense), then the dual basis $\{\mathbf{a}_n\}_{n\geq 0}$ associated with $\{p_n\}_{n\geq 0}$ is explicitly given by

$$\mathbf{a}_n = \frac{p_n}{\langle \mathbf{u}, p_n^2 \rangle} \, \mathbf{u} \ .$$

Then the set equality $\mathcal{P}' = \mathcal{P}^*$ mentioned above allows us to express any linear functional $\mathbf{v} \in \mathcal{P}^*$ in terms of the dual basis $\{\mathbf{a}_n\}_{n\geq 0}$ as

$$\mathbf{v} = \sum_{n=0}^{\infty} \langle \mathbf{v}, p_n \rangle \mathbf{a}_n \ ,$$

in the sense of the weak dual topology in $\mathcal{P}'$. This is a fundamental property in the framework of the algebraic theory of orthogonal polynomials founded by Maroni at the end of the last century. Another important tool is the formal Stieltjes series associated with a given regular linear functional $\mathbf{u} \in \mathcal{P}'$

defined by

$$S_{\mathbf{u}}(z) := -\sum_{n=0}^{\infty} \frac{u_n}{z^{n+1}} \; .$$

$S_{\mathbf{u}}(z)$ is a representation for the moments u_n of $\mathbf{u}$. Thus, since the sequence $\{u_n\}_{n\geq 0}$ characterizes $\mathbf{u}$, then so does $S_{\mathbf{u}}(z)$. Formally, $S_{\mathbf{u}}(z)$ admits the representation

$$S_{\mathbf{u}}(z) = \left\langle \mathbf{u}_x, \frac{1}{x-z} \right\rangle \,,$$

where $\mathbf{u}_x$ means that $\mathbf{u}$ acts on functions of the variable x.

2.2. Semiclassical OP. A functional $\mathbf{u} \in \mathcal{P}^*$ is called semiclassical if it is regular and there exist two nonzero polynomials ϕ and ψ such that

$$\deg \psi \geq 1 \tag{2.1}$$

and $\mathbf{u}$ satisfies the generalized Pearson distributional differential equation

$$D(\phi \mathbf{u}) = \psi \mathbf{u} \; . \tag{2.2}$$

If $\{p_n\}_{n\geq 0}$ is an OPS with respect to a semiclassical functional then $\{p_n\}_{n\geq 0}$ is called a semiclassical OPS. The following is a useful well known criterion.

Proposition 2.1. *Let $\mathbf{u} \in \mathcal{P}^*$. Suppose that $\mathbf{u}$ is regular. Then, $\mathbf{u}$ is semiclassical if and only if there exist two polynomials ϕ and ψ, with at least one of them nonzero, such that (2.2) holds. Moreover, under these conditions, necessarily both ϕ and ψ are nonzero and ψ satisfies (2.1).*

Proof: "Necessity" is obvious, so we only need to prove "sufficiency". Let $\mathbf{u}$ fulfils (2.2), with $\phi, \psi \in \mathcal{P}$. Suppose that $\psi \equiv 0$, so that $D\,(\phi \mathbf{u}) = 0$. If $\phi \not\equiv 0$, setting $r := \deg \phi$ and being $a (\neq 0)$ the leading coefficient of ϕ, we would have $\langle \mathbf{u}, p_r^2 \rangle = a^{-1} \langle \phi \mathbf{u}, p_r \rangle = -a^{-1} \langle D\,(\phi \mathbf{u})\,, \int p_r \rangle = 0$, violating the regularity of $\mathbf{u}$. We conclude that $\psi \equiv 0$ implies $\phi \equiv 0$. Suppose now that $\phi \equiv 0$, so that $\psi \mathbf{u} = 0$. If $\psi \not\equiv 0$, then setting $t := \deg \psi$ and being $b (\neq 0)$ the leading coefficient of ψ, we would have $\langle \mathbf{u}, p_t^2 \rangle = b^{-1} \langle \psi \mathbf{u}, p_t \rangle = 0$, violating again the regularity of $\mathbf{u}$. We conclude that $\phi \equiv 0$ implies $\psi \equiv 0$. Finally, suppose that $\psi \equiv \text{const.} = c \neq 0$. Then $\langle \mathbf{u}, 1 \rangle = c^{-1} \langle \psi \mathbf{u}, 1 \rangle = c^{-1} \langle D(\phi \mathbf{u}), 1 \rangle = 0$, violating once again the regularity of $\mathbf{u}$. Hence ψ satisfies condition (2.1). ■

If $\mathbf{u}$ is a semiclassical functional, the class of $\mathbf{u}$ is the nonnegative integer number

$$s := \min_{(\phi,\psi) \in \mathcal{A}_{\mathbf{u}}} \max\{\deg \phi - 2, \deg \psi - 1\} \,, \tag{2.3}$$

p_n	ϕ	ψ	regularity conditions
H_n	1	$-2x$	
$L_n^{(\alpha)}$	x	$-x+\alpha+1$	$\alpha\neq -n \ , \ n\geq 1$
$P_n^{(\alpha,\beta)}$	$1-x^2$	$-(\alpha+\beta+2)x+\beta-\alpha$	$\alpha\neq -n \ , \ \beta\neq -n \ , \ \alpha+\beta+1\neq -n \ , \ n\geq 1$
$B_n^{(\alpha)}$	x^2	$(\alpha+2)x+2$	$\alpha\neq -n \ , \ n\geq 2$

TABLE 1. Classification and canonical forms of the classical OPS.

where $\mathcal{A}_{\mathbf{u}}$ is the set of all pairs (ϕ,ψ) of nonzero polynomials such that (2.1) and (2.2) hold. The pair $(\phi,\psi)\in\mathcal{A}_{\mathbf{u}}$ where the class of $\mathbf{u}$ is attained is unique. If $\{p_n\}_{n\geq 0}$ is an OPS with respect to a semiclassical functional of class s then $\{p_n\}_{n\geq 0}$ is called a semiclassical OPS of class s. In particular, when $s=0$ (so that $\deg\phi\leq 2$ and $\deg\psi=1$) one obtains, up to an affine change of variables, the four families of classical OPS: Hermite, H_n; Laguerre, $L_n^{(\alpha)}$; Jacobi, $P_n^{(\alpha,\beta)}$; and Bessel, $B_n^{(\alpha)}$. Table 1 summarizes the corresponding canonical forms for the pairs (ϕ,ψ) usually considered in the literature.

Besides the generalized Pearson distributional differential equation (2.2), semiclassical functionals have several characterizations. For our purposes we need the following one, which involves the Stieltjes formal series: $\mathbf{u}\in\mathcal{P}^*$ is semiclassical if and only if it is regular and the associated Stieltjes formal series satisfies formally the first order linear differential equation

$$\phi(z)S_{\mathbf{u}}'(z)=C(z)S_{\mathbf{u}}(z)+D(z)\ , \tag{2.4}$$

where ϕ is a nonzero polynomial, and C and D are polynomials. Moreover, if $\mathbf{u}$ satisfies (2.2), then we may take in (2.4) the same polynomial ϕ appearing in (2.2), being the polynomials C and D given by

$$C=\psi-\phi'\ ,\quad D=(\mathbf{u}\theta_0\psi)-(\mathbf{u}\theta_0\phi)'\ ,$$

where, for any $\pi\in\mathcal{P}$, the polynomials $\theta_0\pi$ and $\mathbf{u}\pi$ are defined by

$$\theta_0\pi(x):=\frac{\pi(x)-\pi(0)}{x}\ ,\quad \mathbf{u}\pi(x):=\left\langle\mathbf{u}_y,\frac{x\pi(x)-y\pi(y)}{x-y}\right\rangle .$$

Furthermore, if the polynomials ϕ, C, and D appearing in (2.4) are co-prime, then the class of $\mathbf{u}$ is given by

$$s=\max\{\deg C-1,\deg D\}\ . \tag{2.5}$$

p_n	C	D
H_n	$-2x$	$-2u_0$
$L_n^{(\alpha)}$	$-x+\alpha$	$-u_0$
$P_n^{(\alpha,\beta)}$	$-(\alpha+\beta)x+\beta-\alpha$	$-(\alpha+\beta+1)u_0$
$B_n^{(\alpha)}$	$\alpha x+2$	$(\alpha+1)u_0$

TABLE 2. C and D corresponding to the canonical forms in Table 1.

Thus, the formal differential equation (2.4) appears as an efficient tool in order to determine the class of a semiclassical functional.

Table 2 gives the polynomials C and D appearing in the differential equation fulfilled by the Stieltjes series for the classical functionals ($s=0$) corresponding to the canonical forms given in Table 1.

Remarks 2.1. *Often, a semiclassical functional is defined requiring the pair (ϕ,ψ) appearing in Pearson's equation (2.2) to be an admissible pair, meaning that, whenever $\deg\phi = 1+\deg\psi$ the leading coefficient of ψ cannot be a negative integer multiple of the leading coefficient of ϕ. We did not included the admissibility condition in the definition of semiclassical functional since there are regular functionals $\mathbf{u}$ fulfilling (2.2) with (ϕ,ψ) a non-admissible pair (see e.g. [1, 22]). We note, however, that for a classical functional the admissibility condition holds necessarily, a fact known as early as the work of Geronimus [12].*

2.3. OP via polynomial mappings. The study of polynomial mappings in the framework of the theory of OP is a very attractive subject. Special attention has been paid to quadratic and cubic transformations (see e.g. [2, 7, 14, 15, 16, 17, 28]). After the important work by Bessis and Moussa [3], the subject has been treated in full generality by Geronimo and Van Assche [11], Charris, Ismail, and Monsalve [5, 6] (using the so-called blocks of OP), and by Peherstorfer [23]. More recently [9, 10] characterization results have been stated in order to ensure that a given OPS becomes obtained from another one via a polynomial mapping. In order to describe this mapping, let $\{p_n\}_{n\geq 0}$ be a monic OPS, characterized by its three-term recurrence relation, convenably expressed in terms of blocks as [5, 6]

$$\begin{gathered}(x-b_n^{(j)})p_{nk+j}(x)=p_{nk+j+1}(x)+a_n^{(j)}p_{nk+j-1}(x)\\ \qquad\qquad (j=0,1,\ldots,k-1\,;\quad n=0,1,2,\ldots)\,,\end{gathered}\tag{2.6}$$

satisfying initial conditions $p_{-1}(x):=0$ and $p_0(x):=1$. Without loss of generality, we take $a_0^{(0)}:=1$. In general, the $a_n^{(j)}$'s and the $b_n^{(j)}$'s are complex numbers with $a_n^{(j)}\neq 0$ for all n and j. With these numbers we may construct determinants $\Delta_n(i,j;x)$ as in [5, 6], so that

$$\Delta_n(i,j;x):=\begin{cases}0 & \text{if } j<i-2\\ 1 & \text{if } j=i-2\\ x-b_n^{(i-1)} & \text{if } j=i-1\end{cases}\tag{2.7}$$

and, if $j\geq i\geq 1$,

$$\Delta_n(i,j;x):=\begin{vmatrix} x-b_n^{(i-1)} & 1 & 0 & \ldots & 0 & 0\\ a_n^{(i)} & x-b_n^{(i)} & 1 & \ldots & 0 & 0\\ 0 & a_n^{(i+1)} & x-b_n^{(i+1)} & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & x-b_n^{(j-1)} & 1\\ 0 & 0 & 0 & \ldots & a_n^{(j)} & x-b_n^{(j)}\end{vmatrix},\tag{2.8}$$

for every $n\in\mathbb{N}_0$. Taking into account that $\Delta_n(i,j;\cdot)$ is a polynomial whose degree may exceed k, and since in (2.6) the $a_n^{(j)}$'s and $b_n^{(j)}$'s are defined only for $0\leq j\leq k-1$, we adopt the convention

$$b_n^{(k+j)}:=b_{n+1}^{(j)}\,,\quad a_n^{(k+j)}:=a_{n+1}^{(j)}\quad (i,j,n\in\mathbb{N}_0)\,,\tag{2.9}$$

and so the following useful equality holds:

$$\Delta_n(k+i,k+j;x)=\Delta_{n+1}(i,j;x)\,.\tag{2.10}$$

Theorem 2.1. [10, Theorem 2.1] *Let $\{p_n\}_{n\geq 0}$ be a monic OPS characterized by the general blocks of recurrence relations (2.6). Fix $r_0\in\mathbb{C}$, $k\in\mathbb{N}$, and $m\in\mathbb{N}_0$, with $0\leq m\leq k-1$. Then, there exist polynomials π_k and θ_m of degrees k and m (resp.) and a monic OPS $\{q_n\}_{n\geq 0}$ such that $q_1(0)=-r_0$ and*

$$p_{kn+m}(x)=\theta_m(x)\,q_n(\pi_k(x))\,,\quad n=0,1,2,\ldots\tag{2.11}$$

if and only if the following four conditions hold:

(i) $b_n^{(m)}$ *is independent of* n *for* $n \geq 0$*;*
(ii) $\Delta_n(m+2, m+k-1; x)$ *is independent of* n *for* $n \geq 0$ *and for every* x*;*
(iii) $\Delta_0(m+2, m+k-1; \cdot)$ *is divisible by* θ_m*, i.e., there exists a polynomial* η_{k-1-m} *with degree* $k-1-m$ *such that*

$$\Delta_0(m+2, m+k-1; x) = \theta_m(x)\, \eta_{k-1-m}(x)\,;$$

(iv) $r_n(x)$ *is independent of* x *for every* $n \geq 1$*, where*

$$\begin{aligned} r_n(x) := {} & a_n^{(m+1)} \Delta_n(m+3, m+k-1; x) - a_0^{(m+1)} \Delta_0(m+3, m+k-1; x) \\ & + a_n^{(m)} \Delta_{n-1}(m+2, m+k-2; x) - a_0^{(m)} \Delta_0(1, m-2; x)\, \eta_{k-1-m}(x)\ . \end{aligned}$$

Under such conditions, the polynomials θ_m *and* π_k *are explicitly given by*

$$\begin{aligned} &\pi_k(x) = \Delta_0(1, m; x)\, \eta_{k-1-m}(x) - a_0^{(m+1)}\, \Delta_0(m+3, m+k-1; x) + r_0\ , \\ &\theta_m(x) := \Delta_0(1, m-1; x) \equiv p_m(x)\ , \end{aligned} \tag{2.12}$$

and the monic OPS $\{q_n\}_{n\geq 0}$ *is generated by the three-recurrence relation*

$$q_{n+1}(x) = (x - r_n)\, q_n(x) - s_n q_{n-1}(x)\,, \quad n = 0, 1, 2, \ldots \tag{2.13}$$

with initial conditions $q_{-1}(x) = 0$ *and* $q_0(x) = 1$*, where*

$$r_n := r + r_n(0)\ , \quad s_n := a_n^{(m)} a_{n-1}^{(m+1)} \cdots a_{n-1}^{(m+k-1)}\ , \quad n = 1, 2, \ldots\ . \tag{2.14}$$

Moreover, for each $j = 0, 1, 2, \ldots, k-1$ *and all* $n = 0, 1, 2, \ldots$*,*

$$\begin{aligned} p_{kn+m+j+1}(x) = {} & \frac{1}{\eta_{k-1-m}(x)} \Big\{ \Delta_n(m+2, m+j; x)\, q_{n+1}(\pi_k(x)) \\ & + \Big(\textstyle\prod_{i=1}^{j+1} a_n^{(m+i)} \Big)\, \Delta_n(m+j+3, m+k-1; x)\, q_n(\pi_k(x)) \Big\}\ . \end{aligned} \tag{2.15}$$

Remarks 2.2. *Notice that for* $j = k-1$*, (2.15) reduces to (2.11).*

3. Polynomial mappings and semiclassical OP

In this section we start by introducing some operators and stating some preliminary lemmas. For fixed $\pi \in \mathcal{P}$, let $\sigma_\pi : \mathcal{P} \to \mathcal{P}$ be the linear operator defined by $\sigma_\pi[f] := f \circ \pi$ for every $f \in \mathcal{P}$, and define $\sigma_\pi^* : \mathcal{P}^* \to \mathcal{P}^*$ by duality. Therefore,

$$\sigma_\pi[f](x) := f(\pi(x))\ , \quad \langle \sigma_\pi^*(\mathbf{u}), f \rangle := \langle \mathbf{u}, \sigma_\pi[f] \rangle\ , \quad f \in \mathcal{P}\ ,\ \mathbf{u} \in \mathcal{P}'\ .$$

Lemma 3.1. [28] *For fixed $\phi, \pi \in \mathcal{P}$ and $\mathbf{u} \in \mathcal{P}'$, the following relations hold:*

$$\phi\, \sigma_\pi^*(\mathbf{u}) = \sigma_\pi^*\big(\sigma_\pi[\phi]\mathbf{u}\big)\,, \quad \sigma_\pi^*\left(D\mathbf{u}\right) = D\left(\sigma_\pi^*(\pi'\mathbf{u})\right)\,. \tag{3.1}$$

Lemma 3.2. *Let $\{p_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ be two monic OPS such that there exists monic polynomials π_k and θ_m of degrees k and m (resp.), with $0 \leq m \leq k-1$, satisfying*

$$p_{nk+m}(x) = \theta_m(x) q_n\left(\pi_k(x)\right)\,, \quad n = 0, 1, 2, \ldots\,.$$

Let $\mathbf{u}$ and $\mathbf{v}$ be the regular functionals in $\mathcal{P}'$ with respect to which $\{p_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ are orthogonal (resp.), and let $\{\mathbf{a}_n\}_{n\geq 0}$ and $\{\mathbf{b}_n\}_{n\geq 0}$ be the associated dual basis. Then the following relations hold:

$$\sigma_{\pi_k}^*\left(\theta_m\, \mathbf{a}_{nk+j}\right) = \delta_{j,m}\mathbf{b}_n \quad (j = 0, 1, \ldots, k-1,\ n = 0, 1, 2, \ldots)\,, \tag{3.2}$$

$$\sigma_{\pi_k}^*\left(\theta_m\, p_j\mathbf{u}\right) = \delta_{j,m}\, v_0^{-1}\left\langle \mathbf{u}, \theta_m^2\right\rangle \mathbf{v} \quad (j = 0, 1, \ldots, k-1)\,. \tag{3.3}$$

Proof: For fixed $j \in \{0, 1, \ldots, k-1\}$ and $n, \ell \in \mathbb{N}_0$, the equalities

$$\begin{aligned}\left\langle \sigma_{\pi_k}^*\left(\theta_m\, \mathbf{a}_{nk+j}\right), q_\ell\right\rangle &= \left\langle \mathbf{a}_{nk+j}, \theta_m \sigma_{\pi_k}[q_\ell]\right)\right\rangle = \left\langle \mathbf{a}_{nk+j}, p_{k\ell+m}\right\rangle \\ &= \delta_{j,m}\langle \mathbf{b}_n, q_\ell\rangle\end{aligned}$$

hold. This proves (3.2), since $\{q_\ell | \ell = 0, 1, 2, \ldots\}$ spans $\mathcal{P}$. Taking $n = 0$ in (3.2) and using the relations (see e.g. [21])

$$\mathbf{a}_j = \frac{p_j}{\langle \mathbf{u}, p_j^2\rangle}\mathbf{u}\,, \quad \mathbf{b}_j = \frac{q_j}{\langle \mathbf{v}, q_j^2\rangle}\mathbf{v} \quad (j \in \mathbb{N}_0)\,,$$

we obtain (3.3). ∎

In [10, Theorem 3.4], considering orthogonality in the positive-definite sense, so that $\{p_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ are orthogonal with respect to some positive Borel measures, the relation between the Stieltjes transforms of these measures has been stated. Without assuming *a priori* orthogonality in the positive-definite sense, we may state the following proposition, which gives the relation between the formal Stieltjes series corresponding to the linear functionals on $\mathcal{P}$ with respect to which $\{p_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ are monic OPS.

Lemma 3.3. *Under the conditions of Theorem 2.1, the formal Stieltjes series $S_{\mathbf{u}}(z) := -\sum_{n=0}^{\infty} u_n/z^{n+1}$ and $S_{\mathbf{v}}(z) := -\sum_{n=0}^{\infty} v_n/z^{n+1}$ associated* *with the regular moment linear functionals* $\mathbf{u}$ *and* $\mathbf{v}$ *with respect to which* $\{p_n\}_{n\geq 0}$ *and* $\{q_n\}_{n\geq 0}$ *are orthogonal (resp.) are related by*

$$S_{\mathrm{u}}(z) = \frac{u_0}{v_0}\,\frac{-v_0\Delta_0(2,m-1;z) + \left(\prod_{j=1}^{m} a_0^{(j)}\right)\,\eta_{k-1-m}(z)\,S_{\mathbf{v}}(\pi_k(z))}{\theta_m(z)}\,. \quad (3.4)$$

Proof: We begin by noticing the following relation (see e.g. [30])

$$\frac{p_m(x)}{x-z} = \frac{p_m(z)}{x-z} + \sum_{j=1}^{m} p_{j-1}(x)p_{m-j}^{(j)}(z)\,,$$

$\{p_n^{(j)}\}_{n\geq 0}$ being the sequence of the associated polynomials of order j, given by

$$p_n^{(j)}(x) := \frac{1}{\langle \mathbf{u}, p_{j-1}^2\rangle}\left\langle p_{j-1}(y)\mathbf{u}_y, \frac{p_{n+j}(x)-p_{n+j}(y)}{x-y}\right\rangle\,.$$

Therefore,

$$\begin{aligned}\left\langle \mathbf{u}_x, \frac{p_m(x)}{x-z}\right\rangle &= p_m(z)\left\langle \mathbf{u}_x, \frac{1}{x-z}\right\rangle + \sum_{j=1}^{m} p_{m-j}^{(j)}(z)\,\langle \mathbf{u}_x, p_{j-1}(x)\rangle \\ &= p_m(z)S_{\mathbf{u}}(z) + p_{m-1}^{(1)}(z)u_0 \\ &= \theta_m(z)S_{\mathbf{u}}(z) + \Delta_0(2,m-1;z)u_0\,.\end{aligned} \quad (3.5)$$

Setting

$$\rho_{k-1}(x,z) := \frac{\pi_k(x)-\pi_k(z)}{x-z} = \sum_{j=0}^{k-1} \alpha_{k-1-j}(z)p_j(x)\,, \quad (3.6)$$

then

$$\begin{aligned}
\left\langle \mathbf{u}_x, \frac{p_m(x)}{x-z} \right\rangle &= \sum_{j=0}^{k-1} \alpha_{k-1-j}(z) \left\langle \mathbf{u}_x, \frac{p_m(x)p_j(x)}{\pi_k(x)-\pi_k(z)} \right\rangle \\
&= \sum_{j=0}^{k-1} \alpha_{k-1-j}(z) \Big(-\sum_{n=0}^{\infty} \frac{\langle \mathbf{u}_x, p_m(x)p_j(x)\pi_k^n(x)\rangle}{\pi_k^{n+1}(z)} \Big) \\
&= \sum_{j=0}^{k-1} \alpha_{k-1-j}(z) \Big(-\sum_{n=0}^{\infty} \frac{\left\langle \sigma_{\pi_k}^* \left(\theta_m p_j \mathbf{u}\right), x^n \right\rangle}{\pi_k^{n+1}(z)} \Big) \qquad (3.7) \\
&= \alpha_{k-1-m}(z) \frac{\left\langle \mathbf{u}, \theta_m^2 \right\rangle}{v_0} \Big(-\sum_{n=0}^{\infty} \frac{\langle \mathbf{v}, x^n \rangle}{\pi_k^{n+1}(z)} \Big) \\
&= \alpha_{k-1-m}(z) \frac{\left\langle \mathbf{u}, \theta_m^2 \right\rangle}{v_0} S_{\mathbf{v}}\left(\pi_k(z)\right) ,
\end{aligned}$$

where in the fourth equality we have made use of relation (3.3). Now, from (3.6), $\alpha_{k-1-j}(z) = \langle \mathbf{u}_x, p_j(x)\rho_{k-1}(x,z)\rangle / \langle \mathbf{u}, p_j^2\rangle$ for all $j = 0, 1, \dots, k-1$, hence

$$\alpha_{k-1-m}(z) = \frac{\langle \mathbf{u}_x, p_m(x)\rho_{k-1}(x,z)\rangle}{\langle \mathbf{u}, p_m^2\rangle} = \frac{u_0 \left(\prod_{j=1}^m a_0^{(j)}\right) \eta_{k-1-m}(z)}{\langle \mathbf{u}, \theta_m^2\rangle} , \qquad (3.8)$$

where the last equality follows immediately after comparing the relation appearing in [10, Lemma 3.3] for $n = 1$, i.e.,

$$p_{k+m-1}^{(1)}(z) - p_{m-1}^{(1)}(z) q_1\left(\pi_k(z)\right) = \left(\prod_{j=1}^m a_0^{(j)}\right) \eta_{k-1-m}(z) ,$$

with the relation (see the proof of [10, Lemma 3.3], p. 2253)

$$p_{k+m-1}^{(1)}(z) = p_{m-1}^{(1)}(z) q_1\left(\pi_k(z)\right) + \frac{1}{u_0} \left\langle \mathbf{u}_x, \theta_m(x) \frac{\pi_k(x)-\pi_k(z)}{x-z} \right\rangle .$$

Therefore, (3.4) follows from (3.5), (3.7), and (3.8). ■

Lemma 3.4. *Let π_k and ϕ be monic polynomials, with $\deg \pi_k = k$, and let $\mathcal{B}_k := \{p_0, p_1, \dots, p_{k-1}\}$ be a simple set of polynomials. Then, to the triple $(\phi, \pi_k, \mathcal{B}_k)$ we may associate k polynomials $\phi_0, \phi_1, \dots, \phi_{k-1}$, with $\deg \phi_j \le \lfloor (\deg \phi)/k \rfloor$ for all $j = 0, 1, \dots, k-1$, such that*

$$\phi = \sum_{j=0}^{k-1} p_j \sigma_{\pi_k}[\phi_j] . \qquad (3.9)$$

Proof: Set $\deg\phi = kq + r$, with $q \in \mathbb{N}_0$ and $0 \le r \le k-1$. Since $\mathcal{P}$ is spanned by the set $\left\{p_j(x)\pi_k^i(x) \,|\, 0 \le j \le k-1; i \in \mathbb{N}_0\right\}$, then there exist uniquely determined scalars $c_{j,i}$ $(0 \le j \le k-1; 0 \le i \le q)$ such that $\phi(x) = \sum_{i=0}^{q}\sum_{j=0}^{k-1} c_{j,i}p_j(x)\pi_k^i(x)$, being $c_{r+1,q} = c_{r+2,q} = \cdots = c_{k-1,q} = 0$ if $r < q$. Therefore, defining

$$\phi_j(x) := \sum_{i=0}^{q} c_{j,i}x^i \quad (j = 0, 1, \ldots, k-1) \, ,$$

we obtain $\phi(x) = \sum_{j=0}^{k-1} p_j(x)\sum_{i=0}^{q} c_{j,i}\pi_k^i(x) = \sum_{j=0}^{k-1} p_j(x)\phi_j\big(\pi_k(x)\big)$, and thus the representation (3.9) follows. ■

Theorem 3.1. *Let $\{p_n\}_{n\ge0}$ and $\{q_n\}_{n\ge0}$ be monic OPS such that there exist monic polynomials π_k and θ_m of degrees k and m (resp.), with $0 \le m \le k-1$, satisfying*

$$p_{nk+m}(x) = \theta_m(x)q_n\left(\pi_k(x)\right) \, , \quad n = 0, 1, 2, \ldots \, . \tag{3.10}$$

Then the following holds:

- (i) *If $\{p_n\}_{n\ge0}$ is semiclassical of class s, then $\{q_n\}_{n\ge0}$ is semiclassical of class $\widetilde{s}$, with $\widetilde{s} \le \lfloor s/k \rfloor$.*
- (ii) *If $\{q_n\}_{n\ge0}$ is semiclassical of class $\widetilde{s}$, then $\{p_n\}_{n\ge0}$ is semiclassical of class s, with $s \le (\widetilde{s}+3)k-3$.*

Proof: Along the proof we denote by $\mathbf{u}$ and $\mathbf{v}$ be the regular functionals with respect to which $\{p_n\}_{n\ge0}$ and $\{q_n\}_{n\ge0}$ are orthogonal, respectively.

(i) Assume that $\{p_n\}_{n\ge0}$ is semiclassical of class s. Then there exist two nonzero polynomials Φ and Ψ, with $\deg\Psi \ge 1$, such that

$$D\left(\Phi\mathbf{u}\right) = \Psi\mathbf{u} \, , \tag{3.11}$$

being $s = \max\{\deg\Phi - 2, \deg\Psi - 1\}$. Set $\ell := 1 + \lfloor s/k \rfloor$ and $p := \ell k - 1 - s$. Then $p \in \mathbb{N}_0$, and multiplying both sides of (3.11) by $x^p\theta_m^2$ we obtain

$$D\left(x^p\theta_m^2\Phi\mathbf{u}\right) = \left\{\begin{array}{ll} \theta_m\left(\theta_m\Psi + 2\,\theta_m'\Phi\right)\mathbf{u} \, , & \text{if} \quad p = 0 \\ x^{p-1}\theta_m\left(x\,\theta_m\Psi + \left(2x\,\theta_m' + p\,\theta_m\right)\Phi\right)\mathbf{u} \, , & \text{if} \quad p \ge 1 \, . \end{array}\right. \tag{3.12}$$

Assume $p \ge 1$ (the proof is similar if $p = 0$). Applying the operator $\sigma_{\pi_k}^*$ to both sides of (3.12), and taking into account Lemma 3.1, we deduce

$$D\left(\sigma_{\pi_k}^*\left(\pi_k'\theta_m^2x^p\Phi\mathbf{u}\right)\right) = \sigma_{\pi_k}^*\left(x^{p-1}\theta_m\left(x\,\theta_m\Psi + \left(2x\,\theta_m' + p\,\theta_m\right)\Phi\right)\mathbf{u}\right) \, . \tag{3.13}$$

Next, Lemma 3.4 ensures the existence of polynomials f_j ($j=0,1,\ldots,k-1$), with each f_j not necessarily of degree j, fulfilling

$$\pi_k'\theta_m x^p\Phi = \sum_{j=0}^{k-1} p_j\sigma_{\pi_k}[f_j]\,. \tag{3.14}$$

Multiplying both sides of (3.14) by θ_m and considering the left multiplication of the resulting polynomials (in both sides) by the functional $\mathbf{u}$, then applying the operator $\sigma_{\pi_k}^*$ and using Lemma 3.2, we obtain

$$\sigma_{\pi_k}^*\left(\pi_k'\theta_m^2x^p\Phi\mathbf{u}\right) = \sum_{j=0}^{k-1}\sigma_{\pi_k}^*\left(\theta_m p_j\sigma_{\pi_k}[f_j]\mathbf{u}\right) = v_0^{-1}\langle\mathbf{u},p_m^2\rangle f_m\mathbf{v}\,. \tag{3.15}$$

Similarly, consider polynomials g_j ($j=0,1,\ldots,k-1$), with each g_j not necessarily of degree j, such that

$$x^{p-1}\big(x\,\theta_m\Psi + (2x\,\theta_m' + p\,\theta_m)\,\Phi\big) = \sum_{j=0}^{k-1} p_j\sigma_{\pi_k}[g_j]\,, \tag{3.16}$$

and proceed as before to deduce

$$\sigma_{\pi_k}^*\left(x^{p-1}\theta_m\left(x\,\theta_m\Psi + (2x\,\theta_m' + p\,\theta_m)\,\Phi\right)\mathbf{u}\right) = v_0^{-1}\langle\mathbf{u},p_m^2\rangle g_m\mathbf{v}\,. \tag{3.17}$$

From (3.13), (3.15), and (3.17), we obtain

$$D\left(f_m\mathbf{v}\right) = g_m\mathbf{v}\,. \tag{3.18}$$

Since $s=\max\{\deg\Phi-2,\deg\Psi-1\}$, then either $\deg\Phi=s+2$, or else $\deg\Phi<s+2$ and $\deg\Psi=s+1$. In the first situation, the polynomial appearing in the left-hand side of (3.14) has degree $(\ell+1)k+m$, hence from the right-hand side of (3.14) we deduce $\deg f_m=\ell+1\geq 2$. In the second situation, the polynomial appearing in the left-hand side of (3.16) has degree $\ell k+m$, hence $\deg g_m=\ell\geq 1$. We conclude that, in any situation, at least one of the polynomials f_m or g_m is different from zero. Thus, since $\mathbf{v}$ is regular and fulfills (3.18), it follows from Proposition 2.1 that $\mathbf{v}$ is semiclassical (being both f_m and g_m different from zero, and $\deg g_m\geq 1$). It remains to prove that the class $\widetilde{s}$ of $\mathbf{v}$ satisfies $\widetilde{s}\leq\lfloor s/k\rfloor$. Notice first that

$$m+k\deg f_m \leq \max_{0\leq j\leq k-1}\{j+k\deg f_j\} = \deg\left\{\pi_k'\,\theta_m\,x^p\,\Phi\right\} \leq (\ell+1)k+m\,,$$

where the equality is due to (3.14) and the last inequality holds since $p = \ell k - 1 - s$ and $\deg\Phi \leq s+2$, hence $\deg f_m \leq \ell+1$. In the same way, using (3.16), we deduce

$$m + k\deg g_m \leq \deg\left\{x^{p-1}\left(x\,\theta_m\Psi + (2x\,\theta_m' + p\,\theta_m)\,\Phi\right)\right\} \leq \ell k + m\;,$$

so $\deg g_m \leq \ell$. But, taking into account the conclusions obtained in the discussion above involving the two possible situations (concerning the degrees of Φ and Ψ), at least one of the equalities $\deg f_m = \ell+1$ or $\deg g_m = \ell$ holds. Therefore,

$$\max\{\deg f_m - 2, \deg g_m - 1\} = \ell - 1 = \lfloor\, s/k\,\rfloor\;, \tag{3.19}$$

and so from (3.18) we obtain $\widetilde{s} \leq \max\{\deg f_m - 2, \deg g_m - 1\} = \lfloor\, s/k\,\rfloor$.

(ii) Assume now that $\{q_n\}_{n\geq 0}$ is semiclassical of class $\widetilde{s}$. Then the associated formal Stieltjes series, $S_{\mathbf{v}}(z) := -\sum_{n=0}^{\infty} v_n z^{-n-1}$, satisfies the (formal) ordinary linear differential equation of the first order

$$\widetilde{\Phi}(z)S_{\mathbf{v}}'(z) = \widetilde{C}(z)S_{\mathbf{v}}(z) + \widetilde{D}(z)\,, \tag{3.20}$$

with $\widetilde{\Phi}$, $\widetilde{C}$, and $\widetilde{D}$ co-prime polynomials, $\widetilde{\Phi}$ nonzero, and $\widetilde{s} = \max\{\deg\widetilde{C} - 1, \deg\widetilde{D}\}$. From Lemma 3.3, we may write

$$S_{\mathbf{v}}(\pi_k(z)) = \frac{v_0\theta_m(z)S_{\mathbf{u}}(z) + A(z)}{B(z)}\,, \tag{3.21}$$

where $A := u_0 v_0\Delta_0(2, m-1, \cdot)$ and $B := u_0\kappa_m\eta_{k-1-m}$, being $\kappa_m := \prod_{j=1}^m a_0^{(j)}$. In (3.20) replacing z by $\pi_k(z)$ and taking into account (3.21), after some computations we see that $S_{\mathbf{u}}(z)$ satisfies

$$\Phi_0(z)S_{\mathbf{u}}'(z) = C_0(z)S_{\mathbf{u}}(z) + D_0(z),$$

where

$$\begin{aligned}
\Phi_0 &:= v_0 B\theta_m^2\sigma_{\pi_k}[\widetilde{\Phi}]\;,\\
C_0 &:= v_0\big(\sigma_{\pi_k}[\widetilde{\Phi}]\mathscr{C}_2 + \sigma_{\pi_k}[\widetilde{C}]\mathscr{C}_3\big)\theta_m\;,\\
D_0 &:= \big(\sigma_{\pi_k}[\widetilde{\Phi}]\mathscr{C}_1 + \sigma_{\pi_k}[\widetilde{D}]\mathscr{C}_3\big)B + \big(\sigma_{\pi_k}[\widetilde{C}]\mathscr{C}_3 + \sigma_{\pi_k}[\widetilde{\Phi}]\mathscr{C}_2\big)A\;,
\end{aligned}$$

being $\mathscr{C}_1 := A\theta_m' - A'\theta_m$, $\mathscr{C}_2 := B'\theta_m - B\theta_m'$, and $\mathscr{C}_3 := B\theta_m\pi_k'$. Using the definitions of A and B, one easily see that the polynomial $\kappa_m\theta_m$ is a common factor of the polynomials Φ_0, C_0 and D_0. Therefore, $S_{\mathbf{u}}(z)$ fulfills

$$\Phi_1(z)S_{\mathbf{u}}'(z) = C_1(z)S_{\mathbf{u}}(z) + D_1(z)\;, \tag{3.22}$$

where the polynomials Φ_1, C_1, and D_1 are given explicitly by

$$\begin{aligned}
\Phi_1 &:= v_0\theta_m\eta_{k-1-m}\sigma_{\pi_k}[\widetilde{\Phi}]\,,\\
C_1 &:= v_0\left(\eta'_{k-1-m}\theta_m - v_0\theta'_m\eta_{k-1-m}\sigma_{\pi_k}[\widetilde{\Phi}] + \eta_{k-1-m}\theta_m\pi'_k\sigma_{\pi_k}[\widetilde{C}]\right)\,,\\
D_1 &:= u_0v_0\left(\Delta_0(2,m-1,\cdot)\eta'_{k-1-m} - \Delta'_0(2,m-1,\cdot)\eta_{k-1-m}\right)\sigma_{\pi_k}[\widetilde{\Phi}]\\
&\qquad +u_0\left(\kappa_m\eta_{k-1-m}\sigma_{\pi_k}[\widetilde{D}] + v_0\Delta_0(2,m-1,\cdot)\sigma_{\pi_k}[\widetilde{C}]\right)\eta_{k-1-m}\pi'_k\,.
\end{aligned}\tag{3.23}$$

It follows that $\mathbf{u}$ is a semiclassical functional. Let us prove that the class s of $\mathbf{u}$ satisfies $s\leq(\widetilde{s}+3)k-3$. Indeed, we have

$$\begin{aligned}
\deg C_1 &\leq \max\{k-2+k\deg\widetilde{\Phi},2(k-1)+k\deg\widetilde{C}\}\leq k(\widetilde{s}+3)-2\,,\\
\deg D_1 &\leq \max\{k-3+k\deg\widetilde{\Phi},2k-3+k\deg\widetilde{C}\}\leq k(\widetilde{s}+3)-3\,.
\end{aligned}$$

Therefore, $s\leq\max\{\deg C_1-1,\deg D_1\}\leq(\widetilde{s}+3)k-3$. ■

Corollary 3.1. *Under the hypothesis of Theorem 3.1, if $\{p_n\}_{n\geq0}$ is a semiclassical OPS of class $s\leq k-1$, then $\{q_n\}_{n\geq0}$ is necessarily a classical OPS.*

Proof: It is an immediate consequence of part (i) in Theorem 3.1. ■

Corollary 3.2. *Under the hypothesis of Theorem 3.1, let $\mathbf{u}$ and $\mathbf{v}$ be the regular functionals with respect to which $\{p_n\}_{n\geq0}$ and $\{q_n\}_{n\geq0}$ are orthogonal, respectively. If there exists nonzero polynomials $\widetilde{\Phi}$ and $\widetilde{\Psi}$ such that $D(\widetilde{\Phi}\mathbf{v})=\widetilde{\Psi}\mathbf{v}$, with $\deg\widetilde{\Psi}\geq1$, then $\mathbf{u}$ fulfills $D(\Phi\mathbf{u})=\Psi\mathbf{u}$, with*

$$\Phi:=\theta_m\eta_{k-1-m}\sigma_{\pi_k}[\widetilde{\Phi}]\,,\quad \Psi:=2\,\eta'_{k-1-m}\theta_m\sigma_{\pi_k}[\widetilde{\Phi}]+\eta_{k-1-m}\theta_m\sigma_{\pi_k}[\widetilde{\Psi}]\pi'_k\,.$$

Proof: The assertion follows immediately from (3.23) in the proof of Theorem 3.1, and taking into account the relation $\Psi=\Phi'+C_1$. ■

Remarks 3.1. *For the particular case $k=3$, part (ii) in Theorem 3.1 recovers* [16, Corollary 2.4]. *The case $k=2$ has been studied in detail in* [14, 15].

4. Semiclassical OP of class at most 2 via cubic transformations

The results contained in [28], for cubic transformations (being therein $k=3$ and $m=0$), may be deduced using the general results proved in the previous section. Indeed, in [28] the authors considered the problem of determining all the semiclassical monic OPS of class 1, $\{p_n\}_{n\geq0}$, such that a cubic decomposition as

$$p_{3n}(x) = q_n(x^3 + qx + r)\ , \quad n = 0, 1, 2, \ldots \tag{4.1}$$

holds, being $\{q_n\}_{n\geq 0}$ a monic OPS and $q, r \in \mathbb{C}$. They proved [28, Proposition 4.2] that such property is fulfilled only if $\{q_n\}_{n\geq 0}$ coincides with some specific family of classical OPS (being special cases of Jacobi polynomials, up to affine changes of the variable). It is clear from Corollary 3.1 that only classical OPS $\{q_n\}_{n\geq 0}$ may appear as solutions of such problem. Moreover, we see immediately that if we consider the analogous problem demanding $\{p_n\}_{n\geq 0}$ to be semiclassical of class 2, then again only classical OPS $\{q_n\}_{n\geq 0}$ may appear fulfilling such transformation. Thus, the following problem arises:

Problem (P). *Determine all monic OPS $\{q_n\}_{n\geq 0}$ and all cubic polynomials $\pi_3(x) \equiv x^3 + qx + r$ such that a semiclassical monic OPS $\{p_n\}_{n\geq 0}$ of class $s \leq 2$ exists fulfilling (4.1). In addition, find explicitly the polynomials Φ and Ψ appearing in the canonical distributional equation $D(\Phi\mathbf{u}) = \Psi\mathbf{u}$ satisfied by the functional $\mathbf{u}$ with respect to which $\{p_n\}_{n\geq 0}$ is an OPS (thus describing all such OPS).*

4.1. Solution to Problem (P).

The solution for Problem (P) is given in Tables 3 and 4, up to affine changes of the variable. In these tables we assume implicitly the conditions on the parameters α and β given in Table 1, ensuring the regularity of the Jacobi or Laguerre functionals in each case. Moreover, in cases (18), (19), (22), (23), and (24), there are some additional regularity conditions, R_{18}, R_{19}, R_{22}, R_{23}, and R_{24}. Their meaning is:

$$\mathrm{R}_{18}: \ P_n^{(\alpha,-1/3)}\big(-1-\tau^3\big) \neq 0\ , \ \ P_n^{(\alpha,-1/3)*}\big(-1;-1-\tau^3\big) \neq 0\ , \ \ n \geq 1\ ;$$

$$\mathrm{R}_{19}: \ P_n^{(-1/3,\beta)}(1-\tau^3) \neq 0\ ; \ \ P_n^{(-1/3,\beta)*}\big(1;1-\tau^3\big) \neq 0\ , \quad n \geq 1\ ,$$

$$\mathrm{R}_{22}: \ P_n^{(\alpha,-1/2)}\big(d^3 - \tfrac{3cd}{2}\big) \neq 0\ , \ \ P_n^{(\alpha,-1/2)*}\big(-1;d^3 - \tfrac{3cd}{2}\big) \neq 0\ , \ \ n \geq 1\ ;$$

$$\mathrm{R}_{23}: \ P_n^{(-1/2,\beta)}\big(d^3 - \tfrac{3cd}{2}\big) \neq 0\ , \ \ P_n^{(-1/2,\beta)*}\big(1;d^3 - \tfrac{3cd}{2}\big) \neq 0\ , \ \ n \geq 1\ ;$$

$$\mathrm{R}_{24}: \ L_n^{(-1/3)}\big(-\tau^3\big) \neq 0\ , \ \ L_n^{(-1/3)*}\big(0;-\tau^3\big) \neq 0\ , \ \ n \geq 1\ ,$$

where $P_n^*(\kappa;\cdot)$ is the kernel polynomial of $K-$parameter κ, given by [8, p. 35]

$$P_n^*(\kappa;x) := \frac{1}{x-\kappa}\Big(P_{n+1}(x) - \frac{P_{n+1}(\kappa)}{P_n(\kappa)}P_n(x)\Big)\ , \quad n \geq 0\,.$$

The need of such conditions is justified by [16, Theorem 2.1].

Case	s	$q_n(x)$	$\pi_3(x)=x^3+qx+r$	$b_0^{(0)}$	$a_0^{(1)}$	Constraints
(1)	0	$\widehat{T}_n$	$x^3-\frac{3c}{2}\,x$	0	c	$c^3=2$
(2)	1	$\widehat{P}_n^{(\alpha,-1/2)}$	$x^3-\frac{3c}{2}\,x$	0	c	$c^3=2\,,\ \alpha\neq-\frac{1}{2}$
(3)		$\widehat{P}_n^{(-1/2,\beta)}$	$x^3-\frac{3c}{2}\,x$	0	c	$c^3=2\,,\ \beta\neq-\frac{1}{2}$
(4)		$\widehat{P}_n^{(\alpha,-1/2)}$	$x^3-\frac{3c}{2}\,x$	$-3c^{-1}$	$-2c$	$c^3=2$
(5)		$\widehat{P}_n^{(-1/2,\beta)}$	$x^3-\frac{3c}{2}\,x$	$3c^{-1}$	$-2c$	$c^3=2$
(6)		$\widehat{P}_n^{(\alpha,-1/3)}$	x^3-1	$-c$	$-2c^{-1}$	$c^3=2$
(7)		$\widehat{P}_n^{(-1/3,\beta)}$	x^3+1	c	$-2c^{-1}$	$c^3=2$
(8)	2	$\widehat{P}_n^{(\alpha,\beta)}$	$x^3-\frac{3c}{2}\,x$	0	$-\frac{c}{2}$	$c^3=2$
(9)		$\widehat{P}_n^{(\alpha,\beta)}$	$x^3-\frac{3c}{2}\,x$	$-3c^{-1}$	$-2c$	$c^3=2\,,\ \beta\neq-\frac{1}{2}$
(10)		$\widehat{P}_n^{(\alpha,\beta)}$	$x^3-\frac{3c}{2}\,x$	$3c^{-1}$	$-2c$	$c^3=2\,,\ \alpha\neq-\frac{1}{2}$
(11)		$\widehat{P}_n^{(\alpha,\beta)}$	$x^3-\frac{3c}{2}\,x$	0	c	$c^3=2\,,\ \alpha\neq-\frac{1}{2}\,,\ \beta\neq-\frac{1}{2}$
(12)		$\widehat{P}_n^{(\alpha,\beta)}$	$x^3-\frac{3c}{2}\,x$	$-4c^{-1}$	$-\frac{9c}{2}$	$c^3=2$
(13)		$\widehat{P}_n^{(\alpha,\beta)}$	$x^3-\frac{3c}{2}\,x$	$4c^{-1}$	$-\frac{9c}{2}$	$c^3=2$
(14)		$\widehat{P}_n^{(\alpha,\beta)}$	x^3-1	$-c$	$-2c^{-1}$	$c^3=2\,,\ \beta\neq-\frac{1}{3}$
(15)		$\widehat{P}_n^{(\alpha,\beta)}$	x^3+1	c	$-2c^{-1}$	$c^3=2\,,\ \alpha\neq-\frac{1}{3}$
(16)		$\widehat{P}_n^{(\alpha,\beta)}$	x^3-1	$-2c$	$-6c^{-1}$	$c^3=2$
(17)		$\widehat{P}_n^{(\alpha,\beta)}$	x^3+1	$2c$	$-6c^{-1}$	$c^3=2$
(18)		$\widehat{P}_n^{(\alpha,-1/3)}$	x^3-1	$-\tau$	$-\tau^2$	$\tau\neq0,\ \tau^3\neq2,\ \mathrm{R}_{18}$
(19)		$\widehat{P}_n^{(-1/3,\beta)}$	x^3+1	τ	$-\tau^2$	$\tau\neq0,\ \tau^3\neq2,\ \mathrm{R}_{19}$
(20)		$\widehat{P}_n^{(\alpha,-1/2)}$	$x^3-3b^2x+2b^3-1$	$-(a+b)$	$2(b-a)^{-1}$	$(a+2b)(a-b)^2=2\,,\ a\neq-b\,,\ a\neq2b$
(21)		$\widehat{P}_n^{(-1/2,\beta)}$	$x^3-3b^2x-2b^3+1$	$a+b$	$2(b-a)^{-1}$	$(a+2b)(a-b)^2=2\,,\ a\neq-b\,,\ a\neq2b$
(22)		$\widehat{P}_n^{(\alpha,-1/2)}$	$x^3-\frac{3c}{2}\,x$	$-(c^{-1}+d)$	$(2c^{-1}+d)(c^{-1}-d)$	$c^3=2\,,\ d^3-\frac{3cd}{2}\neq1\,,\ d\neq c^{-1}\,,\ d\neq-2c^{-1}\,,\ \mathrm{R}_{22}$
(23)		$\widehat{P}_n^{(-1/2,\beta)}$	$x^3-\frac{3c}{2}\,x$	$c^{-1}+d$	$(2c^{-1}+d)(c^{-1}-d)$	$c^3=2\,,\ d^3-\frac{3cd}{2}\neq1\,,\ d\neq c^{-1}\,,\ d\neq-2c^{-1}\,,\ \mathrm{R}_{23}$
(24)		$\widehat{L}_n^{(-1/3)}$	x^3	τ	$-\tau^2$	$\tau\neq0\,,\ \mathrm{R}_{24}$

TABLE 3. Description of the monic OPS $\{q_n\}_{n\geq0}$ and the cubic polynomials $\pi_3(x)$ for all possible semi-classical OPS $\{p_n\}_{n\geq0}$ of class $s\leq2$ obtained via a cubic transformation such that $p_{3n}(x)=q_n(\pi_3(x))$ for all $n\geq0$. This polynomial mapping depends on the choice of the parameters a, b, c, d, and τ, which may be chosen arbitrarily in $\mathbb{C}$ subject to the given constraints.

Case	Φ	Ψ
(1)	x^2-2c	x
(2)	$(x^2-2c)(x+c^{-1})$	$\alpha_{3,7}\,x^2+c^{-1}\alpha_{3,5}\,x-c\alpha_{3,7}$
(3)	$(x^2-2c)(x-c^{-1})$	$\beta_{3,7}\,x^2-c^{-1}\beta_{3,5}\,x-c\beta_{3,7}$
(4)	$(x^2-2c)(x+c^{-1})$	$\alpha_{3,7}\,x^2+c^{-1}\alpha_{3,11}\,x-c\alpha_{3,1}$
(5)	$(x^2-2c)(x-c^{-1})$	$\beta_{3,7}\,x^2-c^{-1}\beta_{3,11}\,x-c\beta_{3,1}$
(6)	x^3-2	$\alpha_{3,8}\,x^2+cx+2c^{-1}$
(7)	x^3+2	$\beta_{3,8}\,x^2-cx+2c^{-1}$
(8)	$\left(x^2-\frac{c}{2}\right)(x^2-2c)$	$3(\alpha_{1,2}+\beta_{1,2})x^3-\frac{3c}{2}(\alpha_{3,4}+\beta_{3,4})x+3(\alpha_{1,0}-\beta_{1,0})$
(9)	$\left(x^2-\frac{c}{2}\right)(x^2-2c)$	$3(\alpha_{1,2}+\beta_{1,2})x^3+3c^{-1}x^2-\frac{3c}{2}(\alpha_{3,5}+\beta_{3,5})x+3(\alpha_{1,0}-\beta_{1,2})$
(10)	$\left(x^2-\frac{c}{2}\right)(x^2-2c)$	$3(\alpha_{1,2}+\beta_{1,2})x^3-3c^{-1}x^2-\frac{3c}{2}(\alpha_{3,5}+\beta_{3,5})x+3(\alpha_{1,2}-\beta_{1,0})$
(11)	$\left(x^2-\frac{c}{2}\right)(x^2-2c)$	$3(\alpha_{1,2}+\beta_{1,2})x^3-\frac{9c}{2}(\alpha_{1,2}+\beta_{1,2})x+3(\alpha_{1,0}-\beta_{1,0})$
(12)	$\left(x^2-\frac{c}{2}\right)(x^2-2c)$	$3(\alpha_{1,2}+\beta_{1,2})x^3+4c^{-1}x^2-\frac{3c}{2}(\alpha_{3,4}+\beta_{3,4})x+\alpha_{3,0}-\beta_{3,4}$
(13)	$\left(x^2-\frac{c}{2}\right)(x^2-2c)$	$3(\alpha_{1,2}+\beta_{1,2})x^3-4c^{-1}x^2-\frac{3c}{2}(\alpha_{3,4}+\beta_{3,4})x+\alpha_{3,4}-\beta_{3,0}$
(14)	$x(x^3-2)$	$3(\alpha_{1,2}+\beta_{1,2})x^3+cx^2+2c^{-1}x-2\beta_{3,4}$
(15)	$x(x^3+2)$	$3(\alpha_{1,2}+\beta_{1,2})x^3-cx^2+2c^{-1}x+2\alpha_{3,4}$
(16)	$x(x^3-2)$	$3(\alpha_{1,2}+\beta_{1,2})x^3+2cx^2+4c^{-1}x-2\beta_{3,2}$
(17)	$x(x^3+2)$	$3(\alpha_{1,2}+\beta_{1,2})x^3-2cx^2+4c^{-1}x+2\alpha_{3,2}$
(18)	$(x-\tau)(x^3-2)$	$\alpha_{3,10}\,x^3-3\tau\alpha_{1,2}\,x^2-4$
(19)	$(x+\tau)(x^3+2)$	$\beta_{3,10}\,x^3+3\tau\beta_{1,2}\,x^2+4$
(20)	$(x-a)(x+2b)(x^2+ax+a^2-3b^2)$	$3\alpha_{1,3}\,x^3+(2b\alpha_{3,8}+a)x^2-(3b^2\alpha_{1,5}-2ab-a^2)x-b^3\alpha_{6,22}+2ba^2-1$
(21)	$(x+a)(x-2b)(x^2-ax+a^2-3b^2)$	$3\beta_{1,3}x^3-(2b\beta_{3,8}+a)x^2-(3b^2\beta_{1,5}-2ab-a^2)x+b^3\beta_{6,22}-2ba^2+1$
(22)	$(x-d)(x+c^{-1})(x^2-2c)$	$3\alpha_{1,3}x^3-\left(d\alpha_{3,5}-3c^{-1}\alpha_{1,3}\right)x^2-\left(c\alpha_{3,11}+c^{-1}d\alpha_{3,5}\right)x+3cd\alpha_{1,1}-4$
(23)	$(x+d)(x-c^{-1})(x^2-2c)$	$3\beta_{1,3}x^3+\left(d\beta_{3,5}-3c^{-1}\beta_{1,3}\right)x^2-\left(c\beta_{3,11}+c^{-1}d\beta_{3,5}\right)x-3cd\beta_{1,1}+4$
(24)	$x+\tau$	$-3x^3-3\tau\,x^2+2$

TABLE 4. The polynomials Φ and Ψ appearing in the canonical distributional differential equation $D(\Phi\mathbf{u})=\Psi\mathbf{u}$ satisfied by the functional $\mathbf{u}$ with respect to which $\{p_n\}_{n\geq 0}$ is an OPS, in accordance with each case described in Table 3. For simplicity, we write $\alpha_{j,n}:=j\alpha+n/2$ and $\beta_{j,n}:=j\beta+n/2$, where α and β are the parameters in the definition of the Jacobi polynomials.

Before explaining how Tables 3 and 4 may be constructed, we point out some special cases described therein.

(i) For $s=0$ we see that there is only one solution to Problem (P), allowing us to recover a well known relation involving Chebyshev polynomials of the first kind:

$$T_{3n}(x)=T_n(T_3(x))\;,\quad n=0,1,2,\dots\;, \tag{4.2}$$

being $T_n(x):=\cos(n\theta)$, $x=\cos\theta$, $0\leq\theta<\pi$. Indeed, denoting by $\widehat{T}_n$ the monic Chebyshev polynomial of degree n, so that $\widehat{T}_n(x)=2^{1-n}T_n(x)$, on the first hand, from case (1) in Table 3 we have

$$p_{3n}(x)=\widehat{T}_n\big(x^3-\tfrac{3c}{2}\,x\big)\;,\quad n=0,1,2,\dots\;,$$

where $c^3=2$; hence, making the (affine) change of variables $y=\frac{c}{2}\,x$, we obtain

$$p_{3n}\big(\tfrac{2y}{c}\big)=\widehat{T}_n\big(4y^3-3y\big)=2^{1-n}T_n\big(T_3(y)\big)\;,\quad n=0,1,2,\dots\;. \tag{4.3}$$

On the other hand, from case (1) in Table 4, $D\big((x^2-2c)\mathbf{u}\big)=x\mathbf{u}$, and so $\mathbf{u}$ is classical. Therefore, defining $\widehat{\mathbf{u}}:=h_{c/2}\,\mathbf{u}$, i.e., $\langle\widehat{\mathbf{u}},x^n\rangle:=(c/2)^n\langle\mathbf{u},x^n\rangle$, $n\geq0$, by using standard arguments we see that $D\big((1-x^2)\widehat{\mathbf{u}}\big)=-x\widehat{\mathbf{u}}$, hence from Table 1 (for $\alpha=\beta=-\frac{1}{2}$) we conclude that the monic OPS with respect to $\widehat{\mathbf{u}}$ is $\{\widehat{T}_n\}_{n\geq0}$. Thus, we deduce

$$p_n(x)=\Big(\frac{2}{c}\Big)^n\widehat{T}_n\Big(\frac{cx}{2}\Big)\;,\quad n=0,1,2,\dots\;.$$

Therefore,

$$p_{3n}\big(\tfrac{2y}{c}\big)=\Big(\frac{2}{c}\Big)^{3n}\widehat{T}_{3n}(y)=2^{1-n}T_{3n}(y)\;,\quad n=0,1,2,\dots\;. \tag{4.4}$$

Combining (4.3) and (4.4) we obtain (4.2).

(ii) For $s=1$, we point out that cases (3), (5), and (7) in Tables 3 and 4 recover [28, Proposition 4.2], up to appropriate affine changes of the variables. Moreover, it is worth mentioning that cases (2), (4), and (6) were missed in [28]. We also remark that the definition considered here (in concordance with the classical literature) for Jacobi polynomials (cf. Table 1) differs from the one used in [28], since the roles of α and β are interchanged. Next we explain why case (3) indeed corresponds (up to an affine change of the variables) to one of the three cases presented in [28, Proposition 4.2]. A similar analysis may be done for cases (5) and (7), giving the other two cases appearing in [28, Proposition 4.2]. From case (3) in Table 3 we have $c^3=2$ and $q=-3c/2\neq0$.

Therefore, setting $\mu_q := -c^{-1}$, it follows that $\mu_q^2 = -q/3$ and so, from Table 4, we obtain

$$\Phi(x) = \Big(x^2 + \tfrac{4q}{3}\Big)(x+\mu_q)\;, \quad \Psi(x) = \tfrac{1}{2}\Big((6\beta+7)(x^2+2q/3) + \mu_q(6\beta+5)x\Big)\;,$$

which agree with the polynomials appearing in formula [28, (4.22)].

(iii) Finally, for $s=2$, we point out that all but one of the possible families $\{q_n\}_{n\geq 0}$ must be a Jacobi OPS (up to affine changes of the variables), being the non-Jacobi family a Laguerre OPS with parameter $\alpha = -1/3$.

4.2. Construction of Tables 3 and 4. Here we briefly describe how we have constructed Tables 3 and 4. We will consider only two illustrative cases. All the other ones may be handled similarly. We start by noticing that by assumption $\{p_n\}_{n\geq 0}$ is semiclassical of class $s\leq 2$, and (4.1) corresponds to a polynomial mapping obtained via a polynomial $\pi_3(x)$ of degree $k=3$. Thus, it follows from Corollary 3.1 that $\{q_n\}_{n\geq 0}$ is a classical monic OPS, hence (up to an affine change of variables) it is one of the families of Hermite, Laguerre, Bessel, or Jacobi, described by the canonical representations appearing in Table 1. So we only need to analyze separately these four possible cases. Before performing this analysis, it is useful to notice that (since $k=3$ and $m=0$) from the general expressions for η_{k-m-1} and π_k appearing in Theorem 2.1 one obtains $\eta_2(x) = \Delta_0(2,2) = x^2 - \big(b_0^{(1)}+b_0^{(2)}\big)x + b_0^{(1)}b_0^{(2)} - a_0^{(2)} = p_2^{(1)}(x)$ and $\pi_3(x) = \Delta_0(1,2) + r_0 = x^3+px^2+qx+r$, where

$$\begin{aligned} &p := -\big(b_0^{(0)}+b_0^{(1)}+b_0^{(2)}\big)\;, \quad q := b_0^{(1)}b_0^{(2)} + b_0^{(0)}(b_0^{(1)}+b_0^{(2)}) - a_0^{(1)} - a_0^{(2)}\;,\\ &r := -b_0^{(0)}\big(b_0^{(1)}b_0^{(2)} - a_0^{(2)}\big) + a_0^{(1)}b_0^{(2)} + r_0\,. \end{aligned} \tag{4.5}$$

According to (4.1), we have $p=0$ and so, setting

$$\tau := b_0^{(0)}\;, \quad k_\tau := a_0^{(1)} + \tau^2\;,$$

after simple computations we obtain

$$\pi_3(x) = x^3+qx+r\;, \quad \eta_2(x) = x^2+\tau x + q + k_\tau = p_2^{(1)}(x)\;. \tag{4.6}$$

Let $\{q_n\}_{n\geq 0}$ be a classical OP. Since $\{p_n\}_{n\geq 0}$ fulfils the cubic decomposition (4.1), it follows from the proof of Theorem 3.1 that its Stieltjes formal series,

$S_{\mathbf{u}}(z)$, satisfies (3.22) with

$$\begin{aligned}\Phi_1 &:= v_0\eta_2\sigma_{\pi_3}[\widetilde{\Phi}]\;,\\ C_1 &:= v_0\left(\eta_2'\sigma_{\pi_k}[\widetilde{\Phi}]+\eta_2\pi_3'\sigma_{\pi_3}[\widetilde{C}]\right)\;,\\ D_1 &:= u_0\eta_2^2\sigma_{\pi_3}[\widetilde{D}]\pi_3'\;,\end{aligned}\tag{4.7}$$

where the polynomials $\widetilde{\Phi}$, $\widetilde{C}$ and $\widetilde{D}$ are those appearing in Tables 1 and 2.

Therefore, as a first illustrative example, let $\{q_n\}_{n\geq 0}$ be the monic OPS of Hermite. We will show that $s\geq 4$, hence Hermite polynomials cannot contribute with solutions to Problem (P). Indeed, writing $\eta_2(x)=(x-a)(x-b)$, it follows from (4.7) that

$$\begin{aligned}\Phi_1(x) &= v_0(x-a)(x-b)\;,\\ C_1(x) &= v_0\left(2x-a-b-2(x-a)(x-b)\pi_3'(x)\pi_3(x)\right)\;,\\ D_1(x) &= -2u_0v_0(x-a)^2(x-b)^2\pi_3'(x)\;.\end{aligned}$$

Thus, since Φ_1, C_1, and D_1 may have at most two common zeros, we obtain

$$s\geq \max\{\deg C_1-1,\deg D_1\}-2\geq 4\;.$$

As a second illustrative example, let $\{q_n\}_{n\geq 0}$ be the monic OPS of Jacobi. Thus, we start with $q_n(x)=\widehat{P}_n^{(\alpha,\beta)}(x)$, for arbitrary parameters α and β, and our goal is to determine which of these parameters contribute with solutions to Problem (P). We will show only how to obtain the solution described in case (13) appearing in Tables 3 and 4. Suppose that $\eta_2(x)=(x-a)^2$, i.e., we are seeking for solutions such that η_2 has a double zero. By (4.7), $v_0(x-a)$ is a common factor of Φ_1, C_1, and D_1. Dividing these three polynomials by this factor gives (for simplicity, we still use Φ_1, C_1, and D_1, although in fact these are the polynomials obtained by dividing the above polynomials by the common factor)

$$\begin{aligned}\Phi_1(x) &:= (x-a)(1-\pi_3^2(x))\;,\\ C_1(x) &:= 2(1-\pi_3^2(x))-\left[(\alpha+\beta)\pi_3(x)+\alpha-\beta\right](x-a)\pi_3'(x)\;,\\ D_1(x) &:= -u_0(\alpha+\beta+1)(x-a)^3\pi_3'(x)\;.\end{aligned}\tag{4.8}$$

Now, we see that if the conditions $\pi_3(a)\neq -1$ and $\pi_3(a)\neq 1$ hold, then a cannot be a common zero of Φ_1, C_1 and D_1, hence, since $(x-a)^3$ is a factor of D_1, we conclude that, in this case, $s\geq 3$, and so no solutions for Problem (P) can be obtained under such conditions. Suppose $\pi_3(a)=-1$.

Then $1+\pi_3(x)=(x-a)(x^2+ax+q+a^2)$. Therefore, the polynomials Φ_1, C_1, and D_1 given by (4.8) have $x-a$ as a common factor, and so dividing these polynomials by $x-a$ we obtain

$$\begin{aligned}
&\Phi_1(x):=(x-a)(x^2+ax+q+a^2)(1-\pi_3(x)) \;, \\
&C_1(x):=2(x^2+ax+q+a^2)(1-\pi_3(x))-\left[(\alpha+\beta)\pi_3(x)+\alpha-\beta\right]\pi_3'(x) \;, \\
&D_1(x):=-u_0(\alpha+\beta+1)(x-a)^2\pi_3'(x) \;.
\end{aligned} \tag{4.9}$$

Since $\pi_3'(a)=-a_0^{(1)}\neq 0$, then $C_1(a)=2(2+\beta)\pi_3'(a)\neq 0$. Let x_0 be a zero of π_3', so that $\pi_3'(x_0)=0$. Since the zeros of π_3' are symmetric, then $\pi_3'(x)=3(x-x_0)(x+x_0)$. If $\Phi_1(x_0)\neq 0$ then Φ_1, C_1, and D_1 have no common zeros, hence, since $\deg D_1=4$, we have $s=4$. Otherwise, if $\Phi_1(x_0)=0$ then $\pi_3(x_0)=-1$ or $\pi_3(x_0)=1$. Suppose that $\pi_3(x_0)=-1$. Then $1+\pi_3(x)$ can be factorized as

$$1+\pi_3(x)=(x-a)(x-x_0)(x+a+x_0)\,. \tag{4.10}$$

Thus, the polynomials Φ_1, C_1, and D_1 given by (4.9) have $x-x_0$ as a common factor, and dividing these polynomials by $x-x_0$ we obtain

$$\begin{aligned}
&\Phi_1(x):=(x-a)(x+a+x_0)(1-\pi_3(x)) \;, \\
&C_1(x):=2(x+a+x_0)(1-\pi_3(x))-3\left[(\alpha+\beta)\pi_3(x)+\alpha-\beta\right](x+x_0) \;, \\
&D_1(x):=-3u_0(\alpha+\beta+1)(x-a)^2(x+x_0) \;.
\end{aligned} \tag{4.11}$$

If $\Phi_1(-x_0)\neq 0$ then Φ_1, C_1, and D_1 have no common factors, hence $s=3$. On the other hand, if $\Phi_1(-x_0)=0$, then $\pi_3(-x_0)=1$ (because $x_0=-a$ or $a=0$ cannot hold, since $\pi'(a)\neq 0$), and so, in this situation,

$$1-\pi_3(x)=-(x+x_0)(x^2-x_0x+q+x_0^2)\,. \tag{4.12}$$

Then we see that $-(x+x_0)$ is a common factor of the polynomials Φ_1, C_1, and D_1 in (4.11). Thus, dividing these polynomials by $-(x+x_0)$ we obtain

$$\begin{aligned}
&\Phi_1(x):=(x-a)(x+a+x_0)(x^2-x_0x+q+x_0^2) \;, \\
&C_1(x):=2(x+a+x_0)(x^2-x_0x+q+x_0^2)+3\left[(\alpha+\beta)\pi_3(x)+\alpha-\beta\right] \;, \\
&D_1(x):=3u_0(\alpha+\beta+1)(x-a)^2 \;.
\end{aligned} \tag{4.13}$$

Since $C_1(a)\neq 0$ we may conclude that Φ_1, C_1, and D_1 given by (4.13) does not share zeros. Therefore, since $\deg D_1=2$, we arrive at $s=2$. Thus, this gives a solution for Problem (P), namely that one appearing as case (13) in Tables 3 and 4. Indeed, from (4.10) and (4.12) we deduce that π_3 admits the representations

$$\pi_3(x) = x^3 - (a^2 + ax_0 + x_0^2)x + a^2x_0 + ax_0^2 - 1 = x^3 + qx + qx_0 + x_0^3 + 1\,.$$

Comparing the coefficients of x we have $q = -a^2 - ax_0 - x_0^2$. Since $\pi_3'(x_0) = 3x_0^2 + q = 0$, we also have $q = -3x_0^2$. Therefore $2x_0^2 - ax_0 - a^2 = 0$, i.e., $(2x_0+a)(x_0-a) = 0$. Thus, since $a \neq x_0$ (because $a = x_0$ would imply $\pi_3'(a) = 0$) we conclude that $x_0 = -a/2$. Using this relation and by comparison of the independent terms in the above equations for $\pi_3(x)$, we deduce $x_0^3 = 1/2$. Therefore,

$$x_0 = -\frac{a}{2} = c^{-1}\,, \tag{4.14}$$

being c chosen so that $c^3 = 2$. By (4.6) we have $a = -\tau/2$, hence we deduce

$$b_0^{(0)} = \tau = 4c^{-1}\,, \quad a_0^{(1)} = 7x_0^2 - \tau^2 = -\frac{9c}{2}\,, \quad \pi_3(x) = x^3 - \frac{3c}{2}x\,.$$

This gives case (13) appearing in Table 3. On the other hand, by (4.13) and taking into account that $\Psi(x) = \Phi'(x) + C_1(x)$, we obtain

$$\Phi_1(x) = \left(x + 2c^{-1}\right)\left(x - c^{-1}\right)\left(x^2 - c^{-1}x - c\right) = \left(x^2 - \tfrac{c}{2}\right)\left(x^2 - 2c\right)$$

$$\Psi_1(x) = 3(\alpha + \beta + 2)x^3 - 4c^{-1}x^2 - \tfrac{3c}{2}(3\alpha + 3\beta + 4) - 3\beta + 3\alpha + 2\,,$$

giving case (13) appearing in Table 4. Finally, we notice that in this case the regularity condition required in [16, Theorem 2.1] are fulfilled, since, using the notation appearing in [16], $c_1 = c_2 = \pi_3(a) = 1$, and so

$$p_n\big(c_1\big) = \widehat{P}_n^{(\alpha,\beta)}(1) = 2^n(1+\alpha)_n/(1+\alpha+\beta+n)_n \neq 0$$

and (cf. case 2 in the proof of [16, Theorem 2.1])

$$p_n^*\big(c_1; c_2\big) = \frac{a_n^{(1)}}{a_0^{(1)}}p_n(c_1) \neq 0\,.$$

Remarks 4.1. In the previous analysis, we started from the knowledge of the monic OPS $\{q_n\}_{n\geq 0}$, and we found the corresponding monic OPS $\{p_n\}_{n\geq 0}$ satisfying the cubic transformation (4.1). We point out that, starting from the knowledge of a monic OPS $\{p_n\}_{n\geq 0}$, e.g., as described by Tables 3 and 4, we can find the corresponding monic OPS $\{q_n\}_{n\geq 0}$ fulfilling (4.1), provided we know *a priori* that such a cubic transformation exists (indeed, this still remains valid for general polynomial mappings). We will illustrate the procedure considering the monic OPS $\{p_n\}_{n\geq 0}$ described in case (24) appearing in Tables 3 and 4 (and so, we know that a cubic transformation exists). In this case, by Table 3, we have $b_0^{(0)} = \tau$ and $a_0^{(1)} = -\tau^2$. Since the polynomials Φ and Ψ satisfy $D(\Phi\mathbf{u}) = \Psi\mathbf{u}$, then $\langle \mathbf{u}, \Psi\rangle = 0$. By Table 4, $\Psi(x) = -3x^3 - 3\tau x^2 + 2$, hence $\langle \mathbf{u}, x^3 + \tau x^2 - \frac{2}{3}\rangle = 0$. Thus, since $p_1(x) = x - \tau$, $p_2(x) = \Delta_0(1,1;x) = x^2 - (b_0^{(1)} + \tau)x + \tau b_0^{(1)} + \tau^2$, and $p_3(x) = \pi_3(x) - r_0 = x^3 - r_0$, we deduce $r_0 = \frac{2}{3}$. Taking into account (4.5) and (4.6), we have

$$b_0^{(2)} = r_0\tau^{-2} = \tfrac{2}{3}\tau^{-2}\,, \quad b_0^{(1)} = -\tau - b_0^{(2)} = -\left(\tau^3 + \tfrac{2}{3}\right)\tau^{-2}\,,$$
$$a_0^{(2)} = b_0^{(1)} b_0^{(2)} = -\left(\tfrac{2}{3}\tau^3 + \tfrac{4}{9}\right)\tau^{-4}\,.$$

Therefore, using (3.14) and (3.16) with $k = 3$ and $m = p = 0$ (notice that while proving (3.14) and (3.16) we considered $p \geq 1$; nevertheless by direct inspection we see that the formulas are valid whenever $p = 0$), we obtain

$$\pi_3'(x)\Phi(x) = 3x^2(x+\tau) = 3\pi_3(x)p_0(x) - \tfrac{2}{\tau}p_1(x) + 3\tau p_2(x)$$
$$\Psi(x) = -3x^3 - 3\tau x^2 + 2 = \left(2 - 3\pi_3(x)\right)p_0(x) + \tfrac{2}{\tau}p_1(x) - 3\tau p_2(x)\,,$$

hence $f_0(x) = 3x$ and $g_0(x) = 2 - 3x$, being f_0 and g_0 the polynomials appearing in (3.18). Thus $\mathbf{v}$ fulfils

$$D(x\,\mathbf{v}) = \left(\tfrac{2}{3} - x\right)\mathbf{v}\,,$$

hence, by Table 1, $q_n = L_n^{(-1/3)}$. This agrees with line (24) in Table 3.

References

[1] W. Al-Salam, W. R. Allaway, R. Askey, R., *Sieved ultraspherical polynomials*, Trans. Amer. Math. Soc. 284 (1984) 39–55.

[2] P. Barrucand, D. Dickinson, *On cubic transformations of orthogonal polynomials*, Proc. Amer. Math. Soc. 17 (1966) 810–814.

[3] D. Bessis, P. Moussa, *Orthogonality properties of iterated polynomial mappings*, Commun. Math. Phys. 88 (1983) 503–529.

[4] C. Brezinski, P. Maroni, *The algebra of linear functionals on polynomials, with applications to Padé approximation*, Numer. Algorithms 11 (1996) 25–33.

[5] J. Charris, M. E. H. Ismail, *On sieved orthogonal polynomials. VII. Generalized polynomial mappings*, Trans. Amer. Math. Soc. 340 (1993) 71–93.

[6] J. Charris, M. E. H. Ismail, S. Monsalve, *On sieved orthogonal polynomials. X. General blocks of recurrence relations*, Pacific J. Math. 163 (1994) 237–267.

[7] T. S. Chihara, *On kernel polynomials and related systems*, Boll. Un. Mat. Ital. 19 (1964) 451–459.

[8] T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach Science Publishers, New York-London-Paris (1978).

[9] C. M. da Fonseca, J. Petronilho, *Explicit inverse of a tridiagonal $k-$Toeplitz matrix*, Numer. Math. 100 (2005) 457–482.

[10] M. N. de Jesus, J. Petronilho, *On orthogonal polynomials obtained via polynomial mappings*, J. Approx. Theory 162 (2010) 2243–2277.

[11] J. Geronimo, W. Van Assche, *Orthogonal polynomials on several intervals via a polynomial mapping*, Trans. Amer. Math. Soc. 308 (1986) 559–581.

[12] Ya. L. Geronimus, *Sur les polynômes orthogonaux relatifs à une suite de nombres donnée et sur le théorème de W. Hahn*, (Russian. French summary), Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4 (1940) 215–228.

[13] P. D. Lax, *Functional Analysis*, Wiley-Interscience [John Wiley & Sons], New York (2002).

[14] F. Marcellán, J. Petronilho, *Eigenproblems for tridiagonal $2-$Toeplitz matrices and quadratic polynomial mappings*, Linear Algebra Appl. 260 (1997)169–208.

[15] F. Marcellán, J. Petronilho, *Orthogonal polynomials and quadratic transformations*, Portugal. Math. 56 (1999) 81–113.

[16] F. Marcellán, J. Petronilho, *Orthogonal polynomials and cubic polynomial mappings I*, Commun. Anal. Theory Contin. Fract. 8 (2000) 88–116.

[17] F. Marcellán, J. Petronilho, *Orthogonal polynomials and cubic polynomial mappings II: The positive-definite case*, Commun. Anal. Theory Contin. Fract. 9 (2001) 11–20.

[18] P. Maroni, *Sur quelques espaces de distributions qui sont des formes linéaires sur l'espace vectoriel des polynômes*, In: Brezinski, C. et al. (eds.), Simposium Laguerre, Bar-le-Duc, Lecture Notes in Math. 1171 (1985) 184–194.

[19] P. Maroni, *Prolégomènes à l'étude des polynômes orthogonaux semiclassiques*, Ann. Mat. Pura Appl. 149 (1987) 165–184.

[20] P. Maroni, *Le calcul des formes linéaires et les polynômes orthogonaux semiclassiques*, In: Alfaro, M. et al. (eds.), Orthogonal Polynomials and Their Applications, Lecture Notes in Math. 1329 (1988) 279–290 .

[21] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Applications aux polynômes orthogonaux semiclassiques*, In: Brezinski, C. et al. (eds.), Orthogonal Polynomials and Their Applications, Proc. Erice 1990, IMACS, Ann. Comp. App. Math. 9 (1991) 95–130.

[22] J. C. Medem, *A family of singular semi-classical functionals*, Indag. Math. 13 (2002) 351–362.

[23] F. Peherstorfer, *Inverse images of polynomial mappings and polynomials orthogonal on them*, J. Comput. Appl. Math. 153 (2003) 371–385.

[24] J. Petronilho, *Topological aspects in the theory of orthogonal polynomials and an inverse problem*, In: Bento, A. et al. (eds.), Proceedings of the Workshop on Analysis, The J. A. Sampaio Martins Anniversary Volume, Textos de Mat. Sér. B Univ. Coimbra 34 (2004) 91–107.

[25] J. Shohat, *A differential equation for orthogonal polynomials*, Duke Math. J. 5 (1939) 401–417.

[26] M. Reed, B. Simon, *Methods of modern mathematical physics. I. Functional analysis.* Second edition. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York (1980).

[27] G. Szegő, *Orthogonal polynomials.* Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I. (1975).

[28] M. I. Tounsi, I. Bouguerra, *Cubic decomposition of a family of semiclassical polynomial sequences of class one*, Integral Transform. Spec. Funct. 26 (2015) 377–394.

[29] F. Trèves, *Topological vector spaces, distributions and kernels.* Academic Press, New York-London (1967)

[30] W. Van Assche, *Ortogonal polynomials, associated polynomials and functions of the second kind*, J. Comput. Appl. Math. 37 (1991) 237–249.

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