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A BOOLEAN EXTENSION OF A FRAME AND A REPRESENTATION OF DISCONTINUITY

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Abstract: Point-free modeling of mappings that are not necessarily continuous has been so far based on the extension of a frame to its frame of sublocales, mimicking the replacement of a topological space by its discretization. This otherwise successful procedure has, however, certain disadvantages making it not quite parallel with the classical theory (see Introduction). We mend it in this paper using a certain extension $\mathsf{S}_\mathfrak{c}(L)$ of a frame L, which is, a.o., Boolean and idempotent. Doing this we do not lose the merits of the previous approach. In particular we show that it yields the desired results in the treatment of semicontinuity. Also, there is no obstacle to using it as a basis of a point-free theory of rings of real functions; the "ring of all real functions" $\mathsf{F}(L) = \mathsf{C}(\mathsf{S}_\mathfrak{c}(L))$ is now order complete.

Keywords: Frame, locale, subfit, regular, sublocale, sublocale lattice, open sublocale, closed sublocale, real function, lower and upper semicontinuities, lower and upper regularizations.

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Introduction

Frame homomorphisms $h\colon M \to L$ represent continuous maps between frames (locales) M, L viewed as generalized spaces (in case of $L = \Omega(X)$, $M = \Omega(Y)$ with classical sober spaces X, Y they represent them precisely). Now if we wish to deal with more general real functions $X \to \mathbb{R}$ (say, the semicontinuous ones, or even the maps not continuous at all) we have to replace the frame L by another one, naturally associated with L, that is "much more discrete". In the classical setting, of course, we can take the same underlying set with the discrete topology; in terms of the frames of open sets this amounts to take the set $\mathfrak{P}(X)$ of all the subsets (subspaces) of X instead of the $\Omega(X)$.

This was done in a nice way in [7], and further developed in papers such as [5, 6], by extending the frame (locale) L by the dual $\mathcal{S}(L) = \mathsf{S}(L)^{\mathrm{op}}$ of the system of all sublocales ($\cong$ the frame of all congruences on L). This

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extension $L \to \reflectbox{$\mathsf{S}$}(L)$ indeed allowed for a very expedient mimicking of the classical theory of embedding the ring of continuous functions into that of all functions, and of related facts.

However, this approach also has certain disadvantages:

(1) $\reflectbox{$\mathsf{S}$}(L)$ can be very big; in particular for spaces $\reflectbox{$\mathsf{S}$}(\Omega(X))$ can be (and typically is) much bigger than the system of all subspaces $\mathfrak{P}(X)$.
(2) Although $\reflectbox{$\mathsf{S}$}(L)$ is very strongly disconnected, it is (again typically) not a Boolean algebra. And for a good analogy we would wish for a natural counterpart of a discrete space – which is a Boolean frame.
(3) The theory is not quite conservative. When applied to semicontinuity in classical spaces it is satisfactory, but the general not necessarily classical functions are represented only by analogy.
(4) The construction is not idempotent, that is, $\reflectbox{$\mathsf{S}$}(\reflectbox{$\mathsf{S}$}(L))$ is typically bigger than $\reflectbox{$\mathsf{S}$}(L)$, as if the discontinuous functions were not discontinuous enough, and needed a further extension to get a representation of "more discontinuous ones" (and again and again).

In this paper we present a variant that might be viewed as a more satisfactory one. Instead of $\reflectbox{$\mathsf{S}$}(L)$ we use a smaller $\mathsf{S}_{\mathfrak{c}}(L)$, a frame of closedly generated sublocales, that occurred before in connection with the study of scatteredness ([4], [15]). Comparing the result with the points above we have that:

(1') If $L = \Omega(X)$ for a T_1-space then $\mathsf{S}_{\mathfrak{c}}(L) \cong \mathfrak{P}(X)$.
(2') If L is subfit (and we are mostly concerned with the much stronger regularity) then $\mathsf{S}_{\mathfrak{c}}(L)$ is Boolean and hence can be viewed as a "discrete cover" of L.
(3') The theory is now conservative, already starting with the representation of general mappings.
(4') $\mathsf{S}_{\mathfrak{c}}(\mathsf{S}_{\mathfrak{c}}(L)) \cong \mathsf{S}_{\mathfrak{c}}(L)$ and hence the discretization is made once for ever.

In representing semicontinuous mappings it is natural to choose another approach instead of extending the locale L. One can keep the L but instead to change the topology of the reals. Thus, one can view the lower semicontinuous functions $X \to \mathbb{R}$ as the continuous functions $X \to \mathbb{R}_u$ where we take the topology of open up-sets on $\mathbb{R}$, and similarly with the upper semicontinuous ones. In the point-free setting, one can replace the point-free reals $\mathfrak{L}\mathbb{R}$ by a naturally modified subframe $\mathfrak{L}_u\mathbb{R}$ and consider the homomorphisms $\mathfrak{L}_u\mathbb{R} \to L$ (and similarly for the upper semicontinuity). Nevertheless, semicontinuous maps are specific (not quite continuous) mappings, and the question naturally arises whether they will also naturally appear as specific homomorphisms $\mathfrak{L}\mathbb{R} \to \mathsf{S}_{\mathfrak{c}}(L)$. And indeed this is the case. We have a completing of the diagram

$$\begin{tikzcd}
\mathfrak{L}_u\mathbb{R} \arrow[r, "h"] \arrow[d, hook] & L \arrow[d, hook, "\mathfrak{o}_L"] \\
\mathfrak{L}\mathbb{R} \arrow[r, "\overline{h}"'] & \mathsf{S}_{\mathfrak{c}}(L)
\end{tikzcd}$$

subjected to a certain assumption $(*)$ on the h, under which one can specify the upper and lower semicontinuous functions by conditions akin to the classical ones. Moreover, the formula for $(*)$ is one that has already appeared in the literature ([12, 8]) in connection with point-free representation of certain function insertion facts, and the result above explains what happened there.

In this article we do not consider the theory of rings of real functions, where the extension of L to $\text{Ƨ}(L)$ plays a fundamental role. Let us just note that the replacement of the $\text{Ƨ}(L)$ by $\mathsf{S}_{\mathfrak{c}}(L)$ does not create obstacles in developing the theory. In fact, in some respects this approach may give better parallels with the classical facts. For instance, similarly like in spaces, by Proposition 1 of Banaschewski-Hong ([3]), the "ring of all real functions" $\mathsf{F}(L) = \mathsf{C}(\mathsf{S}_{\mathfrak{c}}(L))$ is always order complete, unlike the $\mathsf{C}(\text{Ƨ}(L))$.

The paper is organized as follows.

In Preliminaries we introduce the necessary definitions and notation, and the point-free approach to reals. In the next section we recall the techniques of sublocales and introduce the frame $\mathsf{S}_{\mathfrak{c}}(L)$ and those of its properties we will need. In Section 3 we briefly discuss replacing the frame of sublocales $\text{Ƨ}(L)$ by the $\mathsf{S}_{\mathfrak{c}}(L)$. The following section contains the main results, in particular those on semicontinuity which is used to advantage in the framework based on the Boolean $\mathsf{S}_{\mathfrak{c}}(L)$, also with a reformulation of the regularization construction. Sections 3 and 4 cover all the claims formulated above in the points (1') – (4').

1. Preliminaries

1.1. Notation. For a subset A of a poset $(X, \leq)$ we write

$$\uparrow A = \{x \in X \mid x \geq a \text{ for some } a \in A\} \quad \text{and abbreviate } \uparrow\{x\} \text{ to } \uparrow x;$$

if $\uparrow A = A$ we speak of an *up-set.*

$\bigvee \emptyset$, the smallest element of a poset $P = (X, \leq)$ (if it exists) will be denoted by 0_P or simply by 0, and similarly we denote by 1_P or 1 the largest element.

1.2. Pseudocomplements and complements. A *pseudocomplement* of a is the largest x such that $x \wedge a = 0$. If it exists it is uniquely determined; it will be denoted by a^*.

An element b is a *complement* of a if $a \wedge b = 0$ and $a \vee b = 1$. If such a b exists we say that a is *complemented.* In a distributive lattice (our posets will be always such) each complement is a pseudocomplement and we will write a^* also for a complement of a.

1.3. Frames and coframes. A *frame*, resp. *coframe*, is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}, \tag{frm}$$

$$\text{resp.} \quad (\bigwedge A) \vee b = \bigwedge\{a \vee b \mid a \in A\}, \tag{cofrm}$$

for all $A \subseteq L$ and $b \in L$; a *frame* (resp. *coframe*) *homomorphism* preserves all joins and all finite meets (resp. all meets and all finite joins). The lattice $\Omega(X)$ of all open subsets of a topological space X is a typical frame, and a typical frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ is obtained from a continuous $f\colon X \to Y$ by setting $\Omega(f)(U) = f^{-1}[U]$. Thus we have a contravariant functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Frm}$$

where **Frm** designates the category of frames. This functor is a (contravariant) full embedding on the subcategory of sober spaces which justifies taking Ω for a functor extending **Top** to a category $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$ and viewing **Loc** as a category of generalized spaces[1].

[1] It is a (not quite consequently followed) custom to speak of a frame as of a *locale* when emphasizing the geometric interpretation.

1.4. The Heyting resp. co-Heyting structure. The law (frm) states that each of the maps $x \mapsto a \wedge x$ preserves suprema; since our lattice is complete, it is standard that, hence, they are left adjoints, which creates a Heyting operation $\rightarrow$ with

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \rightarrow c.$$

The Heyting structure on a frame will be used in the sequel without further mentioning.

Similarly the rule (cofrm) creates on a coframe a co-Heyting operation $\smallsetminus$ (the *difference*) with

$$a \smallsetminus b \leq c \quad \text{iff} \quad a \leq b \vee c.$$

Note that we have the pseudocomplements in a frame given by the formula $a^* = a \rightarrow 0$.

1.5. Regular and subfit frames. A frame L is *regular* if

$$\forall a \in L,\ a = \bigvee \{x \mid x \prec a\}$$

where $x \prec a$ stands for $x^* \vee a = 1$ (in other words, there is a y such that $y \wedge x = 0$ and $y \vee a = 1$, which formulation is sometimes handier). For a space X and open subsets $U, V \subseteq X$, $U \prec V$ holds iff $\overline{U} \subseteq V$, and hence $\Omega(X)$ is regular in the sense above iff X is regular in the standard topological one.

A frame is *subfit* if

$$a \nleq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \neq b \vee c. \tag{sfit}$$

It is easy to see that this is implied by regularity. The original definition ([9], see 2.4 below) was formulated otherwise, this first order formula is of a later date ([16]).

In spaces, the subfitness of $\Omega(X)$ is slightly weaker than T_1. In fact, $T_1 \equiv$(sfit)&T_D (T_D requires that for each $x \in X$ there is an open $U \ni x$ such that $U \smallsetminus \{x\}$ is open – see [1]).

Subfitness is not inherited in subobjects; the hereditary variant, *fitness* is, somewhat surprisingly, a fairly strong property already close to regularity (see [14]).

1.6. Frames of reals. The *frame of reals* $\mathfrak{L}\mathbb{R}$ ([2]) will be defined without reference to the classical real line $\mathbb{R}$ (but under the Axiom of Choice, $\mathfrak{L}\mathbb{R} \cong \Omega(\mathbb{R})$).

We start with the rational line $\mathbb{Q}$ and obtain $\mathfrak{L}\mathbb{R}$ as the frame generated by all ordered pairs $(p, q) \in \mathbb{Q} \times \mathbb{Q}$ satisfying the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.
(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.
(R4) $\bigvee_{p,q\in\mathbb{Q}}(p, q) = 1$.

Equivalently, $\mathfrak{L}\mathbb{R}$ may be defined as the frame with generators $(p, —)$ and $(—, q)$, $p, q \in \mathbb{Q}$, subject to the following relations:

(r1) $(p, —) \wedge (—, q) = 0$ whenever $p \geq q$.
(r2) $(p, —) \vee (—, q) = 1$ whenever $p < q$.
(r3) $(p, —) = \bigvee_{r>p}(r, —)$, for every $p \in \mathbb{Q}$.
(r4) $(—, q) = \bigvee_{s<q}(—, s)$, for every $q \in \mathbb{Q}$.
(r5) $\bigvee_{p\in\mathbb{Q}}(p, —) = 1$.
(r6) $\bigvee_{q\in\mathbb{Q}}(—, q) = 1$.

With $(p, q) = (p, —) \wedge (—, q)$ one goes back to (R1)-(R4) and defining $(p, —) = \bigvee_{r\in\mathbb{Q}}(p, r)$ and $(—, q) = \bigvee_{r\in\mathbb{Q}}(r, q)$ one recovers the $(p, —)$ and $(—, q)$.

For more details about this alternative approach we refer the reader to [8]. In particular, it yields the frames $\mathfrak{L}_u\mathbb{R}$ and $\mathfrak{L}_l\mathbb{R}$ of upper and lower reals: they are the subframes of $\mathfrak{L}\mathbb{R}$ given by

$$\mathfrak{L}_u\mathbb{R} = \langle\{(p, —) \mid p \in \mathbb{Q}, (p, —) \text{ satisfy (r3) and (r5) for all } p \in \mathbb{Q}\}\rangle,$$
$$\mathfrak{L}_l\mathbb{R} = \langle\{(—, q) \mid q \in \mathbb{Q}, (—, q) \text{ satisfy (r4) and (r6) for all } q \in \mathbb{Q}\}\rangle.$$

Note that in order to define a frame homomorphism $h \colon \mathfrak{L}\mathbb{R} \to L$ it suffices to define it on the generators $(p, —)$ and $(—, q)$ and to check that it turns the defining relations (r1)-(r6) into identities in the frame L.

2. The frame $\mathsf{S}_\mathfrak{c}(L)$

The lattice of sublocales to be discussed in the later part of this section is generally a frame, as stated in the title, but in all the known cases it is *both a frame and a coframe*. In the cases we will be interested in it is even *Boolean*, which is essential for our investigations.

2.1. The coframe of sublocales. A *sublocale* of a frame (locale) L is a subset $S \subseteq L$ such that

(S1) for every $M \subseteq S$ the meet $\bigwedge M$ is in S (hence each sublocale contains $1_L = \bigwedge \emptyset$), and
(S2) for every $s \in S$ and every $x \in L$, $x \to s \in S$.

Note that if we view frames (locales) as generalized spaces, the sublocales are indeed the *sub-locales* (that is, the subobjects in the concrete category **Loc** of locales) in the sense that they are precisely the subsets the embeddings of which are extremal monomorphisms in **Loc**.

The system of all sublocales of L ordered by inclusion,

$$\mathsf{S}(L),$$

is (obviously) a complete lattice with

$$\bigwedge S_i = \bigcap S_i \quad \text{and} \quad \bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\};$$

the least sublocale $\bigvee \emptyset = \{1\}$ is denoted by O and referred to as the *void sublocale*[2]. It is a fundamental fact that

2.1.1. *The lattice* $\mathsf{S}(L)$ *is a coframe.* (see [10] or [13]).

2.2. Open and closed sublocales. With each element $a \in L$ there is associated an *open sublocale*

$$\mathfrak{o}(a) = \{x \in L \mid x = a \to x\} = \{a \to x \mid x \in L\}$$

and a *closed sublocale*

$$\mathfrak{c}(a) = {\uparrow} a.$$

The open resp. closed sublocales precisely correspond to the *open and closed parts* in Isbell's pioneering article [9], and in the spatial case $L = \Omega(X)$ they correspond to open and closed subspaces of X.

2.2.1. Facts. 1. $\mathfrak{o}(a)$ *and* $\mathfrak{c}(a)$ *are complements of each other in* $\mathsf{S}(L)$.
2. *We have* $\mathfrak{o}(0) = \mathsf{O}$, $\mathfrak{o}(1) = L$, $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b)$ *and* $\mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i)$.
3. *We have* $\mathfrak{c}(0) = L$, $\mathfrak{c}(1) = \mathsf{O}$, $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b)$ *and* $\mathfrak{c}(\bigvee a_i) = \bigcap \mathfrak{c}(a_i)$.

[2] This may sound queer but it makes good sense; if L happens to have points P, they are of the form $\{a, 1\}$ with prime $a \neq 1$.

2.2.2. Interior and closure. Since joins of open sublocales are open we have the interior of a sublocale

$$\operatorname{int} S = \bigvee\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S\}$$

and the closure $\overline{S} = \bigwedge\{\mathfrak{c}(a) \mid S \subseteq \mathfrak{c}(a)\}$. Note that for the closure one has a particularly simple formula $\overline{S} = \uparrow \bigwedge S$.

2.3. The frame of sublocales. Besides the coframe of sublocales $\mathsf{S}(L)$ one also uses its dual

$$\mathsf{Z}(L) = \mathsf{S}(L)^{\mathrm{op}}.$$

Note that it is a frame and that we have by 2.2.1.3 an embedding $\mathfrak{c}_L \colon L \to \mathsf{Z}(L)$ that can be viewed as the extension of the *generalized space* L to a *generalized space* $\mathsf{Z}(L)$ with certain useful properties.

The frame $\mathsf{Z}(L)$ is isomorphic to the frame of congruences (or nuclei) on L and hence it starts the well known *tower*

$$L \to \mathsf{Z}(L) \to \mathsf{Z}^2(L) \to \cdots \to \mathsf{Z}^\alpha(L) \to \cdots$$

(see e.g. [10]).

2.4. Subfitness and fitness in terms of the behaviour of sublocales. A frame is subfit (recall 1.5 above) iff

each open sublocale is a join of closed ones

and it is fit iff

each closed sublocale is a meet (*intersection*) of open ones

(in fact those were the original definitions in Isbell's [9], the first order definitions we use now came later – recall 1.5).

2.4.1. Joins of closed sublocales and the frame $\mathsf{S}_{\mathfrak{c}}(L)$. In fact, the characteristic of fitness above has a stronger equivalent, namely that a frame is fit iff

each sublocale is a meet of open ones.

The former statement, however, does not have such an extension (the statement that each sublocale whatsoever is a join of closed ones characterizes the *scattered subfit* frames ([4]), not the general subfit ones). While studying this phenomenon in [15] it was proved that

$$\mathsf{S}_{\mathfrak{c}}(L) = \{S \in \mathsf{S}(L) \mid S \text{ is a join of closed sublocales of } L\}$$

is always a *frame*. Hence we have here

a frame $\mathsf{S}_{\mathfrak{c}}(L)$ *embedded into a* coframe $\mathsf{S}(L)$ *preserving the joins.*

2.4.2. The meets are not generally preserved, but of course if a meet of a system in $\mathsf{S}_{\mathfrak{c}}(L)$ taken in $\mathsf{S}(L)$ happens to be in $\mathsf{S}_{\mathfrak{c}}(L)$ then the two meets coincide.

2.5. Some properties of the frame $\mathsf{S}_{\mathfrak{c}}(L)$. (The proofs are in [15]).

2.5.1. By 2.4 we have for a subfit L a frame embedding

$$\mathfrak{o}_L \colon L \to \mathsf{S}_{\mathfrak{c}}(L).$$

Then, by 2.4.2,

$\mathsf{S}_{\mathfrak{c}}(L)$ *is closed under both the operations of interior and closure in* $\mathsf{S}(L)$.

2.5.2. If L is subfit (in particular if L is regular, which is the case we are mostly interested in this paper) then $\mathsf{S}_{\mathfrak{c}}(L)$ is a Boolean algebra. Hence we have here an extension of L to a Boolean frame, which can be in localic terms interpreted as a cover of the locale L by an (in a generalized way) discrete locale $\mathsf{S}_{\mathfrak{c}}(L)$. Furthermore, $\mathsf{S}_{\mathfrak{c}}(\mathsf{S}_{\mathfrak{c}}(L)) \cong \mathsf{S}_{\mathfrak{c}}(L)$, so that we have such an extension "once for ever".

2.5.3. If X is a T_1-space then

$$\mathsf{S}_{\mathfrak{c}}(\Omega(X)) \cong \mathfrak{P}(X)$$

and $\mathsf{S}_{\mathfrak{c}}(\Omega(X))$ can be naturally viewed as the system of all classical subspaces of X.

2.6. The relation of $\mathcal{Z}(L)$ and $\mathsf{S}_{\mathfrak{c}}(L)$. If L is subfit then $\mathsf{S}_{\mathfrak{c}}(L)$ is the Booleanization of $\mathcal{Z}(L)$. Furthermore, since $\mathfrak{c}\colon L \to \mathcal{Z}(L)$ is universal in among the frame homomorphisms $f\colon L \to M$ with $f[L]$ complemented in M (in the sense that such an f can be lifted to an $h\colon \mathcal{Z}(L) \to M$), there is an $h\colon \mathcal{Z}(L) \to \mathsf{S}_{\mathfrak{c}}(L)$ such that $h \cdot \mathfrak{c} = \mathfrak{o}$ and this h is given by the formula $h(S) = S^*$.

3. Point-free semicontinuity and discontinuity

3.1. Point-free semicontinuity in the literature. Upper and lower semicontinuous real functions in point-free topology were introduced by Li and Wang ([11]) but their approach did not faithfully reflect the classical notions, so they faced some problems in formulating Katětov-Tong insertion theorem in its full generality (see [12]). This was mended in [12] and [8]. There a *lower* *semicontinuous function* on a frame L is defined as a frame homomorphism $f\colon \mathfrak{L}_u\mathbb{R} \to L$ satisfying

$$\bigwedge_{p\in\mathbb{Q}} \mathfrak{o}(f(p,\text{—})) = 0 \tag{3.1.1}$$

in the coframe of sublocales $\mathsf{S}(L)$. Analogously, an *upper semicontinuous function* on L is a frame homomorphism $f\colon \mathfrak{L}_l\mathbb{R} \to L$ satisfying

$$\bigwedge_{q\in\mathbb{Q}} \mathfrak{o}(f(\text{—},q)) = 0. \tag{3.1.2}$$

3.2. Semicontinuities in a more general setting. The treatment of point-free semicontinuities was then placed in a more general and elegant setting, allowing for a treatment of the concept of an *arbitrary* not necessarily (semi)continuous real function on a frame L, by Gutiérrez García-Kubiak-Picado in [7]. The main feature of this improved framework is that now all those continuous and semicontinuous functions have common domains and are therefore members of a unique lattice-ordered ring of real valued functions formed by all frame homomorphisms $\mathfrak{L}\mathbb{R} \to \mathcal{S}(L)$.

A *lower semicontinuous function* on a frame L is a frame homomorphism $f\colon \mathfrak{L}\mathbb{R} \to \mathcal{S}(L)$ such that $f(p,\text{—})$ is a closed sublocale for every $p \in \mathbb{Q}$, and an *upper semicontinuous function* on L is a frame homomorphism $f\colon \mathfrak{L}\mathbb{R} \to \mathcal{S}(L)$ for which $f(\text{—},q)$ is a closed sublocale for every $q \in \mathbb{Q}$. Continuous functions are just the ones in the intersection of the two classes.

3.3. Discretization by $\mathsf{S}_\mathfrak{c}(L)$. The frame $\mathcal{S}(L)$ played the role of a sort of "discretization" of the frame L. The analogy is obvious: if we wish to consider for a moment the general, not necessarily continuous maps $f\colon X \to Y$ between topological spaces, we can consider the (continuous) $f\colon \mathcal{D}X \to Y$ where $\mathcal{D}X$ is the discrete space with the same underlying set as X. Now $\mathcal{S}(L)$ is an extension of L that is "much more discrete" than L (in the sense that it is very strongly disconnected, that is, the complemented elements play a very prominent part).

But the natural counterpart of discrete spaces in point-free topology are the Boolean frames (locales), that is, we would like to have all the elements complemented, and this is not the case with the $\mathcal{S}(L)$. Instead, to follow the analogy with spaces, we will restrict ourselves to the subfit frames and harness the Boolean frame $\mathsf{S}_\mathfrak{c}(L)$.

3.4. Conservativeness, to start with. For a space X, there is no natural correspondence between the general maps $f\colon X \to \mathbb{R}$ and the homomorphisms $\mathfrak{L}\mathbb{R} \to \mathcal{S}(\Omega(X))$ (the frame $\mathcal{S}(\Omega(X))$ is not isomorphic to $\Omega(\mathcal{D}X)$. By 2.5.3 we have, however,

3.4.1. Corollary. *If X is a T_1-space then $\mathsf{S}_\mathfrak{c}(\Omega(X)) \cong \Omega(\mathcal{D}X)$.*

3.4.2. Note. The restriction to T_1-spaces is not a bad loss of generality. It is a (partial) counterpart of our restriction to subfit frames. The full counterpart, a restriction to subfit *spaces* (the concept makes sense in spaces, see 1.5) would yield a quasidiscrete extension which is not what we need.

3.5. Now we have a conservative extension of the concept of a general map. The question naturally arises whether we may not lose the facts of 3.2. The structure of $\mathsf{S}_\mathfrak{c}(L)$ is not as rich as that of $\mathcal{S}(L)$ which helped to establish the relation between the homomorphisms $\mathfrak{L}_u\mathbb{R} \to L$ and $\mathfrak{L}\mathbb{R} \to \mathcal{S}(L)$. But the correspondences will be as desired, as we will show in the following section.

4. Semicontinuities in the new framework

4.1. Lemma. *Let $h\colon \mathfrak{L}_u\mathbb{R} \to L$ be a frame homomorphism and let $\overline{h}\colon \mathfrak{L}\mathbb{R} \to \mathsf{S}_\mathfrak{c}(L)$ be a frame homomorphism such that*

$$\overline{h}(p,\text{—}) = \mathfrak{o}(h(p,\text{—})) \quad \textit{for every } p \in \mathbb{Q}.$$

Then $\overline{h}(\text{—},q) = \bigvee_{s<q} \mathfrak{c}(h(s,\text{—}))$ for every $q \in \mathbb{Q}$.

Proof: "$\leq$": By (r4), $\overline{h}(\text{—},q) = \bigvee_{s<q} \overline{h}(\text{—},s)$. Consider any $s < q$. Since $\overline{h}(\text{—},s) \wedge \overline{h}(s,\text{—}) = 0$ by (r1), then $\overline{h}(\text{—},s) \leq \mathfrak{c}(h(s,\text{—}))$.
"$\geq$": For each $s < q$, $\overline{h}(\text{—},q) \vee \overline{h}(s,\text{—}) = 1$ and thus $\mathfrak{c}(h(s,\text{—})) \leq \overline{h}(\text{—},q)$. ∎

4.2. Proposition. *Let $h\colon \mathfrak{L}_u\mathbb{R} \to L$ be a frame homomorphism, let $j\colon \mathfrak{L}_u\mathbb{R} \to \mathfrak{L}\mathbb{R}$ be the subframe embedding and $\mathfrak{o}_L\colon L \to \mathsf{S}_\mathfrak{c}(L)$ the embedding from 2.5.1. The h can be extended to a frame homomorphism $\overline{h}$ in such a way that the square*

$$\begin{tikzcd}
\mathfrak{L}_u\mathbb{R} \arrow[r, "h"] \arrow[d, hook, "j"'] & L \arrow[d, hook, "\mathfrak{o}_L"] \\
\mathfrak{L}\mathbb{R} \arrow[r, dotted, "\overline{h}"'] & \mathsf{S}_\mathfrak{c}(L)
\end{tikzcd}$$

is commutative if and only if

$$\bigvee_{p\in\mathbb{Q}} \mathfrak{c}(h(p,\text{—})) = 1. \tag{4.2.1}$$

In this case, the $\overline{h}$ *is unique.*

Proof: By the lemma, if there is such $\overline{h}$ then necessarily $\overline{h}(p,\text{—}) = \mathfrak{o}(h(p,\text{—}))$ and $\overline{h}(\text{—},q) = \bigvee_{s<q} \mathfrak{c}(h(s,\text{—}))$ and the uniqueness of $\overline{h}$ is proved. It remains to check that this monotone $\overline{h}$ is indeed a frame homomorphism if and only if (4.2.1) holds. The necessity of (4.2.1) follows immediately from the fact that $\overline{h}$ must turn relation (r6) into an identity in the frame $\mathsf{S}_\mathfrak{c}(L)$. So, let us assume it holds. We need to check that the remaining relations (r1)-(r5) are preserved by $\overline{h}$:

(r1): Let $p \geq q$. Then

$$\begin{aligned}\overline{h}(\text{—},q) \wedge \overline{h}(p,\text{—}) &= \Big(\bigvee_{s<q} \mathfrak{c}(h(s,\text{—}))\Big) \wedge \overline{h}(p,\text{—}) \\ &= \bigvee_{s<q} (\mathfrak{c}(h(s,\text{—})) \wedge \overline{h}(p,\text{—})) \\ &\leq \bigvee_{s<q} (\mathfrak{c}(h(s,\text{—})) \wedge \overline{h}(s,\text{—})) = 0.\end{aligned}$$

(r2): Let $p < q$. Then

$$\overline{h}(p,\text{—}) \vee \overline{h}(\text{—},q) = \mathfrak{o}(h(p,\text{—})) \vee \bigvee_{s<q} \mathfrak{c}(h(s,\text{—})).$$

Take some r such that $p < r < q$. Then, immediately, $\mathfrak{o}(h(p,\text{—}))\vee\bigvee_{s<q} \mathfrak{c}(h(s,\text{—})) \geq \mathfrak{o}(h(r,\text{—})) \vee \mathfrak{c}(h(r,\text{—})) = 1$.

(r3): For each $p \in \mathbb{Q}$, $\overline{h}(p,\text{—}) = \mathfrak{o}(h(p,\text{—})) = \mathfrak{o}(\bigvee_{r>p} h(r,\text{—})) = \bigvee_{r>p} \mathfrak{o}(h(r,\text{—}))$.

(r4): For each $q \in \mathbb{Q}$, $\overline{h}(\text{—},q) = \bigvee_{s<q} \mathfrak{c}(h(s,\text{—}))$. On the other hand, $\bigvee_{s<q} \overline{h}(\text{—},s) = \bigvee_{s<q} \bigvee_{r<s} \mathfrak{c}(h(r,\text{—}))$, which is clearly the same join as $\bigvee_{s<q} \mathfrak{c}(h(s,\text{—}))$.

(r5): $\bigvee_{p\in\mathbb{Q}} \overline{h}(p,\text{—}) = \bigvee_p \mathfrak{o}(h(p,\text{—})) = \mathfrak{o}(\bigvee_p h(p,\text{—})) = \mathfrak{o}(1) = 1.$ ■

4.2.1. Note. Condition (4.2.1) above is equivalent to the (3.1.1) of Gutiérrez García-Picado formulated in the Boolean $\mathsf{S}_\mathfrak{c}(L)$.

4.3. Corollary. *The correspondence* $h \mapsto \overline{h}$ *of the preceding proposition establishes a bijection between the set of all frame homomorphisms* $h\colon \mathfrak{L}_u\mathbb{R} \to L$

satisfying (4.2.1) *and the set of all frame homomorphisms* $\overline{h}\colon \mathfrak{L}\mathbb{R} \to \mathsf{S}_{\mathfrak{c}}(L)$ *such that* $\overline{h}(p,\text{—}) \in \mathfrak{o}(L)$ *for every* $p \in \mathbb{Q}$.

Proof: The correspondence is clearly one-to-one. Let $\overline{h}\colon \mathfrak{L}\mathbb{R} \to \mathsf{S}_{\mathfrak{c}}(L)$ such that, for each $p \in \mathbb{Q}$, $\overline{h}(p,\text{—}) = \mathfrak{o}(a_p)$ for some $a_p \in L$. The map $h\colon \mathfrak{L}_u\mathbb{R} \to L$ given by $(p,\text{—}) \mapsto a_p$ is then a frame homomorphism and by applying Lemma 4.1 we conclude that it satisfies (4.2.1):

$$1 = \bigvee_{q\in\mathbb{Q}} \overline{h}(\text{—},q) = \bigvee_{q}\bigvee_{s<q} \mathfrak{c}(h(s,\text{—})) = \bigvee_{s} \mathfrak{c}(h(s,\text{—})). \quad \blacksquare$$

4.4. This motivates now to introduce general real functions on a frame L as arbitrary frame homomorphisms $\mathfrak{L}\mathbb{R} \to \mathsf{S}_{\mathfrak{c}}(L)$, lower semicontinuous functions on L as frame homomorphisms $f\colon \mathfrak{L}\mathbb{R} \to \mathsf{S}_{\mathfrak{c}}(L)$ such that each $f(p,\text{—})$ is an open sublocale and, similarly, upper semicontinuous functions on a frame L as frame homomorphisms $f\colon \mathfrak{L}\mathbb{R} \to \mathsf{S}_{\mathfrak{c}}(L)$ such that each $f(\text{—},q)$ is an open sublocale. Continuous functions are the ones that are both lower and upper semicontinuous. These classes of morphisms are partially ordered by

$$f \le g \equiv f(p,\text{—}) \le g(p,\text{—}) \text{ for every } p \in \mathbb{Q}.$$

Equivalently, $f \le g$ iff $g(\text{—},p) \le f(\text{—},p)$ for every $p \in \mathbb{Q}$. Indeed, if $f(p,\text{—}) \le g(p,\text{—})$ for every $p \in \mathbb{Q}$, then $g(\text{—},p) = \bigvee_{r<p} g(\text{—},r)$ and, for each such r, $g(\text{—},r) = g(\text{—},r)^{**} \le g(r,\text{—})^{*} \le f(r,\text{—})^{*} \le f(\text{—},p)$.

4.5. Regularization of an arbitrary function. As an illustration of the advantage of the Boolean structure of $\mathsf{S}_{\mathfrak{c}}(L)$ we describe what the important lower and upper regularization[3] procedures of [5] look like now. Let f be a general function $\mathfrak{L}\mathbb{R} \to \mathsf{S}_{\mathfrak{c}}(L)$ and define a mapping f° on the generators of $\mathfrak{L}\mathbb{R}$ as follows:

$$f^{\circ}(p,\text{—}) = \bigvee_{s>p} \operatorname{int} f(s\text{—}) \quad \text{and} \quad f^{\circ}(\text{—},p) = \bigvee_{s<p} (\operatorname{int} f(s,\text{—}))^{*}.$$

It is a straightforward exercise to check that this mapping turns the defining relations (r1)-(r4) of $\mathfrak{L}\mathbb{R}$ into identities on $\mathsf{S}_{\mathfrak{c}}(L)$. Also with (r6):

$$\begin{aligned}\bigvee_{p\in\mathbb{Q}} f^{\circ}(\text{—},p) &= \bigvee_{p}(\operatorname{int} f(p,\text{—}))^{*} = (\bigwedge_{p} \operatorname{int} f(p,\text{—}))^{*}\\ &\ge (\bigwedge_{p} f(p,\text{—}))^{*} = \bigvee_{p} f(p,\text{—})^{*} \ge \bigvee_{p} f(\text{—},p) = 1.\end{aligned}$$

[3]The term *regularization* was coined in [5]. It comes from general topology and has nothing to do with the topological regularity.

But that does not happen necessarily with (r5). Clearly, it happens if and only if

$$\bigvee_{p\in\mathbb{Q}} \operatorname{int} f(p,\text{—}) = 1, \tag{4.5.1}$$

a boundedness condition that guarantees f° to have only values on the reals, not on the extended ones. Therefore, $f^\circ \in \mathrm{LSC}(L)$ whenever (4.5.1) holds.

This constructed f° is the *lower regularization* of the given f. Of course, if f is already lower semicontinuous, $f^\circ = f$. Moreover:

4.5.1. Proposition. *Let* $f, g\colon \mathfrak{L}\mathbb{R} \to \mathsf{S}_{\mathfrak{c}}(L)$ *satisfy* (4.5.1). *Then:*
(1) $f \le g \Rightarrow f^\circ \le g^\circ$.
(2) $f^\circ \le f$.
(3) $f^{\circ\circ} = f^\circ$.
(4) $f^\circ = \bigvee\{h \in \mathrm{LSC}(L) \mid h \le f\}$.

Proof: (1): $f^\circ(p,\text{—}) = \bigvee_{s>p} \operatorname{int} f(s,\text{—}) \le \bigvee_{s>p} \operatorname{int} g(s,\text{—}) = g^\circ(p,\text{—})$.
(2): $f^\circ(p,\text{—}) = \bigvee_{s>p} \operatorname{int} f(s,\text{—}) \le \bigvee_{s>p} f(s,\text{—}) = f(p,\text{—})$.
(3): This is also straightforward:

$$\begin{aligned} f^{\circ\circ}(p,\text{—}) &= \bigvee_{s>p} \operatorname{int} f^\circ(s,\text{—}) = \bigvee_{s>p} \operatorname{int}\big(\bigvee_{r>s} \operatorname{int} f(r,\text{—})\big) \\ &= \bigvee_{s>p}\bigvee_{r>s} \operatorname{int} f(r,\text{—}) = \bigvee_{r>p} \operatorname{int} f(r,\text{—}) = f^\circ(p,\text{—}). \end{aligned}$$

(4) follows immediately from (1) and (2). ■

4.6. Note on the spatial case $L = \Omega(X)$. In [8], the set of lower semicontinuous functions

$$f\colon X \to \mathbb{R}$$

were shown to be in a bijective correspondence with the set of frame homomorphisms

$$h\colon \mathfrak{L}_u\mathbb{R} \to \Omega(X)$$

satisfying a certain boundedness condition. Now we can show that this boundedness condition ((4.6.1) below) is included in our theory as the equality (4.2.1) of Proposition 4.2. (Everything under T_1.)

4.6.1. Proposition. *Let* $h\colon \mathfrak{L}_u\mathbb{R} \to \Omega(X)$ *be a frame homomorphism. If* X *is* T_1*, then condition* (4.2.1) *is equivalent to the following:*

$$\forall x \in X,\ \{p \in \mathbb{Q} \mid x \in h(p,\text{—})\} \textit{ is bounded from above.} \tag{4.6.1}$$

Proof: Let $U_p = h(p,—) \in \Omega(X)$. The U_p's form a cover of X, by (r5), and $U_p \subseteq U_q$ for every $p \geq q$.

Now, suppose X is T_1 and (4.2.1) holds, that is,

$$\bigvee_p {\uparrow}U_p = \Omega(X).$$

If there is some $y \in X$ such that the set $\{p \in \mathbb{Q} \mid y \in U_p\}$ is not bounded from above, that is, $y \in \bigcap_p U_p$, then we would have $\bigvee_p {\uparrow}U_p \neq \Omega(X)$, a contradiction, since the open $X \smallsetminus \{y\}$ would not belong to it. Indeed, if not, otherwise, we would have some $\{U_i\}_i \subseteq \bigcup_p {\uparrow}U_p$ such that $X \smallsetminus \{y\} = \operatorname{int}(\bigcap_p U_i)$. Then, for each i, $U_{p_i} \subseteq U_i$ for some $p_i \in \mathbb{Q}$, y would belong to $\bigcap_p U_p \subseteq \bigcap_i U_{p_i} \subseteq \bigcap_i U_i$ and, consequently, $\bigcap_i U_i$ would be all the X, a contradiction.

Conversely, suppose (4.6.1) holds and let us compute the join

$$\bigvee_p {\uparrow}U_p = \{\bigwedge \mathcal{A} \mid \mathcal{A} \subseteq \bigcup_p {\uparrow}U_p\}.$$

By (4.6.1), for each $x \in X$ there is some $p_x \in \mathbb{Q}$ such that $x \notin U_{p_x}$, that is, $U_{p_x} \subseteq X \smallsetminus \{x\} \in \Omega(X)$. Let U be an arbitrary open in $\Omega(X)$. Since

$$U = \bigcap_{x \notin U} (X \smallsetminus \{x\}) = \bigwedge_{x \notin U} (X \smallsetminus \{x\}),$$

it suffices to take

$$\mathcal{A} = \{X \smallsetminus \{x\} \mid x \notin U\} \subseteq \bigcup_p {\uparrow}U_p$$

in order to conclude that the arbitrary U is in $\bigvee_p {\uparrow}U_p$. Hence $\bigvee_p {\uparrow}U_p = \Omega(X)$. ■

From Proposition 4.6.1 and Corollary 4.3 we get immediately that

4.6.2. Corollary. *Let X be a T_1-space. The set of all lower semicontinuous real functions $f\colon X \to \mathbb{R}$ on X is in bijection with the set of all lower semicontinuous real functions $\mathfrak{L}\mathbb{R} \to \mathsf{S}_\mathfrak{c}(\Omega(X))$ on the frame $\Omega(X)$.* ■

This mends an inaccuracy in Corollary 4.3 of [8], which relies on a mistakenly quoted result used in formula (4.5) that clearly does not hold even for T_1-spaces: the $\mathcal{S}(\Omega(X))$ there should be replaced by our $\mathsf{S}_\mathfrak{c}(\Omega(X))$.

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