Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 17–01

POLYNOMIAL INEQUALITIES AND THE NONNEGATIVITY OF THE COEFFICIENTS OF CERTAIN POWER SERIES II

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ABSTRACT: Define the pseudosymmetric power sum polynomials $s_j = s_j(\underline{p}, \underline{x}) = \sum_{i=1}^n p_i x_i^j$, where the p_i are reals. For $i = 1, \dots, n$, let $h_i = x_i - x_{i+1}$, where $x_{n+1} = 0$; let $m_1, \dots, m_k \geq 1$ be integers; and let $p = s_{m_1} s_{m_2} \cdots s_{m_k}$. Assume p is expressed in the variables h_i. We give formulas expressing the coefficient of the monomial $h_1^{i_1} h_2^{i_2} \dots, h_n^{i_n}$ in p in dependence of $m_1, \dots, m_k$ and $(i_1, \dots, i_n)$. We use this for conjecturing, and partially proving, a strengthening of a series of coefficient inequalities of Holland [Hol] concerning the weighted harmonic mean $(\sum_{i=1}^n p_i(1-x_i t)^{-1})^{-1}$ when developed into a power series in t. The methods here developed extend work in a previous paper where coefficient inequalities of the geometric mean type power series $(1-x_1t)^{\alpha_1} \cdots (1-x_nt)^{\alpha_n}$, due to Laffey where reproved and they should be useful in general for proving inequalities for pseudosymmetric polynomials. As an offspin of our work, a surprising combinatorial identity is uncovered.

KEYWORDS: multivariate polynomial inequalities, power series, derivatives, combinatorial identities, symmetric functions.
MATH. SUBJECT CLASSIFICATION (2000): 26D05,26D15, 05A19, 05E05.

0. Introduction and Motivation

Along this paper we use $x_1, \dots, x_n$ as well for indeterminates as for real numbers in such a way that context will make clear the intended meaning. We sometimes write $\underline{x} = (x_1, \dots, x_n)$, and for $1 \leq i \leq j \leq n$ we may write $x_{i:j} := (x_i, \dots, x_j)$. Also the notation $Sx_{i:j} = x_i + \dots + x_j$ will help to lighten notation. Similar observations go for other letters than x.

We note that it is a trivial matter to rewrite any polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ as a polynomial in the quantities $h_i = x_i - x_{i+1}$, $i = 1, \dots, n-1$, $h_n = x_n$: substitute $x_i = h_i + h_{i+1} + \dots + h_n$ and develop p. In other words, defining $\sigma(\underline{h}) := \sigma(h_1, \dots, h_n) := (Sh_{1:n}, Sh_{2:n}, \dots, Sh_{n-1:n}, h_n)$, develop $(p \circ \sigma)(\underline{h})$.

Received January 9, 2017.
This research was supported by Centro de Matemática da Universidade de Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

The quest for such a development may arise when trying to prove polynomial inequalities; in particular in the cases in which one knows beforehand that it is enough to consider a certain ordering of variables, the method can be very efficient.

For example the most immediate proof for the symmetric polynomial inequality $p(x, y, z) = x^3 + y^3 + z^3 - 3xyz \geq 0$ for $x, y, z \geq 0$ might well be to note that without loss of generality one may assume $x \geq y \geq z \geq 0$ and hence $(x, y, z) = \sigma(\underline{h})$ for some $\underline{h} \in \mathbb{R}^3_{\geq 0}$, and then to compute that

$$(p \circ \sigma)(\underline{h}) = h_1^3 + 3h_1^2 h_2 + 3h_1 h_2^2 + 2h_2^3 + 3h_1^2 h_3 + 3h_1 h_2 h_3 + 3h_2^2 h_3.$$

The nonnegative coefficients turn the nonnegativity of p manifest in a most satisfactory way.

To carry out such a strategy if infinite families of polynomial inequalities are to be established, formulas expressing the coefficients of the monomials in $h_1, \ldots, h_n$ in dependence of the parameters defining the polynomials can obviously be useful.

As an auxiliary means towards this end, in [K] the following is proved and will be reproved for completeness here.

Let $p \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ and consider it as a polynomial in

$$\mathbb{R}[x_1, x_2, \ldots, x_n, T],$$

where $x_{n+1} = T$ is an additional indeterminate. Write $\partial^k_{x_i} p$ or $\partial^k_i p$ for the k-th derivative of p with respect to x_i, and indicate the transformation a polynomial q suffers by taking k times the derivative with respect to x_i, and then putting $x_{i+1} = x_i$ by

$$q \xrightarrow{\partial^k_i,\, x_i = x_{i+1}} \tilde{q},$$

where $\tilde{q}$ is the resulting polynomial.

Theorem 0.1. *Let $p \in \mathbb{R}[x_1, x_2, \ldots, x_n] \subseteq \mathbb{R}[x_1, x_2, \ldots, x_n, T]$. Then:*

a. *p is a finite sum of terms of the form*

$$\frac{\tilde{p}_{i_1 i_2 \ldots i_n}(T)}{i_1! \cdots i_n!}(x_1 - x_2)^{i_1}(x_2 - x_3)^{i_2} \cdots (x_{n-1} - x_n)^{i_{n-1}}(x_n - T)^{i_n},$$

where $\tilde{p} = \tilde{p}_{i_1 i_2 \ldots i_n}(T)$ results from applying a chain of operators (a ∂-reduction) to p as shown in

$$p \xrightarrow{\partial^{i_1}_1,\, x_1 = x_2} \cdot \xrightarrow{\partial^{i_2}_2,\, x_2 = x_3} \cdot \;\cdots\; \cdot \xrightarrow{\partial^{i_{n-1}}_{n-1},\, x_{n-1} = x_n} \cdot \xrightarrow{\partial^{i_n}_n,\, x_n = T} \tilde{p}.$$

b. *In particular, if p is homogeneous of degree s and $i_1 + \cdots + i_n = s$, then $\tilde{p}$ is a real number and the coefficient of $h_1^{i_1} \ldots h_n^{i_n}$ in the development of $(p \circ \sigma)(\underline{h})$ is given by $\tilde{p}(0)/(i_1! \cdots i_n!)$.*

Proof: a. Since each of the operators of the chain is linear, it is enough to show the claim for the case of monomials $p = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$. Developing $x_1^{l_1} = (x_2 + (x_1 - x_2))^{l_1}$ by the binomial theorem, we see

$$p = \sum_{\nu=0}^{l_1} \binom{l_1}{\nu} x_2^{l_1+l_2-\nu} x_3^{l_3} \cdots x_n^{l_n} (x_1 - x_2)^\nu = \sum_{\nu=0}^{l_1} \nu!^{-1} (\partial_{x_1}^\nu p)(x_2, x_{2:n})(x_1 - x_2)^\nu.$$

In the case $n = 1$, we have $p = x_1^{l_1}$ and $x_2 = T$. We then find that the coefficient of $(x_1 - T)^{i_1}$ in p is

$$i_1!^{-1}(\partial_{x_1}^{i_1} p)(T) = i_1!^{-1} \cdot l_1 \cdots (l_1 - i_1 + 1) x_1^{l_1 - i_1}|_{x_1 = T} = i_1!^{-1} \tilde{p}_i,$$

as claimed. Suppose the claim proved for $n - 1$ variables. By above representation, the coefficient of

$$(x_1 - x_2)^{i_1}(x_2 - x_3)^{i_2}(x_3 - x_4)^{i_3} \cdots (x_{n-1} - x_n)^{i_{n-1}}(x_n - T)^{i_n}$$

in p is evidently equal to the coefficient of

$$(x_2 - x_3)^{i_2}(x_3 - x_4)^{i_3} \cdots (x_{n-1} - x_n)^{i_{n-1}}(x_n - T)^{i_n}$$

in $i_1!^{-1}(\partial_{x_1}^{i_1} p)(x_2, x_{2:n})$. Now the latter expression is the result of applying the first arrow in above chain to p and dividing by $i_1!$. As $(\partial_{x_1}^{i_1} p)(x_2, x_{2:n})$ is a polynomial in $n-1$ variables, we can infer by the induction assumption, that applying the remaining arrows to it and multiplying by $(i_1! \cdot i_2! \cdots i_n!)^{-1}$, we obtain the second mentioned coefficient. This yields the theorem.

b. Is an immediate consequence of part a obtained by putting $T = 0$ and putting $h_i = (x_i - x_{i+1})$, $i = 1, \ldots, n$ and $x_{n+1} = 0$. ∎

It is implicit in [K] that, given $\alpha_1, \ldots, \alpha_n \geq 0$ of sum ≤ 1, the polynomials

$$u_k(x_1, \ldots, x_n) = (-1)^k \sum_{j_1 + \cdots + j_n = k} \binom{\alpha_1}{j_1} \binom{\alpha_2}{j_2} \cdots \binom{\alpha_n}{j_n} x_1^{j_1} \cdots x_n^{j_n}$$

can – after the transformation $\underline{x} = \sigma(\underline{h})$ – be written as nonpositive combinations of monomials of the form $h_1^{i_1} \cdots h_n^{i_n}$. It follows by symmetry in particular that, whenever $x_1, \ldots, x_n \geq 0$, and $k \geq 1$, then $u_k(x_1, \ldots, x_n) \leq 0$. By using Cauchy multiplication, one has, equivalently, that the power series in t, $(1 - x_1 t)^{\alpha_1} \cdots (1 - x_n t)^{\alpha_n}$, has only nonpositive coefficients whenever $\underline{x} \in \mathbb{R}_{\geq 0}^n$

(except for the coefficient of t^0). Laffey [L] had proved this same result via quite different considerations and used it in papers with Loewy and Šmigoc [LLS] for advancing the nonnegative inverse eigenvalue problem.

Define the pseudosymmetric power sum polynomials s_j by

$$s_j(\underline{p}, \underline{x}) = \sum_{i\geq 1}^{n} p_i x_i^j.$$

The number of variables play in our investigations only a minor role. We will denote it by n. If various s_j occur in the same polynomial we assume this number equal in all of the polynomials. Polynomials s_j revert by the choice $p_i = 1$ for all i to the usual (symmetric) power sums. Define $Sp_{1:t} = \sum_{i=1}^{t} p_i$ for the t-th partial sum of $\underline{p}$. In [K] we computed, manually, from theorem 0.1 the ∂-reductions of some low-degree products of the s_j and expressed them in terms of the $Sp_{1:i}$. We were unable to produce some general expressions nor did we even have an idea how to implement these calculations on a computer.

The main contribution of this paper are two formulae which, given any polynomial $p(\underline{p}, \underline{x}) = s_{m_1} s_{m_2} \cdots s_{m_k}$ of degree $s = m_1 + \cdots + m_k$, express the coefficient of $h_{i_1} h_{i_2} \cdots h_{i_s}$ in the development of $p(\underline{p}, \sigma(\underline{h}))$. One of these formulae is a polynomial in $Sp_{1:i_1}, Sp_{1:i_2}, \cdots, Sp_{1:i_s}$ with square-free monomials. (Here $i_1, i_2, \ldots, i_s$ are not necessarily distinct.) The other formula is of a very different character and as an offspin these two formulas lead to a very surprising combinatorial identity.

As an example consider the polynomial in $x_1, \ldots, x_6$,

$$q_3 = (-25 \sum_i x_i^3 + 9 \sum_{i\neq j} x_i^2 x_j - 6 \sum_{i<j<k} x_i x_j x_k)/216.$$

This polynomial is the special case of the polynomial

$$q_3(\underline{p}, \underline{x}) = -s_3(\underline{p}, \underline{x}) + 2 s_1(\underline{p}, \underline{x}) s_2(\underline{p}, \underline{x}) - s_1(\underline{p}, \underline{x})^3,$$

arising from putting $\underline{x} = (x_1, x_2, \ldots, x_6)$ and $\underline{p} = \frac{1}{6}(1, 1, 1, 1, 1, 1)$.

Our results allow us to say more generally that for arbitrary number of variables, if $\underline{x} \geq 0$ and $\underline{p}$ is an arbitrary probability vector (i.e. $|\underline{p}| = \sum_i p_i = 1$, $\underline{p} \geq 0$), then the inequality $q_3(\underline{p}, \underline{x}) \leq 0$ will hold since $q_3(\underline{p}, \sigma(\underline{h}))$ will be shown to be a polynomial in $\underline{h}$ with nonpositive coefficients. This in turn will be seen as a consequence of the simple fact that for $0 \leq b \leq c \leq 1$ there holds $-3 + 4b + 2c - 3bc \leq 0$. The rôles os a, b, c herein will be those of certain partial sums of $\underline{p}$.

Similar drastic simplifications hold for other linear combinations of monomials in $s_1, s_2, \ldots$.

It should be noted that pseudosymmetricity of a polynomial is harder to discern than symmetricity. For example, with and $s_i(\underline{p}, \underline{x}) = x^i + 2y^i + 7z^i$, $i = 1, 2, 3$, one has

$$s_1^3 + s_1 s_2 = 2x^3 + 8x^2 y + 14xy^2 + 12y^3 + 28x^2 z + 84xyz + 98y^2 z + 154xz^2 + +308yz^2 + 392z^3.$$

The results reported here where found by trying to strengthen a result of Holland [Hol] who, as a variation of Laffey's result, had shown that the harmonic mean of $(1 - x_1 t), \ldots, (1 - x_n t)$, that is, the power series

$$n((1 - x_1 t)^{-1} + (1 - x_2 t)^{-1} + \cdots + (1 - x_n t)^{-1})^{-1},$$

gives rise to coefficient polynomials $q_l(\underline{x})$ of the t^l, $l \geq 1$ which all assume nonpositive values when $\underline{x} \geq 0$. While Holland's result could probably adapted to show the same for the polynomials $q_l(\underline{p}, \underline{x})$ obtained by considering above $p_i(1 - x_i t)^{-1}$ ($\underline{p}$ a probability vector) in place of $k^{-1}(1 - x_i t)^{-1}$. But his proof does not seem to lend for proving that the polynomials $q_l(\underline{p}, \sigma(\underline{h}))$ have only negative coefficients.

We have yet to give a uniform argument implying this for all l, but our reduction method applicable to any linear combination of products $s_{m_1} s_{m_2} \cdots s_{m_k}$ (and in particular to all symmetric polynomials) allows us convenient experimentation. In particular we were able to confirm the conjecture for every l we tried. We report on this in Section 5.

Timofte [T] and Riener [Rie] have shown that the nonnegativity of an arbitrary symmetric polynomial of degree d defined on an $\mathbb{R}^n$ is guaranteed as soon as its nonnegativity can be established for all $\underline{x}$ with at most $d/2$ distinct entries. This result shares with our's the common feature that in the formulation of the tests to be made to certify nonnegativity the number of variables plays only a minor role. Other researchers in symmetric functions have noted the same. See [MD, p.18].

The organization of this preprint is as follows. In the short Section 1 we note an alternative way for writing the reduction chains of theorem 1 above. It is particularly useful if the number of variables is much larger than the degree of the polynomials involved. We use this notation in Section 2 where we prove one of the formulae mentioned above and which involves chains of multisets; and we use the notation also in Section 3, where we prove the other formula using trees. The short Section 4 derives a surprising combinatorial identity. We conclude in Section 5 by some remarks on the codes we used for implementing the formulae and also how the results might be useful for proving general or symmetric polynomial inequalities.

The reading of sections 2 and 4 is not necessary for following the applications given in Section 5. We conserve these sections here for the sole reason that it is interesting combinatorics that might be useful in advancing the main problem left open here or be useful elsewhere. Also, (the proof of) proposition 2.2 was used only in the construction of a code implementing the formula of theorem 2.1.

1. An alternative way to denote ∂-reductions

The reduction chains of theorem 1 are of the form

$$p \xrightarrow{\partial_1^{i_1}, x_1 = x_2} \cdot \; \cdots \; \cdot \xrightarrow{\partial_n^{i_n}, x_n = T} \tilde{p}.$$

If we apply (as we shall) such a chain to a homogeneous polynomial of degree $s = i_1 + \cdots + i_n$ we get after application of '$\partial_n^{i_n}$' at the right a real number. Hence the operation '$x_n = T$' has no rôle. Write '$\xrightarrow{\text{to } i}$' for saying that the currently existing variables of index $\leq i$ should be mapped to x_i. For example

$$-3x_1 + x_1x_2^2 + x_3 \xrightarrow{\text{to } 2} -3x_2 + x_2^3 + x_3.$$

Then we may substitute every arrow '$\xrightarrow{\partial_l^{i_l}}$' by i_l successive operations $\xrightarrow{\text{to } l, \partial_l}$. Note that in case of polynomials with many variables but of relatively low degree many of the i_ν above may be 0. It is this case above all in which the new notation will be useful and we see that a chain as above can be replaced by a chain of the form

$$\xrightarrow{\text{to } i_1, \partial_{i_1}} \cdot \xrightarrow{\text{to } i_2, \partial_{i_2}} \cdots \xrightarrow{\text{to } i_s, \partial_{i_s}},$$

where we have $i_1 \leq i_2 \leq \cdots \leq i_s$, with i_ν typically differing from the above, but with the same s. The proofs in the next two sections use this notation.

Part b of theorem 0.1 reads in this language as follows:

Corollary 1.1. *Let p be homogeneous of degree s and assume $i_1 + \cdots + i_n = s$. Then the coefficient of $h_1^{i_1} \cdots h_n^{i_n}$ in the development of $(p \circ \sigma)(\underline{h})$ is obtained by applying i_1 operators $\xrightarrow{\text{to } 1, \partial_1}$, followed by i_2 operators $\xrightarrow{\text{to } 2, \partial_2}$, ... followed by i_n operators $\xrightarrow{\text{to } n, \partial_n}$, and dividing the result by $i_1! \cdots i_n!$* ∎

In case a polynomial is not homogeneous, an obvious modification of this corollary can be applied to its homogeneous parts.

Example 1.2. Let $p = p(x_1, x_2, x_3)$ be the polynomial mentioned in Section 0. We have

$$p(x_1, x_2, x_3) \xrightarrow{\text{to } 1, \partial_1} 3x_1^2 - 3x_2x_3 \xrightarrow{\text{to } 1, \partial_1} 6x_1 \xrightarrow{\text{to } 3, \partial_3} 6.$$

The coefficient of $h_1^2 h_3$ in $(p \circ \sigma)(\underline{h})$ therefore is $6/(2!0!1!) = 3$.

2. A formula for the ∂-reduction of $s_{m_1} s_{m_2} \cdots s_{m_k}$ involving multichains of sets

As earlier, let s_m be a shorthand for $s_m(\underline{p}, \underline{x})$. We will have necessity to consider $s_m(i : n) := s_m(p_{i:n}, x_{i:n})$; the notation at the left again is chosen for lightness.

Theorem 2.1. *Let $k \geq 0$, $m_1, \ldots, m_k \geq 1$ and $h \geq 0$ be integers and let $s = \sum_i m_i + h$. Then the result of the computation*

$$*\colon\ s_{m_1} s_{m_2} \cdots s_{m_k} x_0^h \xrightarrow{\text{to } i_1, \partial_{i_1}, \text{to } i_2,\ \partial_{i_2}, \cdots, \text{to } i_s,\ \partial_{i_s}} R$$

is the real number R which can be obtained as follows:

Let $M = \{m_1, m_2, \cdots, m_k\}$ be the multiset of the k integers m_i and consider the family $\mathcal{C}$ of all multichains

$$C : \emptyset =: A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_s \subseteq M$$

of length s of multisubsets A_ν of M. Then with $i_0 = 0$ and

$$s(A_\nu) := \text{sum of the elements of } A_\nu,$$

$$R = \sum_{C \in \mathcal{C}} \prod_{\nu=1}^{s} Sp_{1+i_{\nu-1}:i_\nu}^{|A_\nu| - |A_{\nu-1}|}(s(A_\nu) + h - \nu + 1)).$$

Proof: We prove the claim by induction over s. In case $s = 0$, necessarily $k = 0$, and the left hand side of $*$ is $x^0 = 1$. No computation is done, therefore $R = 1$. At the other hand the only chain of length 0 is the empty one. The product figuring in the expression for R is empty, and hence by usual conventions equal to 1. Thus the sum given is also 1 as we wished to see. Now assume $s=1$. Then the left hand side is either s_1 or x_0^1. In the first case we get

$$s_1 \xrightarrow{\text{to } i_1} Sp_{1:i_1} x_{i_1}^1 + s(1+i_1:n) \xrightarrow{\partial i_1} Sp_{1:i_1}.$$

The set $M=\{1\}$ and we are speaking of chains $C:\emptyset\subseteq A_1\subseteq M=\{1\}$ of length $s=1$. We get either $A_1=\emptyset$ or $A_1=\{1\}$. Now the expression under the sum sign is

$$Sp_{1+i_1-1:i_1}^{|A_1|-|A_0|}(s(A_1)+0-1+1)) = Sp_{1:i_1}^{|A_1|} s(A_1).$$

So since $s(\emptyset)=0$, we find the sum yields $Sp_{1:i_1}$.

In the second case the computation yields

$$x_0 \xrightarrow{\text{to } i_1} x_{i_1} \xrightarrow{\partial i_1} 1.$$

In this case $M=\emptyset$. There is then only one chain , namely $C:\emptyset\subseteq A_1\subseteq\emptyset$. Then the sum has only one term and yields

$$Sp_{1:i_1}^{|A_1|-|A_0|}(0+1-1+1) = Sp_{1:i_1}^0 1 = 1.$$

This concludes the proof of the case $s=1$.

Now suppose the proposition already proved for all expressions $s_{m_1}\cdots s_{m_k}x_0^h$ for which $\sum m_i + h = s' < s$ and assume now $\sum m_i + h = s$. We have the computation

$$s_{m_1}\cdots s_{m_k}x_0^h \xrightarrow{\text{to } i_1} \prod_{\nu=1}^{k}(Sp_{1:i_1}x_{i_1}^{m_\nu} + s_{m_\nu}(1+i_1:n))x_{i_1}^h.$$

With I^c meaning the complement of I in $\{1,2,\ldots,k\}$, the product herein can be written as

$$\sum_{I\subseteq\{1,\ldots,k\}} Sp_{1:i_1}^{|I|} x_{i_1}^{\sum_{\nu\in I} m_\nu} \prod_{\nu\in I^c} s_{m_\nu}(1+i_1:n).$$

Therefore, applying ∂_{i_1} to the right hand side, we obtain

$$s_{m_1}\cdots s_{m_k}x_0^h \xrightarrow{\text{to } i_1,\partial_{i_1}} \sum_{I\subseteq\{1,\ldots,k\}} \left(Sp_{1:i_1}^{|I|}(\sum_{\nu\in I} m_\nu + h) \prod_{\nu\in I^c} s_{m_\nu}(1+i_1:n) \right) x_{i_1}^{\sum_{\nu\in I} m_\nu + h - 1}.$$

By the linearity of the operators $\xrightarrow{\text{to } i_\nu, \partial_{i_\nu}}$, to obtain R we have to compute now for each $I \subseteq \{1, \ldots, k\}$ the results R_I of the computations

$$*_2 : \prod_{\nu \in I^c} s_{m_\nu}(1 + i_1 : n) x_{i_1}^{\sum_{\nu \in I} m_\nu + h - 1} \xrightarrow{\text{to } i_2,\ \partial_{i_2},\ \cdots,\ \text{to } i_s,\ \partial_{i_s}} R_I;$$

to multiply each R_I with $Sp_{1:i_1}^{|I|}(\sum_{\nu \in I} m_\nu + h)$; and to sum the resulting numbers over all $I \subseteq \{1, \ldots, k\}$.

Note that the product at the left of $*_2$ is a product of the form we started with but its degree is $\sum_{\nu \in I^c} m_\nu + \sum_{\nu \in I} m_\nu + h - 1 = s - 1 < s$. So we may apply the induction hypothesis to it, and have to take care only of reindexations.

To each $I \subseteq \{1, \ldots, k\}$ there pertains actually the multiset $\{m_\nu : \nu \in I\}$. Using this multiset instead of I, and calling it A_1 we find that the left hand sides of $*_2$ can be written as

$$\prod_{m \in A_1^c} s_m(1 + i_1 : n) x_{i_1}^{s(A_1) + h - 1}, \qquad A_1 \subseteq M.$$

The application of the operator in $*_2$ (which consist of only $s - 1$ pairs 'to i_ν, ∂_{i_ν}') to this yields by induction hypothesis the sum over all products

$$\prod_{\nu=2}^{s} Sp_{1+i_{\nu-1}:i_\nu}^{|A'_\nu| - |A'_{\nu-1}|}(s(A'_\nu) + (s(A_1) + h - 1) - (\nu - 1) + 1)$$

associated to the family of chains of length $s-1$, $C : \emptyset \subseteq A'_2 \subseteq \cdots \subseteq A'_s \subseteq A_1^c$ of multisets, indexed for convenience below with numbers ranging from 2 to s. If we define $A_\nu = A_1 \uplus A'_\nu$, for $\nu = 2, \ldots, s$, then we can write this as

$$\prod_{\nu=2}^{s} Sp_{1+i_{\nu-1}:i_\nu}^{|A_\nu| - |A_{\nu-1}|}(s(A_\nu) + h - \nu + 1)$$

and the sum is over all chains $A_2 \subseteq A_3 \subseteq \cdots \subseteq A_s$, where $A_1 \subseteq A_2$ and $A_s \subseteq M$.

From these arguments it follows that our searched-for expression for R is

$$\sum_{A_1 \subseteq M} Sp_{1:i_1}^{|A_1|}(s(A_1) + h) \sum_{C \in \mathcal{C}'(A_1)} \prod_{\nu=2}^{s} Sp_{1+i_{\nu-1}:i_\nu}^{|A_\nu| - |A_{\nu-1}|}(s(A_\nu) + h - \nu + 1),$$

where $\mathcal{C}'(A_1)$ is the set of chains $C : A_2 \subseteq \cdots \subseteq A_s \subseteq M$ for which $A_1 \subseteq A_2$. Since this sum can be rewritten precisely in the form the proposition claims, the proof is complete. ∎

We used the ideas in the proof of the following proposition for a computer implementation of the above formula; the fact itself is a special case of [St, Proposition 3.5.1] when formulated for antichains P.

Proposition 2.2. *Let $M = \{m_1, \dots, m_k\}$ be an ordered list of natural numbers and let $\dot{s} > s$ be positive integers. There is a natural bijection C*

$$\{1, \dots, s, \dot{s}\}^k \xrightarrow{C} \{ \text{ chains of length } s,\ C : A_1 \subseteq A_2 \subseteq \cdots \subseteq A_s \subseteq M\}.$$

Proof: Given $\underline{t} = (t_1, \dots, t_k) \in \{1, \dots, s, \dot{s}\}^k$, associate to $l = 1, \dots, s$, the set $A_l := \{m_i : t_i \leq l\}$. Obviously these sets define a chain $C = C(\underline{t})$: $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_s \subseteq M$. Now choose $\underline{t}' \in \{1, \dots, s, \dot{s}\}^k$ with $\underline{t}' \neq \underline{t}$. Let $C' = C(\underline{t}') : A'_1 \subseteq A'_2 \subseteq \cdots \subseteq A'_s \subseteq M$ be the associated chain. There exists an i_0 such that, say, $t_{i_0} < t'_{i_0}$. Then $A'_{t_{i_0}} = \{m_i : t'_i \leq t_{i_0}\}$. It follows that $m_{i_0} \notin A'_{t_{i_0}}$ while $m_{i_0} \in A_{t_{i_0}}$. Hence $C(\underline{t}) \neq C(\underline{t}')$ and the map C is injective.

Now assume a chain $C : A_1 \subseteq A_2 \subseteq \cdots \subseteq A_s \subseteq M$ be given. Define $A_{\dot{s}} := M$ and $t_i = \min\{l : m_i \in A_l\}$, $i = 1, \dots, k$. Obviously $\underline{t} = (t_1, \dots, t_k) \in \{1, \dots, s, \dot{s}\}^k$. So this $\underline{t}$ defines a chain $C(\underline{t})$. We claim $C(\underline{t}) = C$. By definition, the l-th set in $C(\underline{t})$ is $B_l := \{m_i : t_i \leq l\}$, for $l = 1, \dots, s$. Fix an l and consider an i so that $m_i \in B_l$. Then $t_i \leq l$, that is, by definition of $\underline{t}$, $m_i \in A_{t_i}$ and so $m_i \in A_l$. In other words $B_l \subseteq A_l$. Conversely, if i is so that $m_i \in A_l$ then $m_i \in A_{t_i}$ for some $t_i \leq l$ and so $m_i \in B_l$. So $A_l \subseteq B_l$. We see that indeed $B_l = A_l$ for all l. Hence the map C is surjective. ∎

3. A formula for the ∂-reduction of $s_{m_1} s_{m_2} \cdots s_{m_k}$ whose proof involves trees

The result R of the computation

$$*\colon\ s_{m_1} s_{m_2} \cdots s_{m_k} \xrightarrow{\text{to } i_1, \partial_{i_1}, \text{to } i_2,\ \partial_{i_2}, \cdots, \text{to } i_s,\ \partial_{i_s}} R$$

can also be obtained in a very different way. We begin by explaining the construction of certain type of trees. The following particular tree has relevance for computing the result R above when $k = 3$ and m_1, m_2, m_3 are called l, m, r (for left, middle, right, respectively), but a completely analogous construction will yield the corresponding result R for the general case.

The rules for growing the tree are as follows: take $a_1, a_2, \dots$ and x to be symbols and $l, m, r; l', m', r'$ integers.

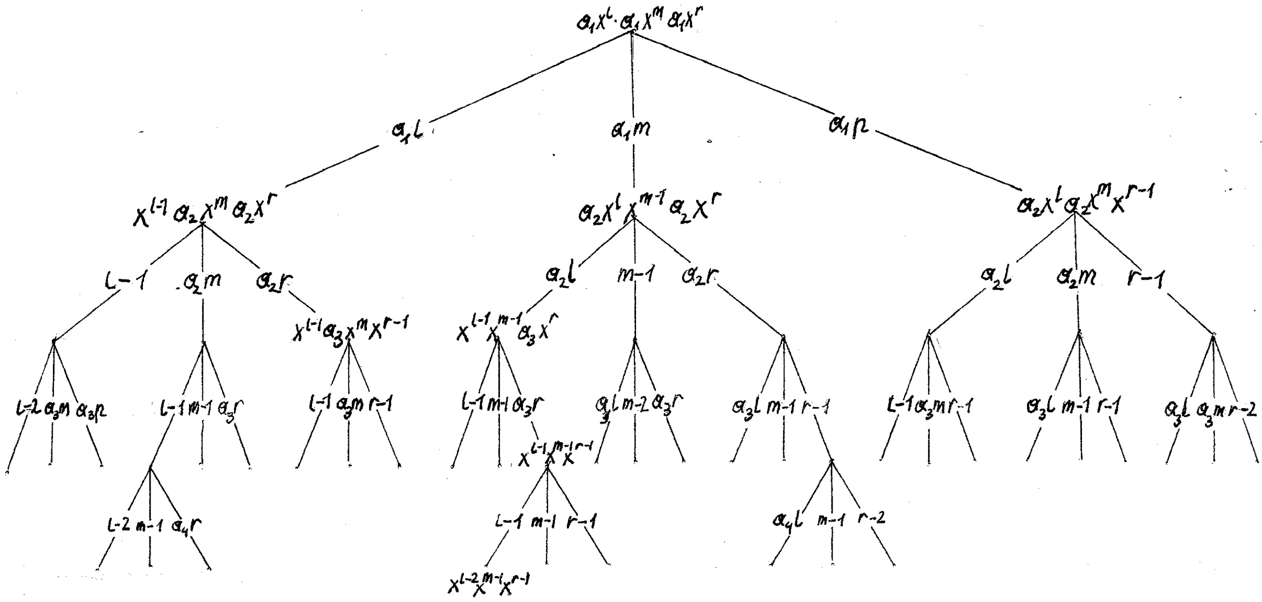
* The root of the tree is labeled $a_1x^l \cdot a_1x^m \cdot a_1x^r$.
* If $ax^{l'} \cdot bx^{m'} \cdot cx^{r'}$ is a vertex with $a, b, c \in \{1, a_i\}$, then it has three sons:
  - the edges to the left/middle/right son are labeled $al'/bm'/cr'$ respectively;
- the left/middle/right sons themselves are labeled

$$1x^{l'-1} \cdot b'x^{m'} \cdot c'x^{r'} \quad / \quad a'x^{l'} \cdot 1x^{m'-1} \cdot c'x^{r'} \quad / \quad a'x^{l'} \cdot b'x^{m'} \cdot 1x^{r'-1},$$

respectively, where we define

$$a' = \begin{cases} 1 & \text{if } a = 1 \\ a_{i+1} & \text{if } a = a_i \end{cases}, \quad b' = \begin{cases} 1 & \text{if } b = 1 \\ a_{i+1} & \text{if } b = a_i \end{cases}, \quad c' = \begin{cases} 1 & \text{if } c = 1 \\ a_{i+1} & \text{if } c = a_i \end{cases}$$

In the following partial picture of the upper part of the tree we suppressed a number of vertex labels for lack of space.



Using traditional designations, the root of a tree has depth 0 and the depth of a vertex is the number of edges on the path from the root to it. Each path can be identified with a word on the alphabet L, M, R in the obvious way. We now assume to construct the tree till all its leaves have the same depth $l + m + r$.

A path from the root to a leaf defines a monomial obtained by taking the product of its edge-labels. The sum of all these monomials defines a polynomial which we wish to characterize.

After a little reflection and tinkering with the rules, the following features are salient:
• The edge to a left/middle/right son always carries exactly one factor of the form $l-t$ $/m-t/r-t$, respectively; if an edge has a factor $l-t$, say, then along a path to the root there occur also the factors $l-t+1,\dots,l-1,l$ (not necessarily uninterruptedly). Similar observations hold for factors $m-t$ and $r-t$.
• Consequently the monomial attached to a root-to-leave path may be zero. For example the monomial obtained by taking always the edge to left sons will be $a_1l(l-1)(l-2)\cdots(l-(l+m+r)+1)$ and hence contain the factor $l-l=0$. We see that a path will define a nonzero monomial if and only if its code has exactly l letters L, m letters M, and r letters R.

We also see that at the first choice to a left son we substitute in the label for the next vertex a letter a by 1; at the first choice to a middle son a letter b by 1; at the first choice to a letter c by 1. Thus:
• If a root-to-leave-path defines a nonzero monomial, then this monomial is of the form $l!m!r!a_1a_ia_j$ with $1<i<j\le l+m+r$. This particular monomial is produced if the *first* times that in the path is taken a descent to a left, middle, or right sun (in any order), are the moments in which descents to vertices of depths 1, i, or j are done. No letters a_* are joined to the product if a descent is of a type that occurred already.

The polynomial that this tree defines is consequently homogeneous of degree 3 in $a_1,\dots,a_{l+m+r}$ and has only positive integer coefficients all of which are divisible by $l!m!r!$. The coefficient of $a_1a_ia_j$ is $l!m!r!$ times the number of words of length $l+m+r$ that have l letters L, m letters M, and r letters R and for which the first occurrences (in any order) of these letters occur in positions $1,i,j$.

Example 3.1. Assume $l=2,m=1,r=1$, and $i=2,j=3$. Then $s=l+m+r=4$. Examples of admissible words are then LMRL, LRML, MRLL. But LLRM is not admitted since none of the letters L,M,R has at position 2 its leftmost occurrence. If we choose $i=2,j=4$ then LRLM and RLLM are admissible, but LRML is not.

We now explain what such considerations have to do with ∂-reductions; in the particular case we consider the reduction of $s_l(\underline{p},\underline{x})s_m(\underline{p},\underline{x})s_r(\underline{p},\underline{x})$. This is a polynomial in $x$es of degree $l+m+r$.

If we apply '$\xrightarrow{\text{to } i_1}$' to $s_l = s_l(\underline{p}, \underline{x})$, we get $Sp_{1:i_1}x_{i_1}^l + s_l(1 + i_1 : n)$ and similar formulas hold for s_m and s_r. So after applying $\xrightarrow{\partial_{i_1}}$ to the product of the transformed formulas, we get the sum of the three products

$(Sp_{1:i_1}l)x_{i_1}^{l-1} \cdot (Sp_{1:i_1}x_{i_1}^m + s_m(1 + i_1 : n))(Sp_{1:i_1}x_{i_1}^r + s_r(1 + i_1 : n))$,
$(Sp_{1:i_1}x_{i_1}^l + s_l(1 + i_1 : n)) \cdot (Sp_{1:i_1}m)x_{i_1}^{m-1} \cdot (Sp_{1:i_1}x_{i_1}^r + s_r(1 + i_1 : n))$,
$(Sp_{1:i_1}x_{i_1}^l + s_l(1 + i_1 : n)) \cdot (Sp_{1:i_1}x_{i_1}^m + s_m(1 + i_1 : n)) \cdot (Sp_{1:i_1}r)x_{i_1}^{r-1}$.

We could store this information in a tree whose root is $s_ls_ms_r$, which has three leaves of depth one, which has edge labels $Sp_{1:i_1}l, Sp_{1:i_1}m, Sp_{1:i_1}r$, and whose leaves are labeled by what remains of the above products after suppressing the edge labels. The information in that tree would be read as: to get the result of $s_ls_ms_r \xrightarrow{\text{to } i_1}\xrightarrow{\partial i_1} R$, multiply each vertex label with the label of the edge leading to it; and then sum these products. By writing a_1 for $Sp_{1:i_1}$ the edge labels turn into those of the tree constructed above. Also note that for knowing the edge labels (which come from applying ∂_{i_1}, the sums $s_l(1 + i_1 : n), s_m(1 + i_1 : n), s_r(1 + i_1 : n)$ above are of no relevance. Also the indices of the x_{i_1} can be suppressed. This justifies to write the simplified notation $a_1x^l \cdot a_1x^m \cdot a_1x^r$ instead of $s_ls_ms_r$ to store all the relevant information.

Next note that

$$\begin{aligned} * :\ Sp_{1:i_1}x_{i_1}^l + s_l(1 + i_1 : n) &\xrightarrow{\text{to } i_2} Sp_{1:i_1}x_{i_2}^l + Sp_{1+i_1:i_2}x_{i_2}^l + s_l(1 + i_2 : n) \\ &= Sp_{1:i_2}x_{i_2}^l + s_l(1 + i_2 : n) \end{aligned}$$

and similar formulae hold for the factors above associated to m and r.

Applying '$\xrightarrow{\text{to } i_2, \partial i_2}$' to the three leaves of the current tree, we see as before using the product rule that three edges will emanate from each of the vertices of depth 1. We again use the factors not depending on x as edge labels. Then writing a_2 for $Sp_{1:i_2}$, we get precisely the edge labels up to depth 2 of the tree shown above.

The relation $*$ remains true if everywhere i_1 and i_2 are substituted by i_ν and $i_{\nu+1}$, respectively. Using this we can continue the above reasoning and see the result R for the case $k = 3$, $(m_1, m_2, m_3) = (l, m, r)$ can be found by using the tree polynomial associated to this triple and substituting a_j by $Sp_{1:j}$.

It is evident that analogous considerations are valid in the general case; so we get

Theorem 3.2. *Let $s = m_1 + m_2 + \cdots + m_k$. Then the result R of the computation*

$$*\colon\ s_{m_1} s_{m_2} \cdots s_{m_k} \xrightarrow{\text{to } i_1, \partial_{i_1}, \text{to } i_2,\ \partial_{i_2}, \cdots, \text{to } i_s,\ \partial_{i_s}} R$$

is a homogeneous polynomial of degree k in the variables $Sp_{1:i_1}, \ldots, Sp_{1:i_s}$. Each monomial is of the form $Sp_{1:i_1} Sp_{1:i_{\nu_2}} \cdots Sp_{1:i_{\nu_k}}$ with $1 = \nu_1 < \nu_2 < \cdots < \nu_k \leq s$. The coefficient of this particular monomial equals $m_1! m_2! \cdots m_k!$ times the number of words of length s with k distinct letters of respective multiplicities $m_1, \ldots, m_k$, and whose leftmost positions are $\nu_1, \nu_2, \ldots, \nu_k$. ■

The following proposition helps to compute the coefficients of the mentioned polynomial.

Proposition 3.3. *Let $M_1, \ldots, M_k$ be distinct letters, and $m_1, \ldots, m_k$ and $1 = \mu_1 < \mu_2 < \ldots < \mu_k \leq \sum m_i$ positive integers. The number of words which we can form using exactly m_i letters M_i (and no others) and which have the property that μ_i is the leftmost position a letter M_i occurs, $i = 1, \ldots, k$ is given by the product*

$$\prod_{i=1}^{k} \binom{m_1 + \cdots + m_i - \mu_i}{m_i - 1}.$$

Proof: Once letter M_k is positioned at place μ_k, there remain the positions $1 + \mu_k, 2 + \mu_k, \ldots, s = (s - \mu_k) + \mu_k$ as admissible for the remaining $m_k - 1$ letters M_k. There are $\binom{s-\mu_k}{m_k-1}$ possible choices for where to put these remaining M_ks. Once a choice is done, we look at the letter M_{k-1}. One of these letters is at position μ_{k-1}, all the other letters M_{k-1} are at the right of it occupying $m_{k-1}-1$ of the still admissible $s-m_k-\mu_{k-1}$ positions. This yields $\binom{s-m_k-\mu_{k-1}}{m_{k-1}-1}$ possible choices. We next have that one letter M_{k-2} takes position μ_{k-2} all the $m_{k-2} - 1$ other letters M_{k-2} lie to the right of it in the still available $s - m_k - m_{k-1} - \mu_{k-2}$ positions. This yields $\binom{s-m_k-m_{k-1}-\mu_{k-2}}{m_{k-2}-1}$ choices. Continuing this way and using the definition of s yields the claim. ■

Corollary 3.4. *Let S_k be the symmetric group acting on $\{1, 2, \ldots, k\}$. Then the coefficient of $Sp_{1:i_1} Sp_{1:i_{\nu_2}} \cdots Sp_{1:i_{\nu_k}}$ in the polynomial R mentioned in the above theorem 3.2 is given by*

$$m_1! m_2! \cdots m_k! \cdot \sum_{\sigma \in S_k} \prod_{i=1}^{k} \binom{m_{\sigma 1} + \cdots + m_{\sigma i} - \nu_i}{m_{\sigma i} - 1} =$$

$$m_1 m_2 \cdots m_k \sum_{\sigma\in S_k} \prod_{i=1}^k (m_{\sigma 1} + \cdots + m_{\sigma i} - \nu_i)^{\underline{m_{\sigma i}-1}}.$$

Proof: The first formula is a consequence of the previous proposition, given that the letters referred in the theorem, let's say $M_1, \ldots, M_k$, may occur at the positions $\nu_1, \ldots, \nu_k$ as leftmost ones of their type in any order. The right formula is a consequence of the definition of the binomial coefficient as $\binom{r}{k} = r^{\underline{k}}/k!$. ∎

Example 3.5. At the beginning of Section 5 we will see that from the above formulas we get

$$s_1 s_2 \xrightarrow{\text{to } i_1, \partial_{i_1}, \text{ to } i_2, \partial_{i_2}, \text{ to } i_3, \partial_{i_3}} 4Sp_{1:i_1} Sp_{1:i_2} + 2Sp_{1:i_1} Sp_{1:i_3}.$$

(This is an example of a reduction computed manually in [K].) It follows from corollary 1.1 that if there are at least variables $x_1, \ldots, x_5$ present, then the coefficient of $h_1 h_3 h_4$ in $s_1(\underline{x}, \sigma(\underline{h})) s_2(\underline{x}, \sigma(\underline{h}))$ will be

$$(1/(1!1!1!))(4Sp_{1:1} Sp_{1:3} + 2Sp_{1:1} Sp_{1:4});$$

and that of $h_1 h_5^2$ will be

$$(1/(1!2!))(4Sp_{1:1} Sp_{1:5} + 2Sp_{1:1} Sp_{1:5}).$$

In case that e.g. $\underline{p}$ (of length of $\underline{x}$) is $\underline{p} = (3, -1, 2, 5, -7, 0, \ldots, 0)$, then the uple of partial sums of $\underline{p}$ is $(3, 2, 4, 9, 2, 2 \ldots, 2)$. So the respective coefficients will be 102 and 18.

4. A surprising identity

From the two ways to compute the reduction R of $s_{m_1} s_{m_2} \cdots s_{m_k}$ we get a surprising identity.

Theorem 4.1. *Assume $M = \{m_1, m_2, \ldots, m_k\}$ to be a multiset of natural numbers and assume $s = m_1 + \cdots + m_k$. For $A \subseteq M$ (understood as a submultiset), let*

$$s(A) := \textit{ sum of the elements of } A.$$

Let $\mathcal{C}$ denote the family of all chains $C : \emptyset \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_s \subseteq M$ of length s of multisubsets A_i of M. Then, with $x_0 = 0$, there holds the identity

$$\sum_{C\in\mathcal{C}}\prod_{\nu=1}^{s}(x_\nu - x_{\nu-1})^{|A_\nu|-|A_{\nu-1}|}(s(A_\nu)-\nu+1)) =$$

$$m_1m_2\cdots m_k \sum_{1=\nu_1<\nu_2<\cdots<\nu_k\le s}\ \sum_{\sigma\in S_k}\prod_{i=1}^{k}(m_{\sigma 1}+\cdots+m_{\sigma i}-\nu_i)^{\underline{m_{\sigma i}-1}}\, x_{\nu_1}x_{\nu_2}\cdots x_{\nu_k}.$$

Proof: Choose $h=0$ in the identity of Theorem and note that $Sp_{1+i_{\nu-1}:i_\nu} = Sp_{1:i_\nu} - Sp_{1:i_{\nu-1}}$. If in the above left hand side we substitute x_ν by $Sp_{1:i_\nu}$, $\nu = 1,\dots,s$, we get R. By the corollary 3.4 we get R also if we substitute the x_{ν_l} by $Sp_{1:i_{\nu_l}}$ in the right hand side. The $Sp_{1:i_\nu}$ can be arbitrary real numbers. But a real multivariate polynomial function in s variables which is identically 0 on $\mathbb{R}^s$, defines the zero polynomial; see [CLS, p.3]. As a consequence we get that the two polynomials in indeterminates $x_1,\dots,x_s$ above are equal. ∎

5. Applications and concluding remarks

We have not found any significant simplification for the formula found for R in Section 3. With '$\longrightarrow$' symbolizing the reduction of R at the beginning of Section 3 we get that $s_1^m \longrightarrow m!Sp_{1:i_1}Sp_{1:i_2}\cdots Sp_{1:i_m}$, and $s_m \longrightarrow m!Sp_{1:i_1}$, but already the reduction of $s_{m_1}s_{m_2}$ is quite unwieldy:

$$s_{m_1}s_{m_2} \longrightarrow$$
$$m_1m_2\sum_{2\le\nu\le m_1+m_2}((m_1-1)!(m_1+m_2-\nu)^{\underline{m_2-1}} + (m_2-1)!(m_1+m_2-\nu)^{\underline{m_1-1}})Sp_{1:i_1}Sp_{1:i_\nu}.$$

Still, in case $m_1+m_2 = 3 = s$ and writing $a = Sp_{1:i_1}$, $b = Sp_{1:i_2}$, $c = Sp_{1:i_3}$, for $1\le i_1\le i_2\le i_3\le 3$ one gets that $s_1^3 \longrightarrow 6abc$, $s_1s_2 \longrightarrow 4ab+2ac$, and $s_3 \longrightarrow 6a$, and consequently

$$q_3 \longrightarrow 2a(-3+4b+2c-3bc) = 2a(-3+2c+b(4-3c))$$

for the polynomial $q_3(\underline{p},\underline{x}) = -s_3 + 2s_1 - s_1^3$, mentioned in the introduction.

The inequalities of Holland, generalized to an arbitrary probability vector $\underline{p}$, read $q_l(\underline{p},\underline{x}) \le 0$ for all $l = 1,2,3,\dots$. and $\underline{x}\ge 0$, where

$$q_l(\underline{p},\underline{x}) = -\,s_l(\underline{p},\underline{x}) + \sum_{l_1+l_2=l} s_{l_1}(\underline{p},\underline{x})s_{l_2}(\underline{p},\underline{x}) -$$
$$\sum_{l_1+l_2+l_3=l} s_{l_1}(\underline{p},\underline{x})s_{l_2}(\underline{p},\underline{x})s_{l_3}(\underline{p},\underline{x}) + \cdots + (-1)^l s_1(\underline{p},\underline{x})^l.$$

In [K] we showed that these polynomials are the coefficient polynomials arising from developing the harmonic mean $(\sum_{i=1}^{n} p_i(1-x_it)^{-1})^{-1}$ into a power series in t.

We conjecture that indeed $q_l(\underline{p}, \sigma(\underline{h}))$ is a polynomial in $h_1, h_2, \ldots$ with nonpositive coefficients.

For $l = 1$ of course we have $q_1 \longrightarrow -a$; for $l = 2$ one finds

$$q_2 \longrightarrow -2a + 2ab = 2a(-1 + b);$$

and it is clear from these observations that the corresponding polynomials in $\underline{h}$ have nonpositive coefficients. for $l = 3$ we found above

$$q_3 \longrightarrow -6a + 8ab + 4ac - 6abc.$$

With the implementation of the formula of Section 3 we similarly found

$$\begin{aligned}
q_4 &\rightarrow (-24a + 40ab + 20ac - 36abc + 12ad - 24abd - 12acd + 24abcd) \\
q_5 &\rightarrow (-120a + 240ab + 120ac - 252abc + 72ad - 168abd - 84acd + \\
&\quad 192abcd + 48ae - 120abe - 60ace + 144abce - 36ade + 96abde + \\
&\quad 48acde - 120abcde)
\end{aligned}$$

We also computed the ∂-reductions for q_6, q_7, q_8. For the reductions of q_l, 2^{l-1} terms can be expected.

The proofs of the nonpositivity of these reductions uses an adaption of the map $\sigma(\underline{h})$ defined in Section 0. For example, to show that the reduction of q_4 is nonpositive for $0 \leq a \leq b \leq c \leq d \leq 1$, note that we may divide by a and then develop the remaining polynomial in terms of $h_3 = (1-d), h_2 = (d-c), h_1 = (c-b)$. That is, put $d = 1-h_3, c = 1-h_3-h_2, b = 1-h_3-h_2-h_1$. As a result one gets a polynomial with only negative coefficients in h_1, h_2, h_3.

More information on the coefficients of the reductions will be necessary to prove negativity of q_l in general. With the computer experiments done till now we have uncovered a curious combinatorial fact. The sums of the coefficients of the homogeneous parts of the reduction of any q_l seems to define a sequence of length l which is equal to $l!(-1)^{j+1}\binom{l-1}{j}$, $j = 0, 1, \ldots, l-1$. For example, for $l = 4$ the sequence is $-24, 72, -72, 24$. We do not know why this holds. With some luck somebody's future publication is able to clarify this issue.

Often symmetric polynomials are given in terms of the elementary symmetric functions. In such cases our procedures are applicable via Warings' formula which expresses elementary symmetric polynomials as polynomials in the power sums; see e.g. [A, Proposition 4.25], or [MD, Formula 2.14']:

$$\text{If } e_l(\underline{x}) = \sum_{1\le i_1<\ldots<i_l\le n} x_{i_1}x_{i_2}\cdots x_{i_l}, \text{ and } s_l(\underline{x}) = \sum_{i=1}^{n} x_i^l,\ l = 1, 2, \ldots, n,$$

then

$$e_l = \sum_{\substack{\underline{b}\in\mathbb{Z}^l_{\ge 0} \\ b_1+2b_2+\cdots+lb_l=l}} \frac{(-1)^{l-|\underline{b}|}}{\prod_i (b_i!\cdot i^{b_i})} s_1^{b_1}s_2^{b_2}\cdots s_l^{b_l}.$$

Other transition formulae between bases for the rings of symmetric functions can be found in [MD].

Mathematica code implementing the main formulae of this article is available from the author.

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