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LAX ORTHOGONAL FACTORISATIONS IN ORDERED STRUCTURES

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ABSTRACT: We give an account of lax orthogonal factorisation systems on order-enriched categories. Among them, we define and characterise the KZ-reflective ones, in a way that mirrors the characterisation of reflective orthogonal factorisation systems. We use simple monads to construct lax orthogonal factorisation systems, such as one on the category of T_0 topological spaces closely related to continuous lattices.

KEYWORDS: lax orthogonal factorization system, lax idempotent monad, order-enriched category, weak factorization system, reflective factorization system, continuous lattice.
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1. Introduction

Weak factorisation systems (WFSs) have been a feature of Homotopy Theory even before Quillen's definition of model categories and the recognition of their importance. WFSs, whose definition can be found in §4.a, can be described as a pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ that satisfy three properties. First, each morphism of the category must be a composition of a morphism from $\mathcal{L}$ followed by one of $\mathcal{R}$ (in a not necessarily unique way). Secondly, each $r \in \mathcal{R}$ must have the *right lifting property* with respect to each $\ell \in \mathcal{L}$; in other words, each commutative square, as displayed, has a (not necessarily unique) diagonal filler.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\ell"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd} \tag{1.1}$$

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Lastly, $(\mathcal{L}, \mathcal{R})$ is, in a precise way, maximal. Each one of Quillen's model categories comes equipped with two WFS (by definition).

Orthogonal factorisations systems (OFS) arose at the same time as WFSs and can be described as WFSs in which the diagonal filler (1.1) not only exists but it is unique. This makes the factorisation of a morphism f as $f = r \cdot \ell$, with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$, unique up to unique isomorphism. Two typical examples of OFSs are the factorisation of a function as a surjection followed by an injection, and of a continuous map between topological spaces as a surjection followed by an embedding (ie an homeomorphism onto its image).

When the ambient category has a terminal object, denoted by 1, there is a case of (1.1) of special interest, namely:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\ell"'] & A \arrow[d] \\ \cdot \arrow[r] \arrow[ur, dotted] & 1 \end{tikzcd} \tag{1.2}$$

If the unique morphism $A \to 1$ has the right (unique) lifting property with respect to ℓ, one says that A is injective with respect (resp., orthogonal to) ℓ. Clearly each OFS $(\mathcal{L}, \mathcal{R})$ gives rise to a class of objects that are orthogonal to each member of $\mathcal{L}$: those objects A such that $A \to 1$ belongs to $\mathcal{R}$. The extent to which $(\mathcal{L}, \mathcal{R})$ is determined by this class of objects is the subject of study of [5]. The OFSs so determined are called *reflective*.

In addition to their widespread use in Homological Algebra, injective objects play a role in many other areas of Mathematics. For example, in the category of metric spaces and non-expansive maps, *hyperconvex* spaces are the objects injective with respect to the family of isometries (see [2] and [15]).

There are examples, as those introduced by D. Scott [29], of squares (1.2) where the diagonal filler is not unique but there exists a smallest one (with respect to an ordering between morphisms). The main example from [29] consists of those topological spaces that arise from endowing *continuous lattices* with the Scott topology. These spaces are characterised by their injectivity with respect to topological embeddings. In fact, if ℓ is a topological embedding and A is a continuous lattice in (1.2), there is a diagonal filler that is the smallest with respect to the (opposite of) the pointwise specialisation order (see §13 for more details).

Another example comes from complete lattices, which can be characterised as those posets that are injective with respect to embeddings of posets. As in the previous example, in the situation (1.2) where A is a complete lattice and ℓ is a poset embedding, there exists a smallest diagonal filler.

Motivated by the above examples, one can generalise the existence of a smallest diagonal filler in the situation (1.2) to the situation of a commutative square (1.1). By doing so, one arrives to the notion of lax orthogonal factorisation system.

The present paper gives an account, in the context of order-enriched categories, of *lax orthogonal factorisation systems* (LOFS), a notion that sits between OFSs and WFSs.

orthogonal factorisation system	$\subset$	lax orthogonal factorisation system	$\subset$	weak factorisation system

LOFSs were introduced and studied in the context of 2-categories by the authors in [7]. We cover here some of the same material in the much simpler framework of order-enriched categories and some completely new results on reflective LOFSs, as well as new examples (see below).

In a LOFS, the existence of a diagonal filler (1.1) is replaced by the existence of a smallest diagonal filler. More precisely, there is a diagonal filler d with the property that $d \leqslant d'$ for any other diagonal filler d'.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\ell"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r] \arrow[ur, bend left, "d"] \arrow[ur, bend right, "d'"'] \arrow[ur, phantom, "\leqslant"] & \cdot \end{tikzcd}$$

Since morphisms between two objects in an order-enriched category form a poset, the above property uniquely defines the smallest diagonal filler. There are, however, advantages in providing these diagonals by means of an algebraic structure, instead of postulating the existence of a smallest diagonal filler. This algebraic structure is provided by the *algebraic weak factorisation systems* (AWFSs), introduced with a different name in [14] and slightly modified in [13]; we use the definition given in the latter.

An AWFS on an order-enriched category $\mathcal{C}$ consists of a locally monotone comonad L and a locally monotone monad R on $\mathcal{C}^{\mathbf{2}}$ interrelated by axioms, and that define a locally monotone functorial factorisation $f = Rf \cdot Lf$. Inspired by the observation of [14] that OFSs correspond to AWFSs whose monad and comonad are idempotent, we defined in [7] LOFSs as AWFSs whose monad and comonad are lax idempotent, or Kock-Zöberlein. We reprise this definition in the context of order-enriched categories, which enables some simplifications.

A fundamental example of LOFS on the order-enriched category of posets factors each morphism as a left adjoint right inverse (or LARI) followed by a split opfibration. This factorisation can be constructed on any order-enriched category with sufficient (finite) limits, and plays a similar role for LOFSs as the factorisation isomorphism–morphism (that factors f as $1_{\mathrm{dom}(f)}$ followed by f) plays for OFSs (§4.d).

We introduce *KZ-reflective* LOFSs as those LOFSs (L, R) that are determined by the restriction of the monad R on $\mathcal{C}^{\mathbf{2}}$ to $\mathcal{C}$ (here $\mathcal{C}$ is viewed as the full subcategory of $\mathcal{C}^{\mathbf{2}}$ with objects of the form $A \to 1$). We characterise KZ-reflective LOFSs in a way that mirrors the characterisation in [5] of reflective OFSs $(\mathcal{L}, \mathcal{R})$ as those with the following property: if $g \cdot f$ and g belong to $\mathcal{L}$, then so does g (§9). For example, the LOFS of LARI–split opfibration mentioned above will be reflective with our definition.

Another contribution of [5] was the construction of reflective OFSs from the so-called simple reflections. The morphisms inverted by them always form a left class of an OFS. We introduce *simple* monads in the order-enriched context, as those satisfying a certain property that allows us to build LOFSs. After providing sufficient conditions for a lax idempotent monad to be simple (§11), we recover the example of topological spaces discussed above in this introduction as a consequence of the simplicity of a certain monad: the *filter monad*, which associates to each topological space the space of filters of its open subsets endowed with a natural topology (§13). The algebras for the filter monad are precisely the continuous lattices (with the Scott topology). The induced LOFS on (T_0) topological spaces has an associated WFS that was considered in [4]. We also provide easy-to-verify conditions guaranteeing that a submonad of a simple lax idempotent monad enjoys these same properties (§12). When applied to the filter monad we obtain LOFSs closely related to continuous Scott domains, stably compact spaces and sober spaces.

Another example that we obtain from a simple monad is a LOFS on the order-enriched category of (skeletal) generalised metric spaces §14. The restriction of this LOFS to the category of metric spaces yields an OFS whose left class of morphisms are the dense inclusions. Further examples are explored in [6] in a very general framework that covers, for example, R. Lowen's *approach spaces* as well as the examples mentioned above.

An appendix §A discusses part of the theory of LOFSs that can be developed in the setting of locally presentable categories, where, under mild hypotheses, there is a reflection between the category of accessible lax idempotent monads and the category of accessible LOFSs. The appendix demands more knowledge of some parts of Category Theory.

2. Order-enriched categories and lax idempotent monads

By an *ordered set* we shall mean what is usually called a *poset*, that is, a pair $(X, \leqslant)$ where X is a set and $\leqslant$ is a relation on X that is reflexive, transitive and antisymmetric. Ordered sets can be identified with small categories with at most one morphism between any two objects and whose isomorphisms are identity morphisms.

We denote by **Ord** the category of ordered sets and monotone maps (functions that preserve $\leqslant$). This is a cartesian closed category, with exponential Y^X defined as the set of all order-morphisms $X \to Y$, and endowed with the pointwise order.

A category enriched in **Ord**, or **Ord**-category, is a locally small category $\mathcal{C}$ whose hom-sets are equipped with an order structure, and whose composition preserves the inequality: if $g \leqslant g'$ then $h \cdot g \leqslant h \cdot g'$ and $g \cdot f \leqslant g' \cdot f$, whenever these compositions are defined. In other words, the composition functions

$$\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z)$$

are monotone maps.

The category **Ord** of ordered sets can be regarded as a full subcategory of the category of small categories **Cat** by regarding ordered sets as small categories, as mentioned above. This means that **Ord**-categories can be regarded as 2-categories, but we do not go to that level of generality.

A locally monotone functor $F\colon \mathcal{C} \to \mathcal{D}$, or **Ord**-*functor*, between **Ord**-categories is an ordinary functor between the underlying ordinary categories that is moreover monotone on homs; ie that each $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is a monotone map.

The category of **Ord**-categories and **Ord**-functors will be denoted by **Ord-Cat**. It is a cartesian closed category.

Example 2.1. The category **Ord** has a canonical structure of an **Ord**-category $\underline{\mathbf{Ord}}$ whose ordered sets are $\underline{\mathbf{Ord}}(X, Y) = Y^X$. Many other categories constructed from **Ord** are **Ord**-enriched, such as the categories of join-semilattices, complete lattices, distributive lattices, and Heyting algebras.

Example 2.2. If X is a topological space, define a preorder on X by $x \leqslant y$ if all the neighbourhoods of y are also neighbourhoods of x, or, equivalently, denoting by $\mathcal{O}X$ the topology of X, $x \in U$ whenever $y \in U$ for every $U \in \mathcal{O}X$; in other words, $x \leqslant y$ if $y \in \overline{\{x\}}$. The opposite of this order is usually called the *specialisation order* and was introduced by D. Scott in [29]. The preorder $(X, \leqslant)$ is an ordered set precisely when X is a T_0 space.

Any continuous function $f\colon X \to Y$ between topological spaces preserves the order $\leqslant$. The category $\mathbf{Top}_0$ of T_0 topological spaces and continuous maps can be endowed with an **Ord**-category structure if we define, for any pair $f, g\colon X \to Y$ of continuous functions, $f \leqslant g$ if $f(x) \leqslant g(x)$ for all $x \in X$.

2.a. Full morphisms and locally full functors.

Definition 2.3. (1) A monotone map f between ordered sets is *full* if it reflects inequalities; ie $f(x) \leqslant f(y)$ implies $x \leqslant y$.

(2) A locally monotone functor $F\colon \mathcal{A} \to \mathcal{B}$ between **Ord**-categories is *locally full* if each monotone map

$$F_{A,B}\colon \mathcal{A}(A, B) \longrightarrow \mathcal{B}(FA, FB)$$

is full.

(3) A morphism $g\colon X \to Y$ in an **Ord**-category $\mathcal{C}$ is *full* if for each $Z \in \mathcal{C}$ the monotone map

$$\mathcal{C}(Z, g)\colon \mathcal{C}(Z, X) \longrightarrow \mathcal{C}(Z, Y)$$

is full.

Full morphisms are necessarily monomorphisms; for if $f\colon X \to Y$ is a full monotone morphism of ordered sets and $f(x) = f(y)$, then we have both $x \leqslant y$ and $y \leqslant x$, so $x = y$.

Lemma 2.4. *Suppose that $F \dashv U\colon \mathcal{B} \to \mathcal{A}$ is an adjunction of locally monotone functors between* **Ord***-categories, with unit $\eta\colon 1_{\mathcal{A}} \Rightarrow UF$. Then F is locally full if each component $\eta_A\colon A \to UFA$ is a full morphism.*

Proof: The naturality of η is expressed by the commutativity of the following diagram.

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[r, "F_{A,B}"] \arrow[drr, "\mathcal{A}(1,\eta_B)"'] & \mathcal{B}(FA,FB) \arrow[r, "U_{FA,FB}"] & \mathcal{A}(UFA,UFB) \arrow[d, "\mathcal{A}(\eta_A,1)"] \\ & & \mathcal{A}(A,UFB) \end{tikzcd}$$

If η_B is full, the diagonal morphism is full and therefore $F_{A,B}$ must be full too. ∎

2.b. Order-enriched (co)limits.

Limits. The category of ordered sets admits the construction of two-dimensional limits, which will be fundamental for us. We denote by $\mathbf{2}$ the order with two elements $0 \leqslant 1$. If X is an ordered set, then the exponential $X^{\mathbf{2}}$ is

$$X^{\mathbf{2}} = \{(x,y) \in X \times X : x \leqslant y\} \subseteq X \times X$$

with the order inherited from $X \times X$. We denote by d_0 and d_1 the two projections from $X^{\mathbf{2}}$ onto X. Slightly more involved is the comma-object of two order morphisms $f\colon X \to Z \leftarrow Y : g$

$$f \downarrow g = \{(x,y) \in X \times Y : f(x) \leqslant g(y)\} \subseteq X \times Y$$

that can equally well be constructed from $Z^{\mathbf{2}}$ by taking the limit of the following diagram.

$$X \xrightarrow{f} Z \xleftarrow{d_0} Z^{\mathbf{2}} \xrightarrow{d_1} Z \xleftarrow{g} Y$$

The constructions of the previous paragraphs can be defined in any **Ord**-category $\mathcal{C}$. If $X \in \mathcal{C}$, then define $X^{\mathbf{2}}$ as an object equipped with two morphisms $d_0 \leqslant d_1\colon X^{\mathbf{2}} \to X$ that induce isomorphisms of orders

$$\mathcal{C}(Z, X^{\mathbf{2}}) \cong \mathcal{C}(Z,X)^{\mathbf{2}}$$

for all $Z \in \mathcal{C}$, in the sense that, for each pair of morphisms $f_0 \leqslant f_1\colon Z \to X$, there exists a unique morphism $h\colon Z \to X^{\mathbf{2}}$ such that $f_0 = d_0 \cdot h$ and $f_1 = d_1 \cdot h$. Furthermore, if $k\colon Z \to X^{\mathbf{2}}$ is another morphism, then the conjunction of $d_0 \cdot h \leqslant d_0 \cdot k$ and $d_1 \cdot h \leqslant d_1 \cdot k$ implies $h \leqslant k$. This universal property guarantees that $X^{\mathbf{2}}$ is unique up to canonical isomorphism.

Similarly, given morphisms $f\colon X \to Z \leftarrow Y : g$ in $\mathcal{C}$, one can define a comma-object $f \downarrow g$ in $\mathcal{C}$ as an object equipped with two morphisms d_0 and

d_1 as shown

$$\begin{tikzcd} & f \downarrow g \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\ X \arrow[dr, "f"'] & \leqslant & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

that induce an order-isomorphism

$$\mathcal{C}(W, f \downarrow g) \cong \mathcal{C}(W, f) \downarrow \mathcal{C}(W, g)$$

for all $W \in \mathcal{C}$. In other words, for each pair of morphisms $h_0 \colon W \to X$ and $h_1 \colon W \to Y$ such that $f \cdot h_0 \leqslant g \cdot h_1$, there exists a unique $h \colon W \to f \downarrow g$ satisfying $d_0 \cdot h = h_0$ and $d_1 \cdot h = h_1$. Furthermore, if $k \colon W \to f \downarrow g$ is another morphism, then the conjunction of $d_0 \cdot h \leqslant d_0 \cdot k$ and $d_1 \cdot h \leqslant d_1 \cdot k$ implies $h \leqslant k$.

Colimits. Let $\mathcal{D}$ be an ordinary category. If $D \colon \mathcal{D} \to \mathcal{C}$ is a functor (ie a diagram in $\mathcal{C}$), we say that an object $C \in \mathcal{C}$ together with a natural transformation $\alpha_X \colon D(X) \to C$ is a colimit of D if

$$\mathcal{C}(\alpha_X, C') \colon \mathcal{C}(C, C') \longrightarrow \mathcal{C}(D(X), C')$$

is a limiting cone in the category **Ord**, for all $C' \in \mathcal{C}$. This is the same as saying that (C, α) is a limit of sets and the bijection $\mathcal{C}(C, C') \cong \lim \mathcal{C}(D-, C')$ is an isomorphism of posets.

It is not hard to verify that *filtered* colimits in **Ord** can be constructed in a completely analogous way to those in the category of sets. Furthermore, it can easily be verified that filtered colimits *commute*, or *distribute*, over finite enriched limits in **Ord**, in the sense that the **Ord**-functor $\lim \colon [\mathcal{F}, \mathbf{Ord}] \to \mathbf{Ord}$ preserves filtered colimits if $\mathcal{F}$ is finite. For example, the functor $(-)^{\mathbf{2}} \colon \mathbf{Ord} \to \mathbf{Ord}$ preserves filtered colimits, as do pullbacks, and therefore comma-objects preserve colimits (since comma-objects can be constructed from $(-)^{\mathbf{2}}$ and pullbacks). This phenomenon is part of the general theory of locally finitely presentable enriched categories developed in [18].

2.c. Adjunctions, extensions and liftings. An adjunction in an **Ord**-category $\mathcal{C}$ consists of two morphisms $f \colon X \to Y$ and $g \colon Y \to X$ in opposite directions with inequalities

$$1_X \leqslant g \cdot f \qquad \text{and} \qquad f \cdot g \leqslant 1_Y .$$

Such an adjunction is usually written $f \dashv g$.

By the usual argument, adjoints are unique up to canonical isomorphism, which in our case, by the antisymmetry of the ordering, means that adjoints are unique. For, if $f \dashv g$ and $f \dashv g'$, then

$$g = 1_X \cdot g \leqslant g' \cdot f \cdot g \leqslant g' \cdot 1_Y = g'$$

and symmetrically, $g' \leqslant g$.

A notion related to adjunctions is that of a *left extension.* If $j \colon X \to Y$ and $f \colon X \to Z$ are morphisms in the **Ord**-category $\mathcal{C}$, we say that an inequality $f \leqslant \mathrm{lan}_j f \cdot j$ exhibits $\mathrm{lan}_j f \colon Y \to Z$ as a left extension of f by j if, for any other $g \colon Y \to Z$ that satisfies $f \leqslant g \cdot j$, the inequality $\mathrm{lan}_j f \leqslant g$ holds.

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \arrow[dl, phantom, "\leqslant"] \\ & Z \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Y \arrow[d, "\mathrm{lan}_j f"] \arrow[d, bend left=60, "g"] \arrow[dl, phantom, "\leqslant"] \arrow[d, phantom, shift left=3ex, "\leqslant"] \\ & Z \end{tikzcd}$$

This universal property makes $\mathrm{lan}_j f$ unique – if it exists.

When j has a right adjoint j^*, there always exists a left extension $\mathrm{lan}_j f$, for any f: the extension is given by $\mathrm{lan}_j f = f \cdot j^*$.

The notion dual to that of a left extension is called *left lifting.* If $j \colon X \to Y$ and $f \colon Z \to Y$ are morphisms in $\mathcal{C}$, we say that an inequality $f \leqslant j \cdot {}^j f$ as depicted exhibits ${}^j f$ as a left lifting of f through j if, for any other morphism g, the inequality $f \leqslant j \cdot g$ implies ${}^j f \leqslant g$.

$$\begin{tikzcd} X \arrow[r, "j"] & Y \\ Z \arrow[u, "g"] \arrow[ur, "f"'] \arrow[ur, phantom, shift left=2ex, "\geqslant"] \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "j"] & Y \\ Z \arrow[u, "{}^j f"'] \arrow[u, bend left=60, "g"] \arrow[u, phantom, shift left=3ex, "\geqslant"] \arrow[ur, "f"'] \arrow[ur, phantom, shift left=2ex, "\geqslant"] \end{tikzcd}$$

When j has a left adjoint j^ℓ, then $j^\ell \cdot f$ is a left lifting of f through j.

2.d. Lax idempotent monads. Before recalling the notion of order-enriched monad, let us remind the reader of the definition of a monad on a category. A monad on a category $\mathcal{A}$ is a triple $\mathsf{T} = (T, \eta, \mu)$ where T is an endofunctor of $\mathcal{A}$ and $\eta \colon 1_{\mathcal{A}} \Rightarrow T \Leftarrow T^2 \colon \mu$ are natural transformations that satisfy the associativity and unit axioms:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, "1"] \\ & T & \end{tikzcd}$$

An algebra for the monad T, or a T-*algebra*, is a pair (A, a) where $a\colon TA \to A$ is a morphism in $\mathcal{A}$ that satisfies two axioms:

$$\begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd}$$

A morphism of T-algebras $(A, a) \to (B, b)$ is a morphism $f\colon A \to B$ in $\mathcal{A}$ that satisfies $b \cdot Tf = f \cdot a$. Algebras and their morphisms form a category T-Alg, that comes equipped with a forgetful functor into $\mathcal{A}$.

Let $\mathcal{C}$ be an **Ord**-category. An order-enriched monad, or **Ord**-*monad*, on $\mathcal{C}$ consists of a monad $\mathsf{T} = (T, \eta, \mu)$ on the ordinary category $\mathcal{C}$ with the additional requirement that T be an **Ord**-functor. When the context is clear, we will refer to **Ord**-monads simply as monads.

Definition 2.5. A monad $\mathsf{T} = (T, \eta, \mu)$ on an **Ord**-category $\mathcal{C}$ is *lax idempotent*, or *Kock–Zöberlein*, if it satisfies any of the following equivalent conditions.

(1) $T\eta \cdot \mu \leqslant 1$.
(2) $1 \leqslant \eta T \cdot \mu$.
(3) For any T-algebra $a\colon TA \to A$, the inequality $1_{TA} \leqslant \eta_A \cdot a$ holds.
(4) A morphism $l : TA \to A$ defines a T-algebra structure (A, l) if and only if $l \dashv \eta_A$ with $l \cdot \eta_A = 1_A$.
(5) $T\eta \leqslant \eta T$.
(6) For any pair of T-algebras (A, a) and (B, b) and all morphisms $f\colon A \to B$ in $\mathcal{C}$, $b \cdot Tf \leqslant f \cdot a$ holds.
(7) For any T-algebra (A, a) and any morphism $f\colon X \to A$ in $\mathcal{C}$, the equality $a \cdot Tf \cdot \eta_X = f$ exhibits $a \cdot Tf$ as a left extension of f along $\eta_X\colon X \to TX$.

The equivalences of the above conditions can be found, in the more general case of 2-categories, in [20]. Morphisms f satisfying condition (6) are called *lax morphisms* of T-algebras, even for a monad T that is not lax idempotent; so condition (6) says that T is lax idempotent if any morphism in $\mathcal{C}$ between T-algebras is a lax morphism of T-algebras.

Definition 2.6. The notion of a lax idempotent comonad $\mathsf{G} = (G, \varepsilon, \delta)$ is a dual one: G is a lax idempotent comonad on $\mathcal{C}$ if $(G^{\mathrm{op}}, \varepsilon^{\mathrm{op}}, \delta^{\mathrm{op}})$, the corresponding monad on $\mathcal{C}^{\mathrm{op}}$, is lax idempotent. We only translate explicitly condition (7) of Definition 2.5: for any G-coalgebra $a\colon A \to GA$ and any morphism $f\colon A \to X$ in $\mathcal{C}$, the equality $f = \varepsilon_X \cdot Gf \cdot a$ exhibits $Gf \cdot a$ as a left lifting of f through ε_X (see §2.c for the definition of left liftings).

$$\begin{tikzcd} GX \arrow[r, "\varepsilon_X"] & X \\ A \arrow[u, "Gf\cdot a"] \arrow[ur, "f"'] \arrow[u, phantom, shift left=4ex, "\geqslant"] & \end{tikzcd}$$

Example 2.7. Given an ordered set X, denote by $P(X)$ the set of down-closed subsets of X, ie the set of those subsets $Y \subseteq X$ satisfying $(x \leqslant y) \wedge (y \in Y) \Rightarrow x \in Y$; the set $P(X)$ is canonically ordered by the inclusion of subsets of X. We denote by $\eta_X\colon X \to P(X)$ the monotone function

$$\eta_X\colon X \longrightarrow P(X) \qquad x \mapsto {\downarrow}x = \{y \in X : y \leqslant x\}.$$

The assignment $X \mapsto P(X)$ can be extended to a functor whose value on a monotone function $f\colon X \to Y$ is

$$P(X) \xrightarrow{f_*} P(Y) \qquad f_*(Z) = \{y \in Y : (\exists x \in Z)(y \leqslant f(x))\} = \cup_{x\in Z}{\downarrow}f(x).$$

Observe that f_* always has a right adjoint $f^*\colon P(Y) \to P(X)$ given by

$$f^*(Z) = \{x \in X : \exists z \in Z \text{ such that } f(x) \leqslant z\}.$$

Clearly, $f_* \leqslant g_*$ if $f \leqslant g$, so P is an **Ord**-functor. It is well-known that $X \mapsto P(X)$ defines a monad on **Ord**, where $P(X)$ is ordered by inclusion, with unit η and multiplication μ given by

$$P^2(X) \longrightarrow P(X) \qquad \big(\mathcal{U} \subseteq P(X)\big) \mapsto \cup\{Y \in \mathcal{U}\} \subseteq X.$$

This **Ord**-monad on the **Ord**-category $\underline{\mathbf{Ord}}$ is lax idempotent, since

$$P\eta_X(Z) = \cup_{x\in Z}{\downarrow}({\downarrow}x) \subseteq {\downarrow}Z = \eta_{P(X)}(Z).$$

The **Ord**-category P-Alg is the category of complete lattices (posets with arbitrary suprema or joins) with morphisms those monotone maps that preserve arbitrary suprema.

Example 2.8. If $\mathbf{Top}_0$ is the category of T_0 topological spaces and $\underline{\mathbf{Top}}_0$ is the associated **Ord**-category, with ordering induced by the opposite of the specialisation order, as in Example 2.2, there is an endo-**Ord**-functor $F\colon \underline{\mathbf{Top}}_0 \to \underline{\mathbf{Top}}_0$ that sends X to the set $F(X)$ of filters of open sets of X, with topology generated by the subsets $U^\# = \{\varphi \in F(X) : U \in \varphi\}$, for

$U \in \mathcal{O}X$. This is in fact the functor part of the lax idempotent *filter monad* on $\underline{\mathbf{Top}}_0$ that will be studied in Section 13.

There is a well-known result about algebras for lax idempotent monads on **Ord**-categories (see [22] and [12]) that can be summarised by saying that algebras are closed under retracts. More precisely:

Lemma 2.9. *If* $\mathsf{T} = (T, \eta, \mu)$ *is a lax idempotent monad on an* **Ord**-*category, the following conditions on an object* A *are equivalent.*

(1) A *admits a (unique)* T-*algebra structure (we simply say that* A *is a* T-*algebra).*
(2) $\eta_A \colon A \to TA$ *has a right inverse.*
(3) A *is a retract of* TA.
(4) A *is a retract of a* T-*algebra.*

Given two monads $\mathsf{S} = (S, \nu, \theta)$ and $\mathsf{T} = (T, \eta, \mu)$ on the category $\mathcal{C}$ we recall that a *monad morphism* $\varphi : \mathsf{S} \to \mathsf{T}$ is a natural transformation such that, for every object X of $\mathcal{C}$, the following diagrams commute.

$$\begin{tikzcd} & TSX \arrow[r, "T\varphi_X"] & \\ SSX \arrow[ur, "\varphi_{SX}"] \arrow[dr, "S\varphi_X"'] \arrow[d, "\theta_X"'] & & TTX \arrow[d, "\mu_X"] \\ & STX \arrow[ur, "\varphi_{TX}"'] & \\ SX \arrow[rr, "\varphi_X"] & & TX \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\nu_X"'] \arrow[dr, "\eta_X"] & \\ SX \arrow[rr, "\varphi_X"] & & TX \end{tikzcd}$$

(There is a more general notion of morphism between monads on different categories, which we will not need.)

Lemma 2.10. *Let* T *and* S *be monads on an* **Ord**-*category. Then there is at most one monad morphism* $\mathsf{T} \to \mathsf{S}$ *if* T *is lax idempotent.*

Proof: Suppose that $\varphi_X \colon TX \to SX$ are the components of a monad morphism. The morphism

$$\psi_X \colon TSX \xrightarrow{\varphi_{SX}} S^2X \xrightarrow{\mu^S_X} SX$$

is a T-algebra structure on SX, and therefore it is uniquely defined as the left adjoint to the unit $SX \to TSX$. Therefore, $\varphi_X = \psi_X \cdot T(\eta^S_X)$ is uniquely defined. ■

3. Orthogonal factorisations and simple reflections, revisited

In this section we revisit some of the material of Cassidy–Hébert–Kelly work on simple reflections [5] from a slightly different perspective, more amenable to generalisation.

Suppose that $T\colon \mathcal{A} \to \mathcal{A}$ is a reflection, with unit $\eta_A\colon A \to TA$, on the category $\mathcal{A}$, which we assume to admit pullbacks. The corresponding reflective subcategory will be denoted by T-Alg, as it consists of the algebras for the idempotent monad T associated to T, whose invertible multiplication we denote by $\mu\colon T^2 \Rightarrow T$.

We say that a morphism f in $\mathcal{A}$ is a *T-isomorphism*, or is *T-invertible*, if Tf is an isomorphism.

Each morphism $f\colon A \to B$ can be factorised through a pullback square, as displayed.

$$f = Rf \cdot Lf \qquad \begin{tikzcd} A \arrow[dr, "Lf"] \arrow[drr, bend left, "\eta_A"] \arrow[ddr, bend right, "f"'] & & \\ & Kf \arrow[r, "q_f"] \arrow[d, "Rf"] \arrow[dr, phantom, "\mathrm{pb}"] & TA \arrow[d, "Tf"] \\ & B \arrow[r, "\eta_B"] & TB \end{tikzcd} \tag{3.1}$$

Remark 3.2. The factorisation $f = Rf \cdot Lf$ is *functorial*, in the sense that, if $(h,k)\colon f \to g$ is a morphism in the arrow category $\mathcal{A}^{\mathbf{2}}$, then there is a morphism $K(h,k)\colon Kf \to Kg$

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "k"] & \cdot \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\ Kf \arrow[r, "{K(h,k)}"] \arrow[d, "Rf"'] & Kg \arrow[d, "Rg"] \\ \cdot \arrow[r, "k"] & \cdot \end{tikzcd}$$

yielding a functor $K\colon \mathcal{A}^{\mathbf{2}} \to \mathcal{A}$.

Remark 3.3. The assignment that sends a morphism $f \mapsto Lf$ is part of an endofunctor on $\mathcal{A}^{\mathbf{2}}$, given on morphisms by

$$f \xrightarrow{(h,k)} g \quad \longmapsto \quad Lf \xrightarrow{(h,K(h,k))} Lg.$$

Furthermore, there is a natural transformation $\Phi\colon L \Rightarrow 1$ with components

$$\Phi_f \quad = \quad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "Rf"] & \cdot \end{tikzcd}$$

Remark 3.4. The assignment $f \mapsto Rf$ underlies a monad on the arrow category $\mathcal{A}^{\mathbf{2}}$. Its unit and multiplication are given by

$$\Lambda_f \quad = \quad \begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd} \qquad \Pi_f \quad = \quad \begin{tikzcd} \cdot \arrow[r, "\pi_f"] \arrow[d, "R^2 f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

where the morphism $\pi_f\colon KRf \to Kf$ is the unique morphism into the pullback Kf such that

$$q_f \cdot \pi_f = \mu_{\mathrm{dom}(f)} \cdot Tq_f \cdot q_{Rf} \quad \text{and} \quad Rf \cdot \pi_f = RRf.$$

One of the contributions of [5] is to introduce a property on the reflection T that guarantees that the factorisation $f = Rf \cdot Lf$ is an orthogonal factorisation system (OFS): the property of being simple.

Definition 3.5. The reflection $\mathsf{T} = (T, \eta)$ is *simple* if Lf is a T-isomorphism.

As pointed out in [5], if T is simple then the factorisation $f = Rf \cdot Lf$ defines an orthogonal factorisation system, with left class of morphisms that of T-isomorphisms. To say only a few words about this fact, any morphism of the form Tf is orthogonal to T-isomorphisms, and so Rf, as a pullback of Tf, is also orthogonal to T-isomorphisms; together with the simplicity hypothesis that Lf be a T-isomorphism, we obtain an orthogonal factorisation.

If we denote by $F^T\colon \mathcal{A} \to \mathsf{T}\text{-Alg}$ the left adjoint of the inclusion $\mathsf{T}\text{-Alg} \subset \mathcal{A}$, then we can consider the full subcategory $\mathsf{T}\text{-Iso} \subset \mathcal{A}^{\mathbf{2}}$ whose objects are those morphisms of $\mathcal{A}$ that are T-isomorphisms (equivalently, those morphisms f such that $F^T(f)$ is an isomorphism) as a pullback.

$$\begin{tikzcd} \mathsf{T}\text{-Iso} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & \mathrm{Iso} \arrow[d] \\ \mathcal{A}^{\mathbf{2}} \arrow[r, "(F^T)^{\mathbf{2}}"] & \mathsf{T}\text{-Alg}^{\mathbf{2}} \end{tikzcd} \tag{3.6}$$

Lemma 3.7. *The subcategory* $\mathsf{T}\text{-Iso} \hookrightarrow \mathcal{A}^{\mathbf{2}}$ *is coreflective if and only if the reflection* T *is simple. In this case, the associated idempotent comonad is* *given by $f \mapsto Lf$ and has counit*

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "Rf"] & \cdot \end{tikzcd}$$

Proof: If T is simple, we know that the T-isomorphisms are the left class of an orthogonal factorisation system, and thus coreflective in $\mathcal{A}^{\mathbf{2}}$. To be more explicit, if $(\mathscr{E}, \mathscr{M})$ is an orthogonal factorisation system in $\mathcal{A}$, and $f = m \cdot e$ with $e \in \mathscr{E}$ and $m \in \mathscr{M}$, then the morphism

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "e"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "m"] & \cdot \end{tikzcd}$$

exhibits e as a coreflection of f into the full subcategory of $\mathcal{A}^{\mathbf{2}}$ defined by $\mathscr{E}$.

Before moving to proving the converse, we make the observation that, for any category $\mathcal{B}$, the full subcategory $\mathrm{Iso} \subset \mathcal{B}^{\mathbf{2}}$ of isomorphisms is coreflective (as well as reflective) with coreflection given by $\Upsilon_f \colon I(f) = 1_{\mathrm{dom}(f)} \to f$

$$\Psi_f \quad = \quad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "1_{\mathrm{dom}(f)}"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

To prove the converse, suppose that the inclusion of T-Iso into $\mathcal{A}^{\mathbf{2}}$ is coreflective, with coreflection given by counits $\Psi_f \colon Gf \to f$ in $\mathcal{A}^{\mathbf{2}}$. Then the pullback diagram (3.6) can be rewritten in the following form, where the categories of coalgebras are those for the respective copointed endofunctors $\Psi \colon G \Rightarrow 1$ and $\Upsilon \colon I \Rightarrow 1$.

$$\begin{tikzcd} (G, \Psi)\text{-Coalg} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & (I, \Upsilon)\text{-Coalg} \arrow[d] \\ \mathcal{A}^{\mathbf{2}} \arrow[r, "(F^T)^{\mathbf{2}}"] & \mathsf{T}\text{-Alg}^{\mathbf{2}} \end{tikzcd}$$

It is well known that, in these circumstances, (G, Ψ) is given by a pullback in the category of endofunctors of $\mathcal{A}^{\mathbf{2}}$

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Psi"'] & (U^T)^{\mathbf{2}} I (F^T)^{\mathbf{2}} \arrow[d, "(U^T)^{\mathbf{2}} \Upsilon (F^T)^{\mathbf{2}}"] \\ 1_{\mathcal{A}^{\mathbf{2}}} \arrow[r, "\eta^{\mathbf{2}}"] & T^{\mathbf{2}} \end{tikzcd}$$

If we apply the domain functor $\mathrm{dom}\colon \mathcal{A}^{\mathbf{2}} \to \mathcal{A}$ to this pullback, we obtain that $\mathrm{dom}(\Psi)$ can be taken to be the identity transformation, since $\mathrm{dom}(U^T \Upsilon_{F^T(f)})$ is an identity morphism for any f. If we apply the codomain functor cod instead, we obtain a pullback square

$$\begin{tikzcd}
\mathrm{cod}(Gf) \arrow[r] \arrow[d, "\mathrm{cod}\,\Psi_f"'] & T\,\mathrm{dom}\,f \arrow[d, "Tf"] \\
\mathrm{cod}(f) \arrow[r, "\eta_{\mathrm{cod}(f)}"'] & T\,\mathrm{cod}(f)
\end{tikzcd}$$

(we have used that $\mathrm{cod}\,U^T I(F^T(f)) = U^T\,\mathrm{cod}(1_{\mathrm{dom}(F^T(f))}) = T\,\mathrm{dom}(f)$). In other words, $\mathrm{cod}\,\Psi_f = Rf$ and $\mathrm{cod}(Gf) = Kf$ as defined in diagram (3.1). From here it is straightforward to verify that $Gf = Lf$. Therefore $Lf \in \mathsf{T}$-Iso, which says that T is a simple reflection, concluding the proof. ∎

The lemma proved above gives a characterisation of simple reflections, so one could define simple reflections as those reflections T on $\mathcal{A}$ such that the full subcategory $\mathsf{T}\text{-Iso} \subset \mathcal{A}^{\mathbf{2}}$ is coreflective. The associated idempotent comonad on $\mathcal{A}^{\mathbf{2}}$ is given by $f \mapsto Lf$.

4. Lax orthogonal factorisations

We now proceed to study lax orthogonal factorisation systems on **Ord**-categories. Before that, we briefly recall basic facts on algebraic weak factorisation systems.

4.a. Weak factorisation systems. This short section recalls the definition of weak factorisation system, a notion that appeared as part of Quillen's definition of model category [27].

We say that a morphism g *has the right lifting property with respect to another* f, and that f *has the left lifting property with respect to* g, if every time we have a commutative square as shown, there exists (a not necessarily unique) diagonal filler.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r] \arrow[ur, dotted] & \cdot
\end{tikzcd}$$

A *weak factorisation system* (WFS) in a category consists of two families of morphisms $\mathcal{L}$ and $\mathcal{R}$ such that:

- $\mathcal{R}$ consists of those morphisms with the right lifting property with respect to each morphism of $\mathcal{L}$.
- $\mathcal{L}$ consists of those morphisms with the left lifting property with respect to each morphism of $\mathcal{R}$.
- Each morphism in the category is equal to the composition of one element of $\mathcal{L}$ followed by one of $\mathcal{R}$.

4.b. Algebraic weak factorisation systems. Algebraic weak factorisation systems (AWFSs) where first introduced by M. Grandis and W. Tholen in [14], with an extra distributivity condition later added by R. Garner in [13]. In this section we shall give the definition of AWFSs on order-enriched categories, which is the case we will need, even though the definitions remain virtually unchanged.

Definition 4.1. An **Ord**-functorial factorisation on an **Ord**-category $\mathcal{C}$ consists of a factorisation

$$\mathrm{dom} \overset{\lambda}{\Rightarrow} E \overset{\rho}{\Rightarrow} \mathrm{cod}$$

in the category of locally monotone functors $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ of the natural transformation $\mathrm{dom} \Rightarrow \mathrm{cod}$ with component at $f \in \mathcal{C}^{\mathbf{2}}$ equal to $f\colon \mathrm{dom}(f) \to \mathrm{cod}(f)$. It is important that in this factorisation E should be a locally monotone functor.

As in the case of functorial factorisations on ordinary categories, an **Ord**-functorial factorisation as the one described in the previous paragraph can be equivalently described as:

- A copointed endo-**Ord**-functor $\Phi\colon L \Rightarrow 1_{\mathcal{C}^{\mathbf{2}}}$ on $\mathcal{C}^{\mathbf{2}}$ with $\mathrm{dom}(\Phi) = 1$.
- A pointed endo-**Ord**-functor $\Lambda\colon 1_{\mathcal{C}^{\mathbf{2}}} \Rightarrow R$ on $\mathcal{C}^{\mathbf{2}}$ with $\mathrm{cod}(\Lambda) = 1$.

The three descriptions of an **Ord**-functorial factorisation are related by:

$$\mathrm{dom}(\Lambda_f) = Lf = \lambda_f \qquad \mathrm{cod}(\Phi_f) = Rf = \rho_f. \tag{4.2}$$

Definition 4.3. An *algebraic weak factorisation system*, abbreviated AWFS, on an **Ord**-category $\mathcal{C}$ consists of a pair (L, R), where $\mathsf{L} = (L, \Phi, \Sigma)$ is an **Ord**-comonad and $\mathsf{R} = (R, \Lambda, \Pi)$ is an **Ord**-monad on $\mathcal{C}^{\mathbf{2}}$, such that (L, Φ) and (R, Λ) represent the same **Ord**-functorial factorisation on $\mathcal{C}$ (ie, the equalities (4.2) hold), plus a distributivity condition that we proceed to explain.

The unit axiom $\Pi \cdot (\Lambda R) = 1$ of the monad R implies, since $\mathrm{cod}(\Lambda) = 1$, that $\mathrm{cod}(\Pi) = 1$; dually $\mathrm{dom}(\Sigma) = 1$, so these transformations have components that look like:

$$\Sigma_f = \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "L^2f"] \\ \cdot \arrow[r, "\sigma_f"] & \cdot \end{tikzcd} \quad \text{and} \quad \Pi_f = \begin{tikzcd} \cdot \arrow[r, "\pi_f"] \arrow[d, "R^2f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

One can form a transformation

$$\Delta\colon LR \Longrightarrow RL \qquad \Delta_f = \begin{tikzcd} Kf \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] \arrow[dr, dotted, "1"] & KLf \arrow[d, "RLf"] \\ KRf \arrow[r, "\pi_f"] & Kf \end{tikzcd}$$

The distributivity axiom requires Δ to be a mixed distributive law between the comonad L and the monad R; this amounts to the commutativity of the following diagrams.

$$\begin{tikzcd} LR^2 \arrow[r, "\Delta R"] \arrow[d, "L\Pi"'] & RLR \arrow[r, "R\Delta"] & R^2L \arrow[d, "\Pi L"] \\ LR \arrow[rr, "\Delta"] & & RL \end{tikzcd} \qquad \begin{tikzcd} LR \arrow[rr, "\Delta"] \arrow[d, "\Sigma R"'] & & RL \arrow[d, "R\Sigma"] \\ L^2R \arrow[r, "L\Delta"] & LRL \arrow[r, "\Delta L"] & RL^2 \end{tikzcd} \tag{4.4}$$

(The two axioms of a mixed distributive law that involve the unit of the monad and the counit of the comonad automatically hold.)

Example 4.5. Each OFS $(\mathscr{E}, \mathscr{M})$ on $\mathcal{C}$ gives rise (upon choosing an $(\mathscr{E}, \mathscr{M})$-factorisation for each morphism) to an AWFS (L, R), where L is the idempotent comonad associated to the coreflective subcategory $\mathscr{E} \subset \mathcal{C}^{\mathbf{2}}$ and R is the idempotent monad associated to the reflective inclusion $\mathscr{M} \subset \mathcal{C}^{\mathbf{2}}$. Conversely, an AWFS (L, R) with both L and R idempotent induces an OFS. This was first shown in [14, Thm. 3.2], and [3, Prop. 3] further shows that it suffices that either L or R be idempotent.

If (L, R) is an AWFS on $\mathcal{C}$, an L-coalgebra structure on f and an R-algebra structure on g can be depicted by commutative squares

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "f"'] & \cdot \arrow[d, "Lf"] \\ \cdot \arrow[r, "s"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "Rg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

and the (co)algebra axioms can be written in the following way (where the morphisms σ_f and π_g are those described in Definition 4.3).

$$\begin{aligned} Rf \cdot s &= 1 & K(1,s)\cdot s &= \sigma_f \cdot s \\ p \cdot Lg &= 1 & p\cdot K(p,1) &= p \cdot \pi_g \end{aligned}$$

A morphism of L-coalgebras $(f,s) \to (f',s')$ is a morphism $(h,k)\colon f \to f'$ in $\mathcal{C}^{\mathbf{2}}$ that is compatible with the coalgebra structures in the usual way:

$$K(h,k)\cdot s = s' \cdot k.$$

Similarly, a morphism of R-algebras $(g,p) \to (g',p')$ is a morphism $(u,v)\colon g \to g'$ such that

$$p' \cdot K(u,v) = u \cdot p.$$

With the obvious composition and identities we obtain categories L-Coalg and R-Alg, equipped with forgetful functors into $\mathcal{C}^{\mathbf{2}}$. These are **Ord**-categories by stipulating that the ordering of morphisms of (co)algebras is inherited from the ordering of morphisms in $\mathcal{C}^{\mathbf{2}}$; as a consequence, the forgetful functors from L-Coalg and R-Alg to $\mathcal{C}^{\mathbf{2}}$ become **Ord**-enriched.

4.c. Underlying WFSs. Each AWFS (L,R) (enriched or not) has an *underlying* WFS $(\mathcal{L},\mathcal{R})$. The class $\mathcal{L}$ consists of all those morphisms that admit a structure of coalgebra over the copointed endofunctor (L,Φ) that underlies L; similarly, $\mathcal{R}$ consists of all those morphisms that admit a structure of an algebra over the pointed endofunctor (R,Λ) that underlies R.

4.d. LARIs **and** AWFSs. One of the most important examples of AWFSs for us will be provided by the so-called LARIs.

Definition 4.6. A *left adjoint right inverse*, or LARI, in an **Ord**-category is a morphism f that is part of an adjunction $f \dashv g$ with $1 = g \cdot f$. In the same situation, we say that g is a *right adjoint left inverse*, or RALI.

Suppose given another adjunction $f' \dashv g'$ with $1 = g' \cdot f'$, and morphisms h and k as in the displayed diagram.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, shift right, "f"'] \arrow[d, phantom, "\dashv"] & X' \arrow[d, shift right, "f'"'] \arrow[d, phantom, "\dashv"] \\ Y \arrow[r, "k"'] \arrow[u, shift right, "g"'] & Y' \arrow[u, shift right, "g'"'] \end{tikzcd}$$

We say that (h,k) is a *morphism of* LARIs $f \to f'$, and that (h,k) is a *morphism of* RALIs $g \to g'$, if $f' \cdot h = k \cdot f$ and $g' \cdot k = h \cdot g$. With the obvious notion of composition, LARIs and RALIs form categories that come equipped with forgetful functors into $\mathcal{C}^{\mathbf{2}}$. Furthermore, if $\mathcal{C}$ is an **Ord**-category, there are **Ord**-categories $\mathrm{Lari}(\mathcal{C})$ and $\mathrm{Rali}(\mathcal{C})$ with objects and morphisms described above, and ordering between morphisms those of $\mathcal{C}^{\mathbf{2}}$.

Example 4.7. Consider the *free (split) opfibration monad* M on **Ord**, given on $f\colon X \to Y$ by $M(f)$

$$Kf = f \downarrow 1_{\mathrm{cod}(f)} = \big\{(x,y) \in X \times Y : f(x) \leqslant y\big\} \xrightarrow{Mf} Y \qquad (x,y) \mapsto y$$

with ordering inherited from that of $X \times Y$. Furthermore, M is a locally monotone endofunctor of $\mathbf{Ord}^{\mathbf{2}}$. The category M-Alg of algebras for this monad has objects the *(split) opfibrations*, ie monotone functions $f\colon X \to Y$ with a choice for each $x \in X$ and $y \in Y$ that satisfy $f(x) \leqslant y$, of an $x_y \in X$ such that: $x \leqslant x_y$, $f(x_y) = y$, and $(x \leqslant x') \wedge (f(x') = y)$ implies $x_y \leqslant x'$. As an aside comment, we note that there is no difference between the notions of an opfibration and of a split opfibration in **Ord** due to the antisymmetry property satisfied by the orderings.

Any monotone function $f\colon X \to Y$ can be factorised as

$$f : X \xrightarrow{Ef} Kf \xrightarrow{Mf} Y$$

where $Ef(x) = (x, f(x)) \in f \downarrow Y = f \downarrow 1_Y$. This is in fact part of an AWFS, as we proceed to show. As the functorial factorisation is the one just described, the locally monotone endofunctor E of $\mathbf{Ord}^{\mathbf{2}}$ has a copoint $\Phi_f = (1_X, Mf)\colon Ef \to f$. The monotone function $Ef\colon X \to f \downarrow Y$ has a right adjoint $r_f\colon f \downarrow Y \to X$, given by $r_f(x,y) = x_y$. We can define

$$\sigma_f\colon Kf = f \downarrow Y \longrightarrow KEf = Ef \downarrow Kf \qquad (x,y) \mapsto (r_f(x,y), (x,y))$$

and morphisms Σ_f that form the comultiplication of a comonad $\mathsf{E} = (\mathsf{E}, \Phi, \Sigma)$.

$$\Sigma_f\colon Ef \longrightarrow E^2 f \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[d, "Ef"'] & X \arrow[d, "E^2 f"] \\ Kf \arrow[r, "\sigma_f"] & KE(f) \end{tikzcd}$$

The morphism $ME(f)\colon KEf \to Kf$ is a left adjoint to σ_f, as can be easily verified. Furthermore, $\Phi_{Ef} \dashv \Sigma_f$, which means that the comonad E is lax idempotent. The distributivity axiom of AWFSs can be verified by hand, or, alternatively, one can appeal to Theorem 7.2.

We conclude with the observation that the endofunctors E and M preserve filtered colimits; equivalently, the functor $K\colon \mathbf{Ord}^{\mathbf{2}} \to \mathbf{Ord}$ preserves filtered colimits. This is so because K is constructed by means of comma-objects and the comments at the end of §2.b.

Example 4.8. Precisely the same construction can be carried out in any **Ord**-category that admits comma-objects (see §2.b); for example, in any **Ord**-category that admits cotensor products with **2** and pullbacks. The morphism Mf is a projection in the comma-object depicted.

$$\begin{tikzcd} Kf \arrow[r, "r_f"] \arrow[d, "Mf"'] \arrow[dr, phantom, "\geqslant"] & X \arrow[d, "f"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

The left part of the factorisation $Ef\colon X \to Kf$ is the unique morphism defined by the conditions

$$Mf \cdot Ef = f \quad \text{and} \quad r_f \cdot Ef = 1_X.$$

It is not hard to show that $Ef \dashv r_f$.

The endo-**Ord**-functor $f \mapsto Mf$ is part of the free (split) opfibration monad on $\mathcal{C}$. The endo-**Ord**-functor E is part of a comonad with counit $\Phi^E_f = (1, Mf)\colon Ef \to f$ and comultiplication $\Sigma_f = (1, \sigma_f)\colon Ef \to E^2 f$ defined by

$$r_{Ef} \cdot \sigma_f = r_f \quad \text{and} \quad MEf \cdot \sigma_f = 1_{Kf}.$$

Lemma 4.9. *Suppose that* $\mathcal{C}$ *is an* **Ord***-category with comma-objects and* (E, M) *the AWFS constructed in the previous example. If* $\Phi^E\colon E \Rightarrow 1$ *is the underlying copointed endofunctor of the comonad* E*, then:*

(1) There is an isomorphism $\mathrm{Lari}(\mathcal{C}) \cong \mathsf{E}\text{-}\mathrm{Coalg}$ *over* $\mathcal{C}^{\mathbf{2}}$.

(2) The forgetful functor

$$\mathsf{E}\text{-}\mathrm{Coalg} \longrightarrow (E, \Phi^E)\text{-}\mathrm{Coalg} \tag{4.10}$$

is an isomorphism.

Proof: This proof follows a direction not suggested by the statement. We shall first prove that there is an isomorphism between $\mathrm{Lari}(\mathcal{C})$ and (E, Φ^E)-Coalg and then show that (4.10) is an isomorphism. The reason the lemma is stated in the present form is that this form extends to 2-categories [7].

Suppose given a morphism in $\mathcal{C}^{\mathbf{2}}$ as depicted.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[d, "Ef"] \\ B \arrow[r, "s"] & Kf \end{tikzcd} \tag{4.11}$$

The morphism $s\colon B \to Kf = f \downarrow B$ corresponds to a pair of morphisms $r\colon B \to A$ and $u\colon B \to B$ that satisfy $f \cdot r \leqslant u$. The morphisms r and u are the composition of s with, respectively, the projections $f \downarrow B \to A$ and $Mf\colon f \downarrow B \to B$. The commutativity of (4.11) translates into $r \cdot f = 1$ and $u \cdot f = f$.

Now suppose that (4.11) is a morphism of (E, Φ^E)-coalgebras, ie that $Mf \cdot s = 1$. By definition of u, this is equivalent to saying that $u = 1$. Therefore, to give an (E, Φ^E)-coalgebra structure on f is equivalent to giving a morphism $r\colon B \to A$ such that $f \cdot r \leqslant 1$ and $r \cdot f = 1$. In other words, an (E, Φ^E)-coalgebra structure on f is the same as a LARI structure on f.

To conclude the proof, we show that any (E, Φ^E)-coalgebra structure $(1, s)\colon f \to Ef$ is an E-coalgebra, ie it satisfies the coassociativity equality

$$\sigma_f \cdot s = K(1, s) \cdot s. \tag{4.12}$$

The codomain of the morphisms at both sides of the equality is KEf, so (4.12) holds precisely when it does after composing with the projections $MEf\colon KEf \to Kf$ and $r_{Ef}\colon KEf \to X$. One of these equalities is obvious, since

$$MEf \cdot \sigma_f \cdot s = 1 \cdot s = s = s \cdot 1 = s \cdot Mf \cdot s = MEf \cdot K(1, s) \cdot s.$$

The second equality holds by the following string of equalities, the first of which uses the definition of σ_f and the last uses $r_{Ef} \cdot K(1, s) = r_f$.

$$r_{Ef}\sigma_f \cdot s = r_f \cdot s = r_{Ef} \cdot K(1, s) \cdot s.$$

This completes the proof of the lemma. ∎

4.e. Lax orthogonal factorisation systems.

Definition 4.13. An AWFS (L, R) on an **Ord**-category $\mathcal{C}$ is a *lax orthogonal factorisation system* (abbreviated LOFS) if either of the following equivalent conditions holds:

- The comonad L is lax idempotent.

- The monad R is lax idempotent.

Before proving the equivalence between the above properties we describe more explicitly what it means for (L, R) to be lax orthogonal.

According to our notation, the unit and multiplication of R and the counit and comultiplication of L are depicted as morphisms in $\mathcal{C}^{\mathbf{2}}$ as follows.

$$\begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[dr, phantom, "\Lambda_f"] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "\pi_f"] \arrow[d, "R^2 f"'] \arrow[dr, phantom, "\Pi_f"] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] \arrow[dr, phantom, "\Phi_f"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "Rf"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] \arrow[dr, phantom, "\Sigma_f"] & \cdot \arrow[d, "L^2 f"] \\ \cdot \arrow[r, "\sigma_f"'] & \cdot \end{tikzcd}$$

Then, (L, R) is lax orthogonal if and only if any of the following conditions hold (the equivalence of these conditions will be shown in Proposition 4.16):

$$K(Lf, 1) \cdot \pi_f \leqslant 1 \qquad 1 \leqslant LRf \cdot \pi_f \qquad 1 \leqslant \sigma_f \cdot RLf \qquad \sigma_f \cdot K(1, Rf) \leqslant 1. \tag{4.14}$$

In terms of R-algebras and L-coalgebras, the lax idempotency of (L, R) is described as follows. If (f, s) is an L-coalgebra and (g, p) is an R-algebra, as displayed below,

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "(f,s)"] & \cdot \arrow[d, "Lf"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "Rg"'] \arrow[dr, phantom, "(g,p)"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

then the AWFS is lax orthogonal if and only if any of the following two equivalent conditions hold, for all (f, s) and (g, p) (again, the equivalence will be shown in Proposition 4.16):

$$1 \leqslant s \cdot Rf \quad \text{and} \quad 1 \leqslant Lg \cdot p. \tag{4.15}$$

Proposition 4.16. *If (L, R) is an* AWFS *on an* **Ord**-*category $\mathcal{C}$, then L is lax idempotent if and only if R is lax idempotent.*

Proof: In this proof we use the following general property of AWFSs, whose details can be found in [3, §2.8]. If (L, R) is an AWFS on an (ordinary) category and f, g are two composable morphisms each one of which carries an L-coalgebra structure, then their composition $g \cdot f$ carries a canonical L-coalgebra structure. We regard morphisms of the form Lf as L-coalgebras with structure given by the comultiplication $\Sigma_f = (1, \sigma_f) \colon Lf \to L^2 f$. Furthermore, we use the following fact, whose proof can be found in [3, §3.1]:

the morphism $(1, \pi_f)$ depicted is a morphism of L-coalgebras from $LRf \cdot Lf$ to Lf.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "Lf"'] & A \arrow[dd, "Lf"] \\ Kf \arrow[d, "LRf"'] & \\ KRf \arrow[r, "\pi_f"] & Kf \end{tikzcd}$$

Assuming that L is lax idempotent, we shall show that R is lax idempotent by exhibiting an inequality $R\Lambda \cdot \Pi \leqslant 1$, where Λ and Π are the unit and multiplication of the monad. The converse, namely that L is lax idempotent if R is so, is not necessary to prove, as it follows by a duality argument, more specifically, by taking the opposite **Ord**-category.

Let $f \colon A \to B$ be a morphism of $\mathcal{C}$, and consider the composition of the morphisms $(1_A, \pi_f) \colon LRf \cdot Lf \to Lf$ with $L\Lambda_f = (Lf, K(Lf, 1)) \colon Lf \to LRf$, as depicted.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "Lf"'] & A \arrow[r, "Lf"] \arrow[dd, "Lf"] & Kf \arrow[dd, "LRf"] \\ Kf \arrow[d, "LRf"'] & & \\ KRf \arrow[r, "\pi_f"] & Kf \arrow[r, "K(Lf,1)"] & KRf \end{tikzcd} \tag{4.17}$$

The composition of this diagram with the counit $\Phi_{Rf} = (1, R^2 f)$ equals the morphism $(Lf, R^2 f) \colon LRf \cdot Lf \to Rf$, depicted on the right below, since

$$R^2 f \cdot K(Lf, 1) \cdot \pi_f = Rf \cdot \pi_f = R^2 f. \qquad \begin{tikzcd} A \arrow[r, "Lf"] \arrow[d, "Lf"'] & Kf \arrow[dd, "Rf"] \\ Kf \arrow[d, "LRf"'] & \\ KRf \arrow[r, "R^2 f"] & B \end{tikzcd} \tag{4.18}$$

Since L is lax idempotent, the L-coalgebra morphism (4.17) is a left lifting of (4.18) through Φ_{Rf} (see Definition 2.6).

On the other hand, the morphism in $\mathcal{C}^{\mathbf{2}}$ depicted below is also equal to (4.18) upon composition with the counit Φ_{Rf}

$$\begin{tikzcd} A \arrow[r, "Lf"] \arrow[d, "Lf"'] & Kf \arrow[dd, "LRf"] \\ Kf \arrow[d, "LRf"'] & \\ KRf \arrow[r, equal] & KRf \end{tikzcd} \tag{4.19}$$

and by the universal property of liftings we deduce that (4.17) is less or equal than (4.19), so $K(Lf,1)\cdot\pi_f \leqslant 1_{KRf}$. It remains to prove that this defines an inequality in $\mathcal{C}^{\mathbf{2}}$ with identity codomain component; in other words, that the inequality becomes an equality upon composition with R^2f. But this holds, since both sides become equal:

$$R^2f\cdot K(Lf,1)\cdot\pi_f = Rf\cdot\pi_f = R^2f,$$

concluding the proof. ∎

Example 4.20. The AWFS (E,M) of Example 4.7, for which M-algebras are opfibrations and E-coalgebras are LARIs, is lax orthogonal. Indeed, the monad M is well-known to be lax idempotent.

4.f. Categories of AWFSs. There is a category $\mathbf{AWFS}(\mathcal{C})$ whose objects are AWFSs on the **Ord**-category $\mathcal{C}$. A morphism $(\mathsf{L},\mathsf{R}) \longrightarrow (\mathsf{L}',\mathsf{R}')$ is a natural family of morphisms φ_f that make the following diagrams commute.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "L'f"] \\ Kf \arrow[r, "\varphi_f"] \arrow[d, "Rf"'] & K'f \arrow[d, "R'f"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd} \tag{4.21}$$

Furthermore, the morphisms $(1,\varphi_f)\colon Lf \to L'f$ must form a comonad morphism $\mathsf{L}\to\mathsf{L}'$, and the morphisms $(\varphi_f,1)\colon Rf\to R'f$ must form a monad morphism $\mathsf{R}\to\mathsf{R}'$.

There is a full subcategory $\mathbf{LOFS}(\mathcal{C})$ of $\mathbf{AWFS}(\mathcal{C})$ consisting of the LOFSs.

Lemma 4.22. $\mathbf{LOFS}(\mathcal{C})$ *is a preorder.*

Proof: If the morphisms φ_f as in (4.21) form a morphism from (L,R) to $(\mathsf{L}',\mathsf{R}')$, then the morphisms $(\varphi_f,1)\colon Rf\to R'f$ define a morphism of monads. There can only be one such, by Lemma 2.10. ∎

5. Lifting operations

In this section we introduce KZ lifting operations and explain the motivation behind the definition of lax orthogonal factorisation systems. Before all that, we must say something about how lifting operations work in relation to AWFSs on **Ord**-categories.

5.a. Lifting operations on Ord-categories. Suppose that $U\colon \mathcal{A}\to\mathcal{C}^{\mathbf{2}}\leftarrow\mathcal{B}\colon V$ are locally monotone functors between **Ord**-categories. A *lifting operation* from U to V can be described as a choice of a diagonal filler $\phi_{a,b}(h,k)$ for each morphism $(h,k)\colon Ua\to Vb$ in $\mathcal{C}^{\mathbf{2}}$.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[d, "Vb"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dotted, "\phi_{a,b}(h,k)"] & \cdot \end{tikzcd}$$

These diagonal fillers must satisfy a naturality condition with respect to morphisms in $\mathcal{A}$ and $\mathcal{B}$. If $\alpha\colon a'\to a$ and $\beta\colon b\to b'$ are morphisms in $\mathcal{A}$ and $\mathcal{B}$ respectively, then

$$\phi_{a',b'}\big(\operatorname{dom}V\beta\cdot h\cdot\operatorname{dom}U\alpha,\operatorname{cod}V\beta\cdot k\cdot\operatorname{cod}U\alpha\big)=(\operatorname{dom}V\beta)\cdot\phi_{a,b}(h,k)\cdot(\operatorname{cod}U\alpha)$$

as depicted in the following diagram.

$$\begin{tikzcd} \cdot \arrow[r, "\operatorname{dom}U\alpha"] \arrow[d, "Ua'"'] & \cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[r, "\operatorname{dom}V\beta"] \arrow[d, "Vb"] & \cdot \arrow[d, "Vb'"] \\ \cdot \arrow[r, "\operatorname{cod}U\alpha"'] \arrow[urrr, dotted] & \cdot \arrow[r, "k"'] \arrow[ur, dotted] & \cdot \arrow[r, "\operatorname{cod}V\beta"'] & \cdot \end{tikzcd}$$

So far, the definition of lifting operation is the one given in [13], but our categories are enriched in **Ord** and the functors U and V are locally monotone, so we require that the diagonal filler satisfies: if (h,k) and $(h',k')\colon Ua\to Vb$ are commutative squares in $\mathcal{C}$ with $(h,k)\leqslant(h',k')$ (ie $h\leqslant h'$ and $k\leqslant k'$) then

$$\phi_{a,b}(h,k)\leqslant\phi_{a,b}(h',k').$$

5.b. Lifting operations from Ord-functorial factorisations. The idea of a functorial factorisation $\mathrm{dom} \Rightarrow E \Rightarrow \mathrm{cod}$, as defined in Definition 4.1, is that it induces a canonical lifting operation between the forgetful **Ord**-functors U and V

$$U\colon (L, \Phi)\text{-Coalg} \longrightarrow \mathcal{C}^{\mathbf{2}} \longleftarrow (R, \Lambda)\text{-Alg}\colon V.$$

Here $\Phi\colon L \Rightarrow 1_{\mathcal{C}^{\mathbf{2}}}$ and $\Lambda\colon 1_{\mathcal{C}^{\mathbf{2}}} \Rightarrow R$ are, respectively, the copointed endo-**Ord**-functor and the pointed endo-**Ord**-functor on $\mathcal{C}^{\mathbf{2}}$ associated to the given **Ord**-functorial factorisation.

A coalgebra for (L, Φ) can be depicted as the commutative square on the left below, while an algebra for (R, Λ) is a commutative square on the right

$$(f, s) \quad = \quad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "f"'] & \cdot \arrow[d, "Lf"] \\ \cdot \arrow[r, "s"] & \cdot \end{tikzcd} \qquad (g, p) \quad = \quad \begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "Rg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

satisfying $Rf \cdot s = 1$ and $p \cdot Lg = 1$. Given a commutative square $(h, k)\colon f \to g$, there is a canonical diagonal filler

$$\phi_{(f,s),(g,p)}(h, k) = p \cdot E(h, k) \cdot s.$$

It is immediate to see that these diagonal fillers form a lifting operation from U to V.

Remark 5.1. Even though an (**Ord**-)functorial factorisation $f = Rf \cdot Lf$ as the one discussed in the previous paragraphs yields a lifting operation of (L, Φ)-coalgebras against (R, Λ)-algebras, there is no guarantee of being able to find a diagonal filler for a commutative diagram of the form

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "Lf"'] & \cdot \arrow[d, "Rg"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

since Lf may not support an (L, Φ)-coalgebra structure, and Rg may not support an (R, Λ)-algebra structure. A natural way of endowing Lf and Rg with the corresponding structures is to require that (L, Φ) extends to a comonad and (R, Λ) extends to a monad; in this way, Lf is a (cofree) coalgebra and Rg is a (free) algebra. This one of the reasons for the form that the definition of AWFS takes (see Definition 4.3).

There is an useful fact that is worth including at this point, and will be useful in the proof of Theorem 5.6.

Lemma 5.2. *For any* AWFS (L,R), *the diagonals* $\phi_{Lf,Rf}(Lf,Rf)$ *are identity morphisms.*

$$\begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, "Rf"'] \arrow[ur, "1"] & \cdot \end{tikzcd}$$

Proof: If we write the commutative square of the statement as a pasting of two commutative squares $(1,Rf)$ and $(Lf,1)$, as displayed, we can easily compute the diagonal filler.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[r, "Lf"] \arrow[d, "f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, "Rf"'] & \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

$\phi_{Lf,Rf}(Lf,Rf) = \pi_f \cdot K(Lf,Rf) \cdot \sigma_f = \pi_f \cdot K(Lf,1) \cdot K(1,Rf) \cdot \sigma_f = 1 \cdot 1 = 1.$

Remark 5.3. As pointed out in [3, §2.5], the commutativity of the two diagrams (4.4) that express the fact that $\Delta\colon LR \Rightarrow RL$ is a mixed distributive law is equivalent to the requirement that the diagonal filler of the displayed square be $\sigma_f \cdot \pi_f$.

$$\begin{tikzcd} Kf \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] & KLf \arrow[d, "RLf"] \\ KRf \arrow[r, "\pi_f"'] \arrow[ur, dotted, "\sigma_f \cdot \pi_f"] & Kf \end{tikzcd}$$

5.c. KZ lifting operations. In the previous section we saw that each AWFS canonically induced a lifting operation. It is logical to expect that lifting operations that arise from lax orthogonal AWFSs carry extra structure. In this section we identify this structure.

Definition 5.4. Suppose given a lifting operation ϕ from $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$ to $V\colon \mathcal{B} \to \mathcal{C}^{\mathbf{2}}$ on an **Ord**-category $\mathcal{C}$ as defined in §5.a. We say that ϕ is a KZ*-lifting operation* if, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and each commutative diagram as on the left, the inequality on the right holds.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[d, "Vb"] \\ \cdot \arrow[r, "k"'] \arrow[ur, "d"] & \cdot \end{tikzcd} \qquad \Longrightarrow \qquad \phi_{a,b}(h,k) \leqslant d$$

In other words, the diagonal filler given by the lifting operation ϕ is a lower bound of all possible diagonal fillers.

Example 5.5. Consider the **Ord**-functor $0\colon \mathbf{1} \to \mathbf{2}$ that includes the terminal ordered set as the initial element of the ordered set $\mathbf{2} = (0 \leqslant 1)$. There is a bijection between opfibration structures on a morphism $g\colon X \to Y$ in **Ord** and KZ lifting operations on g against the morphism 0. To see this, first notice that a commutative square

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d, "0"'] & X \arrow[d, "g"] \\ \mathbf{2} \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

is equally well given by an element $x \in X$ and an element $y \in Y$ such that $g(x) \leqslant y$. The existence of a diagonal filler is the existence of an element $x_y \in X$ with $x \leqslant x_y$ and $g(x_y) = y$. This diagonal filler is a lower bound if for any other $x \leqslant \bar{x}$ with $g(\bar{x}) = y$ there is an inequality $x_y \leqslant \bar{x}$. The element x_y is unique and the assignment $(x, y) \mapsto x_y$ defines a split opfibration structure on g.

Theorem 5.6. *The following conditions are equivalent for an* AWFS (L, R) *on an* **Ord***-category* $\mathcal{C}$*.*

(1) The AWFS *is a* LOFS*.*
(2) The lifting operation from the forgetful functor $U\colon \mathsf{L}\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$ *to the forgetful functor* $V\colon \mathsf{R}\text{-Alg} \to \mathcal{C}^{\mathbf{2}}$ *is a* KZ*-lifting operation.*

Proof: Assume that (L, R) is lax orthogonal, (f, s) is an L-coalgebra and (g, p) is an R-algebra. Given a diagonal filler d as depicted, we must show $\phi_{(f,s),(g,p)}(h, k) \leqslant d$.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "k"'] \arrow[ur, "d"] & \cdot \end{tikzcd}$$

Using the inequalities $1 \leqslant s \cdot Rf$ and $1 \leqslant Lg \cdot p$ from (4.15), we obtain

$$\bigl(\phi_{(f,s),(g,p)}(h, k) = p \cdot K(h, k) \cdot s \leqslant d\bigr) \Leftrightarrow \bigl(K(h, k) \leqslant Lg \cdot d \cdot Rf\bigr).$$

There is a morphism $(Lg \cdot d \cdot Rf, k)\colon Rf \to Rg$ in $\mathcal{C}^{\mathbf{2}}$, as shown by the diagram below, which precomposed with the unit $\Lambda_f = (Lf, 1)\colon f \to Rf$ of R equals

$\Lambda_g \cdot (h, k) = (Lg \cdot h, k) \colon f \to Rg$.

$$\begin{tikzcd}
\cdot \arrow[r, "Rf"] \arrow[d, "Rf"'] & \cdot \arrow[r, "d"] \arrow[drr, "k"] & \cdot \arrow[r, "Lg"] \arrow[dr, "g"] & \cdot \arrow[d, "Rg"] \\
\cdot \arrow[rrr, "k"'] & & & \cdot
\end{tikzcd}$$

On the other hand, by the lax idempotency of R, we have that $K(h, k)$ is a left extension of $\Lambda_g \cdot (h, k)$ along Λ_f, so there exists $K(h, k) \leqslant Lg \cdot d \cdot Rf$, as desired.

Conversely, assume that the lifting operation ϕ induced by the AWFS is KZ, and consider the commutative square

$$\begin{tikzcd}
\cdot \arrow[rr, "LRf"] \arrow[dd, "LRf"'] \arrow[dr, dotted, "1"] & & \cdot \arrow[dd, "R^2 f"] \\
& \cdot \arrow[ur, "LRf"'] \arrow[dr, dotted, "Rf"'] & \\
\cdot \arrow[ur, "\pi_f"'] \arrow[rr, "R^2 f"'] & & \cdot
\end{tikzcd}$$

By Lemma 5.2, ϕ provides the diagonal filler $\phi_{LRf, R^2 f}(LRf, R^2 f) = 1$, so we have an inequality $1 \leqslant LRf \cdot \pi_f$ as required. ∎

Theorem 5.7. *Let (L, R) be a LOFS on an* **Ord***-category $\mathcal{C}$. Then, the following statements about a morphism f of $\mathcal{C}$ are equivalent:*

(1) f has an (unique) R-algebra structure (we simply say that f is an R-algebra).
(2) f is injective with respect to L-coalgebras, in the sense that any commutative square

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\ell"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

with $\ell \in \mathsf{L}\text{-Coalg}$ has a diagonal filler.
(3) f admits a (non-necessarily unique) (R, Λ)-algebra structure.
(4) f is a retract in $\mathcal{C}^{\mathbf{2}}$ of an R-algebra.

The WFS that underlies (L, R) has as left part those morphisms in the image of the forgetful functor $\mathsf{L}\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$ and as right part those morphisms in the image of the forgetful functor $\mathsf{R}\text{-Alg} \to \mathcal{C}^{\mathbf{2}}$.

Proof: We have seen in §5.b that (1) implies (2). To prove that (2) implies (3), consider the diagonal filler below, which shows that $(p, 1)\colon Rf \to f$ is is an (R, Λ)-algebra structure.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "Rf"'] \arrow[ur, dotted, "p"] & \cdot \end{tikzcd}$$

The implications (3)⇒(4)⇒(1) are particular instances of part of Lemma 2.9, since R is lax idempotent.

As mentioned in §4.c, the underlying WFS $(\mathcal{L}, \mathcal{R})$ of (L, R) has as right class the algebras for the pointed endofunctor (R, Λ). Then, $f \in \mathcal{R}$ (or, by duality, $f \in \mathcal{L}$) precisely when f is an R-algebra (an L-coalgebra). ∎

6. Horizontally ordered double categories and LOFSs

6.a. Horizontally ordered double categories. Double categories, introduced by C. Ehresmann [9], can be succinctly described as internal categories in the cartesian category of categories. They consist of an internal graph of categories and functors $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ (domain and codomain) with an identity functor $\mathrm{id}\colon \mathcal{G}_0 \to \mathcal{G}_1$ and a composition functor $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ that satisfy the usual associativity and identity axioms. The morphisms of $\mathcal{G}_0$ will be represented as horizontal arrows. The objects of $\mathcal{G}_1$ have a domain and a codomain that are objects of $\mathcal{G}_0$, and will be represented as vertical morphisms. Morphisms of $\mathcal{G}_1$ will be represented as squares; for example a morphism $\alpha\colon x \to y$ in $\mathcal{G}_1$ will be represented as

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "x"'] & \cdot \arrow[d, "y"] \\ \cdot \arrow[r] \arrow[ur, phantom, "\alpha"] & \cdot \end{tikzcd}$$

Objects of $\mathcal{G}_1$, ie vertical arrows, can be vertically composed, as well as squares as the one above.

Definition 6.1. A *horizontally ordered double category* is an internal category in the cartesian category **Ord-Cat** of **Ord**-categories and **Ord**-functors. This means that in a horizontally ordered double category we can speak of inequalities between horizontal morphisms and between squares. A *monotone double functor* between two horizontally ordered double categories is a double functor that preserves the inequalities between horizontal morphisms and between squares.

Example 6.2. Let $\mathcal{C}$ be an **Ord**-category. The horizontally ordered double category $\mathbb{S}\mathrm{q}(\mathcal{C})$ has underlying graph $\mathrm{dom}, \mathrm{cod}\colon \mathcal{C}^{\mathbf{2}} \rightrightarrows \mathcal{C}$, so both horizontal and vertical morphisms are morphisms of $\mathcal{C}$, and squares are commutative squares in $\mathcal{C}$. The inequality between horizontal morphisms is the inequality between morphisms of $\mathcal{C}$. One square is less or equal than another, as depicted,

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "x"'] & \cdot \arrow[d, "y"] \\ \cdot \arrow[r, "k"] & \cdot \end{tikzcd} \qquad \leqslant \qquad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "x"'] & \cdot \arrow[d, "y"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

if and only if $h \leqslant u$ and $k \leqslant v$.

Example 6.3. LARIs form a horizontally ordered double category. If $f\colon A \to B$ and $g\colon B \to C$ are LARIs, with respective right adjoints f^* and g^*, then their composition $g \cdot f\colon A \to B$ is also a LARI with right adjoint $f^* \cdot g^*$. This composition of LARIs is clearly associative and has identities, namely the identity morphisms.

6.b. Lifting operations. If $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ is an **Ord**-functor, there is an **Ord**-category $\mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$ over $\mathcal{C}^{\mathbf{2}}$ whose objects are morphisms f of $\mathcal{C}$ with a KZ-lifting operation against U, ie with a RALI structure on each

$$\phi_{-,f}\colon \mathcal{C}(\mathrm{cod}\, Uj, \mathrm{dom}\, f) \longrightarrow \mathcal{C}^{\mathbf{2}}(Uj, f). \tag{6.4}$$

A morphism is a morphism in $\mathcal{C}^{\mathbf{2}}$ that is compatible with these RALI structures in the obvious way. The ordering of morphisms is that of $\mathcal{C}^{\mathbf{2}}$. The forgetful **Ord**-functor

$$U^{\pitchfork_{\mathrm{KZ}}}\colon \mathcal{J}^{\pitchfork_{\mathrm{KZ}}} \longrightarrow \mathcal{C}^{\mathbf{2}} \tag{6.5}$$

is injective on objects, since (6.4) can be a RALI in a unique way.

The construction $(\mathcal{J}, U) \mapsto (\mathcal{J}^{\pitchfork_{\mathrm{KZ}}}, U^{\pitchfork_{\mathrm{KZ}}})$ is part of a functor

$$(-)^{\pitchfork_{\mathrm{KZ}}}\colon (\mathbf{Cat}/\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \longrightarrow \mathbf{CAT}/\mathcal{C}^{\mathbf{2}}.$$

Explicitly, if $S\colon \mathcal{J} \to \mathcal{I}$ is an **Ord**-functor over $\mathcal{C}^{\mathbf{2}}$

$$\begin{tikzcd} \mathcal{J} \arrow[rr, "S"] \arrow[dr, "U"'] & & \mathcal{I} \arrow[dl, "V"] \\ & \mathcal{C}^{\mathbf{2}} & \end{tikzcd}$$

then there is an **Ord**-functor

$$S^{\pitchfork_{\mathrm{KZ}}}\colon \mathcal{J}^{\pitchfork_{\mathrm{KZ}}} \longrightarrow \mathscr{J}^{\pitchfork_{\mathrm{KZ}}}$$

defined by the obvious observation that if the morphism on the left hand side of (6.6) is a RALI, then so is the one on the right hand side, since $Uj = VSj$.

$$\mathcal{C}(\operatorname{cod} Vi, \operatorname{dom} f) \longrightarrow \mathcal{C}^{\mathbf{2}}(Vi, f) \qquad \mathcal{C}(\operatorname{cod} Uj, \operatorname{dom} f) \longrightarrow \mathcal{C}^{\mathbf{2}}(Uj, f). \quad (6.6)$$

Proposition 6.7. *Given an* **Ord***-functor* $U\colon \mathscr{J} \to \mathcal{C}^{\mathbf{2}}$*, there is a horizontally ordered double category with:*

- *objects, those of* $\mathcal{C}$*;*
- *vertical morphisms those morphisms of* $\mathcal{C}$ *that are objects of* $\mathscr{J}^{\pitchfork_{\mathrm{KZ}}}$*;*
- *horizontal morphisms, the morphisms of* $\mathcal{C}$*;*
- *squares, commutative squares in* $\mathcal{C}$*.*

We denote this horizontally ordered category by $\mathscr{J}^{\pitchfork_{\mathrm{KZ}}}$*. Moreover,* U *defines an identity on objects double functor* $\mathscr{J}^{\pitchfork_{\mathrm{KZ}}} \to \mathbb{S}\mathrm{q}(\mathcal{C})$*.*

Proof: We have to prove the following: (a) if f and g are two composable morphisms and both are in $\mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$, then their composition $g\cdot f$ is also in $\mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$; (b) this composition is associative; (c) that any identity morphism is an object of $\mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$; (d) identity morphisms are identities for the composition of part (a).

The first observation is that (b) and (d) are automatic because (6.5) is injective on objects, so we only need to prove (a) and (c).

(a) Suppose that f and g are composable objects of $\mathscr{J}^{\pitchfork_{\mathrm{KZ}}}$, with lifting operations that we denote, respectively, $\phi_{-,f}$ and $\phi_{-,g}$. If $j \in \mathscr{J}$, then $\theta_j(h,k) \coloneqq \phi_{j,f}(h, \phi_{j,g}(f\cdot h, k))$ provides a diagonal filler for the solid square $(h,k)\colon Uj \to g\cdot f$, as displayed.

$$\begin{tikzcd}
\cdot \arrow[r, "h"] \arrow[dd, "Uj"'] & \cdot \arrow[d, "f"] \\
 & \cdot \arrow[d, "g"] \\
\cdot \arrow[uur, dotted, "{\phi_{j,f}(h,\phi_{j,g}(f\cdot h,k))}"] \arrow[ur, dotted, "{\phi_{j,g}(f\cdot h,k)}"'] \arrow[r, "k"'] & \cdot
\end{tikzcd}$$

To prove that the lifting operation θ is a KZ-lifting operation we have to prove that $\theta_j(h,k)$ is the least diagonal filler. Suppose that d is another diagonal filler of the square. This implies that $f\cdot d$ is a diagonal filler of the square

$(f \cdot h, k) \colon Uj \to g$, and therefore $\phi_{j,g}(f \cdot h, k) \leqslant f \cdot d$. We now have two morphisms in $\mathcal{C}^{\mathbf{2}}$, namely

$$(h, \phi_{j,g}(f \cdot d, k)) \leqslant (h, f \cdot d) \colon Uj \longrightarrow f$$

from where we obtain the required inequality

$$\theta_j(h,k) = \phi_{j,f}(h, \phi_{j,g}(f \cdot d, k)) \leqslant \phi_{j,f}(h, f \cdot d) \leqslant d; \tag{6.8}$$

the first inequality in (6.8) above arises from the fact that the lifting operation ϕ is **Ord**-enriched (see §5.a), while the second inequality exists because d is a diagonal filler of $(h, f \cdot d) \colon Uj \to f$.

(c) It remains to prove that identity morphisms are in $\mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$, for which we note that there is only one possible diagonal filler for a square of the form

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "Uj"'] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "k"'] & \cdot \end{tikzcd}$$

namely, k itself. This completes the proof. ∎

Given an **Ord**-functor $U \colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$, there is another

$$^{\pitchfork_{\mathrm{KZ}}}U \colon {}^{\pitchfork_{\mathrm{KZ}}}\mathcal{J} \longrightarrow \mathcal{C}^{\mathbf{2}}$$

that is constructed dually to $\mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$. More explicitly, $^{\pitchfork_{\mathrm{KZ}}}\mathcal{J}$ has objects $(f, \phi_{f,-})$ where $f \in \mathcal{C}^{\mathbf{2}}$ and ϕ is a KZ-lifting operation from f to U.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "Uj"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dotted, "\phi_{f,j}(h,k)"] & \cdot \end{tikzcd}$$

The KZ-lifting operation $\phi_{f,-}$ is a RALI structure on the monotone morphisms $\mathcal{C}(\mathrm{cod}(f), \mathrm{dom}\, Uj) \to \mathcal{C}^{\mathbf{2}}(f, Uj)$.

Theorem 6.9. *Suppose given* **Ord**-*functors*

$$\mathcal{J} \xrightarrow{U} \mathcal{C}^{\mathbf{2}} \xleftarrow{V} \mathcal{I}.$$

There is a bijection between:

- *KZ-lifting operations from U to V;*
- **Ord**-*functors $\mathcal{I} \to \mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$;*
- **Ord**-*functors $\mathcal{J} \to {}^{\pitchfork_{\mathrm{KZ}}}\mathcal{I}$.*

These correspondences yield a contravariant adjunction in **Ord-Cat**$/\mathcal{C}^{\mathbf{2}}$ *between* ${}^{\pitchfork_{\mathrm{KZ}}}(-)$ *and* $(-)^{\pitchfork_{\mathrm{KZ}}}$*.*

6.c. LOFSs and KZ lifting operations. Suppose that (L,R) is a LOFS on the **Ord**-category $\mathcal{C}$. There is an **Ord**-functor

$$\mathsf{R}\text{-Alg} \longrightarrow \mathsf{L}\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}} \tag{6.10}$$

introduced in [7], that equips each R-algebra with its canonical KZ-lifting operation against L-coalgebras (see Theorem 5.6). Using [3, §6.3] one could deduce that (6.10) is an isomorphism. We prefer, however, to give a self-contained proof.

Theorem 6.11. *The* **Ord**-*functor* (6.10) *induced by a* LOFS (L,R) *is an isomorphism.*

Proof: Supposing that $(g,\phi_{-,g})$ is a KZ-lifting operation against the forgetful **Ord**-functor $U\colon \mathsf{L}\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$, we want to construct an R-algebra structure on $g\colon A \to B$. There is a KZ-diagonal filler $p = \phi_{Lg,g}(1,Rg)$ as depicted below.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "Lg"'] & A \arrow[d, "g"] \\
Kg \arrow[r, "Rg"'] \arrow[ur, dotted, "p"] & B
\end{tikzcd}$$

Then $(p,1)\colon Rg \to g$ will be our candidate for an algebra structure. By the lax idempotency of R, we only have to show $(p,1) \dashv \Lambda_g = (Lg,1)$ (see §4.e). We know that $p \cdot Lg = 1$, and it remains to show $1 \leqslant Lg \cdot p$. The commutativity the following diagram shows that $Lg \cdot p$ is a diagonal filler of the square $(Lg,Rg)\colon Lg \to Rg$.

$$\begin{tikzcd}
A \arrow[rr, "Lg"] \arrow[dr, "1"] \arrow[dd, "Lg"'] & & Kg \arrow[dd, "Rg"] \\
& A \arrow[ur, "Lg"] \arrow[dr, "g"] & \\
Kg \arrow[ur, "p"] \arrow[rr, "Rg"'] & & B
\end{tikzcd}$$

The canonical KZ-lifting operation, exhibited in Theorem 5.6, chooses the identity morphism as the diagonal filler of the outer square, by Lemma 5.2, so we deduce $1 \leqslant Lg \cdot p$. This completes the proof that $(p,1)\colon Rg \to g$ is an algebra structure.

The next part of the proof is the verification that the assignment

$$\operatorname{ob}\bigl(\mathsf{L}\text{-}\mathrm{Alg}^{\pitchfork_{\mathrm{KZ}}}\bigr) \longrightarrow \operatorname{ob}\bigl(\mathsf{R}\text{-}\mathrm{Alg}\bigr) \tag{6.12}$$

constructed in the previous paragraph is an inverse of the effect of (6.10) on objects. Both (6.10) and (6.12) commute with the injective forgetful assignments from $\operatorname{ob}\bigl(\mathsf{L}\text{-}\mathrm{Alg}^{\pitchfork_{\mathrm{KZ}}}\bigr)$ and $\operatorname{ob}\bigl(\mathsf{R}\text{-}\mathrm{Alg}\bigr)$ to $\operatorname{ob}\bigl(\mathcal{C}^{\mathbf{2}}\bigr)$. This immediately implies that (6.12) is the inverse of (6.10) on objects.

It remains to prove that (6.10) is fully faithful, in the **Ord**-enriched sense. Suppose that $(h,k)\colon (f,\phi_{-,f}) \to (g,\phi_{-,g})$ is a morphism in $\mathsf{L}\text{-}\mathrm{Coalg}^{\pitchfork_{\mathrm{KZ}}}$, and let $p_f\colon Rf \to f$ and $p_g\colon Rg \to g$ be the associated algebra structures. We have the following string of equalities

$$h\cdot p_f = h\cdot\phi_{Lf,f}(1,Rf) = \phi_{Lf,g}(h,k\cdot Rf) = \phi_{Lg,g}(1,Rg)\cdot K(h,k) = p_g\cdot K(h,k),$$

which are a result of the definition of lifting operations.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[r, "h"] \arrow[d, "f"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "Rf"'] \arrow[ur, dotted, "p_f"] & \cdot \arrow[r, "k"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[rr, "h"] \arrow[d, "Lf"'] & & \cdot \arrow[d, "g"] \\ \cdot \arrow[rr, "k\cdot Rf"'] \arrow[urr, dotted, "\phi_{Lf,g}(Lf{,}k\cdot Rf)"] & & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Lf"'] & \cdot \arrow[r, equal] \arrow[d, "Lg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "K(h{,}k)"'] & \cdot \arrow[r, "Rg"'] \arrow[ur, dotted, "p_g"] & \cdot \end{tikzcd}$$

This shows that (6.10) is full on morphisms. That is faithful and full on 2-cells, or inequalities, follows from the fact (6.10) commutes with the forgetful **Ord**-functors into $\mathcal{C}^{\mathbf{2}}$ and these forgetful **Ord**-functors are faithful and full on inequalities. ∎

Corollary 6.13. *For any* LOFS (L,R)*, the* **Ord***-categories* L-Coalg *and* R-Alg *are the object of the arrow part of horizontally ordered categories that we denote by* $\mathsf{L}\text{-}\mathbb{C}\mathrm{oalg}$ *and* $\mathsf{R}\text{-}\mathbb{A}\mathrm{lg}$*. Furthermore, the respective* **Ord***-functors into* $\mathcal{C}^{\mathbf{2}}$ *are the arrow part of horizontally monotone double functors into* $\mathbb{S}\mathrm{q}(\mathcal{C})$*.*

Proof: We use the isomorphism of Theorem 6.11 to transfer the structure of a horizontally ordered double category from $\mathsf{L}\text{-}\mathrm{Coalg}^{\pitchfork_{\mathrm{KZ}}}$ to R-Alg; see Proposition 6.7. The statement about L-coalgebras is dual. ∎

A straightforward modification of [3, Thm. 6] yields the following theorem.

Theorem 6.14. *A horizontally monotone double functor* $\mathbb{U} = (U, U_0)\colon \mathbb{D} \to \mathbb{S}\mathrm{q}(\mathcal{C})$ *is isomorphic over* $\mathbb{S}\mathrm{q}(\mathcal{C})$ *to* $\mathsf{R}\text{-}\mathbb{A}\mathrm{lg} \to \mathbb{S}\mathrm{q}(\mathcal{C})$ *for a* LOFS (L,R) *if and only if*

- U *is monadic and the induced* **Ord***-monad is lax idempotent.*

- *for each vertical arrow f in $\mathbb{D}$ the following square is in the image of U.*

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "f"'] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

We conclude the section with a result on morphisms of LOFSs.

Proposition 6.15. *Suppose that* (L,R) *and* $(\mathsf{L}',\mathsf{R}')$ *are* LOFS *on the* **Ord**-*category* $\mathcal{C}$, *and* $\varphi_f\colon Kf \to K'f$ *a natural family of morphisms. Then, there is a bijection between the following sets, which, moreover, can have at most one element.*

(a) Morphisms of LOFSs $(\mathsf{L},\mathsf{R}) \longrightarrow (\mathsf{L}',\mathsf{R}')$.
(b) Comonad morphisms $\mathsf{L} \to \mathsf{L}'$.
(c) Monad morphisms $\mathsf{R} \to \mathsf{R}'$.

Proof: First, there is at most one morphism of the kind in (a), (b) and (c) by Lemma 4.22, Lemma 2.10 and its dual form (ie, the version for comonads). Clearly, if there is a morphism as in (a), then there are morphisms as in (b) and (c), just by definition of morphism of AWFSs (§4.f).

Suppose there is a morphism of comonads Q from L to L', with components $Q_f\colon Lf \to L'f$. Due to the counit axiom, $(1, R'f)\cdot Q_f = (1, Rf)$, we have that Q_f is of the form $(1,\varphi_f)$ for a morphism $\varphi_f\colon Kf \to K'f$. Let

$$Q_*\colon \mathsf{L}\text{-Coalg} \longrightarrow \mathsf{L}'\text{-Coalg}$$

be the **Ord**-functor induced by Q; it commutes with the forgetful **Ord**-functors into $\mathcal{C}^{\mathbf{2}}$. Applying the functor $(-)^{\pitchfork_{\mathrm{KZ}}}$ to Q_* and employing the isomorphisms (6.10) (Theorem 6.11) we obtain an **Ord**-functor, depicted by a dashed arrow.

$$\begin{tikzcd} \mathsf{R}'\text{-Alg} \arrow[r, dashed] \arrow[d, "\cong"'] & \mathsf{R}\text{-Alg} \arrow[d, "\cong"] \\ \mathsf{L}'\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}} \arrow[r, "Q_*^{\pitchfork_{\mathrm{KZ}}}"] & \mathsf{L}\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}} \end{tikzcd}$$

The vertical isomorphisms were described in the proof of Theorem 6.11, and this description can be used to describe the dashed arrow. If $(p,1)\colon R'f \to f$ is an R'-algebra structure, the associated KZ-lifting operation $\phi_{-,f}$ defines a diagonal filler for each commutative square

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "\ell"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dotted, "\phi_{\ell,f}(h,k)"] & \cdot \end{tikzcd} \qquad \phi_{\ell,f}(h,k) = p \cdot K(h,k) \cdot s$$

for any L'-coalgebra $(1,s)\colon \ell \to L\ell$. Uppon applying $Q_*^{\pitchfork_{\mathrm{KZ}}}$ we obtain a KZ-lifting operation $\psi_{-,f}$ of f against all L-coalgebras. If $(1,t)\colon g \to Lg$ is an L-coalgebra, its image under Q_* is

$$g \xrightarrow{(1,t)} Lg \xrightarrow{(1,\varphi_g)} L'g$$

and therefore $\psi_{g,f}(h,k)$ is the form

$$\psi_{g,f}(h,k) = \phi_{Q_*g,f}(h,k) = p \cdot K'(h,k) \cdot \varphi_g \cdot t = p \cdot \varphi_f \cdot K(h,k) \cdot t.$$

We now obtain the R-algebra structure on f by $\psi_{Lf,f}(1,Rf)$,

$$\psi_{Lf,f}(1,Rf) = p \cdot \varphi_f \cdot K(1,Rf) \cdot \sigma_f = p \cdot \varphi_f.$$

In conclusion, the dashed arrow in page 37 represents the **Ord**-functor that sends an R'-algebra $(p,1)\colon R'f \to f$ to the R-algebra $(p \cdot \varphi_f, 1)\colon Rf \to f$. This implies that $(\varphi_f, 1)\colon Rf \to R'f$ is a monad morphism, and the set (c) is non-empty.

We have seen that (c) has a member if (b) has a member. By a duality argument, ie by taking the opposite **Ord**-category of $\mathcal{C}$, we deduce the converse: (b) has a member if (c) does. Furthermore, from the construction of the previous paragraph, we know that if $(1,\varphi_f)\colon Lf \to L'f$ is a comonad morphism, then the monad morphism must be of the form $(\varphi_f,1)\colon Rf \to R'f$, and vice versa. Therefore, the existence of a comonad morphism $\mathsf{L} \to \mathsf{L}'$, or the existence of a monad morphism $\mathsf{R} \to \mathsf{R}'$, are equivalent to the existence of a unique $\varphi_f\colon Kf \to K'f$ such that $(1,\varphi_f)\colon Lf \to L'f$ is a comonad morphism and $(\varphi_f,1)\colon Rf \to R'f$ is a monad morphism. In other words, equivalent to the existence of a unique morphism of AWFSs $(\mathsf{L},\mathsf{R}) \to (\mathsf{L}',\mathsf{R}')$. ∎

The above proposition is a reminder of the differences that exist between general AWFSs and those enriched over **Ord**. In the general case, the proposition does not hold; see [28, Lemma 6.9] or [3, Prop. 2].

7. The definition of LOFS revisited

Lax orthogonal factorisation systems on **Ord**-categories were defined in §4.e as **Ord**-enriched AWFSs (L,R) whose comonad L is lax idempotent, or equivalently, by Proposition 4.16, whose monad R is lax idempotent. The definition of AWFS includes a mixed distributive law $\Delta\colon LR \Rightarrow RL$, with components $(\sigma_f, \pi_f)\colon LRf \to RLf$. The axioms of a mixed distributive law in this case amount to the commutativity of the diagrams in (4.4), and they are equivalent, as mentioned in Remark 5.3, to the requirement that the diagonal filler of the square below be $\sigma_f \cdot \pi_f$.

$$\begin{tikzcd} Kf \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] & KLf \arrow[d, "RLf"] \\ KRf \arrow[r, "\pi_f"'] \arrow[ur, dotted, "\sigma_f\cdot\pi_f"] & Kf \end{tikzcd} \tag{7.1}$$

The main result of the section is the following.

Theorem 7.2. *In the definition of* LOFS*, the distributive law axiom is redundant. More precisely, the following suffices to define a* LOFS*: a domain-preserving* **Ord***-comonad* L *and a codomain-preserving monad* R *on* $\mathcal{C}^2$ *that define the same* **Ord***-functorial factorisation* $f = Rf \cdot Lf$*; both* L *and* R *should be lax idempotent.*

Proof: All we need to show is that $\sigma_f{\cdot}\pi_f$ is the diagonal filler of the square (7.1). The existence of a KZ-lifting operations for R-algebras against L-coalgebras does not depend on the distributivity axiom but it only suffices that both L and R be lax idempotent. Then, we only need to show that

$$\sigma_f \cdot \pi_f \leqslant d \tag{7.3}$$

for the KZ-diagonal filler d of the square (7.1), for, in this case, the inequality is necessarily an equality. There are adjunctions $\sigma_f \dashv K(1, Rf)$ and $K(Lf, 1) \dashv \pi_f$ since L and R are lax idempotent. Thus, the inequality (7.3) is equivalent to $1 \leqslant K(1, Rf) \cdot d \cdot K(Lf, 1)$, due to the inequalities (4.14) of §4.e. Consider the following diagram, where $(Lf, K(Lf, 1)) = L(Lf, 1)$ is a morphism of L-coalgebras and $(K(1, Rf), Rf) = R(1, Rf)$ is a morphism of

R-algebras.

$$\begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "Lf"'] & \cdot \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] & \cdot \arrow[r, "K(1,Rf)"] \arrow[d, "RLf"] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, "K(Lf,1)"'] & \cdot \arrow[r, "\pi_f"'] \arrow[ur, "d"] & \cdot \arrow[r, "Rf"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, "Rf"'] \arrow[ur, dotted, "1"] & \cdot \end{tikzcd}$$

By the naturality of the diagonal fillers with respect to morphisms of L-coalgebras and morphism of R-algebras, we deduce that $K(1, Rf)\cdot d\cdot K(Lf, 1)$ is the diagonal filler of the square on the right hand side, and hence equal to the identity morphism (see Lemma 5.2). Therefore the inequality (7.3) holds, completing the proof. ■

We can summarise the theorem above and Proposition 4.16 in the following way: given a domain-preserving **Ord**-comonad L and a codomain-preserving **Ord**-monad R on $\mathcal{C}^{\mathbf{2}}$ that induce the same **Ord**-functorial factorisation $f = Rf \cdot Lf$, the following two statements are equivalent, and when they hold we are in the presence of a LOFS.

- One of L, R is lax idempotent and the distributive law axiom holds.
- Both L and R are lax idempotent.

8. Embeddings with respect to a monad

Embeddings with respect to a lax idempotent monad were extensively exploited in [11, 12] and in [10], where topological embeddings were exhibited as an example (more on this in §13). In this section we begin our analysis of the interplay between these embeddings and LOFSs.

Definition 8.1. If $S\colon \mathcal{C} \to \mathcal{B}$ is a locally monotone functor between **Ord**-categories, an S-*embedding* structure on a morphism f in $\mathcal{C}$ is a LARI structure in Sf in $\mathcal{B}$. Recall that LARI structures on a morphism in an **Ord**-category are unique, which one usually rephrases by saying that being a LARI is a *property* of a morphism. Therefore, being an S-embedding in an **Ord**-category is a property of morphisms.

The **Ord**-category of S-embeddings, denoted by S-Emb, is the category whose objects are pairs (f, r) where f is a morphism in $\mathcal{C}$ and $Sf \dashv r$ is a LARI in $\mathcal{B}$. A morphism $(f, r) \to (g, t)$ in this category is a morphism $(h, k)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$ satisfying $Sh \cdot r = t \cdot Sk$. There is an obvious forgetful functor $S\text{-Emb} \to \mathcal{C}^{\mathbf{2}}$ given on objects by $(f, r) \mapsto f$. We make S-Emb into an **Ord**-category by declaring $(h, k) \leqslant (h', k')$ if this inequality holds in $\mathcal{C}^{\mathbf{2}}$;

this makes the forgetful functor U into a locally monotone functor that fits in a pullback square.

$$\begin{tikzcd} S\text{-Emb} \arrow[r] \arrow[d, "U"'] \arrow[dr, phantom, "\mathrm{pb}"] & \mathrm{Lari}(\mathcal{B}) \arrow[d] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "S^{\mathbf{2}}"'] & \mathcal{B}^{\mathbf{2}} \end{tikzcd} \tag{8.2}$$

Lemma 8.3. *S-embeddings in $\mathcal{C}$ are the vertical morphisms of a horizontally ordered double category, with objects those of $\mathcal{C}$, horizontal morphisms the morphisms of $\mathcal{C}$ and squares those commutative squares in $\mathcal{C}$ that represent morphisms of S-embeddings. Furthermore, the pullback diagram displayed above is part of a pullback diagram of horizontally ordered double categories.*

$$\begin{tikzcd} S\text{-}\mathbb{E}\mathrm{mb} \arrow[r] \arrow[d, "U"'] \arrow[dr, phantom, "\mathrm{pb}"] & \mathbb{L}\mathbf{ari}(\mathcal{B}) \arrow[d] \\ \mathbb{S}\mathrm{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathrm{q}(S)"'] & \mathbb{S}\mathrm{q}(\mathcal{B}) \end{tikzcd} \tag{8.4}$$

Proof: At the level of **Ord**-categories of objects, the square of the statement has identity vertical arrows and $\operatorname{ob} S \colon \operatorname{ob} \mathcal{C} \to \operatorname{ob} \mathcal{D}$ as horizontal arrows. Hence, it is a pullback at the level of **Ord**-categories of objects. At the level of **Ord**-categories of arrows, the square is precisely the pullback square (8.2). Therefore, S-Emb $\rightrightarrows \mathcal{C}$ has a unique internal category structure that makes (8.4) a pullback square of internal categories. ■

Lemma 8.5. *The forgetful* **Ord**-*functor S-*Emb $\to \mathcal{C}^{\mathbf{2}}$ *creates colimits, provided that $\mathcal{C}$ has and S preserves colimits.*

Proof: In the pullback diagram (8.2), the leftmost vertical **Ord**-functor creates any colimit that is preserved by S (and thus by $S^{\mathbf{2}}$), since the rightmost vertical **Ord**-functor creates colimits. ■

Definition 8.6. If T is an **Ord**-monad on $\mathcal{C}$, we shall call F^{T}-embeddings T-*embeddings*, and denote the **Ord**-category F^{T}-Emb by T-Emb.

Lemma 8.7. *Let T be an* **Ord**-*monad on $\mathcal{C}$ and $F^{\mathsf{T}} \dashv V^{\mathsf{T}} \colon \mathsf{T}\text{-Alg} \to \mathcal{C}$ the associated Eilenberg-Moore adjunction. If V^{T} is locally full, ie if $V^{\mathsf{T}} f \leqslant V^{\mathsf{T}} g$ implies $f \leqslant g$ for parallel morphism of algebras f and g, then T-embeddings coincide with T-embeddings.*

For example, the above lemma applies when T is lax idempotent.

Proposition 8.8. *Let* T *be a lax idempotent monad on an* **Ord**-*category with a terminal object. The obvious* **Ord**-*functor*

$$\mathsf{T}\text{-Emb} \longrightarrow {}^{\pitchfork_{\mathrm{KZ}}}(\mathsf{T}\text{-Alg}/1) \tag{8.9}$$

is an isomorphism.

Proof: We define the **Ord**-functor (8.9) and show that it is bijective on objects at the same time by showing that a morphism f of $\mathcal{C}$ is a T-embedding if and only if it has a right KZ-lifting operation against morphisms $A \to 1$ for all T-algebras A.

The forgetful **Ord**-functor $V\colon \mathsf{T}\text{-Alg} \to \mathcal{C}$ can be composed with the inclusion $\mathcal{C} \to \mathcal{C}^{\mathbf{2}}$ that sends X to $(X \to 1)$, and then consider the ${}^{\pitchfork_{\mathrm{KZ}}}(-)$ of the resulting functor into $\mathcal{C}^{\mathbf{2}}$. An object of ${}^{\pitchfork_{\mathrm{KZ}}}(\mathsf{T}\text{-Alg}/1)$ is a morphism $f\colon X \to Y$ of $\mathcal{C}$ with a RALI structure on

$$\mathcal{C}(Y, V(A)) = \mathcal{C}^{\mathbf{2}}(f, V1_A) \longrightarrow \mathcal{C}^{\mathbf{2}}(f, (VA \to 1)) = \mathcal{C}(X, VA) \tag{8.10}$$

In other words, each morphism $X \to A$ can be extended along f and this extension is a left Kan extension.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

The morphism (8.10) can be written as

$$\mathcal{C}(Y, V(A)) \cong \mathsf{T}\text{-Alg}(F^{\mathsf{T}}Y, A) \xrightarrow{\mathsf{T}\text{-Alg}(F^{\mathsf{T}}f,1)} \mathsf{T}\text{-Alg}(F^{\mathsf{T}}X, A) \cong \mathcal{C}(X, V(A)) \tag{8.11}$$

which has a RALI structure, for all T-algebras A, if and only if $F^{\mathsf{T}}f$ has a LARI structure. This defines a bijection between the objects of the domain and codomain of (8.9).

It remains to define (8.9) on morphisms and to verify that it is bijective on these morphisms, and locally full on inequalities. Suppose that f and g are T-embeddings. A morphism $(h, k)\colon f \to g$ is a morphism in the codomain of (8.9) if it is compatible with the RALI structures on the morphisms (8.10) corresponding to f and g; in other words, if (h, k) induces a morphism of RALIs. This is equivalent to requiring that (h, k) should induce a morphism of RALIs between the RALIs (8.11) that correspond to f and g. By Yoneda lemma, this means that (h, k) is a morphism of T-embeddings. This defines a functor (8.9) that is bijective on morphisms.

It remains to show that (8.9) is locally full on morphisms, but this is easy and left to the reader. ∎

Proposition 8.12. *Let* T *be a lax idempotent monad on an* **Ord**-*category with a terminal object. The obvious* **Ord**-*functor*

$$\mathsf{T}\text{-Alg}/1 \longrightarrow (\mathsf{T}\text{-Emb})^{\pitchfork_{\mathrm{KZ}}}_{1} \subset (\mathsf{T}\text{-Emb})^{\pitchfork_{\mathrm{KZ}}}$$

is an isomorphism between T-Alg *and the fiber of* $\mathrm{cod}\colon (\mathsf{T}\text{-Emb})^{\pitchfork_{\mathrm{KZ}}} \longrightarrow \mathcal{C}$ *over* 1.

Proof: We will show that a morphism $A \to 1$ is in $(\mathsf{T}\text{-Emb})^{\pitchfork_{\mathrm{KZ}}}$ if and only if A is a T-algebra.

The components $\eta_X \colon X \to TX$ of the unit of the monad T are T-embeddings due to the adjunction $T\eta_X \dashv \mu_X$. Furthermore, for any morphism $u\colon X \to Y$, there is a morphism $(u, Tu)\colon \eta_X \to \eta_Y$ in T-Emb because $Tu \cdot \mu_X = \mu_Y \cdot T^2 u$.

Suppose that $A \to 1$ has a KZ-lifting operation against T-embeddings, which provides a diagonal filler to the square displayed below.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\eta_A"'] & A \arrow[d] \\ TA \arrow[r] \arrow[ur, dotted, "a"] & 1 \end{tikzcd}$$

We will show that a is a T-algebra structure.

It is not hard to verify that the diagonal filler of the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\eta_A"'] & TA \arrow[d] \\ TA \arrow[r] & 1 \end{tikzcd}$$

is the identity morphism, where TA is equipped with the KZ-lifting operation induced by its free T-algebra structure. On the other hand, $\eta_A \cdot a$ is another diagonal filler, so there is a inequality $1_{TA} \leqslant \eta_A \cdot a$. Thus, $a \dashv \eta_A$ which is equivalent to saying that a is a T-algebra structure on A.

We leave to the reader the verification that the **Ord**-functor of the statement if full and faithful. ∎

Corollary 8.13. *In the conditions of Proposition 8.12, the unit of the component at* T-Emb *of the adjunction of Theorem 6.9*

$$\mathsf{T}\text{-Emb} \longrightarrow {}^{\pitchfork_{\mathrm{KZ}}}(\mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}})$$

is an isomorphism.

Proof: Continuing with the notation used in Proposition 8.12, the inclusion of $\mathsf{T}\text{-Emb}_1^{\pitchfork_{\mathrm{KZ}}}$ into $\mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}}$ induces an **Ord**- functor in the opposite direction

$$ {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}}\big) \longrightarrow {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Emb}_1^{\pitchfork_{\mathrm{KZ}}}\big). $$

We can form a morphism from right to left, displayed below, where the two isomorphisms are those given by the Propositions 8.8 and 8.12.

$$ \mathsf{T}\text{-Emb} \xrightarrow{\cong} {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Alg}/1\big) \xrightarrow{\cong} {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Emb}_1^{\pitchfork_{\mathrm{KZ}}}\big) \longleftarrow {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}}\big) $$

The resulting **Ord**-functor

$$ {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}}\big) \longrightarrow \mathsf{T}\text{-Emb} \tag{8.14} $$

commutes with the forgetful **Ord**-functors into $\mathcal{C}^{\mathbf{2}}$. Since these forgetful functors are injective on objects and on morphisms, and full on inequalities between morphisms, we deduce that (8.14) is necessarily an inverse for the component of the unit of the statement. ∎

Corollary 8.15. *If* (L,R) *is a* LOFS *on an* **Ord**-*category with a terminal object, then there is a canonical* **Ord**-*functor*

$$ \mathsf{L}\text{-Coalg} \longrightarrow \mathsf{R}_1\text{-Emb} $$

where R_1 *is the* **Ord**-*monad on* $\mathcal{C} \cong \mathcal{C}/1$ *that is the restriction of* R.

Proof: The inclusion of $\mathsf{R}_1\text{-Alg} \hookrightarrow \mathsf{R}\text{-Alg}$, given by $A \mapsto (A \to 1)$, induces the unlabelled arrow in the following string of **Ord**-functors over $\mathcal{C}^{\mathbf{2}}$,

$$ \mathsf{L}\text{-Coalg} \cong {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{R}\text{-Alg}\big) \longrightarrow {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{R}_1\text{-Alg}/1\big) \cong \mathsf{R}_1\text{-Emb} $$

where the last isomorphism is provided by Proposition 8.8. ∎

The **Ord**-functor of Corollary 8.15 may be described more explicitly. If $f\colon X \to Y$ is an L-coalgebra, then the corresponding R_1-embedding structure is given by the adjunction $R_1 f \dashv r\colon R_1Y \to R_1X$ where r is the unique morphism of R_1-algebras that composed with the unit $\eta_Y\colon Y \to R_1Y$ equals the KZ-lifting corresponding to the square displayed below.

$$ \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & R_1A \arrow[d, "!=R(!)"] \\ Y \arrow[r, "!"'] \arrow[ur, dotted, "r\cdot\eta_Y"] & 1 \end{tikzcd} $$

9. KZ-reflective LOFSs

We begin by summarising the most basic definitions of [5] around reflective factorisation systems.

An OFS $(\mathscr{E}, \mathscr{M})$ (or even a pre-factorisation system, which is similar to a OFS but without the requirement that each morphism should be a composition of one in $\mathscr{E}$ followed by one in $\mathscr{M}$) on a category with a terminal object $\mathcal{C}$, induces a reflective subcategory of $\mathcal{C}$ formed by those objects X for which $X \to 1$ belongs to $\mathscr{M}$. In the other direction, each reflective subcategory $\mathcal{B} \subseteq \mathcal{C}$ induces a pre-factorisation system $(\mathscr{E}, \mathscr{M})$ whose $\mathscr{E}$ is formed by all the morphisms that are orthogonal to each object of $\mathcal{B}$. With an obvious ordering on reflective subcategories and pre-factorisation systems, these two constructions form an adjunction (a Galois correspondence). Those pre-factorisation systems obtained from reflective subcategories are called *reflective*, and are characterised as those for which $g \cdot f \in \mathscr{E}$ and $g \in \mathscr{E}$ implies $f \in \mathscr{E}$.

In this section we consider the analogous notion of KZ-reflective LOFS and find a characterisation that mirrors the case of OFSs.

Definition 9.1. We say that the **Ord**-monad T on $\mathcal{C}$ is *fibrantly* KZ*-generating* if the forgetful **Ord**-functor $\mathsf{T}\text{-Emb} \to \mathcal{C}^{\mathbf{2}}$ has a right adjoint (in the **Ord**-enriched sense).

Proposition 9.2. *Assume that $\mathcal{C}$ is a cocomplete and finitely complete* **Ord***-category. Then T is fibrantly* KZ*-generating if and only if there exists an* **Ord***-enriched* AWFS (L, R) *for which $\mathsf{L}\text{-Coalg} \cong \mathsf{T}\text{-Emb}$ over $\mathcal{C}^{\mathbf{2}}$. Furthermore, this* AWFS *is lax orthogonal.*

Proof: The implication in one direction is clear; indeed, if T-Emb is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to L-Coalg then the condition of Definition 9.1 holds.

Assume that T is fibrantly KZ-generating. The forgetful **Ord**-functor $\mathrm{Lari}(\mathcal{C}) \to \mathsf{T}\text{-Alg}^{\mathbf{2}}$ is comonadic by Lemma 4.9. The **Ord**-functor $\mathsf{T}\text{-Emb} \to \mathcal{C}^{\mathbf{2}}$ is a pullback of the comonadic **Ord**-functor mentioned, therefore, it satisfies all the hypotheses of (the **Ord**-enriched version) of Beck's comonadicity theorem, except perhaps for the hypothesis of being a left adjoint. Together with Definition 9.1, we deduce that T-Emb is comonadic over $\mathcal{C}^{\mathbf{2}}$.

The **Ord**-category of T-embeddings forms part of a horizontally ordered double category $\mathsf{T}\text{-}\mathbb{E}\mathrm{mb}$, as in Lemma 8.3. We will be able to apply the dual of Theorem 6.14 if we show the following: if f is a T-embedding, then the square on the left is a morphism of T-embeddings $1 \to f$. This is equivalent to saying that the square on the right is a morphism of LARIs $1 \to F^T f$, which is easily seen to hold.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "1"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "1"'] & \cdot \arrow[d, "F^T f"] \\ \cdot \arrow[r, "F^T f"] & \cdot \end{tikzcd}$$

We deduce, by a dual form of Theorem 6.14, that T-Emb is L-Coalg for an AWFS (L, R).

It remains to show that this AWFS is a LOFS, for which we appeal to the dual version of [25, Cor. 6.9], which we explain here without proof. By definition of T-Emb, there is a pullback diagram

$$\begin{tikzcd} \mathsf{T}\text{-Emb} \arrow[r] \arrow[d, "U"'] & \mathsf{E}\text{-Coalg} \arrow[d] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "(F^T)^{\mathbf{2}}"] & \mathsf{T}\text{-Alg}^{\mathbf{2}} \end{tikzcd}$$

where $\mathcal{C}$ is cocomplete and the free algebra **Ord**-functor F^T is a left adjoint. The comonad E on $\mathsf{T}\text{-Alg}^{\mathbf{2}}$ is the one of §4.d and exists since $\mathcal{C}$, and thus T-Alg, has finite limits. We are in the dual conditions of Corollaries 6.9 and 6.10 of [25], which guarantees that the comonad corresponding to the comonadic U is lax idempotent. ■

Definition 9.3. The **Ord**-category of lax idempotent monads on the **Ord**-category $\mathcal{C}$, denoted by $\mathbf{LIMnd}(\mathcal{C})$, has morphisms $\mathsf{T} \to \mathsf{S}$ natural transformations that are compatible with the multiplication and unit of the monads, in the usual manner.

We will denote by $\mathbf{LIMnd}_{\mathrm{fib}}(\mathcal{C})$ the full sub-**Ord**-category of $\mathbf{LIMnd}(\mathcal{C})$ consisting of those monads that are fibrantly KZ-generating, in the sense of Definition 9.1.

When $\mathcal{C}$ is cocomplete and finitely complete, we have a situation that can be summarised by the following diagram of **Ord**-functors.

$$\begin{tikzcd} \mathbf{LOFS}(\mathcal{C}) \arrow[d, hook, "(-)\text{-Coalg}"'] \arrow[drr, "\tilde{\Phi}"] & & \mathbf{LIMnd}_{\mathrm{fib}}(\mathcal{C}) \arrow[ll, dotted, "\Psi"'] \arrow[d, hook, "I"] \\ \mathbf{Ord\text{-}Cat}/\mathcal{C}^{\mathbf{2}} & & \mathbf{LIMnd}(\mathcal{C}) \arrow[ll, "\tilde{\Psi}"'] \end{tikzcd} \tag{9.4}$$

The vertical **Ord**-functors are full and faithful, the one on the right being just an inclusion. The one on the left sends each lax orthogonal AWFS on $\mathcal{C}$ to the **Ord**-category L-Coalg over $\mathcal{C}^{\mathbf{2}}$. The **Ord**-functor $\tilde{\Psi}$ sends a lax idempotent monad T on $\mathcal{C}$ to the category $^{\pitchfork_{\mathrm{KZ}}}(\mathsf{T}\text{-Alg}/1)$ over $\mathcal{C}^{\mathbf{2}}$, and has a lifting to an **Ord**-functor Ψ that sends a fibrantly KZ-generating T to the LOFS (L,R) on $\mathcal{C}$ that satisfies $\mathsf{L}\text{-Coalg} \cong \mathsf{T}\text{-Emb}$ – see Proposition 9.2. Finally, $\tilde{\Phi}$ sends (L,R) to R_1, the restriction of R to the slice $\mathcal{C}/1 \cong \mathcal{C}$.

It will be convenient to use the following relaxed notion of adjunction. Suppose given a diagram of functors and a natural transformation, that may be enriched as needed, as displayed.

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "F"'] \arrow[r, phantom, ""] & \mathcal{B} \arrow[l, "G"'] \arrow[d, "I"] \arrow[dl, phantom, "\overset{\theta}{\Rightarrow}"] \\
 & \mathcal{D}
\end{tikzcd}$$

Definition 9.5. Following [31, §2], we say that θ *exhibits* G *as a* I*-right adjoint of* F, and F *as a* I*-left adjoint of* G denoted by $F \dashv_I G$, if

$$\mathcal{A}(A, G(B)) \xrightarrow{F} \mathcal{D}(F(A), FG(B)) \xrightarrow{\mathcal{D}(1,\theta_B)} \mathcal{D}(F(A), I(B))$$

is invertible.

It is easy to prove that if $I\colon \mathcal{B} \to \mathcal{D}$ is fully faithful and θ is an isomorphism, then G is fully faithful.

Theorem 9.6. *In the situation of the diagram (9.4), the* **Ord***-functor* $\tilde{\Phi}$ *is a* I*-left adjoint of* Ψ*. Moreover,* Ψ *is fully faithful.*

Proof: We have to exhibit a natural bijection

$$\mathbf{LIMnd}(\mathcal{C})(\mathsf{R}_1, \mathsf{T}) \cong \mathbf{LOFS}(\mathcal{C})((\mathsf{L},\mathsf{R}), \Psi(\mathsf{T}))$$

using our knowledge of the existence of natural isomorphisms

$$\mathbf{LIMnd}(\mathcal{C})(\mathsf{R}_1, \mathsf{T}) = \mathbf{Mnd}(\mathcal{C})(\mathsf{R}_1, \mathsf{T}) \cong \mathbf{Ord\text{-}Cat}/\mathcal{C}\big(\mathsf{T}\text{-Alg}, \mathsf{R}_1\text{-Alg}\big)$$
$$\mathbf{LOFS}((\mathsf{L},\mathsf{R}), \Psi(\mathsf{T})) = \mathbf{AWFS}((\mathsf{L},\mathsf{R}), \Psi(\mathsf{T})) \cong \big(\mathbf{Ord\text{-}Cat}/\mathcal{C}^{\mathbf{2}}\big)\big(\mathsf{L}\text{-Coalg}, \mathsf{T}\text{-Emb}\big).$$

Suppose that $H\colon \mathsf{L}\text{-Coalg} \to \mathsf{T}\text{-Emb}$ is an **Ord**-functor over $\mathcal{C}^{\mathbf{2}}$. From this data we have to produce a monad morphism $\mathsf{R}_1 \to \mathsf{T}$, or what is equivalent, an **Ord**-functor

$$\mathsf{T}\text{-Alg} \longrightarrow \mathsf{R}_1\text{-Alg} \tag{9.7}$$

where the notation on the right means the **Ord**-category of R-algebras with codomain 1. We can use H, the adjunction between ${}^{\pitchfork_{\mathrm{KZ}}}(-)$ and $(-)^{\pitchfork_{\mathrm{KZ}}}$, and Theorem 6.11 to define an **Ord**-functor over $\mathcal{C}^{\mathbf{2}}$

$$\mathsf{T}\text{-Alg}/1 \longrightarrow \big({}^{\pitchfork_{\mathrm{KZ}}}(\mathsf{T}\text{-Alg}/1)\big)^{\pitchfork_{\mathrm{KZ}}} \cong \big(\mathsf{T}\text{-Emb}\big)^{\pitchfork_{\mathrm{KZ}}} \xrightarrow{H^{\pitchfork_{\mathrm{KZ}}}} \big(\mathsf{L}\text{-Coalg}\big)^{\pitchfork_{\mathrm{KZ}}} \cong \mathsf{R}\text{-Alg} \tag{9.8}$$

that assigns to each T-algebra A an R-algebra of the form $A \to 1$. This is the **Ord**-functor (9.7) we seek.

In addition, the adjunction between ${}^{\pitchfork_{\mathrm{KZ}}}(-)$ and $(-)^{\pitchfork_{\mathrm{KZ}}}$ implies that for any $N \colon \mathsf{T}\text{-Alg}/1 \to \mathsf{R}\text{-Alg}$ over $\mathcal{C}^{\mathbf{2}}$ there exists a unique $H \colon \mathsf{L}\text{-Coalg} \to \mathsf{T}\text{-Emb}$ over $\mathcal{C}^{\mathbf{2}}$ such that (9.8) equals N. This means that we have established the necessary bijection.

If $(\mathsf{L}, \mathsf{R}) = \Psi(\mathsf{T})$, the counit θ of the I-adjunction,

$$\begin{tikzcd} \mathbf{LOFS}(\mathcal{C}) \arrow[dr, "\tilde{\Phi}"'] & \mathbf{LIMnd}_{\mathrm{fib}}(\mathcal{C}) \arrow[l, "\Psi"'] \arrow[d, hook, "I"] \arrow[d, phantom, shift right=3ex, "\overset{\theta}{\Rightarrow}"] \\ & \mathbf{LIMnd}(\mathcal{C}) \end{tikzcd}$$

has component at T the morphism of monads

$$\theta_{\mathsf{T}} \colon \tilde{\Phi}\Psi(\mathsf{T}) \longrightarrow \mathsf{T}$$

corresponding in the construction of the previous paragraphs to the **Ord**-functor H that is the isomorphism $\mathsf{L}\text{-Coalg} \cong \mathsf{T}\text{-Emb}$. It follows from (9.8) that θ_{T} is an isomorphism provided that

$$\mathsf{T}\text{-Alg}/1 \longrightarrow \big(\mathsf{T}\text{-Emb}\big)^{\pitchfork_{\mathrm{KZ}}}$$

is an isomorphism, which was proved in Proposition 8.12. As mentioned above the present theorem, the invertibility of θ implies that Ψ is fully faithful. ■

Definition 9.9. We call a LOFS *KZ-reflective* if it is isomorphic to one of the form $\Psi(\mathsf{T})$, for a fibrantly KZ-generating lax idempotent monad T.

Proposition 9.10. *For a reflective* LOFS (L, R) *on an* **Ord***-category with terminal object, there is an isomorphism* $\mathsf{L}\text{-Coalg} \cong \mathsf{R}_1\text{-Emb}$ *over* $\mathcal{C}^{\mathbf{2}}$ *and* $(\mathsf{L}, \mathsf{R}) \cong \Psi(\mathsf{R}_1)$.

Proof: Suppose that $(\mathsf{L}, \mathsf{R}) \cong \Psi(\mathsf{T})$ for a lax idempotent monad T. By hypothesis, $\mathsf{L}\text{-Coalg} \cong \mathsf{T}\text{-Emb}$ for an **Ord**-monad T on $\mathcal{C}^{\mathbf{2}}$. On the other hand,

$\mathsf{R}\text{-Alg} \cong \mathsf{L}\text{-Coalg}^{\pitchfork_{\text{KZ}}}$ for any LOFS, as we saw in Theorem 6.11. Therefore,

$$\mathsf{R}_1\text{-Alg} = \mathsf{R}\text{-Alg}_1 \cong \mathsf{T}\text{-Emb}_1^{\pitchfork_{\text{KZ}}} \cong \mathsf{T}\text{-Alg}$$

where the subscript 1 denotes the fiber of the various categories fibered over $\mathcal{C}$ via the codomain functor. The last isomorphism of the sequence is the one provided by Proposition 8.12. Since the isomorphism $\mathsf{R}_1\text{-Alg} \cong \mathsf{R}\text{-Alg}$ constructed is over $\mathcal{C}$, we obtain an isomorphism between R_1 and T. ∎

Notation 9.11. In this section we will denote by (E, M) the LOFS on $\mathcal{C}$ whose E-coalgebras are LARIs in $\mathcal{C}$ and whose M-algebras are split opfibrations in $\mathcal{C}$.

Definition 9.12. We will refer to those LOFSs (L, R) that admit a morphism $(\mathsf{E}, \mathsf{M}) \to (\mathsf{L}, \mathsf{R})$ as *sub-LARI LOFSs*. If such morphism exists, it is unique.

Not all LOFSs are sub-LARI. For example, the initial AWFS (the one that factors a morphism f as $f = Rf \cdot Lf$ with $Lf = 1_{\mathrm{dom}(f)}$ and $Rf = f$) is orthogonal and, thus, lax orthogonal. Coalgebras for the associated comonad are the invertible morphisms in $\mathcal{C}$. It is clear that not every LARI is an isomorphism, so this LOFS is not sub-LARI.

Proposition 9.13. *KZ-reflective LOFSs are sub-LARI.*

Proof: By definition, L-Coalg is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to T-Emb, for a certain T. We have to show that there exists a (unique) **Ord**-functor

$$\mathrm{Lari}(\mathcal{C}) \longrightarrow \mathsf{T}\text{-Emb}$$

over $\mathcal{C}^{\mathbf{2}}$. By definition of T-Emb as a pullback (see Definition 8.1) it suffices to exhibit a commutative square

$$\begin{tikzcd} \mathrm{Lari}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Lari}(\mathsf{T}\text{-Alg}) \arrow[d] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathsf{T}}"'] & \mathsf{T}\text{-Alg}^{\mathbf{2}} \end{tikzcd}$$

where the vertical arrows are the obvious forgetful **Ord**-functors. The **Ord**-functor F^{T} obviously induces another $\mathrm{Lari}(\mathcal{C}) \to \mathrm{Lari}(\mathsf{T}\text{-Alg})$ that makes the diagram commutative, since any **Ord**-functor preserves LARIs. ∎

Definition 9.14. We shall be interested in LOFS (L, R) that satisfy the following cancellation properties:

- If g and $g \cdot f$ are L-coalgebras, then f is an L-coalgebra.
- If, in the following diagram, g, g', $g \cdot f$ and $g' \cdot f'$ are L-coalgebras and (v, w) and (u, w) are morphisms of L-coalgebras, then (u, v) is a morphism of L-coalgebras.

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "f'"] \\
\cdot \arrow[r, "v"] \arrow[d, "g"'] & \cdot \arrow[d, "g'"] \\
\cdot \arrow[r, "w"] & \cdot
\end{tikzcd}$$

We call these LOFSs *cancellative.*

The definition of cancellative LOFS regards being a LARI as a property. As a result, it does not extend from **Ord**-categories to 2-categories without modification.

Example 9.15. For LOFSs that are OFSs on a category, or in other words, when both the comonad and the monad of the LOFSs are idempotent, the second condition of the definition above is superfluous. Therefore, cancellative OFSs are precisely the reflective OFS, as shown in [5, Thm. 2.3]. This is the result that we will generalise in Theorem 9.17.

Lemma 9.16. *The* LOFS (E, M) *is cancellative.*

Proof: Recall that E-coalgebras are the same as LARIs. Suppose that f and g are composable morphisms and that $g \dashv r$ and $(g \cdot f) \dashv t$ are LARI structures. Defining $s = t \cdot g$, we have that $s \cdot f = t \cdot g \cdot f = 1$. It remains to prove that $f \cdot s = f \cdot t \cdot g \leqslant 1$, which is equivalent to $g \cdot f \cdot t \cdot g \leqslant g$, and this inequality holds since $g \cdot f \cdot t \leqslant 1$.

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "f"'] & \bullet \arrow[d, "f'"] \\
\bullet \arrow[r, "v"] \arrow[d, shift right, "g"'] \arrow[d, phantom, "\dashv"] & \bullet \arrow[d, shift left, "g'"] \arrow[d, phantom, "\vdash"] \\
\bullet \arrow[r, "w"] \arrow[u, shift right, "r"'] \arrow[uu, bend left=60, "t"] & \bullet \arrow[u, shift left, "r'"] \arrow[uu, bend right=60, "t'"']
\end{tikzcd}$$

Now suppose given morphisms of LARIs $(u, w) \colon g \cdot f \to g' \cdot f'$ and $(v, w) \colon g \to g'$, as depicted. We have to show that $(u, v) \colon f \to f'$ is a morphism of LARIs, ie that $u \cdot t \cdot g = t' \cdot g' \cdot v$, which holds by the following string of equalities

$$u \cdot t \cdot g = t' \cdot w \cdot g = t' \cdot g' \cdot v$$

completing the proof. ■

Theorem 9.17. *For a sub-*LARI *LOFS* (L,R) *on a finitely complete* **Ord***-category, the following statements are equivalent:*

(1) It is cancellative.
(2) It is reflective.

Proof: When L-Coalg is isomorphic to T-Emb for some lax idempotent T, it always satisfies the cancellation properties of Definition 9.14 since LARIs do: if g and $g \cdot f$ are T-embeddings, ie if Tg and $T(g \cdot f) = Tg \cdot Tf$ are LARIs, then Tf is a LARI, which is to say that f is a T-embedding; and similarly for morphisms. See Lemma 9.16.

Conversely, suppose that (L,R) is cancellative (Definition 9.14) and there is a morphism of AWFSs $(\mathsf{E},\mathsf{M}) \to (\mathsf{L},\mathsf{R})$, or equivalently, there is an **Ord**-functor Lari$(\mathcal{C}) \to \mathsf{L}$-Coalg over $\mathcal{C}^{\mathbf{2}}$. We shall show that the **Ord**-functor L-Coalg $\to \mathsf{R}_1$-Emb of Corollary 8.15 is an isomorphism, so $(\mathsf{L},\mathsf{R}) \cong \Psi(\mathsf{R}_1)$ is reflective.

If $f\colon X \to Y$ is an R_1-embedding, then consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "L!"'] & Y \arrow[d, "L!"] \\
R_1X \arrow[r, "R_1f"] \arrow[d] & R_1Y \arrow[d] \\
1 \arrow[r, equal] & 1
\end{tikzcd}$$

The morphisms $L!$ are cofree L-coalgebras while R_1f is a LARI and therefore an L-coalgebra. So, $L! \cdot f$ is an L-coalgebra and f is an L-coalgebra by the cancellation hypothesis. This means that each R_1-embedding is an L-coalgebra, and all that remains to prove is that morphisms of R_1-embeddings are morphisms of L-coalgebras.

Let $(u,v)\colon f \to f'$ be a morphism of R_1-embeddings, so $(R_1u, R_1v)\colon R_1f \to R_1f'$ is a morphism of LARIs, and, therefore, a morphisms of L-coalgebras.

It follows that $(u, R_1 v)$, depicted on the left below, is a morphism of L-coalgebras.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "L!"'] & X' \arrow[d, "L!"] \\ R_1 X \arrow[r, "R_1 u"] \arrow[d, "R_1 f"'] & R_1 X' \arrow[d, "R_1 f'"] \\ R_1 Y \arrow[r, "R_1 v"] & R_1 Y' \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "v"] \arrow[d, "L!"'] & Y' \arrow[d, "L!"] \\ R_1 Y \arrow[r, "R_1 v"] & R_1 Y' \end{tikzcd}$$

On the other hand, $(v, R_1 v)$ is a morphism of L-coalgebras, being the image under L of the morphism $(v, 1) \colon (Y \to 1) \to (Y' \to 1)$. By the second part of Definition 9.14, we deduce that (u, v) is a morphism of L-coalgebras, as required. This shows that L-Coalg $\to$ R_1-Emb is an isomorphism, completing the proof. ∎

10. Simple adjunctions

In §3 we saw that a reflection T on $\mathcal{C}$ is simple if and only if T-Iso $\to \mathcal{C}^{\mathbf{2}}$ is comonadic. In this section we generalise that result in three directions. First, we work with **Ord**-enriched categories, **Ord**-enriched functors and so on. Secondly, the 2-dimensional aspect introduced by the enrichment over **Ord** allows us to substitute isomorphisms by LARIs and T-isomorphisms by T-embeddings. Thirdly, even though §3 speaks of reflections, the constructions therein only need an adjunction (not necessarily a reflection) and this is the framework we choose.

Definition 10.1. Let $S \dashv G \colon \mathcal{B} \to \mathcal{C}$ be an adjunction between locally monotone functors on **Ord**-categories, of which we require $\mathcal{C}$ to have pullbacks and $\mathcal{B}$ to have comma-objects. We can always construct a monad R on $\mathcal{C}^{\mathbf{2}}$ by considering the comma-object $Kf = GSf \downarrow \eta_Y$ and defining $Rf \colon Kf \to Y$ as the second projection.

$$\begin{tikzcd} X \arrow[dr, "Lf"] \arrow[drr, bend left, "\eta_X"] \arrow[ddr, bend right, "f"'] & & \\ & Kf \arrow[r, "q_f"] \arrow[d, "Rf"'] \arrow[dr, phantom, "\geqslant"] & GSX \arrow[d, "GSf"] \\ & Y \arrow[r, "\eta_Y"'] & GSY \end{tikzcd}$$

The **Ord**-functorial factorisation $f = Rf \cdot Lf$ has an associated locally monotone copointed endofunctor $\Phi\colon L \Rightarrow 1$, where the component Φ_f is provided by the commutative square displayed.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "Rf"] & \cdot \end{tikzcd}$$

We continue with the notation of previous sections, where (E, M) denotes the LOFS whose E-coalgebras are the LARIs.

Remark 10.2. The comma-square of Definition 10.1 can be obtained by pulling back along η_Y the image under G of the projection $M(Sf)\colon Sf \downarrow SY \to SY$.

$$\begin{tikzcd} Kf \arrow[r] \arrow[d, "Rf"'] \arrow[dr, phantom, "\mathrm{pb}"] & G(Sf \downarrow SY) \arrow[r] \arrow[d, "G(MSf)"] \arrow[dr, phantom, "\geqslant"] & GSX \arrow[d, "GSf"] \\ Y \arrow[r, "\eta_Y"'] & GSY \arrow[r, equal] & GSY \end{tikzcd}$$

Lemma 10.3. *There is a pullback square of locally monotone endofunctors of $\mathcal{C}^{\mathbf{2}}$, as depicted on the left. There is a pullback of* **Ord***-categories, as depicted on the right.*

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\mathrm{pb}"] & G^{\mathbf{2}}ES^{\mathbf{2}} \arrow[d, "G^{\mathbf{2}}\Phi^E S^{\mathbf{2}}"] \\ 1_{\mathcal{C}^{\mathbf{2}}} \arrow[r, "\eta^{\mathbf{2}}"] & G^{\mathbf{2}}S^{\mathbf{2}} \end{tikzcd} \qquad \begin{tikzcd} (L,\Phi)\text{-Coalg} \arrow[r] \arrow[d, "U"'] \arrow[dr, phantom, "\mathrm{pb}"] & (E,\Phi^E)\text{-Coalg} \arrow[d] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "S^{\mathbf{2}}"'] & \mathcal{B}^{\mathbf{2}} \end{tikzcd}$$

Proof: In order to obtain a pullback square as on the left hand side of the statement, we need to give two pullback squares: one corresponding to the domain component and another corresponding to the codomain component. We define the domain component of $L \to G^{\mathbf{2}}ES^{\mathbf{2}}$ to be the unit $\eta\colon 1 \to GS$; this is possible since $\operatorname{dom} E = 1$. The resulting has horizontal morphisms both equal to η and vertical morphisms equal to the identity, since $\operatorname{dom} \Phi^E = 1$. This square is manifestly a pullback. The codomain component we choose is the pullback square of Remark 10.2.

The fact that there is a pullback of **Ord**-functors as on the right hand side of the statement follows easily, and it is a well-known fact (see, eg, [16, Prop. 9.2]). ∎

As a consequence of the previous lemma, the pullback square in (8.2) that defines S-Emb factors as two pullback squares, as depicted.

$$\begin{tikzcd}
S\text{-Emb} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pb}"] & \mathsf{E}\text{-Coalg} \arrow[d, "\cong"] \\
(L,\Phi)\text{-Coalg} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pb}"] & (E,\Phi^E)\text{-Coalg} \arrow[d] \\
\mathcal{C}^{\mathbf{2}} \arrow[r, "S^{\mathbf{2}}"] & \mathcal{B}^{\mathbf{2}}
\end{tikzcd}$$

The isomorphism E-Coalg $\cong (E,\Phi^E)$-Coalg, which is just the inclusion, was exhibited in Lemma 4.9. The **Ord**-functor S-Emb $\to (L,\Phi)$-Coalg is an isomorphism, being the pullback of an isomorphism. The remark that follows describes this functor and its inverse in more explicit terms.

Remark 10.4. Suppose that $f\colon X \to Y$ has a structure of (L,Φ)-coalgebra, given by $(1,s)\colon f \to Lf$, where $s\colon Y \to Kf$. This structure corresponds bijectively to an $r_f\colon SY \to SX$ in $\mathcal{B}$ with $r_f \cdot Sf = 1$ and $Sf \cdot r_f \leqslant 1$, in a way that can be explicitly described: $r_f\colon SY \to SX$ is the morphism whose transpose under the adjunction $S \dashv G$ is $q_f \cdot s\colon Y \to Kf \to GSX$, ie

$$r_f = \big(SY \xrightarrow{Ss} SKf \xrightarrow{Sq_f} SGSX \xrightarrow{\varepsilon_{SX}} SX\big).$$

and

$$Rf \cdot s = 1 \qquad q_f \cdot s = \big(Y \xrightarrow{\eta_Y} GSY \xrightarrow{Gr_f} GSX\big).$$

Definition 10.5. We say that the adjunction $S \dashv G$ is *simple* (or *simple with respect to* (E,M)) if, for each $f\colon X \to Y$ in $\mathcal{C}$, the morphism Lf has an S-embedding structure given by

$$\big(SX \xrightarrow{SLf} SKf\big) \dashv \big(SKf \xrightarrow{Sq_f} SGSX \xrightarrow{\varepsilon_{SX}} SX\big).$$

where ε is the counit of $S \dashv G$. This amounts to the existence of the inequality $SLf \cdot \varepsilon_{SX} \cdot Sq_f \leqslant 1$.

The following theorem is an analogue to the characterisation of simple reflections of §3.

Theorem 10.6. *The following statements are equivalent.*

(1) The adjunction $S \dashv G$ is simple.

(2) *The locally monotone forgetful functor* $U\colon S\text{-Emb} \to \mathcal{C}^{\mathbf{2}}$ *has a right adjoint and the induced comonad has underlying functor* L *and counit* $\Phi\colon L \Rightarrow 1_{\mathcal{C}^2}$.

(3) *The locally monotone copointed endofunctor* $\Phi\colon L \Rightarrow 1_{\mathcal{C}^2}$ *admits a comultiplication* $\Sigma\colon L \Rightarrow L^2$ *making* $\mathsf{L} = (\mathsf{L}, \Phi, \Sigma)$ *into a comonad whose category of coalgebras is isomorphic to* S-Emb *over* $\mathcal{C}^{\mathbf{2}}$.

Proof: Clearly (3) implies (2). The opposite implication holds if U is comonadic, which is if it has a right adjoint, by Beck's Theorem 10.10 and Lemma 10.11, showing that (2) implies (3).

Let us now prove that (3) implies (1). Let $f\colon X \to Y$ be a morphism of $\mathcal{C}$. The comultiplication $\Sigma_f\colon Lf \to L^2 f$ is of the form $\Sigma_f = (1, \sigma_f)$ for $\sigma_f\colon Kf \to KLf$. One of the counit axioms of the comonad says

$$1 = \big(Kf \xrightarrow{\sigma_f} KLf \xrightarrow{K(1,Rf)} Kf\big)$$

and upon composing with the projection $q_f\colon Kf \to GSX$ we have

$$q_f = q_f \cdot K(1, Rf) \cdot \sigma_f = q_{Lf} \cdot \sigma_f = Gr_{Lf} \cdot \eta_X \tag{10.7}$$

where we have used, first the definition of K as a comma-object (Definition 10.1), and then the fact that σ_f is an (L, Φ)-coalgebra structure on Lf together with the explicit description of the isomorphism $S\text{-Emb} \cong (L, \Phi)\text{-Coalg}$ (Remark 10.4); as before, $r_{Lf}\colon SKf \to SX$ denotes the right adjoint retract that endows Lf with an S-embedding structure. By adjointness, the equality (10.7) is equivalent to $r_{Lf} = \varepsilon_{SX} \cdot Sq_f$, which is precisely saying that $S \dashv G$ is simple.

Finally, we prove that (1) implies (2). For each $g\colon X \to Y$, the morphism $Lg\colon X \to Kg$ has an S-embedding structure, given by

$$r_{Lg} = \varepsilon_{SX} \cdot Sq_g\colon SKg \longrightarrow SX. \tag{10.8}$$

This defines a functor $J\colon \mathcal{C}^{\mathbf{2}} \to S\text{-Emb}$, since the image of any morphism $(h,k)\colon f \to g$ is compatible with the right adjoints r_{Lf} and r_{Lg}. To wit,

$$r_{Lg}\cdot SK(h,k) = \varepsilon_{SZ}\cdot Sq_g \cdot SK(h,k) = \varepsilon_{SZ}\cdot SGSh\cdot Sq_f = Sh\cdot\varepsilon_{SX}\cdot Sq_f = Sh\cdot r_{Lf}.$$

It is clear that J is a locally monotone functor. We shall show that it is a right adjoint to the forgetful functor $U\colon S\text{-Emb} \to \mathcal{C}^{\mathbf{2}}$.

Given an S-embedding (f, r_f) in $\mathcal{C}$, consider its associated (L, Φ)-coalgebra structure, as described in Remark 10.4:

$$(1, s_f)\colon (f, r_f) \longrightarrow (Lf, r_{Lf}) \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "Lf"] \\ Y \arrow[r, "s_f"] & Kf \end{tikzcd} \tag{10.9}$$

where s_f is defined by the equalities

$$Rf \cdot s_f = 1_X \qquad q_f \cdot s_f = Gr_f \cdot \eta_Y \colon Y \to GSY \to GSX.$$

If we equip Lf with the S-embedding structure r_{Lf} of (10.8), then $(1, s_f)$ becomes a morphism in S-Emb, since

$$r_{Lf} \cdot Ss_f = \varepsilon_{SX} \cdot Sq_f \cdot Ss_f = \varepsilon_{SX} \cdot SGr_f \cdot S\eta_Y = r_f \cdot \varepsilon_{SY} \cdot S\eta_Y = r_f.$$

Furthermore, (10.9) are the components of a natural transformation $\Psi\colon 1_{S\text{-Emb}} \Rightarrow JU$. To see this, if $(h, k)\colon f \to g$ is a morphism in S-Emb, where $g\colon Z \to W$, we have to show the equality $K(h, k) \cdot s_f = s_g \cdot k$. This holds since we have

$$\begin{aligned} q_g \cdot K(h,k) \cdot s_f = GSh \cdot q_f \cdot s = GSh \cdot Gr_f \cdot \eta_Y = \\ = Gr_g \cdot GSk \cdot \eta_Y = Gr_g \cdot \eta_W \cdot k = q_g \cdot s_g \cdot k \end{aligned}$$

$$Rg \cdot K(h,k) \cdot s_f = k \cdot Rf \cdot s = k = Rg \cdot s_g \cdot k.$$

To complete the proof, we show that the transformation Ψ with components (10.9) is the unit of an adjunction $U \dashv J$ with counit $\Phi\colon JU = L \Rightarrow 1_{\mathcal{C}^2}$. The triangle identity $\Phi_{U(f,r_f)} \cdot U\Psi_f = 1$ holds, since it amounts to $Rf \cdot s_f = 1$. The other triangle identity, $J\Phi_f \cdot \Psi_{Jf} = 1$, requires a bit more of work. The morphism of S-embeddings Ψ_{Jf} has the form $(1, \sigma_f)\colon Lf \to L^2 f$, and is defined by $RLf \cdot \sigma_f = 1$ and

$$q_{Lf} \cdot \sigma_f = \big(Kf \xrightarrow{q_f} GSX\big).$$

The morphism $J\Psi_f$ equals $(1, K(1, Rf))$, so the triangular equality translates into $K(1, Rf) \cdot \sigma_f = 1$. Both sides are equal to Rf upon composing with Rf, so it remains to show that $q_f \cdot K(1, Rf) \cdot \sigma_f = q_f$. This equality follows easily from what we already know about σ_f.

$$q_f \cdot K(1, Rf) \cdot \sigma_f = q_{Lf} \cdot \sigma_f = q_f.$$

This completes the proof of the statement (2), and so, the proof of the theorem. ∎

Theorem 10.10 (Beck)**.** *A functor $U\colon \mathcal{T} \to \mathcal{A}$ is comonadic if and only if*

(1) It has a right adjoint.
(2) U creates equalisers of parallel pairs of morphisms in $\mathcal{T}$ whose image under U has an absolute equaliser in $\mathcal{A}$.

Lemma 10.11. *In a pullback diagram of functors, as displayed, U satisfies condition (2) of Beck's Theorem 10.10 if V does so.*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "Q"] \arrow[d, "U"'] & \mathcal{S} \arrow[d, "V"] \\ \mathcal{A} \arrow[r, "S"] & \mathcal{B} \end{tikzcd}$$

Remark 10.12. Even if $U\colon S\text{-Emb} \to \mathcal{C}^{\mathbf{2}}$ is comonadic, the requirement that the associated comonad has underlying copointed endofunctor (L, Φ) is necessary for Theorem 10.6 to hold. This can be seen at the same time as exploring what the theorem means in the case that the **Ord**-categories $\mathcal{C}$ and $\mathcal{B}$ are ordinary categories. In this case, a LARI in $\mathcal{B}$ is an isomorphism, so S-Emb is the full subcategory $S\text{-Emb} \subset \mathcal{C}^{\mathbf{2}}$ of morphisms inverted by S. It may very well be the case that $S\text{-Emb} \subset \mathcal{C}^{\mathbf{2}}$ is a coreflective subcategory while the adjunction $S \dashv G$ is not simple. For example, if $\mathcal{C}$ has finite limits and intersection of all strong monomorphisms [5, Thm. 3.3].

11. Simple monads

Definition 11.1. Let $\mathcal{C}$ be an **Ord**-category that admits comma-objects and pullbacks. A monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{C}$ whose functor part T is locally monotone (ie, **Ord**-enriched) is *simple* if the free T-algebra adjunction is simple in the sense of Definition 10.5.

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F^{\mathsf{T}}"] \arrow[r, phantom, "\perp"] & \mathsf{T}\text{-Alg} \arrow[l, shift left, "U^{\mathsf{T}}"] \end{tikzcd}$$

Explicitly, T is simple when, for each $f\colon X \to Y$ in $\mathcal{C}$, the morphism $F^{\mathsf{T}}(L^{\mathsf{T}} f)$ is a right adjoint of $\varepsilon^{\mathsf{T}}_{F^{\mathsf{T}}X} \cdot F^{\mathsf{T}} q^{\mathsf{T}}_f$, with these morphisms defined by the following diagram, where the square is a comma-object.

$$\begin{tikzcd} X \arrow[dr, "Lf"] \arrow[drr, bend left, "\eta_X"] \arrow[ddr, bend right, "f"'] & & \\ & Kf \arrow[r, "q_f"] \arrow[d, "Rf"'] \arrow[dr, phantom, "\geqslant"] & TX \arrow[d, "Tf"] \\ & Y \arrow[r, "\eta_Y"'] & TY \end{tikzcd} \tag{11.2}$$

We will be specially interested in simple monads that are lax idempotent.

Lemma 11.3. *A lax idempotent* **Ord**-*monad* T *on* $\mathcal{C}$ *is simple if and only if there is an adjunction* $T(Lf) \dashv \mu_X \cdot Tq_f$*, where* μ_X *is the multiplication of* T.

Proof: The simplicity of T is the existence of an inequality

$$F^{\mathsf{T}}Lf \cdot \varepsilon_{F^{\mathsf{T}}X} \cdot F^{\mathsf{T}}q_f \leqslant 1. \tag{11.4}$$

Applying the forgetful **Ord**-functor U^{T} one obtains

$$TLf \cdot \mu_X \cdot Tq_f \leqslant 1 \tag{11.5}$$

and thus the adjunction of the statement. All this holds for a general **Ord**-monad T. If T is lax idempotent, the forgetful **Ord**-functor $U^{\mathsf{T}}\colon \mathsf{T}\text{-Alg} \to \mathcal{C}$ is locally full and in particular it reflects inequalities between morphisms. It follows that (11.5) implies (11.4). ∎

Corollary 11.6. *A lax idempotent* **Ord**-*monad* T *on* $\mathcal{C}$ *is simple if and only if* $TLf \cdot q_f \leqslant \eta_{Kf}$.

Proof: By lax idempotency of T, the left extension of $TLf \cdot q_f : Kf \to TKf$ along η_{Kf} is $\mu_{Kf} \cdot T(TLf \cdot q_f) = TLf \cdot \mu_X \cdot Tq_f$; see Definition 2.5 (7). Therefore, (11.5) holds if, and only if, $TLf \cdot q_f \leqslant \eta_{Kf}$. ∎

Putting together Theorem 10.6 and Definition 9.1, we have:

Corollary 11.7. *Simple lax idempotent monads on* **Ord**-*categories with comma-objects are fibrantly generating.*

This means that, if $\mathcal{C}$ has comma-objects, each simple lax idempotent monad T induces a LOFS (L, R) with L-Coalg isomorphic to T-Emb over $\mathcal{C}^{\mathbf{2}}$.

Proposition 11.8. *The monad* P *on* **Ord** *described in Example 2.7 is simple.*

Proof: The proof uses Corollary 11.6, for which we shall need the description of the comma-object Kf of (11.2) as

$$Kf = \big\{(W,y)\in P(X)\times Y : f_*(W)\subseteq {\downarrow}y\big\} = \big\{(W,y)\in P(X)\times Y : W\subseteq f^*({\downarrow}y)\big\}$$

and of the morphism $Lf\colon X\to Kf$ as $Lf(x) = ({\downarrow}x, f(x))$.

We must show that

$$(Lf)_*\cdot q_f \leqslant \eta_{Kf}.$$

Evaluating on $(W,y)\in Kf$, we have

$$(Lf)_*\cdot q_f(W,y) = (Lf)_*(W)\subseteq \eta_{Kf}(W,y) = {\downarrow}(W,y)$$

if and only if

$$W\subseteq (Lf)^*\big({\downarrow}(W,y)\big) = \big\{x\in X : ({\downarrow}x, f(x))\leqslant (W,y)\big\}.$$

This last inequality always holds, since, for $w\in W$, the inclusion ${\downarrow}w\subseteq W$ always holds, and $f(w)\leqslant y$, because $f_*(W)\subseteq{\downarrow}y$. ∎

For each morphism $f\colon X\to Y$ there is a "comparison" morphism

$$\kappa\colon T(Tf\downarrow\eta_Y)\longrightarrow T^2f\downarrow T\eta_Y$$

induced by the universal property of comma-objects. More explicitly, κ is a morphism, as displayed in the diagram below, unique with the property of making the triangles $\boxed{1}$ and $\boxed{2}$ commutative.

$$\begin{tikzcd}
TX \arrow[rrr, "T\eta_X"] \arrow[ddd, "Tf"'] & & & T^2X \arrow[ddd, "T^2f"] \\
& TKf \arrow[ul, "TLf"'] \arrow[urr, "Tq_f"] \arrow[dr, "\kappa"] \arrow[ddl, "TRf"'] \arrow[r, phantom, "\boxed{1}"] \arrow[d, phantom, "\boxed{2}"] & {} & \\
& {} & T^2f\downarrow T\eta_Y \arrow[uur, "p_f"'] \arrow[dll, "p_Y"] \arrow[dr, phantom, "\geqslant"] & \\
TY \arrow[rrr, "T\eta_Y"'] & & & TTY
\end{tikzcd}$$

Proposition 11.9. *A lax idempotent* **Ord***-monad* T *is simple provided that, for every f and $u : Kf\to TKf$, $u\leqslant\eta_{Kf}$ whenever $\kappa\cdot u\leqslant\kappa\cdot\eta_{Kf}$, where κ is the comparison morphism $TKf\to T^2f\downarrow T\eta_Y$.*

Proof: From

$$p_f\cdot\kappa\cdot TLf\cdot q_f = T\eta_X\cdot q_f\leqslant \eta_{TX}\cdot q_f = p_f\cdot\kappa\cdot\eta_{Kf}$$
$$p_Y\cdot\kappa\cdot TLf\cdot q_f = Tf\cdot q_f\leqslant \eta_Y\cdot Rf = p_Y\cdot\kappa\cdot\eta_{Kf}$$

and the definition of comma-object one has $\kappa \cdot TLf \cdot q_f \leqslant \kappa \cdot \eta_{Kf}$, and the conclusion follows from the hypothesis and Corollary 11.6. ■

For example, the above proposition applies in the cases when κ is a full morphism.

12. Submonads of simple monads

The aim of the present section is to provide easy criteria that will allow us to recognise simple submonads of simple lax idempotent monads. These results will be later used in Corollary 13.3 of §13.

Lemma 12.1. *Let* T *be an* **Ord**-*monad. If* T *is lax idempotent, then* T-*embeddings are full if and only if the components of the unit* $X \to TX$ *are full.*

Proof: By definition of lax idempotent monad, the unit components $\eta_X\colon X \to TX$ are T-embeddings, and, hence, they are full provided that T-embeddings are full.

Conversely, suppose that $f\colon X \to Y$ is a T-embedding. Then, $\eta_Y \cdot f = Tf \cdot \eta_X$ is full, being a composition of the LARI Tf and the full morphism η_X. Therefore, f is full. ■

Proposition 12.2. *Suppose that* $\varphi\colon \mathsf{S} \to \mathsf{T}$ *is a monad morphism between* **Ord**-*monads and that its components* φ_X *are* T-*embeddings. If* T *is lax idempotent and the components of the unit* $\eta_X : X \to TX$ *are full, then* S *is lax idempotent, with full unit components* $e_X : X \to SX$.

Proof: That S is lax idempotent follows from the following calculations and fullness of $T\varphi_X \cdot \varphi_{SX} = \varphi_{TX} \cdot S\varphi_X$:

$$\varphi_{TX} \cdot S\varphi_X \cdot Se_X = T\eta_X \cdot \varphi_X \leqslant \eta_{TX} \cdot \varphi_X = \varphi_{TX} \cdot e_{TX} \cdot \varphi_X = \varphi_{TX} \cdot S\varphi_X \cdot e_{SX}.$$

Moreover, with $\eta_X = \varphi_X \cdot e_X$ full, also e_X is full. ■

We say that a morphism $f\colon X \to Y$ is a *pullback-stable* T-*embedding* if the pullback of f along any morphism into Y is a T-embedding.

Theorem 12.3. *Suppose that* $\varphi\colon \mathsf{S} \to \mathsf{T}$ *is a monad morphism between* **Ord**-*monads whose components are pullback-stable* T-*embeddings, and that* T-*embeddings are full. If* T *is lax idempotent, then* S *is simple whenever* T *is so.*

Proof: Let us denote the unit of S by $e\colon 1 \Rightarrow S$, and the **Ord**-functorial factorisations obtained from S and T following the construction of the comma-object (11.2), respectively, by

$$\big(X \xrightarrow{L_S f} K_S f \xrightarrow{R_S f} Y\big) = \big(X \xrightarrow{f} Y\big) = \big(X \xrightarrow{L_T f} K_T f \xrightarrow{R_T f} Y\big)$$

Consider the following diagram where $K_T f = Tf \downarrow \eta_Y$, $K_S f = Sf \downarrow e_Y$, and $L_T f = \overline{\varphi}_f \cdot L_S f$, and note that $\boxed{1}$ is a pullback.

$$\begin{tikzcd}
X \arrow[rr, "e_X"] \arrow[dr, "L_S f"] \arrow[ddd, "f"'] & & SX \arrow[r, "\varphi_X"] \arrow[d, phantom, "\boxed{1}"] & TX \arrow[ddd, "Tf"] \\
& K_S f \arrow[ur, "t_f"] \arrow[dr, "\overline{\varphi}_f"] \arrow[ddl, "R_S f"] & {} & \\
& & K_T f \arrow[uur, "q_f"'] \arrow[ddll, "R_T f"] \arrow[dr, phantom, "\geqslant"] & \\
Y \arrow[rr, "e_Y"'] & & SY \arrow[r, "\varphi_Y"'] & TY
\end{tikzcd}$$

By Corollary 11.6 to conclude that S is simple it is enough to show that $SL_S f \cdot t_f \leqslant e_{K_S f}$. And this inequality follows from the following calculations, using the fullness of $T\overline{\varphi_f} \cdot \varphi_{K_S f}$.

$$\begin{aligned}
T\overline{\varphi}_f \cdot \varphi_{K_S f} \cdot SL_S f \cdot t_f = T\overline{\varphi}_f \cdot TL_S f \cdot \varphi_X \cdot t_f = T\overline{\varphi}_f \cdot TL_S f \cdot q_f \cdot \overline{\varphi}_f \leqslant \\
\leqslant \eta_{Kf} \cdot \overline{\varphi}_f = T\overline{\varphi}_f \cdot \varphi_{K_S f} \cdot e_{K_S f} \quad \blacksquare
\end{aligned}$$

Corollary 12.4. *Suppose that $\varphi\colon \mathsf{S} \to \mathsf{T}$ is a monad morphism between* **Ord***-monads whose components are T-embeddings, and where T is lax idempotent and simple, with full unit components $X \to TX$. Then:*

(1) S is lax idempotent and simple, with full unit components $X \to SX$;
(2) every S-embedding is a T-embedding;
(3) S-embeddings are full.

Proof: (1) follows from Proposition 12.2, while (3) follows directly from (2) and our assumptions. To show (2), first note that the unit components $e_X : X \to SX$ are T-embeddings since both $\eta_X = \varphi_X \cdot e_X$ and φ_X are. Now let $f : X \to Y$ be an S-embedding. As a LARI, Sf is a T-embedding, and so is f because both e_Y and $e_Y \cdot f = Sf \cdot e_X$ are T-embeddings. ■

13. Filter monads

In this section we exhibit AWFSs on the **Ord**-category of T_0 topological spaces arising from simple lax idempotent **Ord**-monads. These factorisations were constructed in [4].

As mentioned in Example 2.2 each T_0 topological space X carries an order given by

$$x \leqslant y \text{ if and only if } y \in \overline{\{x\}} \tag{13.1}$$

– this is the opposite of what is usually called the *specialisation order*. This induces an order structure on each hom-set $\mathbf{Top}_0(X, Y)$ by defining $f \leqslant g$ if $f(x) \leqslant g(x)$, for all $x \in X$, making $\mathbf{Top}_0$ into an **Ord**-enriched category.

A comma-object $f \downarrow g$ in $\mathbf{Top}_0$ can be described as the subspace of $X \times Y$ defined by the subset $\{(x, y) \in X \times Y \colon f(x) \leqslant g(y)\}$.

$$\begin{tikzcd} f \downarrow g \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\leqslant"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

Denote by $\mathsf{F}\colon \mathbf{Top}_0 \to \mathbf{Top}_0$ the filter monad. If X is a T_0 space, FX is the set of filters of open sets of X, with topology generated by the subsets $U^\# = \{\varphi \in FX : U \in \varphi\}$, where $U \in \mathcal{O}(X)$. The (opposite of the) specialisation order on FX results in the opposite of the inclusion of filters. In particular, FX is a poset. If $f\colon X \to Y$ is continuous, then Ff is defined by $Ff(\varphi) = \{V \in \mathcal{O}(Y) : f^{-1}(V) \in \varphi\}$. The unit of the monad has components $\eta_X\colon X \to FX$, where $\eta_X(x)$ is the *principal filter* generated by x, that is $\eta_X(x) = \{U \in \mathcal{O}(X) : x \in U\}$. The multiplication of the monad has components $\mu_X\colon F^2X \to FX$, given by $\mu_X(\Theta) = \{U \in \mathcal{O}(X) : U^\# \in \Theta\}$.

Observe that η_X is a full morphism. It is in fact an *embedding* meaning a topological embedding, in the usual sense: a continuous function that is an homeomorphism onto its image, where the latter is equipped with the subspace topology.

It was shown in [8] that the category of algebras for this monad is isomorphic to the category whose objects are continuous lattices [29] and morphisms poset maps that preserve directed sups and arbitrary infs. Our choice of the (opposite of the) specialisation order on spaces, which is the opposite of the order used in [8], grants a few comments as a way of avoiding confusion. A space $X \in \mathbf{Top}_0$ has an F-algebra structure precisely when the opposite of the poset $(X, \leqslant)$ is a continuous lattice, where $\leqslant$ is the order (13.1). The topology of the space X can be recovered as the Scott topology of the continuous lattice $(X, \leqslant)^{\mathrm{op}}$. A morphism of F-algebras $f\colon X \to Y$ is a continuous function that preserves arbitrary suprema, as a poset map $(X, \leqslant) \to (Y, \leqslant)$ [8, Thm. 4.4].

The filter monad F was shown to be lax idempotent in [12], where it is also proved that a continuous function f between T_0 spaces is an embedding if and only if Ff is a LARI. In other words, F-embeddings are precisely the topological embeddings.

Theorem 13.2. *The* **Ord**-*monad* F *is simple.*

Proof: We verify the hypothesis of Proposition 11.9. For any pair of continuous maps $f\colon X \to Z$ and $g\colon Y \to Z$, the comparison morphism

$$\kappa\colon F(f \downarrow g) \longrightarrow Ff \downarrow Fg \subset FX \times FY$$

sends a filter φ on $f \downarrow g$ to the pair of filters (ψ_0, ψ_1)

$$\psi_0 = \{U \in \mathcal{O}(X) : d_0^{-1}(U) \in \varphi\} \qquad \psi_1 = \{V \in \mathcal{O}(Y) : d_1^{-1}(V) \in \varphi\}$$

where d_0 and d_1 are the projections from $f \downarrow g$ to X and Y, respectively. Given $(x, y) \in f \downarrow g$, recall that its image under the unit is

$$\eta_{f\downarrow g}(x, y) = \{W \in \mathcal{O}(f \downarrow g) : (x, y) \in W\}$$

We have $(Fd_0)\eta_{f\downarrow g}(x, y) = \eta_X d_0(x, y) = \eta_X(x)$, and similarly, $(Fd_1)\eta_{f\downarrow g}(x, y) = \eta_Y(y)$.

The hypothesis of Proposition 11.9 will be satisfied if we show that $\kappa \cdot u \leqslant \kappa \cdot \eta_{f\downarrow g}$ implies $u \leqslant \eta_{f\downarrow g}$; or, in terms of filters, if we show that, given $\varphi \in F(f \downarrow g)$, $(x, y) \in f \downarrow g$ as above, the inequalities $\psi_0 \leqslant \eta_X(x)$ and $\psi_1 \leqslant \eta_Y(y)$ imply $\varphi \leqslant \eta_{f\downarrow g}(x, y)$. By definition of the (opposite) specialisation order, we need to show the two inclusions

$$\{U \in \mathcal{O}(X) : d_0^{-1}(U) \in \varphi\} \supseteq \{U \in \mathcal{O}(X) : x \in U\}$$
$$\{V \in \mathcal{O}(Y) : d_1^{-1}(V) \in \varphi\} \supseteq \{V \in \mathcal{O}(Y) : y \in V\}$$

imply $\varphi \supseteq \{W \in \mathcal{O}(f \downarrow g) : (x, y) \in W\}$. Given $x \in U \in \mathcal{O}(X)$, $y \in V \in \mathcal{O}(Y)$, then

$$(U \times V) \cap (f \downarrow g) = d_0^{-1}(U) \cap d_1^{-1}(V) \in \varphi.$$

But any neighbourhood W of (x, y) contains another of the form $(U \times V) \cap (f \downarrow g)$, so $W \in \varphi$, completing the proof. ∎

Since every principal filter is completely prime, and so in particular prime and proper, and $\mu_X(\Theta)$ is completely prime (resp. prime, proper) whenever Θ is so, the functors F_1, F_ω and F_Ω that assign to each space X the space of proper (resp. prime, completely prime) filters are the functor part of submonads F_1, F_ω and F_Ω of the filter monad, with the monad morphisms defined pointwise by the corresponding embeddings. Hence, using Corollary 12.4, we can immediately conclude:

Corollary 13.3. *The* **Ord***-monads of proper filters, of prime filters and of completely prime filters are lax idempotent and simple.*

Therefore these monads induce LOFSs $(\mathsf{L}_\alpha, \mathsf{R}_\alpha)$, with $\alpha = 0, 1, \omega, \Omega$ (denoting F by F_0), with associated weak factorisation systems $(\mathcal{L}_\alpha, \mathcal{R}_\alpha)$, where $\mathcal{L}_0$ is the class of embeddings, $\mathcal{L}_1$ is the class of dense embeddings, $\mathcal{L}_\omega$ is the class of flat embeddings, and $\mathcal{L}_\Omega$ is the class of completely flat embeddings [11, 12, 4]. Moreover, $\mathcal{R}_\alpha$ is the class of morphisms which are injective with respect to $\mathcal{L}_\alpha$ (see [4] for details).

14. Metric spaces

It is an insight of Bill Lawvere [23, 24] that metric spaces can be regarded as enriched categories and that, from this point of view, completeness can be interpreted in terms of "modules." The necessary base of enrichment is the category of *extended real numbers* $\bar{\mathbb{R}}_+$.

The category $\bar{\mathbb{R}}_+$ has objects the real non-negative numbers plus an extra object ∞, and has one morphism $\alpha \to \beta$ if and only if $\alpha \geqslant \beta$; ∞ is an initial object and 0 a terminal object. One can use the addition of real numbers to define a symmetric monoidal structure on $\bar{\mathbb{R}}_+$, with the convention that adding ∞ always produces ∞. The unit object of this tensor product is 0. Furthermore, $\bar{\mathbb{R}}_+$ is closed, with internal hom $[\alpha, \beta]$ equal to $\beta - \alpha$ if this difference is non-negative, and equal to zero otherwise, with the convention that $[\alpha, \infty] = \infty$, $[\infty, \infty] = 0$ and $[\infty, \alpha] = 0$.

A small $\bar{\mathbb{R}}_+$-category can be described as a set A with a distance function $A(-,-)\colon A \times A \to \bar{\mathbb{R}}_+$ that satisfies $A(a,a) = 0$ for all $a \in A$ and the triangular inequality. In general, it may very well happen that $A(a,b) = 0$ even if $a \neq b$; the distance may not be symmetric, ie $A(a,b) \neq A(b,a)$, and, the distance between two points may be ∞. We regard $\bar{\mathbb{R}}_+$-categories as generalised metric spaces and think of $A(a,b) \in \bar{\mathbb{R}}_+$ as the "distance" from a to b.

For example, $\bar{\mathbb{R}}_+$ itself is a generalised metric space with distance from α to β given by $[\alpha, \beta]$.

Each generalised metric space A has an *opposite* A^{op} with the same points and distance $A^{\mathrm{op}}(a, b) = A(b, a)$. We will concentrate on *skeletal* generalised metric spaces, ie those spaces A for which $A(a, b) = 0 = A(b, a)$ implies $a = b$. For example, $\bar{\mathbb{R}}_+$ is skeletal.

$\bar{\mathbb{R}}_+$-enriched functors $f\colon A \to B$ are identified with functions $A \to B$ that are non-expansive: $A(a, b) \geqslant B(f(a), f(b))$. It is easy to verify that there exists a unique $\bar{\mathbb{R}}_+$-natural transformation $f \Rightarrow g\colon A \to B$ if and only if $0 = B(f(a), g(a))$ for all $a \in A$. In this way we obtain an **Ord**-category $\mathbf{Met}_{\mathrm{sk}}$ of skeletal generalised metric spaces, with objects the skeletal $\bar{\mathbb{R}}_+$-categories, morphisms the $\bar{\mathbb{R}}_+$-functors and inequality $f \leqslant g$ between two of them given by the existence of a $\bar{\mathbb{R}}_+$-natural transformation $f \Rightarrow g$. Observe that $\mathbf{Met}_{\mathrm{sk}}(A, B)$ is not only a preorder but a poset, because B is skeletal.

There is a notion of colimit suited to enriched categories, known as *weighted colimit* (or *indexed colimit* in older texts); see [17, 19] for a standard reference. Each family of weights induces a lax idempotent **Ord**-monad on $\mathbf{Met}_{\mathrm{sk}}$ whose algebras are the skeletal generalised metric spaces that admit colimits with weights in the family (see [21, Theorems 6.1 and 6.3]). This monad is in fact simple (§11), as shown in the more general context in [7, §12]. It follows from the theory developed herein that there is a LOFS on $\mathbf{Met}_{\mathrm{sk}}$ whose left morphisms are the embeddings with respect to that monad and whose fibrant objects are the skeletal generalised metric spaces that admit all Φ-colimits (see Proposition 9.2 and Corollary 11.7). The rest of the section is occupied by the example of a particular class of colimits that admit an explicit description.

The class of absolute colimits, ie the weights whose associated colimits are preserved by any $\bar{\mathbb{R}}_+$-functor whatsoever, generates a simple lax idempotent monad Q on $\mathbf{Met}_{\mathrm{sk}}$. Putting together [23] and [30] one can give a description of Q in terms of Cauchy sequences.

Cauchy sequences in a skeletal generalised metric space A are defined in the same way as for classical metric spaces. Two Cauchy sequences (a_n) and (b_n) are *equivalent* if both $A(a_n, b_n)$ and $A(b_n, a_n)$ have limit 0. Denote by QA the set of equivalence classes of Cauchy sequences in A with distance $QA([a_n], [b_n]) = \lim_n A(a_n, b_n)$. It is not hard to see that QA is a skeletal generalised metric space.

The assignment $A \mapsto QA$ is part of an **Ord**-monad Q on $\mathbf{Met}_{\mathrm{sk}}$, with unit $A \to QA$ the map that sends $a \in A$ to the constant sequence on a, that we denote by c_a.

Convergence of a sequence (x_n) to a point a in generalised metric space A differs from ordinary convergence in metric spaces only in that we have to require that both $A(a, x_n)$ and $A(x_n, a)$ converge to 0 in $\bar{\mathbb{R}}_+$. The following assertions are equivalent for a skeletal generalised metric space A: it is an algebra for Q; the canonical isometry $A \to QA$ has a left adjoint; A is a retract of a space of the form QB; every Cauchy sequence in A converges. Spaces that satisfy these equivalent properties are known as *Cauchy-complete*.

If $(\mathsf{L}_Q, \mathsf{R}_Q)$ is the KZ-reflective LOFS on $\mathbf{Met}_{\mathrm{sk}}$ generated by Q, the L_Q-coalgebras, or left maps of the factorisation, are the Q-embeddings and can be characterised as follows.

Proposition 14.1. *A non-expansive map $f\colon A \to B$ between skeletal genen-eralised spaces is a Q-embedding if and only if it is an isometry and for each $b \in B$ the non-expansive function $B(f-, b)\colon A^{\mathrm{op}} \to B$ can be written as $B(f-, b) = \lim_n A(-, x_n)$ for a Cauchy sequence (x_n) in A.*

Proof: First, if Qf has a retract r, then Qf is an isometry and thus f is an isometry; for, $B(f(a), f(a')) = QB(c_{f(a)}, c_{f(a')}) = QB(Qf(c_a), Qf(c'_a)) = QA(c_a, c_{a'}) = A(a, a')$.

If r is moreover a right adjoint of Qf, and, for a given $b \in B$, $r(c_b)$ has an associated Cauchy sequence (x_n) in A, we must have

$$B(f(a), b) = QB\big(c_{f(a)}, c_b\big) = QB\big(Qf(c_a), c_b\big) = QA\big(c_a, r(c_b)\big) = \lim_n A(a, x_n)$$

for all $a \in A$.

Conversely, suppose that f is an isometry and $B(f-, b) = \lim_n A(-, x_n)$. We must define an equivalence class of Cauchy sequences $r[b_n] \in QA$ for each $[b_n] \in QB$ in a way such that $QB([f(a_n)], [b_n]) = QA([a_n], r[b_n])$. Since any Cauchy sequence is a limit of constant sequences (eg, $b_n = \lim_n c_{b_n}$), it suffices to define r and to verify this equality for constant sequences; ie we have to give $r[c_b] \in QA$ such that $B(f(a), b) = QA(c_a, r[c_b])$. Since we know that $B(f-, b) = \lim_n A(-, x_n)$, we may set $r[c_b] = [x_n]$ and the equality holds. In this way we prove that there is an adjunction $Qf \dashv r\colon QB \to QA$. It remains to prove that $r \cdot Qf = 1$, but f is an isometry, which implies that Qf is an isometry and therefore one-to-one, so the equality follows from the adjunction triangle equation $Qf \cdot r \cdot Qf = Qf$. ∎

It follows from the general theory that, given a Q-embedding $f\colon A \to B$ and a non-expansive function $h\colon A \to C$ into Cauchy-complete skeletal generalised metric space C, there is an extension d.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \\ B \arrow[ur, dotted, "d"'] & \end{tikzcd}$$

Furthermore, Cauchy-complete skeletal generalised metric spaces are precisely those injective with respect to the Q-embeddings. In terms of sequences, the extension d is given by $d(b) = \lim_n h(x_n)$, where (x_n) is a Cauchy sequence in A such that $B(f-, b) = \lim_n A(-, x_n)$.

Corollary 14.2. *Let $f\colon A \to B$ be a non-expansive function between skeletal generalised metric spaces, and assume that B is a metric space. Then, f is a Q-embedding if and only if it is a dense isometry.*

Proof: If f is a Q-embedding and $b \in B$, there is a Cauchy sequence (x_n) in A such that $\lim_n A(-, x_n) = \lim_n B(f-, b)$. Given $\varepsilon > 0$, there is a n_0 such that $A(x_n, x_m) < \varepsilon/2$ if $n, m \geqslant n_0$. Thus, for $m \geqslant n_0$ we have

$$B(f(x_m), b) = \lim_n B(f(x_m), f(x_n)) = \lim_n A(x_m, x_n) \leqslant \varepsilon/2 < \varepsilon.$$

It follows that $(f(x_m))$ converges to b, and f is dense. Observe that we have used that the distance of B is symmetric.

Conversely, if f is a dense isometry, any $b \in B$ is $\lim_n f(x_n)$ for some sequence (x_n) in A, which is Cauchy since f preserves distances and $(f(x_n))$ converges. Then $B(f(a), b) = \lim_n A(a, x_n)$ for all $a \in A$, and Proposition 14.1 applies. ∎

The definition of QA given in terms of Cauchy sequences immediately tells us that if A is a metric space then QA is a metric space too; ie, its distance function is symmetric. We deduce:

Corollary 14.3. *The* LOFS $(\mathsf{L}_Q, \mathsf{R}_Q)$ *restricts to an* OFS *on the category of metric spaces. Its left maps are the dense isometries.*

Appendix A. Accessible AWFSs

In §9 we characterised those LOFSs "fibrantly generated" by a lax idempotent monad. In this section we explore what more can be said in the case when the base **Ord**-category is locally presentable and all the monads and comonads involved are accessible. We confine our discussion to this appendix, as we will assume familiarity with the basic theory of accessible and locally presentable categories, for which the standard references are [26] and [1].

We start with a result about ordinary (instead of enriched) accessible AWFSs. These are AWFS whose comonad and monad are accessible functors; in fact, it suffices that only one of them should be accessible. See [3] for details.

Proposition A.1. *Let F be a left adjoint functor between a locally presentable category $\mathcal{C}$ and an accessible category $\mathcal{A}$, and (G, S) be an accessible AWFS on $\mathcal{A}$. Given the following pullback of double categories*

$$\begin{tikzcd} \mathbb{L} \arrow[r] \arrow[d] & \mathsf{G}\text{-}\mathbb{C}\mathrm{oalg} \arrow[d] \\ \mathbb{S}\mathrm{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathrm{q}(F)"] & \mathbb{S}\mathrm{q}(\mathcal{A}) \end{tikzcd}$$

there exists an accessible (L, R) on $\mathcal{C}$ such that $\mathsf{L}\text{-}\mathbb{C}\mathrm{oalg} \cong \mathbb{L}$ over $\mathbb{S}\mathrm{q}(\mathcal{C})$ and the vertical category of $\mathbb{L}$ is locally presentable.

Proof: If suffices to prove that the functor $U\colon \mathcal{L} \to \mathcal{C}^{\mathbf{2}}$ is comonadic (see [3, Prop. 4]). By the dual version of Lemma 10.11, it suffices to show that it has a left adjoint. Being the pullback of a functor that creates colimits (indeed, comonadic) along a cocontinuous functor, U creates colimits too, so $\mathcal{L}$ is cocomplete and U cocontinuous. On the other hand, $\mathcal{L}$ is accessible, being the limit of a diagram of accessible categories and accessible functors (see [26, Thm. 5.1.6]). It follows that $\mathcal{L}$ is locally presentable, and therefore the cocontinuous functor U is a left adjoint. ■

Definition A.2. Ord-enriched categories or functors will be called accessible or locally presentable if their underlying (ordinary) categories or functors are so. An AWFS (L, R) on an accessible **Ord**-category is accessible if one of the following equivalent conditions holds: the endofunctor L is accessible; the endofunctor R is accessible; the category of L-coalgebras is accessible; the category of R-algebras is accessible.

In what follows we maintain the terminology and notations of §9. Split opfibrations in an **Ord**-category with comma-objects $\mathcal{C}$ are the algebras for the monad M on $\mathcal{C}^{\mathbf{2}}$ given by $M(f) = (f \downarrow 1)$ (see Notation 9.11).

Lemma A.3. *Split opfibrations in* **Ord***-categories are full morphisms.*

Proof: Recall from §2 that a morphism $p\colon X \to Y$ in an **Ord**-category $\mathcal{A}$ is full if the monotone morphism $\mathcal{A}(Z,p)\colon \mathcal{A}(Z,X) \to \mathcal{A}(Z,Y)$ between posets is full in the usual sense. If p is a split opfibration, then $\mathcal{A}(Z,p)$ is a split opfibration of posets. Then, it suffices to prove that split opfibrations of posets are full. This is an easy verification: if $p : X \to Y$ is a split opfibration and $p(x) \leqslant p(y)$, then there is an opcartesian lifting $x \leqslant \tilde{y}$ with $p(\tilde{y}) = p(y)$, and $\tilde{y} \leqslant y$. Thus $x \leqslant y$. ■

In this section we will make explicit the distinction between **Ord**-enriched categories, functors and monads and their ordinary counterparts by adding to the latter the subscript $(-)_\circ$; this is the same notation employed in [17, 19] and elsewhere.

There is a theory of locally finitely presentable enriched categories, developed in detail in [18]. Furthermore, much of this theory carries over to locally presentable categories enriched in a locally finitely presentable symmetric monoidal closed category (in our case, **Ord**). There will be very few facts about locally presentable **Ord**-categories that we shall need, so we point the reader to [18, 7.4] for some guidance about the overall theory.

Definition A.4. Let κ be a regular cardinal. An object X of a cocomplete **Ord**-category is κ-presentable if $\mathcal{C}(X,-)\colon \mathcal{C}_\circ \to \mathbf{Ord}$ preserves κ-filtered colimits. We say that $\mathcal{C}$ is a *locally κ-presentable* **Ord**-category if it is cocomplete (in the **Ord**-enriched sense) and has a small full sub-**Ord**-category $\mathcal{G} \subseteq \mathcal{C}$ consisting of κ-presentable objects and such that the associated "nerve" functor $\mathcal{C} \to [\mathcal{G}^{\mathrm{op}}, \mathbf{Ord}]$ reflects isomorphisms. A locally presentable **Ord**-category is one that is κ-presentable for some κ.

The first thing we need to mention is that if $\mathcal{C}$ is a locally presentable **Ord**-category, then it is automatically complete and its underlying category $\mathcal{C}_\circ$ is locally presentable in the usual sense (with the same accessibility exponent). An **Ord**-functor between locally presentable **Ord**-categories is said to be *accessible* when its underlying functor is accessible in the usual sense; this is because preservation of conical colimits is just preservation of those colimits by the underlying functor. An **Ord**-monad is accessible if its underlying functor is so. If T is an accessible **Ord**-monad on the locally presentable **Ord**-category $\mathcal{C}$, then T-Alg is locally presentable.

Remark A.5. In locally κ-presentable category $\mathcal{C}$, finite limits commute with κ-filtered colimits. In fact all that is necessary is the existence of a family of

κ-presentable objects $\{G_i\}$ such that the functors $\mathcal{C}(G_i, -)\colon \mathcal{C}_0 \to \mathbf{Ord}$ are jointly conservative (ie, a morphism f is an isomorphism if each $\mathcal{C}(G_i, f)$ is an isomorphism).

Definition A.6. An **Ord**-enriched AWFS (L, R) on a locally presentable **Ord**-category $\mathcal{C}$ is *accessible* if its underlying ordinary AWFS on the accessible ordinary category $\mathcal{C}_\circ$ is accessible.

Theorem A.7. *Let $\mathcal{C}$ be a locally presentable* **Ord***-category. Then, accessible lax idempotent monads on $\mathcal{C}$ are fibrantly* KZ*-generating. The* LOFS $\Psi(\mathsf{T})$ *generated by an accessible lax idempotent monad T is accessible.*

Proof: We have to show that there is an **Ord**-enriched AWFS (L, R) for which L-Coalg $\cong$ T-Emb. We first show Lari(T-Alg)$_\circ$ is an accessible category. Even though we know that the category T-Alg$_\circ$ is accessible by [26, Thm. 5.1.6], it is not enough for our purposes, as our proof involves **Ord**-enriched (co)limits, and we have to argue as follows.

The existence of limits in the **Ord**-category $\mathcal{C}$ ensures the same for T-Alg. By hypothesis, $\mathcal{C}$ is locally κ-presentable and T preserves κ'-filtered colimits, but we may assume $\kappa = \kappa'$ by raising the accessibility exponent (see [26]). Then T-Alg has κ-filtered colimits and the family $\{T(G) : G \in \mathcal{G}\}$ satisfies the conditions of Remark A.5, so finite limits commute with κ-filtered colimits in T-Alg (the latter can be shown to be cocomplete but we do not need it here). The comonad E on T-Alg$^{\mathbf{2}}$ whose coalgebras are LARIs (Lemma 4.9) was described in §4.d by means of finite limits (specifically, comma-objects) and therefore preserves κ-filtered colimits. In particular, Lari(T-Alg)$_\circ$ is accessible and comonadic over T-Alg$_\circ^{\mathbf{2}}$.

We next show that that there is an accessible ordinary AWFS (L, R) with an isomorphism of categories L-Coalg $\cong$ T-Emb$_\circ$ over $\mathcal{C}_\circ^{\mathbf{2}}$ by applying Proposition A.1, whose hypotheses we now verify. We have an accessible AWFS (E, M) on T-Alg$_\circ$, by the previous paragraph. By definition, T-Emb is the pullback of Lari(T-Alg)$_\circ$ $\to$ T-Alg$_\circ^{\mathbf{2}}$ along $(F^{\mathsf{T}})_\circ^{\mathbf{2}}\colon \mathcal{C}_\circ^{\mathbf{2}} \to \mathsf{T}$-Alg$_\circ^{\mathbf{2}}$. An application of Proposition A.1 produces the required accessible AWFS on $\mathcal{C}_\circ$.

All that remains is to show that it is an **Ord**-enriched AWFS, or equivalently, that the comonad L (whose category of coalgebras is T-Emb$_\circ$) is **Ord**-enriched. Or, equivalently still, that $U\colon \mathsf{T}$-Emb $\to \mathcal{C}^{\mathbf{2}}$ has an **Ord**-enriched right adjoint. We have shown above that the ordinary functor $U_\circ$

has a right adjoint, say W. All we have to show is that the monotone map

$$\mathsf{T}\text{-Emb}(f, Wg) \xrightarrow{U} \mathcal{C}^{\mathbf{2}}(Uf, UWg) \xrightarrow{\mathcal{C}^{\mathbf{2}}(1,(1,Rg))} \mathcal{C}^{\mathbf{2}}(Uf, g) \tag{A.8}$$

is not only an isomorphism of sets but also an isomorphism of posets. This amounts to showing that it is a full morphism of posets. Before doing so, we need the following observation.

The functor $\mathsf{E}_\circ$-Coalg $\to$ L-Coalg that expresses the fact that each LARI is canonically a T-embedding, induces a morphism of AWFS $(\mathsf{E}_\circ, \mathsf{M}_\circ) \to (\mathsf{L}, \mathsf{R})$, and thus a morphism of monads $\mathsf{M}_\circ \to \mathsf{R}$; in this argument we have used [3, Prop. 2] twice. It follows that each R-algebra is an M-algebra, ie a split opfibration.

Returning to (A.8), the first arrow is full because an inequality between morphims of T-embeddings is, by Definition 8.1, an inequality between them as morphisms in $\mathcal{C}^{\mathbf{2}}$. The second morphism in (A.8) is also full, because Rg is a split opfibration (see the previous paragraph) and Lemma A.3. Therefore, W extends to an **Ord**-enriched adjoint to U, completing the proof. ∎

Theorem A.9. *If $\mathcal{C}$ is a locally presentable* **Ord***-category, the fully faithful* **Ord***-functor*

$$\Psi\colon \mathbf{LIMnd}_{\mathrm{acc}}(\mathcal{C}) \longrightarrow \mathbf{LOFS}_{\mathrm{acc}}(\mathcal{C})$$

exhibits the **Ord***-category of accessible lax idempotent monads as a reflective full sub-***Ord***-category of the category of accessible* LOFS*s. Its replete image consists of all cancellative sub-*LARI LOFS*s that are accessible.*

Proof: The **Ord**-functor Ψ from $\mathbf{LIMnd}_{\mathrm{fib}}(\mathcal{C})$ to **LOFS** restricts to the subcategories of accessible lax idempotent monads and accessible LOFSs, by Theorem A.7 yielding an **Ord**-functor as in the statement. We know from Proposition 9.13 that $\Psi(\mathsf{T})$ is always sub-LARI.

Clearly, the monad $\Phi(\mathsf{L}, \mathsf{R}) = \mathsf{R}_1$ is accessible if (L, R) is an accessible AWFS, so we obtain a left adjoint Φ to the fully faithful **Ord**-functor Ψ of the statement. Its unit

$$\varpi\colon (\mathsf{L}, \mathsf{R}) \longrightarrow \Psi\Phi(\mathsf{L}, \mathsf{R}) = \Psi(\mathsf{R}_1)$$

is the morphism of AWFSs that corresponds to the **Ord**-functor that is the inclusion of L-Coalg into R_1-Emb, and the former is invertible if and only if the latter is so. We may now apply Theorem 9.17 to deduce that (L, R) is cancellative precisely when the unit ϖ is invertible, which is another way of saying that (L, R) is in the replete image of Φ. ∎

Example A.10. There are accessible monads that are not simple, as exhibited below. This means that, even though the monad induces an LOFS, it cannot be obtained through the methods of §10 and §11. One example that involves only ordinary categories, which we may regard as locally discrete **Ord**-categories, is [5, Example 4.2], where the monad D on the category of abelian groups **Ab** is given by $A \mapsto A/2A$ (quotient by $2A = \{2a : a \in A\}$). If $f\colon 0 \to D(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ is the unique possible morphism, then the comma-object Kf is the pullback of f along the quotient map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. In other words, this pullback is the inclusion $2\mathbb{Z} \hookrightarrow \mathbb{Z}$. The morphism $Lf\colon 0 \to 2\mathbb{Z}$ is the unique possible, and $D(Lf)$ is not an isomorphism (equivalently, a LARI) since $D(2\mathbb{Z}) \ncong 0$.

This example can be modified to show that, for example, the monads on the **Ord**-categories of (commutative) monoids in **Ord** that sends a monoid $(V, e, \otimes)$ to the coequalizer of the pair of morphisms $V \to V$ that are $x \mapsto (x \otimes x)$ and $x \mapsto e$, is not simple. Nonetheless, this monad gives rise to a LOFS, by Theorem A.9.

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