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ON SKEW-SYMMETRIC MATRICES RELATED TO THE VECTOR CROSS PRODUCT IN $\mathbb{R}^7$

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ABSTRACT: A study of real skew-symmetric matrices of orders 7 and 8, defined through the vector cross product in $\mathbb{R}^7$, is presented. More concretely, results on matrix properties, eigenvalues, (generalized) inverses and rotation matrices are established.

KEYWORDS: vector cross product, skew-symmetric matrix, matrix properties, eigenvalues, (generalized) inverses, rotation matrices.
MATH. SUBJECT CLASSIFICATION (2010): 15A72, 15B57, 15A18, 15A09, 15B10.

1. Introduction

A classical result known as the generalized Hurwitz Theorem asserts that, over a field of characteristic different from 2, if $\mathcal{A}$ is a finite dimensional composition algebra with identity, then its dimension is equal to $1, 2, 4$ or 8. Furthermore, $\mathcal{A}$ is isomorphic either to the base field, a separable quadratic extension of the base field, a generalized quaternion algebra or a generalized octonion algebra, [8].

A consequence of the cited theorem is that the values of n for which the Euclidean spaces $\mathbb{R}^n$ can be equippped with a binary vector cross product, satisfying the same requirements as the usual one in $\mathbb{R}^3$, are restricted to 1 (trivial case), 3 and 7. A complete account on the existence of r-fold vector cross products for d-dimensional vector spaces, where they are used to construct exceptional Lie superalgebras, is in [3].

The interest in octonions, seemingly forgotten for some time, resurged in the last decades, not only for their intrinsic mathematical relevance but also because of their applications, as well as those of the vector cross product in $\mathbb{R}^7$. This product was used for the implementation of the seven-dimensional vector analysis method in [15], to estimate the amount of abnormalities in algorithms that provide accurate feedback in rehabilitation.

Moreover, as it is mentioned in [9], the octonions play an important role in Physics. Namely, they led up to the theory of fundamental particles known

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as eightfold way. More recently, in [11], it was shown that if the fundamental particles, the fermions, are assumed to have seven time-spatial dimensions, then the so called hierarchy problem, concerning the unknown reason for the weak force to be stronger than the gravity force, could be solved.

In this work, we extend the results, devoted to the vector cross product in $\mathbb{R}^3$ and real skew-symmetric matrices of order 4, in [16]. Concretely, we study real skew-symmetric matrices of orders 7 and 8 defined through the vector cross product in $\mathbb{R}^7$. These are denoted, for any $a, b \in \mathbb{R}^7$, by S_a and $M_{a,b}$, respectively.

The latter ones, called hypercomplex matrices in [9], can be used to write the coordinate matrix of the left multiplication by an octonion. The particular case $b = a$ leads to $M_{a,a}$, an orthogonal design which, according to [14] and references therein, can be used in the construction of space time block codes for wireless transmissions. Furthermore, if $b = a = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$ then $I_8 + M_{a,a}$ is a Hadamard matrix of skew-symmetric type.

For completeness, in section 2 we recall some definitions and results related to the binary vector cross product in $\mathbb{R}^7$, inverses and skew-symmetric matrices. Throughout the work, for simplicity, we omit the word binary.

In section 3 we approach the vector cross product in $\mathbb{R}^7$ from a matrix point of view. For this purpose, we consider the matrices S_a and establish some related properties.

As far as section 4, we devote it to the eigenvalues of S_a and $M_{a,b}$. We obtain the characteristic polynomials of these matrices, using adequate Schur complements in the latter case.

In section 5, we deduce either the inverse or the Moore-Penrose inverse of $M_{a,b}$ depending on its determinant. The Moore-Penrose inverse of S_a is presented in section 3.

We dedicate section 6, the last one of this work, to the generation of rotation matrices from the Cayley transforms and the exponentials of the skew-symmetric matrices S_a and $M_{a,b}$.

2. Preliminaries

Throughout this work, $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. With $n = 1$, we identify $\mathbb{R}^{m\times 1}$ with $\mathbb{R}^m$. With $m = n = 1$, we identify $\mathbb{R}^{1\times 1}$ with $\mathbb{R}$.

Consider the usual real vector space $\mathbb{R}^8$, with canonical basis $\{e_0, \dots, e_7\}$, equipped with the multiplication $*$ given by $e_i * e_i = -e_0$ for $i \in \{1, \dots, 7\}$, being e_0 the identity, and the below Fano plane, where the cyclic ordering of each three elements lying on the same line is shown by the arrows.

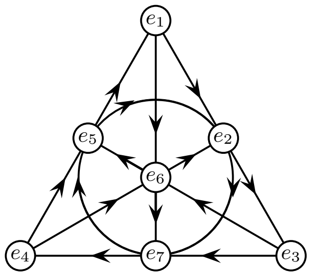

FIGURE 1. Fano plane for $\mathbb{O}$.

Then $\mathbb{O} = (\mathbb{R}^8, *)$ is the real (non-split) octonion algebra. Every element $\underline{x} \in \mathbb{O}$ may be represented by

$$\underline{x} = x_0 + x, \text{ where } x_0 \in \mathbb{R} \text{ and } x = \sum_{i=1}^{7} x_i e_i \in \mathbb{R}^7$$

are, respectively, the *real part* and the *pure part* of the octonion $\underline{x}$.

The multiplication $*$ can be written in terms of the Euclidean inner product and the vector cross product in $\mathbb{R}^7$, hereinafter denoted by $\langle \cdot, \cdot \rangle$ and $\times$, respectively. Concretely, as in [9], we have

$$\underline{x} * \underline{y} = x_0 y_0 - \langle x, y \rangle + x_0 y + y_0 x + x \times y.$$

A formula for the *double vector cross product* in $\mathbb{R}^7$ is

$$x \times (y \times z) = \langle x, z \rangle y - \langle x, y \rangle z + \frac{1}{3} J(x, y, z),$$

[10]. Here J stands for the *Jacobian*, the alternate application defined by

$$J(x, y, z) = x \times (y \times z) + y \times (z \times x) + z \times (x \times y).$$

For $(\mathbb{R}^3, \times)$, a Lie algebra, the well known formula for the double vector cross product in $\mathbb{R}^3$ arises since, for any $x, y, z \in \mathbb{R}^3$, $J(x, y, z) = 0$.

Let $A \in \mathbb{R}^{m \times n}$.

A matrix $A^- \in \mathbb{R}^{n \times m}$ is a *generalized inverse* of A if $AA^-A = A$.

The *Moore-Penrose inverse* of A is the unique matrix $A^\dagger \in \mathbb{R}^{n \times m}$ satisfying

$$AA^\dagger A = A,\ A^\dagger A A^\dagger = A^\dagger,\ (A^\dagger A)^T = A^\dagger A \text{ and } (AA^\dagger)^T = AA^\dagger,$$

[1]. In particular, if u is a nonzero vector in $\mathbb{R}^{m \times 1}$, then its Moore-Penrose inverse is given by

$$u^\dagger = \frac{u^T}{||u||^2},$$

where, hereinafter, $||\cdot||$ denotes the Euclidean norm.

In the remaining part of this section, assume that $m = n$.

The matrix A is a *rotation matrix* if A is orthogonal ($A^T A = I$) and $\det A = 1$.

From now on, assume also that A is a skew-symmetric matrix. Hence, according to a classical result on skew-symmetric matrices, the eigenvalues of A are purely imaginary or null.

If n is odd, then $\det A = 0$. If n is even, then $\det A = (\mathrm{pf}\ A)^2$, where $\mathrm{pf}\ A$ denotes the *Pfaffian* of A. This is the homogeneous polynomial of degree $\frac{n}{2}$, in the entries of A, defined by

$$\mathrm{pf}(A) = \frac{1}{N!2^N} \sum_{\sigma \in S_{2N}} sgn(\sigma) \prod_{i=1}^{N} a_{\sigma(2i-1)\sigma(2i)}$$

where $n = 2N$. For the numeric and symbolic evaluation of the pfaffian, see, for instance, [6].

Due to the skew-symmetry of A, $I_n + A$ is invertible. The *Cayley transform* of A is the matrix given by $\mathcal{C}(A) = (I_n + A)^{-1}(I_n - A)$. It is well known that $\mathcal{C}(A)$ is a rotation matrix and, as $I_n - A = 2I_n - (I_n + A)$,

$$\mathcal{C}(A) = 2(I_n + A)^{-1} - I_n.$$

This is one of the Cayley formulas in [5], that allow to establish a one-to-one correspondence between the skew-symmetric matrices and the orthogonal matrices that do not have the eigenvalue -1.

As it is known, $R = \mathrm{e}^A$ is the rotation matrix, called *exponential* of A, defined by the absolute convergent power series

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Conversely, given a rotation matrix $R \in \mathbf{SO}(n)$, there exists a skew-symmetric matrix A such that $R = \mathrm{e}^A$, [5]. The combination of these two facts is equivalent to saying that the map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$, from the Lie algebra $\mathfrak{so}(n)$ of skew-symmetric $n \times n$ matrices to the Lie group $\mathbf{SO}(n)$, is surjective, [7].

3. Matrix properties of S_a

In the present section, following [9] and [10], we consider a matrix representation of the Maltsev algebra $(\mathbb{R}^7, \times)$ in terms of particular cases of hypercomplex matrices. If $a \in \mathbb{R}^7$, then let S_a be the matrix in $\mathbb{R}^{7\times 7}$ defined by

$$S_a x = a \times x$$

for any $x \in \mathbb{R}^7$. Hence, for $a = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T$, S_a is the skew-symmetric matrix

$$\begin{bmatrix} 0 & -a_3 & a_2 & -a_5 & a_4 & -a_7 & a_6 \\ a_3 & 0 & -a_1 & -a_6 & a_7 & a_4 & -a_5 \\ -a_2 & a_1 & 0 & a_7 & a_6 & -a_5 & -a_4 \\ a_5 & a_6 & -a_7 & 0 & -a_1 & -a_2 & a_3 \\ -a_4 & -a_7 & -a_6 & a_1 & 0 & a_3 & a_2 \\ a_7 & -a_4 & a_5 & a_2 & -a_3 & 0 & -a_1 \\ -a_6 & a_5 & a_4 & -a_3 & -a_2 & a_1 & 0 \end{bmatrix}.$$

We now establish some properties related to S_a.

Proposition 3.1. *Let $a, c \in \mathbb{R}^7$ and $\alpha, \gamma \in \mathbb{R}$. Then:*

(i) $S_{\alpha a+\gamma c} = \alpha S_a + \gamma S_c$;
(ii) $S_a c = -S_c a$;
(iii) S_a *is singular;*
(iv) $S_a^2 = aa^T - a^T a I_7$;
(v) $S_a^3 = -a^T a S_a$;
(vi) $S_a^\dagger = \begin{cases} 0 & \text{if } a = 0 \\ -\frac{1}{a^T a} S_a & \text{if } a \neq 0 \end{cases}$;
(vii) $S_{S_a b} = \frac{3}{2}(ba^T - ab^T) - \frac{1}{2}[S_a, S_b]$, *where $[\cdot,\cdot]$ denotes the matrix commutator.*

Proof: Properties (i) and (ii) are direct consequences of the bilinearity and of the skew-symmetry of $\times$.

As far as (iii), on the one hand, if $a = 0$ then $S_a = 0$, being S_a singular. On the other hand, if $a \neq 0$ then, from (ii), we have $S_a a = 0$. If S_a was invertible then $a = 0$, a contradiction.

Regarding (iv), for any $x \in \mathbb{R}^7$, we have

$$S_a S_a x = a \times (a \times x) = \langle a, x\rangle a - \langle a, a\rangle x = (aa^T)x - (a^T a)x = (aa^T - a^T a I_7)x.$$

Concerning (v), note that $aa^T S_a = -a(S_a a)^T = 0$ by (ii). Hence, by (iv), $S_a^3 = S_a^2 S_a = -a^T a S_a$.

To obtain (vi), since the case $a = 0$ is trivial, assume that $a \neq 0$. By (v),

$$S_a\left(-\tfrac{1}{a^Ta}S_a\right)S_a = -\tfrac{1}{a^Ta}S_a^3 = S_a.$$

Taking into account the skew-symmetry of S_a,

$$\left(S_a\tfrac{-1}{a^Ta}S_a\right)^T = -\tfrac{1}{a^Ta}S_a^TS_a^T = S_a\tfrac{-1}{a^Ta}S_a.$$

The remaining equalities of the Moore-Penrose inverse definition can be proved in a similar way.

By (ii), for any $x \in \mathbb{R}^7$, we get

$$S_{S_ab}x = -S_xS_ab = -x \times (a \times b) = -\langle x, b\rangle a + \langle x, a\rangle b - \tfrac{1}{3}J(x, a, b).$$

As $J(x, a, b) = x \times (a \times b) + a \times (b \times x) + b \times (x \times a) = -S_{a\times b}x + [S_a, S_b]x$, then we obtain

$$S_{S_ab}x = (ba^T - ab^T + \tfrac{1}{3}S_{S_ab} - \tfrac{1}{3}[S_a, S_b])x$$

and (vii) follows. ■

4. Eigenvalues of S_a and $M_{a,b}$

In this section and in the following ones, we consider real skew-symmetric matrices of order 8 written as bordered matrices in the partitioned form

$$M_{a,b} = \begin{bmatrix} S_a & b \\ -b^T & 0 \end{bmatrix},$$

with $b = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \end{bmatrix}^T$, $a = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T \in \mathbb{R}^{7\times 1}$. These 8×8 matrices constitute a generalization of the 4×4 matrices in [16].

Theorem 4.1. *The determinant of $M_{a,b}$ is*

$$\det(M_{a,b}) = (a^Ta)^2(a^Tb)^2.$$

Proof: Using the Mathematica implementation in [6], the Pfaffian of $M_{a,b}$ is

$$\mathrm{pf}(M_{a,b}) = -(a^Ta)(a^Tb)$$

and the stated result follows. ■

Before proceeding to the problem of determining the eigenvalues of S_a and $M_{a,b}$, we recall a result related to block determinants.

Proposition 4.2. [13] *Let $E \in \mathbb{R}^{r\times r}$, $F \in \mathbb{R}^{r\times s}$, $G \in \mathbb{R}^{s\times r}$ and $H \in \mathbb{R}^{s\times s}$.*

$$\det\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{cases} \det(E)\det(H - GE^{-1}F) & \text{when } E^{-1} \text{ exists} \\ \det(H)\det(E - FH^{-1}G) & \text{when } H^{-1} \text{ exists} \end{cases},$$

where $H - GE^{-1}F$ and $E - FH^{-1}G$ are the Schur complements of E and H, respectively.

Theorem 4.3. *The characteristic polynomial of $M_{a,b}$ is*

$$p_{M_{a,b}}(\lambda) = (\lambda^2 + a^T a)^2(\lambda^4 + \lambda^2(a^T a + b^T b) + (a^T b)^2).$$

Proof: The characteristic polynomial of $M_{a,b}$ is given by

$$p_{M_{a,b}}(\lambda) = \det(M_{a,b} - \lambda I_8) = \det \begin{bmatrix} S_a - \lambda I_7 & b \\ -b^T & -\lambda \end{bmatrix}.$$

If $\lambda = 0$ then $p_{M_{a,b}}(0) = \det(M_{a,b}) = (a^T a)^2(a^T b)^2$. Assume that $\lambda \neq 0$. Then $S_a - \lambda I_7$ is invertible. Since the adjugate and, using Mathematica, the determinant of this matrix are, respectively,

$$(\lambda^2 + a^T a)^2(\lambda(S_a + \lambda I_7) + aa^T) \text{ and } -\lambda(\lambda^2 + a^T a)^3,$$

then

$$(S_a - \lambda I_7)^{-1} = -\frac{1}{\lambda^2 + a^T a}\left(S_a + \lambda I_7 + \frac{1}{\lambda} aa^T\right).$$

By Proposition 4.2,

$$\det(M_{a,b} - \lambda I_8) = \det(S_a - \lambda I_7)(-\lambda + b^T(S_a - \lambda I_7)^{-1}b).$$

As $b^T aa^T b = (a^T b)^2$ and, by (ii) of Proposition 3.1, $b^T S_a b = 0$, we arrive at $(\lambda^2 + a^T a)^2(\lambda^4 + \lambda^2(a^T a + b^T b) + (a^T b)^2)$. ■

Corollary 4.4. *The eigenvalues of S_a are 0 and $\pm||a||i$.*

Proof: A consequence of the proof of Theorem 4.3 since the characteristic polynomial of S_a is $-\lambda(\lambda^2 + a^T a)^3$. ■

Corollary 4.5. *The eigenvalues of $M_{a,b}$ are the purely imaginary numbers*

$$\pm||a||\, i \text{ and } \pm\sqrt{\tfrac{1}{2}(||a||^2 + ||b||^2 \pm ||a-b||||a+b||)}\, i.$$

Proof: From Theorem 4.3, putting $\lambda^2 = x$ in $p_{M_{a,b}}(\lambda)$, we obtain

$$(x + a^T a)^2(x^2 + (a^T a + b^T b)x + (a^T b)^2) = 0.$$

Thus,

$$x = -a^T a \text{ or } x_{1,2} = -\frac{a^T a + b^T b}{2} \pm \frac{\sqrt{(a^T a + b^T b)^2 - 4(a^T b)^2}}{2},$$

We have $x_1 + x_2 = -(a^T a + b^T b)$ and $x_1 x_2 = (a^T b)^2$. So, invoking Girard-Newton-Viète laws, $x_1 \leq 0$ and $x_2 \leq 0$. Finally, a straightforward computation leads to the result since

$$x = -a^T a \text{ or } x_{1,2} = -\frac{a^T a + b^T b}{2} \pm \frac{\sqrt{(a-b)^T(a-b)(a+b)^T(a+b)}}{2}. \qquad \blacksquare$$

Remark 4.6. *Assume that a and b are orthogonal vectors. So, $||a||^2 + ||b||^2 = ||a+b||^2$. By Corollary 4.5, the eigenvalues of $M_{a,b}$ are $\pm ||a||\ i, 0$ and $\pm ||a+b||\ i$. Invoking Gerschgorin's Theorem in* [13], *we obtain*

$$||a+b|| \leq \max\{r_i : i \in \{1, \ldots, 8\}\},$$

$$\textit{where } r_t = \sum_{\substack{s=1 \\ s \neq t}}^{7} |a_s| + |b_t| \textit{ for } t \in \{1, \ldots, 7\},\ r_8 = \sum_{k=1}^{7} |b_k|.$$

Taking $a = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}^T$ *and* $b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$, *we see that this upper bound can be sharper than $||a|| + ||b||$, the one given by the triangle inequality. Concretely, we get* $\max\{3, 4\} < 2 + \sqrt{7}$.

5. Inverses of $M_{a,b}$

The Moore-Penrose inverse of S_a was characterized in section 3. Depending on the determinant of $M_{a,b}$, either the inverse or the Moore-Penrose inverse of $M_{a,b}$ may be determined. For this purpose, we recall the following result where *, $R(A)$ and $N(A)$ stand for the conjugate transpose of a matrix, the column space of A and the nullspace of A, respectively.

Theorem 5.1. [12] *Let T denote the complex bordered matrix*

$$\begin{bmatrix} A & c \\ d^* & \alpha \end{bmatrix}$$

where A is $m \times m$, c and d are columns, and α is a scalar. Let $k = A^\dagger c$, $h^* = d^* A^\dagger$, $u = (I - AA^\dagger)c$, $v = (I - A^\dagger A)d$, $w_1 = 1 + k^* k$, $w_2 = 1 + h^* h$ and $\beta = \alpha - d^* A^\dagger c$. Then

(i) *$rank(T) = rank(A) + 2$ if and only if $c \notin R(A)$ and $d \notin R(A^*)$,*

(ii) *$rank(T) = rank(A)$ if and only if $c \in R(A), d \in R(A^*)$ and $\beta = 0$.*

The Moore-Penrose inverse of T is as follows.

(i) *When $rank(T) = rank(A) + 2$,*

$$T^\dagger = \begin{bmatrix} A^\dagger - ku^\dagger - v^{*\dagger} h^* - \beta v^{*\dagger} u^\dagger & v^{*\dagger} \\ u^\dagger & 0 \end{bmatrix}.$$

(ii) *When* $rank(T) = rank(A)$,

$$T^\dagger = \begin{bmatrix} A^\dagger - w_1^{-1}kk^*A^\dagger - w_2^{-1}A^\dagger hh^* & w_2^{-1}A^\dagger h \\ w_1^{-1}k^*A^\dagger & 0 \end{bmatrix} + \frac{k^*A^\dagger h}{w_1 w_2}\begin{bmatrix} k \\ -1 \end{bmatrix}\begin{bmatrix} h^* & -1 \end{bmatrix}.$$

Proposition 5.2. *Consider the matrix*

$$M_{a,b} = \begin{bmatrix} S_a & b \\ -b^T & 0 \end{bmatrix}.$$

Following the notation in Theorem 5.1, let:

$$\begin{array}{llll} k = S_a^\dagger b, & h = -(S_a^\dagger)^T b, & w_1 = 1 + k^T k, & w_2 = 1 + h^T h, \\ \alpha = 0, & \beta = b^T S_a^\dagger b, & u = (I_7 - S_a S_a^\dagger)b, & v = -(I_7 - S_a^\dagger S_a)b. \end{array}$$

Then $\beta = 0$ *and the subsequent equalities hold:*

$$k = h = \begin{cases} 0 & \text{if } a = 0 \\ \frac{1}{a^Ta}S_a^T b & \text{if } a \neq 0 \end{cases},$$

$$w_1 = w_2 = \begin{cases} 1 & \text{if } a = 0 \\ 1 + \frac{(a^Ta)(b^Tb) - (a^Tb)^2}{(a^Ta)^2} & \text{if } a \neq 0 \end{cases}, \quad u = -v = \begin{cases} b & \text{if } a = 0 \\ \frac{a^Tb}{a^Ta}a & \text{if } a \neq 0 \end{cases}.$$

Proof: The equalities hold trivially when $a = 0$. So, assume that $a \neq 0$. By the properties of S_a in Proposition 3.1, we have

$$k = S_a^\dagger b = -\tfrac{1}{a^Ta}S_a b = \tfrac{1}{a^Ta}S_a^T b,$$

$$h = (S_a^\dagger)^T(-b) = -\tfrac{1}{a^Ta}S_a^T(-b) = k,$$

$$\begin{aligned} w_1 &= 1 + k^T k \\ &= 1 - \tfrac{1}{(a^Ta)^2} b^T S_a^2 b \\ &= 1 + \tfrac{b^T(a^Ta)b - (a^Tb)^T(a^Tb)}{(a^Ta)^2} \\ &= 1 + \tfrac{(a^Ta)(b^Tb) - (a^Tb)^2}{(a^Ta)^2}, \end{aligned}$$

$$w_2 = 1 + h^T h = 1 + k^T k = w_1,$$

$$\beta = b^T S_a^\dagger b = -\tfrac{1}{a^Ta}b^T S_a b = \tfrac{1}{a^Ta}b^T S_b a = -\tfrac{1}{a^Ta}(S_b b)^T a = 0,$$

$$u = (I_7 - S_a S_a^\dagger)b = (I_7 + \tfrac{1}{a^Ta}S_a^2)b = (I_7 + \tfrac{1}{a^Ta}(aa^T - a^TaI_7))b = \tfrac{a^Tb}{a^Ta}a,$$

$$v = -(I_7 - S_a^\dagger S_a)b = -(I_7 - S_a S_a^\dagger)b = -u. \qquad \blacksquare$$

Theorem 5.3. *Consider the matrix* $M_{a,b} = \begin{bmatrix} S_a & b \\ -b^T & 0 \end{bmatrix}$.

(1) *If* $a = 0$ *and* $b \neq 0$ *then*

$$M_{a,b}^{\dagger} = -\frac{1}{b^T b}\begin{bmatrix} 0_7 & b \\ -b^T & 0 \end{bmatrix}.$$

(2) *If* $a = b = 0$ *then* $M_{a,b}^{\dagger} = 0_8$.

(3) *If* $a \neq 0$ *and* $a^T b \neq 0$ *then*

$$M_{a,b}^{-1} = -\frac{1}{a^T b}\begin{bmatrix} \frac{a^T b}{3a^T a} S_a + \frac{2}{3} S_b - \frac{1}{3a^T a}[S_a, S_{a\times b}] & a \\ -a^T & 0 \end{bmatrix}.$$

(4) *If* $a \neq 0$ *and* $a^T b = 0$ *then*

$$M_{a,b}^{\dagger} = -\frac{1}{a^T a + b^T b}\begin{bmatrix} (1 + \frac{b^T b}{3a^T a}) S_a - \frac{1}{3a^T a}[S_b, S_{a\times b}] & b \\ -b^T & 0 \end{bmatrix}$$

and a generalized inverse of $M_{a,b}$ *is*

$$M_{a,b}^{-} = \begin{bmatrix} S_a^{\dagger} & a \\ -a^T & 0 \end{bmatrix}.$$

Proof: Suppose now that $a = 0$ and $b \neq 0$. Then $rank(S_a) = 0$ and $rank(M_{a,b}) = 2$. So, the case 1. is a consequence of (i) in Theorem 5.1 and Proposition 5.2.

The case 2. is straightforward.

As far as 3., assume that $a \neq 0$ and $a^T b \neq 0$. Hence, by Proposition 5.2 and Theorem 4.1, $u \neq 0$ and $\det(M_{a,b}) \neq 0$. Consequently, b does not belong to the column space of S_a and, so, $-b$ does not belong to the column space of S_a^T. By Theorem 5.1, we have $rank(M_{a,b}) = rank(S_a) + 2$. Thus, $rank(S_a) = 6$. Also by the cited theorem,

$$M_{a,b}^{-1} = \begin{bmatrix} S_a^{\dagger} - ku^{\dagger} - (v^T)^{\dagger} h^T & (v^T)^{\dagger} \\ u^{\dagger} & 0 \end{bmatrix}.$$

Invoking Proposition 5.2, we conclude that:

$u^{\dagger} = \dfrac{u^T}{u^T u} = \frac{1}{a^T b} a^T,$

$ku^{\dagger} = \frac{1}{a^T a} S_a^T b \frac{1}{a^T b} a^T = -\frac{1}{(a^T a)(a^T b)} S_a b a^T,$

$(v^T)^{\dagger} = (-u^T)^{\dagger} = -(u^{\dagger})^T = -\frac{1}{a^T b} a,$

$(v^T)^\dagger h^T = -\frac{1}{a^Tb} a \frac{1}{a^Ta} b^T S_a = -\frac{1}{(a^Ta)(a^Tb)} ab^T S_a.$

From these equalities, we arrive at

$$M_{a,b}^{-1} = -\frac{1}{a^Tb}\begin{bmatrix} -a^TbS_a^\dagger - \frac{1}{a^Ta}(S_aba^T + ab^TS_a) & a \\ -a^T & 0 \end{bmatrix}.$$

Applying the properties of S_a in Proposition 3.1, we obtain

$$\begin{aligned} S_aba^T + ab^TS_a &= S_aba^T - ab^TS_a^T \\ &= \tfrac{2}{3}S_{S_aS_ab} + \tfrac{1}{3}[S_a, S_{S_ab}] \\ &= \tfrac{2}{3}S_{aa^Tb - a^Tab} + \tfrac{1}{3}[S_a, S_{a\times b}] \\ &= \tfrac{2}{3}(a^TbS_a - a^TaS_b) + \tfrac{1}{3}[S_a, S_{a\times b}]. \end{aligned}$$

Therefore, $(a^Tb)S_a^\dagger + \frac{1}{a^Ta}(S_aba^T + ab^TS_a) = -\frac{a^Tb}{3a^Ta}S_a - \frac{2}{3}S_b + \frac{1}{3a^Ta}[S_a, S_{a\times b}]$ and 3. follows.

In order to prove 4., suppose now that $a \neq 0$ and $a^Tb = 0$. Thus, $\det(M_{a,b}) = 0$. By Proposition 5.2, we have

$k = h = -\frac{1}{a^Ta}S_ab,$

$w_1 = w_2 = 1 + \frac{b^Tb}{a^Ta},$

$u = v = 0.$

Moreover, $b \in N(a^T)$ and $N(a^T) = R(S_a)$ since $R(S_a) = (N(S_a^T))^\perp = \langle a\rangle^\perp$. Consequently, $rank(M_{a,b}) = rank(S_a)$ and, by Theorem 5.1, we get

$$M_{a,b}^\dagger = \begin{bmatrix} S_a^\dagger - w_1^{-1}kk^TS_a^\dagger - w_2^{-1}S_a^\dagger hh^T & w_2^{-1}S_a^\dagger h \\ w_1^{-1}k^TS_a^\dagger & 0 \end{bmatrix} + \frac{k^TS_a^\dagger h}{w_1w_2}\begin{bmatrix} k \\ -1 \end{bmatrix}\begin{bmatrix} h^T & -1 \end{bmatrix}.$$

Taking into account the properties in Proposition 3.1 and in Proposition 5.2, we have

$$\frac{k^TS_a^\dagger h}{w_1w_2} = \frac{h^TS_a^\dagger h}{w_1^2} = -\frac{b^TS_aS_a^\dagger S_ab}{(a^Ta + b^Tb)^2} = -\frac{b^TS_ab}{(a^Ta + b^Tb)^2} = 0,$$

$$k^TS_a^\dagger = h^TS_a^\dagger = -\frac{1}{(a^Ta)^2}b^TS_a^2 = \frac{-1}{(a^Ta)^2}b^T(aa^T - a^TaI_7) = \frac{1}{a^Ta}b^T,$$

$$S_a^\dagger h = S_a^\dagger k = (k^T(S_a^\dagger)^T)^T = -(k^TS_a^\dagger)^T = -\frac{1}{a^Ta}b,$$

$$-kk^TS_a^\dagger - S_a^\dagger hh^T = \frac{1}{a^Ta}(-kb^T + bk^T) = \frac{1}{(a^Ta)^2}(S_abb^T + bb^TS_a).$$

We also obtain

$$\begin{aligned} S_abb^T + bb^TS_a &= S_abb^T - bb^TS_a^T \\ &= \tfrac{2}{3}S_{S_bS_ab} + \tfrac{1}{3}[S_b, S_{S_ab}] \end{aligned}$$

$$= \frac{2}{3}S_{-S_bS_ba} + \frac{1}{3}[S_b, S_{a\times b}]$$
$$= \frac{2}{3}S_{b^Tba - bb^Ta} + \frac{1}{3}[S_b, S_{a\times b}]$$
$$= \frac{2}{3}b^TbS_a + \frac{1}{3}[S_b, S_{a\times b}],$$

$$S_a^\dagger + \frac{1}{(a^Ta)^2w_1}(S_abb^T + bb^TS_a) = \frac{-1}{a^Ta+b^Tb}\left(\frac{a^Ta+b^Tb}{a^Ta}S_a - \frac{1}{a^Ta}(S_abb^T + bb^TS_a)\right)$$
$$= \frac{-1}{a^Ta+b^Tb}\left(S_a + \frac{b^Tb}{a^Ta}S_a - \frac{2b^Tb}{3a^Ta}S_a - \frac{1}{3a^Ta}[S_b, S_{a\times b}]\right)$$
$$= -\frac{1}{a^Ta+b^Tb}\left((1 + \frac{b^Tb}{3a^Ta})S_a - \frac{1}{3a^Ta}[S_b, S_{a\times b}]\right),$$

$$w_2^{-1}S_a^\dagger h = -\frac{1}{w_1a^Ta}b = -\frac{1}{a^Ta+b^Tb}b,$$
$$w_1^{-1}k^TS_a^\dagger = \frac{1}{w_1a^Ta}b^T = \frac{1}{a^Ta+b^Tb}b^T.$$

Hence, the first part of 4. follows. To finish the proof, observe that

$$\begin{bmatrix} S_a & b \\ -b^T & 0 \end{bmatrix}\begin{bmatrix} S_a^\dagger & a \\ -a^T & 0 \end{bmatrix}\begin{bmatrix} S_a & b \\ -b^T & 0 \end{bmatrix} = \begin{bmatrix} S_aS_a^\dagger S_a - S_aab^T & S_aS_a^\dagger b \\ -b^TS_a^\dagger S_a & -b^TS_a^\dagger b \end{bmatrix} = M_{a,b}$$

since

$$S_aS_a^\dagger S_a - S_aab^T = S_a,$$
$$S_aS_a^\dagger b = \frac{-1}{a^Ta}S_a^2b = \frac{-1}{a^Ta}(aa^Tb - a^Tab) = b,$$
$$-b^TS_a^\dagger S_a = \frac{1}{a^Ta}b^TS_a^2 = \frac{1}{a^Ta}(b^Taa^T - b^Ta^Ta) = -b^T,$$
$$-b^TS_a^\dagger b = \frac{1}{a^Ta}b^TS_ab = -\frac{1}{a^Ta}b^TS_ba = \frac{1}{a^Ta}(S_bb)^Ta = 0.$$

■

6. Rotation matrices from S_a and $M_{a,b}$

Possible representations for rotation operators are the ones in the form of rotation matrices. In particular, the Cayley transform and the exponential of a skew-symmetric matrix may be considered.

Let us begin with the Cayley transform of S_a and with the Cayley transform of $M_{a,b}$, writing the latter in terms of the former one. With this purpose in mind, we first recall the following result.

Proposition 6.1. [13] *Let $E \in \mathbb{R}^{r\times r}$, $F \in \mathbb{R}^{r\times s}$, $G \in \mathbb{R}^{s\times r}$ and $H \in \mathbb{R}^{s\times s}$. If E and $J = H - GE^{-1}F$, the Schur complement of E, are invertible, then*

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} + E^{-1}FJ^{-1}GE^{-1} & -E^{-1}FJ^{-1} \\ -J^{-1}GE^{-1} & J^{-1} \end{bmatrix}.$$

Theorem 6.2. *The Cayley transform of $M_{a,b}$ is the rotation matrix*

$$\mathcal{C}(M_{a,b}) = \begin{bmatrix} \mathcal{C}(S_a) - \dfrac{2}{s} S^{-1} b b^T S^{-1} & -\dfrac{2}{s} S^{-1} b \\ \\ \dfrac{2}{s} b^T S^{-1} & \dfrac{2}{s} - 1 \end{bmatrix},$$

where S stands for $S_a + I_7$, s is the Schur complement of S in $I_8 + M_{a,b}$ and $\mathcal{C}(S_a)$ is the Cayley transform of S_a given by the rotation matrix

$$\mathcal{C}(S_a) = \frac{1}{1 + a^T a}(-2S_a + 2aa^T + (1 - a^T a) I_7).$$

Proof: Let us denote $S_a + I_7$ by S. Invoking the proof of Theorem 4.3, we have

$$S^{-1} = -\frac{1}{1 + a^T a}(S_a - I_7 - aa^T).$$

As $C(S_a) = 2S^{-1} - I_7$, then the stated formula for $\mathcal{C}(S_a)$ follows. Furthermore, the Schur complement $1 + b^T S^{-1} b$ of S in $I_8 + M_{a,b}$ is equal to

$$s = \frac{1 + a^T a + b^T b + (a^T b)^2}{1 + a^T a}.$$

and, so, is invertible. By Proposition 6.1, we obtain

$$(I_8 + M_{a,b})^{-1} = \frac{1}{s} \begin{bmatrix} sS^{-1} - S^{-1} b b^T S^{-1} & -S^{-1} b \\ b^T S^{-1} & 1 \end{bmatrix}.$$

Taking into account that $\frac{2}{s}(sS^{-1} - S^{-1} b b^T S^{-1}) - I_7 = \mathcal{C}(S_a) - \frac{2}{s} S^{-1} b b^T S^{-1}$ and that $\mathcal{C}(M_{a,b}) = 2(I_8 + M_{a,b})^{-1} - I_8$, we arrive at the stated matrix for $\mathcal{C}(M_{a,b})$. ∎

An explicit expression for computing the exponential of an order 3 skew-symmetric matrix B is given by the Rodrigues' formula, a consequence of $B^3 = -\alpha^2 B$ for a certain scalar α. Although this does not hold in general for an order $n \geq 4$, hypercomplex matrices are an exception, [9]. Moreover, a generalization of the Rodrigues' formula that allows to compute the exponential of a skew-symmetric matrix of order $n \geq 3$ was proposed in [4].

Theorem 6.3. [9] *Let $\underline{a} = a_0 + a \in \mathbb{O}$ with $||a|| = \alpha \neq 0$. Then*

$$\mathrm{e}^{tS_a} = I\cos(\alpha t) + S_a \frac{\sin(\alpha t)}{\alpha} + \frac{1 - \cos(\alpha t)}{\alpha^2} aa^T.$$

Theorem 6.4. [4] *Given any non-null skew-symmetric $n \times n$ matrix B, where $n \geq 3$, if $\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$ is the set of distinct eigenvalues of B, where $\theta_j > 0$ and each $i\theta_j$ (and $-i\theta_j$) has multiplicity $k_j \geq 1$, there are p unique skew-symmetric matrices $B_1, \ldots, B_p$ such that*

$$B = \theta_1 B_1 + \ldots + \theta_p B_p, \quad B_i B_j = B_j B_i = 0_n (i \neq j), \quad B_i^3 = -B_i$$

for all i, j with $1 \leq i, j \leq p$, and $2p \leq n$. Furthermore,

$$\mathrm{e}^B = \mathrm{e}^{\theta_1 B_1 + \ldots \theta_p B_p} = I_n + \sum_{i=1}^{p} (\sin\theta_i B_i + (1 - \cos\theta_i) B_i^2).$$

Theorem 6.5. *Let $a, b \in \mathbb{R}^7$ such that $a \neq 0_{7\times 1}$. The exponentials of S_a and of $M_{a,b}$ are, respectively, the rotation matrices*

$$\mathrm{e}^{S_a} = I_7 + \frac{\sin||a||}{||a||} S_a + \frac{1 - \cos||a||}{||a||^2} S_a^2$$

and

$$\mathrm{e}^{M_{a,b}} = I_8 + \sum_{k=1}^{p} (\sin\theta_k M_{a,b,k} + (1 - \cos\theta_k) M_{a,b,k}^2),$$

where $p = \begin{cases} 2 & \text{if } a^T b = 0 \\ 3 & \text{if } a^T b \neq 0 \end{cases}$,

$$\{\theta_j : 1 \leq j \leq p\} = \begin{cases} \{||a||, ||a+b||\} & \text{if } p = 2 \\ \left\{||a||, \sqrt{\frac{1}{2}(||a||^2 + ||b||^2 \pm ||a-b|| \, ||a+b||)}\right\} & \text{if } p = 3 \end{cases}$$

and the p unique skew-symmetric matrices $M_{a,b,k}$ can be obtained through the solution of a $28p \times 28p$ linear equations system deduced from

$$M_{a,b} = \sum_{k=1}^{p} \theta_k M_{a,b,k}, \qquad M_{a,b}^3 = -\sum_{k=1}^{p} \theta_k^3 M_{a,b,k}, \ \ldots \ ,$$

$$M_{a,b}^{2p-1} = (-1)^{p-1} \sum_{k=1}^{p} \theta_k^{2p-1} M_{a,b,k}.$$

Proof: Let $a, b \in \mathbb{R}^7$ such that $a \neq 0_{7\times 1}$.

From (iv) in Proposition 3.1, we have $aa^T = S_a^2 + ||a||^2 I_7$. Hence, by Theorem 6.3, we obtain the stated Rodrigues-like formula for the exponential of S_a.

By Theorem 6.4, we obtain the stated formulas for the exponential of $M_{a,b}$ and its odd powers, where $\{\pm\theta_j i : \theta_j > 0, 1 \leq j \leq p\}$ is the set of distinct non-null eigenvalues of $M_{a,b}$. From Theorem 4.1, we have $\det(M_{a,b}) = (a^Ta)^2(a^Tb)^2$. If $a^Tb = 0$ then $M_{a,b}$ has, at least, an eigenvalue equal to 0 and $b \neq -a$. By Corollary 4.5, we obtain $\theta_1 = ||a||$ and $\theta_2 = ||a+b||$. Hence, $p = 2$. If $a^Tb \neq 0$ then all eigenvalues of $M_{a,b}$ are different from 0. Thus, $p = 3$. Concretely, once again by Corollary 4.5, we get

$$\theta_1 = ||a||, \quad \theta_2 = \sqrt{\frac{1}{2}(||a||^2 + ||b||^2 - ||a-b||||a+b||)},$$
$$\theta_3 = \sqrt{\frac{1}{2}(||a||^2 + ||b||^2 + ||a-b||||a+b||)}. \quad \blacksquare$$

The generalization in [4] is theoretically interesting, however, according to [2], its computational cost seems prohibitive unless n is small. See [2] for details on effective methods for performing the computation of the exponential of a skew-symmetric matrix.

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