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NONCROSSING PARTITIONS, NONCROSSING GRAPHS AND q-PERMANENTAL FORMULAS

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ABSTRACT: First we characterize a noncrossing permutation, in terms of the lengths (number of inversions) of its cycles. Then we use derivative formulas for the q-permanent, a polynomial that interpolates the determinant and the permanent of a matrix, to characterize several structures of noncrossing kind, for example: the digraphs with noncrossing permutation subdigraphs, the noncrossing graphs [with eventual restrictions on cycles and edges], noncrossing forests. We use the derivative formulas to prove two particular cases of a conjecture on the q-monotonicity of the q-permanent of a Hermitian positive definite matrix A.

KEYWORDS: graphs, permutations, permanent, polynomial identities.
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1. Introduction

Since the pioneering work of G. Kreweras [15], research on noncrossing partitions and other noncrossing structures had a remarkable development, well documented in the excellent survey [21] by R. Simion, where it is shown the ubiquitous character of the concepts involved. Besides those referenced in [21], we refer, *e.g.*, [11, 19, 20] on the enumerative side of the problem, and [1, 4] which are closer to our work. All this is well known in combinatorics; not so noted tough is the other protagonist of this paper, the *q-permanent* of an n-square matrix $A = (a_{ij})$ given by

$$\operatorname{per}_q A = \textstyle\sum_{\sigma\in\mathscr{S}_n} q^{\ell(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}, \tag{1}$$

where $\mathscr{S}_n$ is the symmetric group of order n, and $\ell(\sigma)$ is the number of inversions of the permutation σ, here called *length* of σ according to the tradition in the theory of Coxeter groups [5, 13]. The permanent and the determinant are specializations of the q-permanent, which, by way of the quantum parameter q, interpolates the former two well-known functions. Contrarily to

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determinant and permanent, the q-permanent does not show up so frequently in combinatorics. Nevertheless, when A's entries are independent variables, $\operatorname{per}_q A$ may be seen as a table of the length function in $\mathscr{S}_n$; moreover, $\operatorname{per}_q J$, where J is the matrix with all entries 1, is the generating function enumerating the permutations with respect to their length [6, 22]. The q-permanent (also known as q-determinant and μ-permanent) appeared around the 1990's in the area of quantum groups and quantum algebras [18, 23, 26]. In [24, 25] the reader will find a generalization to multiple quantum parameters. In relation to a problem of mathematical physics, it has been proved by M. Bozejko and R. Speicher [7, 8], that $\sigma \rightsquigarrow q^{\ell(\sigma)}$ is a positive definite function on $\mathscr{S}_n$, for $q \in]-1,1[$; it follows that $\operatorname{per}_q A > 0$ if A is positive definite and $q \in [-1,1]$; this lead R. Bapat [2] to an interesting conjecture, to be considered in section 6, asserting that the $\operatorname{per}_q A$ is strictly increasing for $q \in [-1,1]$, if A is positive definite and non-diagonal. For other interesting results on the q-permanent we refer [16, 3, 17]. Here, we focus on formulas of the kind

$$\tfrac{\partial}{\partial q} \operatorname{per}_q A = \textstyle\sum_c \ell(c)\ q^{\ell(c)-1} \prod_i a_{c_i c_{i+1}}\ \operatorname{per}_q A(c), \tag{2}$$

inspired by a theorem of A. Lal [16], whose terms will be explained later; the sum is extended to cycles c, and $A(c)$ denotes the principal submatrix obtained by elimination of the rows and columns of A corresponding to c. Such formulas were designed to eventually produce a positive q-derivative of $\operatorname{per}_q A$ whenever A is positive definite. Equation (2) does not hold for all matrices, so it is tempting to characterize the graphs G such that it holds for any matrix A with graph G. In this sense (2) characterizes the noncrossing graphs with no edge-under-edge (an expression to be explained in section 4). In section 5 we alter $A(c)$ in a natural manner, and show that the modified (2) characterizes the family of noncrossing graphs. Moreover, if the right hand side of (2) is truncated to sum only over transpositions, we enter the realm of forests, and characterize those that are noncrossing (and with no edge-under-edge).

2. Basic concepts and notations

For the basics on graphs, digraphs and matrices we follow mainly the traditions as can be seen in [9, 12], for example. Here, graphs and digraphs have vertex set $[n] = \{1, \ldots, n\}$, so they are labeled, and labeling is crucial. By an *interval*, with notations as $[r, s]$, $]r, s[$, etc., we mean an integer interval.

Two subsets P, Q of $[n]$ are said to be *crossing sets* if there exist $i, j \in P$ and $r, s \in Q$, satisfying $i < r < j < s$ or $r < i < s < j$; otherwise we say they are *noncrossing* sets. We say that G is a *noncrossing graph* if no pair of distinct edges cross. A partition of $[n]$ is said to be a *noncrossing partition* if no two distinct parts of it cross. $\sigma \in \mathscr{S}_n$ is said to be a *noncrossing permutation* if the orbits of σ form a noncrossing partition of $[n]$.

All our matrices are square over a field $\mathbb{F}$ of characteristic zero. The *digraph of A*, denoted $D(A)$, has (i, j) as arc iff $a_{ij} \neq 0$. The *graph of A*, denoted $G(A)$, has $\{i, j\}$ as edge iff $i \neq j$ and either $a_{ij} \neq 0$ or $a_{ji} \neq 0$. Note that $G(A)$ has no loops, so it tells nothing about the diagonal entries of A. The *underlying digraph* of a graph G is the digraph D_G of which (i, j) is an arc iff $i = j$ or $\{i, j\}$ is an edge of G. So an underlying digraph is a symmetric digraph with loops at all vertices.

A matrix A is said to be *generic* if its nonzero entries are independent variables over the base field. A is said to be a *generic matrix with graph G* if A is generic and $a_{ij} \neq 0$ iff: $i = j$ or $\{i, j\}$ is an edge of G. When we say that a given system of algebraic equations holds for every matrix (over $\mathbb{F}$) with a given digraph D, this is equivalent to saying that the generic matrix with digraph D satisfies the given system over the ring $\mathbb{F}[q, \{a_{ij}\}]$. This principle holds for infinite fields, which is the case of $\mathbb{F}$. (See [14, pp. 235 ff] for a proof and refinements for finite fields.)

A *cyclic permutation*, or *cycle* $c \in \mathscr{S}_n$ may be represented in cycle notation, $c = (c_1 c_2 \dots c_k)$, where k is the *order of* c, the c_i are distinct vertices, and $c(c_i) = c_{i+1}$, with i read modulo k. We denote by $\mathscr{C}_n$ the set of all (oriented) cycles of the complete digraph on $[n]$; the number of vertices of such cycle is called the *order* of the cycle. The cycles of $\mathscr{C}_n$ and those of $\mathscr{S}_n$, of orders > 1 will be naturally identified, and the cyclic notation will be used in both cases. Given a matrix A, the *weight of a cycle c in A* and the *total weight of a permutation σ in A* are defined by

$$\mathrm{wt}_c(A) = \textstyle\prod_{i=1}^{k} a_{c_i c_{i+1}} \quad \text{and} \quad \mathrm{twt}_\sigma(A) = \textstyle\prod_{i=1}^{n} a_{i\sigma_i}.$$

This may be simplified to $\mathrm{wt}_c, \mathrm{twt}_\sigma$. Clearly $\mathrm{twt}_c(A)/\,\mathrm{wt}_c(A)$ is a product of diagonal entries of A, whenever $\mathrm{twt}_c \neq 0$. Define $\mathrm{Mov}(\sigma)$ as the set of indices moved by a permutation σ, that is, the complement of the set of fixed points of σ. Let

$$\sigma = \omega_1 \cdots \omega_r$$

be the factorization of σ into (pairwise) disjoint cycles, called the *cycles of* σ; in such expression 1-*cycles are not considered.* For a cycle ω, the smallest integer interval containing $\mathrm{Mov}(\omega)$ will be denoted by M_ω. With this notation define

$$M_\sigma = \bigcup_{i=1}^{r} M_{\omega_i}.$$

For a cycle c, $A(c)$ denotes the submatrix obtained by eliminating the rows and columns of A with indices in $\mathrm{Mov}(c)$. And we let $A_c^{\checkmark}$ be the $n \times n$ matrix obtained from A by zeroing out all A's rows and columns with indices in $\mathrm{Mov}(c)$, except the diagonal entries $a_{c_i c_i}$ which are replaced by 1's. Note that the digraph of $A_c^{\checkmark}$ is obtained from D by removing all arcs incident on c's vertices, and adding a loop at each vertex of c. An example for $n = 5$:

$$A_{(14)}^{\checkmark} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & a_{25} \\ 0 & a_{32} & a_{33} & 0 & a_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & a_{52} & a_{53} & 0 & a_{55} \end{bmatrix}.$$

3. Noncrossing permutations

In this section we borrow some expressions from the theory of Coxeter groups, *e.g.*, [1, 5, 13]. To my knowledge, [4] is the first to study in detail the noncrossing permutations as related to the absolute order in $\mathscr{S}_n$, as described in [1]; the results of [4, 1] seem to not interfere with this section, because here we are working with the traditional Bruhat order.

Denote by T the set of all transpositions and, for a permutation σ, let $T_L(\sigma) = \{t \in T : \ell(t\sigma) < \ell(\sigma)\}$ and $T_R(\sigma) = \{t \in T : \ell(\sigma t) < \ell(\sigma)\}$. It is well-known that

$$\ell(\sigma\tau) = \ell(\sigma) + \ell(\tau) \;\Leftrightarrow\; T_R(\sigma) \text{ and } T_L(\tau) \text{ are disjoint.} \tag{3}$$

There are other statements equivalent to $\ell(\sigma\tau) = \ell(\sigma) + \ell(\tau)$; one of such statements is $\sigma \leqslant_R \sigma\tau$, where $\leqslant_R$ denotes the weak right Bruhat order in $\mathscr{S}_n$; another one is $T_L(\sigma) \subseteq T_L(\sigma\tau)$ [5, pp. 23-ff], but we choose to work with (3). We let L_σ $[R_\sigma]$ be the set of all j, such that $(ji) \in T_L(\sigma)$ [resp., $(ji) \in T_R(\sigma)$] for some i.

Lemma 3.1. (a) *For $\sigma \in \mathscr{S}_n$ consider the indices u, w, such that $u < w$ and $\sigma(u) > \sigma(w)$. Then $T_R(\sigma)$ is the set of the corresponding transpositions (uw), and $T_L(\sigma)$ is the set of the transpositions $(\sigma(u)\ \sigma(w))$.*

(*b*) $L_\sigma = R_\sigma = M_\sigma$.

Proof: (*a*) Let $\overline{\sigma} := \sigma(uw)$. In complete notation, write $\sigma = \sigma_1\sigma_2\dots\sigma_n$, and $\overline{\sigma} = \overline{\sigma}_1\overline{\sigma}_2\dots\overline{\sigma}_n$. We show that $\ell(\sigma) > \ell(\overline{\sigma})$ by a simple inversions counting. We have $\overline{\sigma}_u = \sigma_w$, $\overline{\sigma}_w = \sigma_u$, and $\overline{\sigma}_i = \sigma_i$ otherwise.

Define $\ell_k(\sigma) := \sharp\{i : i > k, \sigma_i < \sigma_k\}$. Clearly $\ell(\sigma) = \sum_k \ell_k(\sigma)$. Omitting some details we have

$$
\begin{aligned}
&\ell_k(\sigma) = \ell_k(\overline{\sigma}), \text{ for } k \notin [u,w];\\
&\ell_u(\sigma) = \ell_u(\overline{\sigma}) + \sharp\{i \in]u,w[: \sigma_i \in]\sigma_w,\sigma_u[\} + \sharp\{i > w : \sigma_i \in]\sigma_w,\sigma_u[\} + 1;\\
&\ell_w(\sigma) = \ell_w(\overline{\sigma}) - \sharp\{i > w : \sigma_i \in]\sigma_w,\sigma_u[\};\\
&\textstyle\sum_{k\in]u,w[}\big(\ell_k(\sigma) - \ell_k(\overline{\sigma})\big) = \sharp\{k \in]u,w[: \sigma_k \in]\sigma_w,\sigma_u[\}.
\end{aligned}
$$

Therefore $\ell(\sigma) - \ell(\overline{\sigma}) = 2\sharp\{i \in]u,w[: \sigma_i \in]\sigma_w,\sigma_u[\} + 1 > 0$. This proves $(uw) \in T_R(\sigma)$. That no other transpositions lie in $T_R(\sigma)$ is a consequence of $\sharp T_R(\sigma) = \ell(\sigma)$, a well-known fact of Coxeter theory. The claim on $T_L(\sigma)$ follows from $T_L(\sigma) = \sigma T_R(\sigma)\sigma^{-1}$.

(*b*) Clearly σ has an expression $\sigma = s_1s_2\dots s_m$, where each s_i is a Coxeter generator (*i.e.*, a transposition of consecutive indices) which transposes indices of M_σ. As any such expression contains a reduced subword, we may assume that $s_1s_2\dots s_m$ is already reduced. If t is a transposition such that $\ell(\sigma t) < \ell(\sigma)$, then t is a product of some of the s_k's, and therefore t moves two indices of M_σ. See [5, p. 16-17]. This proves $R_\sigma \subseteq M_\sigma$.

To prove the reverse inclusion pick any m in M_σ. Then $m \in M_\omega$ for some cycle ω of σ. Represent σ in complete (one-line) notation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, and let $\mathrm{Mov}(\sigma) = \{\sigma_{i_1},\dots,\sigma_{i_k}\}$, with $i_1 < \cdots < i_k$. In case $m \in \mathrm{Mov}(\sigma)$ there obviously exists $i_r \in \mathrm{Mov}(\sigma)$ such that the pair (m, i_r) is inverted by σ. In case m is fixed by σ, the word $\sigma_1\cdots\sigma_n$ has a $\sigma_{i_r} > m$ on the left of m, otherwise, all $\sigma_1,\dots,\sigma_{m-1}$ would be smaller than m, and so $[m] \cap \mathrm{Mov}(\omega)$ would be a σ-invariant, nonempty, proper subset of $\mathrm{Mov}(\omega)$, in violation of the fact that $\mathrm{Mov}(\omega)$ is an orbit of σ. So in any case we have a transposition $t = (m\ \sigma_{i_r})$, where m and σ_{i_r} are inverted by σ. This implies $t \in T_R(\sigma)$, and hence $m \in R_\sigma$. We just proved that $R_\sigma = M_\sigma$. As $T_L(\sigma) = T_R(\sigma^{-1})$ and $M_\sigma = M_{\sigma^{-1}}$, the lemma follows. ■

Theorem 3.2. *For disjoint permutations σ and τ, $\ell(\sigma\tau) = \ell(\sigma) + \ell(\tau)$ if and only if no orbit of σ crosses an orbit of τ.*

Proof: Assume $\ell(\sigma\tau) < \ell(\sigma) + \ell(\tau)$. By (3), choose $(vw) \in T_R(\sigma) \cap T_L(\tau)$. The indices v, w cannot be both fixed by σ, otherwise (vw) would not lie in $T_R(\sigma)$; therefore, one of v, w, say v, is moved by σ; and the other, w, is moved by τ by a similar reason. Then v is fixed by τ and w is fixed by σ. Assume that $v < w$ (the case $v > w$ is similar). The condition $\ell(\sigma(vw)) < \ell(\sigma)$ means that in the word $\sigma_1 \ldots \sigma_n$, the symbols σ_v, σ_w occur inverted, *i.e.*, $\sigma(v) > \sigma(w) = w$; and $\ell((vw)\tau) < \ell(\tau)$ means that, in $\tau_1 \ldots \tau_n$, the indices v, w occur inverted, that is $\tau^{-1}(w) < \tau^{-1}(v) = v$. So we have $\tau^{-1}(w) < v < w < \sigma(v)$, and therefore the orbit of σ containing v crosses the orbit of τ containing w.

For the converse, assume that an orbit of σ crosses an orbit of τ; so there exists a cycle ω of σ and a cycle ν of τ such that $\mathrm{Mov}(\omega)$ crosses $\mathrm{Mov}(\nu)$. Write ω and ν in cycle notation, $\omega = (\cdots \omega_i\, \omega_{i+1} \cdots)$, and $\nu = (\cdots \nu_j\, \nu_{j+1} \cdots)$, where $\omega_{i+1} = \sigma(\omega_i)$ and $\nu_{j+1} = \tau(\nu_j)$, the indices i and j being read modulo the orders of ω and ν, respectively. By the crossing condition one of the following configurations occurs:

$$\text{(I) } \nu_{j_1} < \omega_{i_1} < \nu_{j_2} < \omega_{i_2} \qquad \text{or} \qquad \text{(II) } \omega_{i_1} < \nu_{j_1} < \omega_{i_2} < \nu_{j_2}.$$

Let $\max_\omega$ and $\min_\omega$ be the maximum and minimum of $\mathrm{Mov}(\omega)$, and define $\max_\nu$ and $\min_\nu$ likewise. As the roles of σ and τ may be interchanged, we shall assume that $\max_\nu < \max_\omega$. Any of (I)-(II) gives rise to a configuration of type (I)

$$\min_\nu < \omega_i < \max_\nu < \max_\omega .$$

Not all elements $\omega_i, \omega_{i+1}, \omega_{i+2}, \ldots$ lie in the interval $[\min_\nu, \max_\nu]$; so let m be the largest integer $\geqslant i$ such that $\omega_i, \omega_{i+1}, \ldots, \omega_m$ all lie in $[\min_\nu, \max_\nu]$; then we have one of the following two cases

$$\omega_{m+1} < \min_\nu < \omega_m < \max_\nu < \max_\omega \tag{4}$$

$$\min_\nu < \omega_m < \max_\nu < \omega_{m+1} \leqslant \max_\omega . \tag{5}$$

In case (4), choose k so that $\nu_k = \max_\nu$, and let j be the largest index $\geqslant k$ such that all $\nu_k, \nu_{k+1}, \ldots, \nu_j$ are greater than ω_m; then we get

$$\omega_{m+1} < \nu_{j+1} < \omega_m < \nu_j. \tag{6}$$

Let t be the transposition $(\nu_{j+1}\ \omega_m)$. On one hand the inequality

$$\big[\sigma(\nu_{j+1}) - \sigma(\omega_m)\big](\nu_{j+1} - \omega_m) = (\nu_{j+1} - \omega_{m+1})(\nu_{j+1} - \omega_m) < 0$$

and lemma 3.1 show that $t \in T_R(\sigma)$. On the other hand, we have

$$\left[\tau^{-1}(\nu_{j+1}) - \tau^{-1}(\omega_m)\right](\nu_{j+1} - \omega_m) = (\nu_j - \omega_m)(\nu_{j+1} - \omega_m) < 0;$$

therefore, from lemma 3.1 we get $t \in T_R(\tau^{-1}) = T_L(\tau)$. In the current case $T_R(\sigma) \cap T_L(\tau) \neq \varnothing$, and (3) implies $\ell(\sigma\tau) < \ell(\sigma) + \ell(\tau)$.

In case (5), choose p so that $\nu_p = \min_\nu$, and let s be the largest index $\geqslant p$ such that all $\nu_p, \nu_{p+1}, \ldots, \nu_s$ are less than ω_m. We obtain the inequalities

$$\nu_s < \omega_m < \nu_{s+1} < \omega_{m+1},$$

which are kind of dual to (6). The same argument used in case (4), now applied to $t = (\omega_m\ \nu_{s+1})$, leads to the same conclusion of that case, namely $\ell(\sigma\tau) < \ell(\sigma) + \ell(\tau)$. ■

Corollary 3.3. *Let $\omega_1, \ldots, \omega_r$ be the disjoint cycles of σ. Then σ is a noncrossing permutation if and only if $\ell(\sigma) = \ell(\omega_1) + \cdots + \ell(\omega_r)$.* ■

4. Submatrices and restricted noncrossing graphs

Given σ factorized into disjoint cycles, say $\sigma = \omega_1 \cdots \omega_r$, and a generic matrix A with digraph D, the condition $a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n} \neq 0$, means that the ω_i are cycles of D, and D has a loop at each vertex fixed by σ; we follow [9, §9.1] in saying that σ determines a *permutation subdigraph of D*. (So a permutation subdigraph is a spanning collection of disjoint cycles, including loops, called *linear sub[di]graph* in [10].) A permutation digraph is said to be *noncrossing* if the permutation itself is noncrossing. Figure 1 depicts

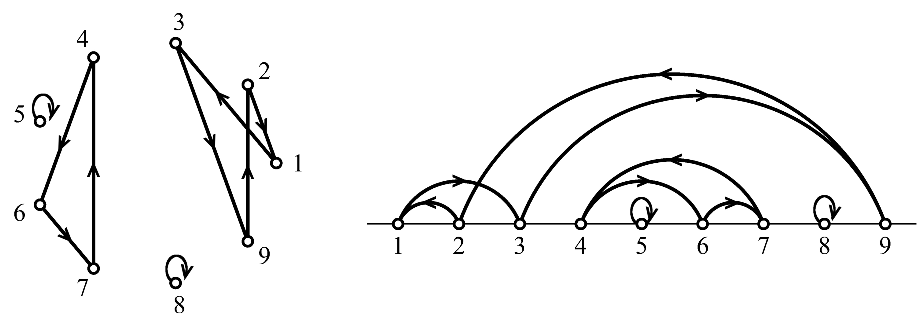

FIGURE 1. The permutation digraph of (1392)(467).

a permutation digraph in the two traditional ways [21], namely, in circular presentation and in linear presentation. The permutation (1392)(467) is noncrossing; the crossings inside each cycle are irrelevant, as for the crossing property only the orbits matter. Besides crossings, we wish to forbid another kind of configuration in a permutation subdigraph. When two disjoint cycles of σ, say ω_1, ω_2, of orders $\geqslant 2$, satisfy the property $M_{\omega_1} \subset M_{\omega_2}$, we say that ω_1 *lies under* ω_2, and that σ has a *cycle-under-cycle* configuration. The term *under* is suggested by the linear representation of figure 1, where the cycle (467) lies under (1392). Note that while the "crossing" property is invariant under cyclic relabeling (to replace i by $i+1$ modulo n), the "lie under" relation depends heavily on the position of label 1.

The disjoint cycles of σ determine the set of intervals $\mathfrak{M}_\sigma = \{M_{\omega_1}, \ldots, M_{\omega_r}\}$ partially ordered by inclusion.

Proposition 4.1. (a) *σ is noncrossing if and only if incomparable elements of $\mathfrak{M}_\sigma$ are disjoint, and $M_{\omega_i} \subset M_{\omega_j}$ implies $M_{\omega_i} \cap \mathrm{Mov}(\omega_j) = \varnothing$.* ($b$) *$\sigma$ is noncrossing with no cycle-under-cycle if and only if all elements of $\mathfrak{M}_\sigma$ are pairwise disjoint.*

Proof: The *only if part* of (a). If two incomparable intervals I, J intersect and have no common extreme points, then obviously $\{\min I, \max I\}$ and $\{\min J, \max J\}$ are crossing sets. On the other hand, if $M_{\omega_i} \subset M_{\omega_j}$ and $v \in M_{\omega_i} \cap \mathrm{Mov}(\omega_j)$, then $\mathrm{Mov}(\omega_i)$ and $\mathrm{Mov}(\omega_j)$ cross, because $\min M_{\omega_i} < v < \max M_{\omega_i} < \max M_{\omega_j}$.

The *if part* of (a). Assume σ is a crossing permutation, and incomparable elements of $\mathfrak{M}_\sigma$ are disjoint. There exist ω_i, ω_j such that $\mathrm{Mov}(\omega_i)$ and $\mathrm{Mov}(\omega_j)$ cross; therefore M_{ω_i} and M_{ω_j} are comparable, say $M_{\omega_i} \subset M_{\omega_j}$. There exists $v \in \mathrm{Mov}(\omega_j)$ that lies in between two elements of $\mathrm{Mov}(\omega_i)$; therefore $v \in M_{\omega_i} \cap \mathrm{Mov}(\omega_j)$, and so this set in nonempty, which finishes this proof part. Item (b) is now trivial. ∎

Lemma 4.2. *Let $\alpha \in \mathscr{S}_n$, and let $W \subseteq [n]$ be a set of points which are individually fixed by α. Let α' be the restriction of α to $[n] \smallsetminus W$. Then $\ell(\alpha) = \ell(\alpha') + 2J_W$, where J_W is the cardinality of $\{(i, w) : w \in W,\ i < w < \alpha(i)\}$.*

Proof: Note that $J_W = \sum_{w \in W} J_{\{w\}}$. In a linear presentation as that of figure 1 (where $J_{\{5\}} = 2$ and $J_{\{8\}} = 1$), $J_{\{w\}}$ is the number of jumps that α shows from the left to the right of w; clearly $J_{\{w\}}$ is also the number of jumps from the right to the left of w. So $2J_{\{w\}}$ is the number of inversions of α involving the vertex w. The lemma follows by simple induction, extracting from $[n]$ the vertices of W one after another. ∎

Theorem 4.3. *D is a digraph in which all permutation subdigraphs are noncrossing and have no cycle-under-cycle, if and only if any matrix A with* *digraph D satisfies*

$$\frac{\partial}{\partial q}\operatorname{per}_q A = \sum_{c\in\mathscr{C}_n} \ell(c)\, q^{\ell(c)-1}\ \mathrm{wt}_c(A)\ \operatorname{per}_q A(c). \tag{7}$$

Proof: The *if part*. Suppose that (7) holds. We may assume that A is a generic matrix with digraph D. We shall get a contradiction from the existence of a permutation α such that $\mathrm{twt}_\alpha \neq 0$ and one of the following conditions holds: (i) α is a crossing permutation, or (ii) α has a cycle-under-cycle. For that purpose, consider the permutation matrix P_α defined by

$$\big[P_\alpha\big]_{ij} = \delta_{\alpha(i)\,j} \quad \text{— so that } \operatorname{per}_q P_\alpha = q^{\ell(\alpha)}. \tag{8}$$

Note that P_α is a *specialization* of A, (*i.e.*, it is the result of replacing some of the *nonzero* entries of A). So we replace A with P_α, and compare the right hand side of (7) with the value of the left hand side which is $\frac{\partial}{\partial q}\operatorname{per}_q P_\alpha = \ell(\alpha)q^{\ell(\alpha)-1}$.

Case (i): *α is crossing.* Let $\alpha = \omega_1\cdots\omega_r$ be the factorization of α into disjoint cycles. The sum in (7) has r relevant terms, one for each cycle ω_i. As $A(\omega_i)$ is a permutation matrix, $\operatorname{per}_q A(\omega_i) = q^{z_i}$ for some integer $z_i \geqslant 0$. Therefore, (7) reads

$$\ell(\alpha)q^{\ell(\alpha)-1} = \textstyle\sum_{i=1}^r \ell(\omega_i)\, q^{\ell(\omega_i)-1+z_i}. \tag{9}$$

But this is impossible since, by corollary 3.3, $\ell(\alpha) < \ell(\omega_1)+\cdots+\ell(\omega_r)$.

Case (ii): *α has a cycle-under-cycle.* By (i) we assume α is noncrossing. Two of α's cycles, say ω_u, ω_v, satisfy $M_{\omega_u} \subset M_{\omega_v}$. We may assume that M_{ω_v} is maximal in $\mathfrak{M}_\alpha$. By proposition 4.1, $M_{\omega_u} \cap \mathrm{Mov}(\omega_v)$ is empty (as in figure 1, where (467) plays the role of ω_u). Equation (9) still holds, and we take a closer look at the values z_u, z_v. We have

$$z_u = \ell((\omega_1\cdots\widehat{\omega}_u\cdots\omega_r)') = \ell(\omega_1\cdots\widehat{\omega}_u\cdots\omega_r) - 2J_u \tag{10}$$

$$z_v = \ell((\omega_1\cdots\widehat{\omega}_v\cdots\omega_r)') = \ell(\omega_1\cdots\widehat{\omega}_v\cdots\omega_r) - 2J_v \tag{11}$$

where the following notations are used: $\widehat{\omega}_i$ expresses the elimination of ω_i from the product $\omega_1\cdots\omega_r$; $(\chi)'$ denotes, in (10), the restriction of χ to $[n]\smallsetminus\mathrm{Mov}(\omega_u)$, and in (11) the restriction of χ to $[n]\smallsetminus\mathrm{Mov}(\omega_v)$; and the J_i's denote numbers of inversions according to lemma 4.2. Clearly J_u is positive because $\mathrm{Mov}(\omega_u) \subset M_{\omega_v}$. Suppose $J_v > 0$; then for some $a \neq v$ we would have a configuration of the kind $i < w < \omega_a(i)$, with $w \in \mathrm{Mov}(\omega_v)$. So $M_{\omega_a}\cap M_{\omega_v} \neq \varnothing$ and, by proposition 4.1 and the maximality of M_{ω_v}, we would have $M_{\omega_a} \cap \mathrm{Mov}(\omega_v) = \varnothing$. This contradicts the fact that $w \in M_{\omega_a} \cap \mathrm{Mov}(\omega_v)$. Therefore $J_v = 0$. So (9) transforms into

$$\ell(\alpha)q^{\ell(\alpha)-1} = \ell(\omega_v)\, q^{\ell(\alpha)-1} + \ell(\omega_u)\, q^{\ell(\alpha)-1-2J_u} + \textstyle\sum_{i\neq u,v} \ell(\omega_i)\, q^{\ell(\omega_i)-1+z_i}, \quad (12)$$

which is clearly impossible. This finishes the *if part* of the proof.

The *only if part*. Suppose $\mathrm{twt}_\sigma(A) \neq 0$. In both sides of (7) all monomials are of the kind twt_τ; so we are supposed to show that twt_σ occurs in both sides with the same coefficient. In the left hand side, the coefficient is $\ell(\sigma)q^{\ell(\sigma)-1}$. If $\omega_1, \dots, \omega_r$ are the cycles of σ, in the right hand side of (7) twt_σ occurs in r summands, corresponding to $c = \omega_s$ for $s = 1, \dots, r$. To find the summand corresponding to ω_s, we must search in the defining expansion of $\mathrm{per}_q\, A(\omega_s)$ for the occurrence of the monomial $m_s := \mathrm{twt}_\sigma \,/\, \mathrm{wt}_{\omega_s}$. It turns out that m_s is precisely the total weight of $(\omega_1 \cdots \widehat{\omega}_s \cdots \omega_r)'$ (notation as in (10)-(11)) as a permutation of the set $[n] \smallsetminus M_{\omega_s}$, with respect to the submatrix $A(\omega_s)$. The disjointness of the elements of $\mathfrak{M}_\sigma$ (*cf.* proposition 4.1) implies $\ell((\omega_1 \cdots \widehat{\omega}_s \cdots \omega_r)') = \ell(\omega_1 \cdots \widehat{\omega}_s \cdots \omega_r)$. So the contribution of ω_s to the right hand side of (7) is $\ell(\omega_s)q^{\ell(\sigma)-1}\, \mathrm{twt}_\sigma$. Therefore, twt_σ occurs in the right hand side with coefficient $\ell(\sigma)q^{\ell(\sigma)-1}$, because $\ell(\sigma) = \ell(\omega_1) + \cdots + \ell(\omega_r)$. ■

Given a graph G, a permutation subdigraph of the underlying digraph D_G is called a *permutation subgraph of* G. We call *edge-under-edge* to a configuration of two edges of G, $\{i, j\}, \{r, s\}$, such that: $r < i < j < s$. The reduction from digraphs to graphs comes with pleasant obvious simplifications, due to the presence of a generous supply of loops in D_G:

> *All permutation subgraphs of G are noncrossing if and only if G is a noncrossing graph.*
> *All permutation subgraphs of G have no cycle-under-cycle if and only if G has no edge-under-edge.*

The application of theorem 4.3 to a graph G will be handled by bounding the sum in (7), to a symmetric set of cycles of the complete digraph on $[n]$. More precisely, we say that $\Omega \subseteq \mathscr{C}_n$ is *symmetric* if all elements of Ω have orders > 1, and Ω is closed for inversion when envisaged as a subset of $\mathscr{S}_n$. When we say that an edge $\{i, j\}$ lies in Ω we of course mean that (i, j) and (j, i) are members of Ω, and when we say that a (non-oriented) cycle of G lies in Ω, we mean that the two corresponding oriented cycles of D_G are members of Ω.

Theorem 4.4. *G is a noncrossing graph, with no edge-under-edge and has all cycles and edges in Ω, if and only if any [symmetric, Hermitian] matrix A with graph G satisfies* (7)*, with the summation extended to $c \in \Omega$.*

In this and in other statements to come, the expression *"any [symmetric, Hermitian] matrix"* means we have three statements: one with *any matrix*, another with *any symmetric matrix*, and a third one with *any Hermitian matrix.*

Proof: The *only if* part follows trivially from theorem 4.3.

The *if part.* Let φ be a cycle of D_G (φ may represent an edge or a cycle of G). Suppose that (7), with summation over $c \in \Omega$, holds for a generic matrix A with graph G. As $\operatorname{twt}_\varphi(A) \neq 0$ occurs in the left hand side of (7), it also occurs on the right hand side. So there exists $c \in \Omega$ such that $\operatorname{twt}_\varphi(A)$ occurs in $\operatorname{wt}_c(A) \operatorname{per}_q A(c)$, that is $\operatorname{twt}_\varphi(A) = \operatorname{wt}_c(A) \operatorname{twt}_\eta(A(c))$, for some permutation η of $[n] \smallsetminus \operatorname{Mov}(c)$. Of course we may view η as an element of $\mathscr{S}_n$ disjoint from c, and we get

$$\operatorname{twt}_\varphi(A) = \operatorname{twt}_{c\eta}(A). \tag{13}$$

This implies $\varphi = c\eta$, and therefore $\varphi = c$. So φ lies in Ω.

Next let A be a generic *symmetric* matrix with graph G. The proof that $\varphi \in \Omega$ may go the previous way until (13). However, while in the case of a generic A the identity $\operatorname{twt}_\sigma(A) = \operatorname{twt}_\tau(A)$ implies $\sigma = \tau$, for a generic *symmetric* matrix this is no longer true; the reader may enjoy solving on the fly the following neat exercise, where A is understood as a generic symmetric matrix:

Exercise

1. $\operatorname{twt}_\sigma$ *has the form* $a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}$ *where* $(i_1, \dots, i_n, j_1, \dots, j_n)$ *has all members of* $[n]$ *repeated; and any such form is a* twt_τ *for some* τ*.*
2. $\operatorname{twt}_\sigma = \operatorname{twt}_\tau$ *if and only if* τ *results from* σ *by inverting some of* σ*'s cycles. So the number of* τ*'s such that* $\operatorname{twt}_\sigma = \operatorname{twt}_\tau$ *is* 2^k*, where* k *is the number of cycles of* σ *of orders* $\geqslant 3$*.*
   Hint. Paint a copy of $[n]$ red, and another blue; use $\{i_1, j_1\}, \dots, \{i_n, j_n\}$ to build a perfect matching from blue to red; then switch colors in chosen cycles.

This being secured, (13) implies $\varphi = c$ or $\varphi = c^{-1}$. Hence φ lies in Ω.

In case (7) (with $c \in \Omega$) holds for all Hermitian matrices, then it holds for all real symmetric matrices, and the previous argument shows that all cycles and edges of G lie in Ω.

Assume now that G has edges $\{v, w\}$, $\{\bar{v}, \bar{w}\}$, satisfying (i) $v < \bar{v} < w < \bar{w}$, or (ii) $v < \bar{v} < \bar{w} < w$. Then let α be $(vw)(\bar{v}\bar{w})$, and define P_α as in (8). Clearly P_α is a symmetric permutation and, as we have seen above, the transpositions (vw) and $(\bar{v}\bar{w})$ lie in Ω. Therefore, we may follow the path in the proof of theorem 4.3 (with easier details) to get contradictions in both cases (i) and (ii). So G is a noncrossing graph with no edge-under-edge. ∎

If in theorem 4.4 we let $\Omega = T$, the set of all transpositions, we get:

Corollary 4.5. *G is a noncrossing forest with no edge-under-edge if and only if any [symmetric, Hermitian] matrix A having G as graph satisfies*

$$\frac{\partial}{\partial q}\operatorname{per}_q A = \sum_{i<j} \ell((ij))\, q^{\ell((ij))-1}\, a_{ij}a_{ji}\, \operatorname{per}_q A((ij)). \tag{14}$$

∎

Formula (14) appears in [C. Fonseca, On a conjecture about the μ-permanent, *Linear and Multilinear Algebra*, 53(2005), pp. 225-230] as valid for Hermitian matrices and *any tree*, which we proved is not true. The flaw resides in the proof of p. 228 of that paper, when it is assumed that in the first row of A, all zero entries may be pushed to the right. This is usually achieved by a permutation similarity, but the q-permanent is not invariant for this kind of transformation.

If in theorem 4.4 we let Ω be the set of edges of a given graph H, we get:

Corollary 4.6. *G is a noncrossing forest, with no edge-under-edge and is a subgraph of H, if and only if any [symmetric, Hermitian] matrix A having G as graph satisfies* (14) *with the sum restricted to the edges of H.* ∎

5. Characterizations of noncrossing graphs and forests

Theorem 5.1. *D is a digraph in which all permutation subdigraphs are noncrossing, if and only if any matrix A with digraph D satisfies*

$$\tfrac{\partial}{\partial q}\operatorname{per}_q A = \sum_{c\in\mathscr{C}_n} \ell(c)\, q^{\ell(c)-1}\, \operatorname{wt}_c(A)\, \operatorname{per}_q A_c^{\checkmark}. \tag{15}$$

Proof: The *if part* follows the method used for case (i) of the proof of theorem 4.3. The *only if part* is a bit easier than the *only if part* of the proof of theorem 4.3. We use the same notation. If $\operatorname{twt}_\sigma \neq 0$, we determine the contribution of $c = \omega_s$ in the right hand side of (15). The monomial $\operatorname{twt}_\sigma / \operatorname{wt}_{\omega_s}$ is the total weight of the permutation $\omega_1 \cdots \widehat{\omega}_s \cdots \omega_r$ in $A_{\omega_s}^{\checkmark}$. So the summand corresponding to ω_s is

$$\ell(\omega_s) q^{\ell(\omega_s)-1} q^{\ell(\sigma)-\ell(\omega_s)} \operatorname{twt}_\sigma = \ell(\omega_s) q^{\ell(\sigma)-1} \operatorname{twt}_\sigma .$$

To end up the proof, we sum this expression for $s = 1, \dots, r$, and apply corollary 3.3. ∎

The following two corollaries are given with no proofs. For such proofs the reader may adapt the techniques introduced in previous results. The technical differences arising upon the change from $A(c)$ to $A_c^{\checkmark}$ have already been treated in the proofs of the main theorems 4.3 and 5.1. We adopt the perspective of the previous section of considering graphs with cycles and edges in a symmetric set Ω.

Corollary 5.2. *A graph G is noncrossing and has all cycles and edges in Ω, if and only if any [symmetric, Hermitian] matrix A having G as graph satisfies the identity* (15)*, with the summation extended to $c \in \Omega$.* ∎

Split Ω into a set E of transpositions (2-cycles) and a set C of cycles of orders $k \geqslant 3$. As C is closed for inversion, C is a union of doubletons $\{c, c^{-1}\}$. We let R be a set of representatives of those doubletons; thus $C = R \cup R^{-1}$, $R \cap R^{-1} = \varnothing$. With this notation, the identity (15), with the summation extended to $c \in \Omega$, may be expanded as follows in case A is Hermitian:

$$\frac{\partial}{\partial q} \operatorname{per}_q A = \sum_{i<j,\, \{i,j\} \in E} \ell((ij))\, q^{\ell((ij))-1} \, |a_{ij}|^2 \, \operatorname{per}_q A_{(ij)}^{\checkmark} + 2 \sum_{c \in R} \ell(c)\, q^{\ell(c)-1} \, \operatorname{Re}[\mathrm{wt}_c(A)] \, \operatorname{per}_q A_c^{\checkmark}, \tag{16}$$

because $\ell(c^{-1}) = \ell(c)$, $A_{c^{-1}}^{\checkmark} = A_c^{\checkmark}$ and $\mathrm{wt}_{c^{-1}} = \overline{\mathrm{wt}}_c$.

Corollary 5.3. *A graph G is a noncrossing forest if and only if any [symmetric, Hermitian] matrix A having G as graph satisfies*

$$\frac{\partial}{\partial q} \operatorname{per}_q A = \sum_{i<j} \ell((i\,j))\, q^{\ell((i\,j))-1} \, a_{ij} a_{ji} \, \operatorname{per}_q A_{(ij)}^{\checkmark}. \tag{17}$$

∎

6. The case of Hermitian matrices

In [7, 8], M. Bozejko and R. Speicher proved, for $q \in]-1,1[$, that $\sigma \rightsquigarrow q^{\ell(\sigma)}$ is a positive definite function on $\mathscr{S}_n$. Then, in [2], R. Bapat showed how that result implies $\operatorname{per}_q A > 0$, for $q \in [-1,1]$, if A is positive definite; he conjectured that $\operatorname{per}_q A$ *is strictly increasing with $q \in [-1,1]$, if A is a non-diagonal positive definite matrix*, and proved his conjecture for $n \leqslant 3$. In

[16], A. Lal proved that conjecture for tridiagonal matrices. More specifically, A. Lal established the formula

$$\frac{\partial}{\partial q} \operatorname{per}_q A = \sum_{i=1}^{n-1} |a_{i,i+1}|^2 \operatorname{per}_q A((i\ i+1)), \tag{18}$$

for a Hermitian tridiagonal matrix. On a closer look, Lal has in fact proved (14) for the graph with edges $\{i, i+1\}$, for $i \in [n-1]$.

The following two theorems are based on the derivative formulas of section 5, specially (16) (those of section 4 give weaker results, due to the severe cycle-under-cycle restriction). The techniques of the omitted proofs use the following simple facts: if a cycle c of G has even order, then $\ell(c)$ is odd and so $q^{\ell(c)-1} > 0$ for $q \in [-1, 1]$, $q \neq 0$; and the same happens with the edges $\{i, j\}$ of G, together with $|a_{ij}|^2 > 0$. To get $\operatorname{Re}[\operatorname{wt}_c(A)] > 0$ the easy way is to postulate $A \geqslant 0$ entrywise. Finally, as $A_c^{\checkmark}$ is positive definite, Bapat's corollary to the cited Bozejko-Speicher result [7, 2] guarantees $\operatorname{per}_q A_c^{\checkmark} > 0$.

Theorem 6.1. *Let G is a noncrossing forest with at least one edge. For any positive definite matrix A with graph G, $\operatorname{per}_q A$ is strictly increasing with $q \in [-1, 1]$.* ■

Theorem 6.2. *Let A be a non-diagonal real symmetric matrix, which is positive definite and entrywise nonnegative. If $G(A)$ is a noncrossing graph and all its cycles have even order, then $\operatorname{per}_q A$ is strictly increasing with $q \in [-1, 1]$.* ■

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