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CHARACTERIZATION OF TRIEBEL-LIZORKIN-TYPE SPACES WITH VARIABLE EXPONENTS VIA MAXIMAL FUNCTIONS, LOCAL MEANS AND NON-SMOOTH ATOMIC DECOMPOSITIONS

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Abstract: In this paper we study the maximal function and local means characterization and the non-smooth atomic decomposition characterization of the Triebel-Lizorkin-type spaces with variable exponents $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. These spaces were recently introduced by Yang et all and cover the Triebel-Lizorkin spaces with variable exponents $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ as well as the classical Triebel-Lizorkin spaces $F^{s}_{p,q}(\mathbb{R}^n)$, even the case when $p=\infty$, and also the Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ with constant exponents which, in turn covers the Triebel-Lizorkin-Morrey spaces. As an application we obtain a pointwise multiplier assertion for those spaces.

Keywords: Triebel-Lizorkin spaces, variable exponents, Peetre maximal operator, local means, atoms, pointwise multipliers.

1. Introduction

Spaces of variable integrability, also known as variable exponent function spaces $L_{p(\cdot)}(\mathbb{R}^n)$, can be traced back to Orlicz [20] 1931, but the modern development started with the papers [12] of Kováčik and Rákosník as well as [4] of Diening. The interest on these spaces comes not only from the theoretically point of view but also in view of their applications: in the theory and modeling of electrorheological fluids, in differential equations and in image restoration, for instance. We refer to [5] and the references therein for a more complete overview.

The concept of function spaces with variable smoothness and the concept of variable integrability were firstly mixed up by Diening, Hästö and Roudenko

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in [6], where the authors defined Triebel-Lizorkin spaces with variable exponents $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. From the trace theorem on $\mathbb{R}^{n-1}$ proved there and stated as follows ([6, Theorem 3.13])

$$\operatorname{Tr} F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\cdot)-\frac{1}{p(\cdot)}}_{p(\cdot),p(\cdot)}(\mathbb{R}^{n-1}), \text{ with } s(\cdot)-\frac{1}{p(\cdot)} > (n-1)\max\left(\frac{1}{p(\cdot)}-1,0\right),$$

one can easily understand the necessity of taking s and q variable if p is not constant. A similar interplay between smoothness and integrability can also be verified in [33], where Vybíral established Sobolev embeddings for these spaces. Moreover, Almeida and Hästö also introduced in [3] Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with all three indices variable and showed a Sobolev embedding for these spaces.

More generally, 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability were introduced in [9, 10] and provide a unified approach that covers not only the classical Besov and Triebel-Lizorkin spaces, but also many spaces related with variable smoothness and generalized smoothness. Many results have been studied regarding these spaces, such as characterizations by local means, smooth and non-smooth atoms, molecules, ball means of differences and Peetre maximal functions (we refer to [1], [2], [11], [8]).

On the other hand, another class of generalized Besov and Triebel-Lizorkin spaces, the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$, were introduced in [34]. Within this class of function spaces one can find the classical Besov and Triebel-Lizorkin spaces as well as the Triebel-Lizorkin-Morrey spaces introduced by Tang and Xu in [27]. These function spaces, including some of their special cases related to Q spaces, have been used to study the existence and the regularity of solutions of some partial differential equations such as (fractional) Navier-Stokes equations; see, for example [13].

Based on $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$, recently Yang, Zhuo and Yuan introduced in [37, 36] a generalized scale of function spaces considering variable exponents $s(\cdot)$, $p(\cdot)$ and $q(\cdot)$ and a measurable function ϕ on $\mathbb{R}^{n+1}_+$. Denoted by $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, these spaces then include the Besov and Triebel-Lizorkin-type spaces (with constant exponents) and also Besov and Triebel-Lizorkin spaces with variable exponents. They characterized those function spaces by means of atomic decompositions and derived the trace on hyperplanes.

In this paper we characterize $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ by maximal functions and, as a corollary, we also get a characterization by local means. Moreover, we use these results to derive a non-smooth atomic decomposition for the spaces in consideration.

Our characterization by maximal functions and local means extends results from [11] and [35]. Regarding the non-smooth atomic decomposition, as far as we know the result is new for this scale of function spaces, even in the case of constant exponents, in particular to $F^{s,\tau}_{p,q}(\mathbb{R}^n)$. The idea is to show that the usual atoms used in smooth atomic decompositions as in [36] or [37] can be replaced by more general ones, meaning weaker assumptions on the smoothness, and, nevertheless, we keep all the crucial information when comparing to smooth atomic decompositions. This modification appeared first in [31], where Triebel and Winkelvoß suggested the use of these more relaxed conditions to define classical Besov and Triebel-Lizorkin spaces intrinsically on domains. More recently, Scharf in [24] defined even more general atoms and proved a non-smooth atomic characterization for the spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$. As far as the scale of spaces with variable exponents is concerned, we refer to [8], where the authors showed the corresponding result for the scale of 2-microlocal spaces with variable integrability. Following [8], we then make use of the characterizations via smooth atoms and local means to derive the more general atomic decomposition result for Triebel-Lizorkin-type spaces with variable exponents.

Having the non-smooth atomic decomposition theorem, we are able to deal easily with pointwise multipliers in the function spaces $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, as stated in the last section.

2. Notation and definitions

We start by collecting some general notation used throughout the paper.

As usual, we denote by $\mathbb{N}$ the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^n$, $n \in \mathbb{N}$, the n-dimensional real Euclidean space with $|x|$, for $x \in \mathbb{R}^n$, denoting the Euclidean norm of x. By $\mathbb{Z}^n$ we denote the lattice of all points in $\mathbb{R}^n$ with integer components. For $\beta := (\beta_1, \cdots, \beta_n) \in \mathbb{Z}^n$, let $|\beta| := |\beta_1| + \cdots + |\beta_n|$. If $a, b \in \mathbb{R}$, then $a \vee b := \max\{a, b\}$. We denote by c a generic positive constant which is independent of the main parameters, but its value may change from line to line. The expression $A \lesssim B$ means that $A \leq c\, B$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded.

If E is a measurable subset of $\mathbb{R}^n$, we denote by χ_E its characteristic function and by $|E|$ its Lebesgue measure. By supp f we denote the support of the function f.

For each cube $Q \subset \mathbb{R}^n$ we denote its center by c_Q and its side length by $\ell(Q)$ and, for $a \in (0,\infty)$ we denote by aQ the cube concentric with Q having the side length $a\ell(Q)$. For $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, we denote by $Q(x,r)$ the cube centered at x with side lenght r, whose sides are parallel to the axes of coordinates.

Given $k \in \mathbb{N}_0$, $C^k(\mathbb{R}^n)$ is the space of all functions $f : \mathbb{R}^n \to \mathbb{C}$ which are k-times continuously differentiable (continuous in $k = 0$) such that

$$\|f \mid C^k(\mathbb{R}^n)\| := \sum_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha f(x)| < \infty.$$

The Hölder space $\mathscr{C}^s(\mathbb{R}^n)$ with index $s > 0$ is defined as the set of all functions $f \in C^{\lfloor s\rfloor^-}(\mathbb{R}^n)$ with

$$\|f \mid \mathscr{C}^s(\mathbb{R}^n)\| := \|f \mid C^{\lfloor s\rfloor^-}(\mathbb{R}^n)\| + \sum_{|\alpha|=\{s\}^+} \sup_{x,y\in\mathbb{R}^n, x\ne y} \frac{|f(x)-f(y)|}{|x-y|^{\{s\}^+}} < \infty,$$

where $\lfloor s\rfloor^- \in \mathbb{N}_0$ and $\{s\}^+ \in (0,1]$ are uniquely determined numbers so that $s = \lfloor s\rfloor^- + \{s\}^+$. If $s = 0$ we set $\mathscr{C}^0(\mathbb{R}^n) := L_\infty(\mathbb{R}^n)$.

By $\mathscr{S}(\mathbb{R}^n)$ we denote the usual Schwartz class of all infinitely differentiable rapidly decreasing complex-valued functions on $\mathbb{R}^n$ and $\mathscr{S}'(\mathbb{R}^n)$ stands for the dual space of tempered distributions. The Fourier transform of $f \in \mathscr{S}(\mathbb{R}^n)$ or $f \in \mathscr{S}'(\mathbb{R}^n)$ is denoted by $\widehat{f}$ while its inverse transform is denoted by $f^\vee$.

Now we give a short survey on variable exponents. For a measurable function $p : \mathbb{R}^n \to (0,\infty)$, let

$$p^- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p^+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

In this paper we denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p : \mathbb{R}^n \to (0,\infty)$ (called variable exponents) such that $0 < p^- \le p^+ < \infty$. For $p \in \mathcal{P}(\mathbb{R}^n)$ and a measurable set $E \subset \mathbb{R}^n$, the space $L_{p(\cdot)}(E)$ is defined to be the set of all (complex or real-valued) measurable functions f such that

$$\|f \mid L_{p(\cdot)}(E)\| := \inf\left\{\lambda \in (0,\infty) : \int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \leq 1\right\} < \infty.$$

It is known that $L_{p(\cdot)}(E)$ is a quasi-Banach space, a Banach space when $p^- \geq 1$. If $p(\cdot) \equiv p$ is constant, then $L_{p(\cdot)}(E) = L_p(E)$ is the classical Lebesgue space.

For later use we recall that $L_{p(\cdot)}(E)$ has the lattice property. Moreover, we have

$$\|f \mid L_{p(\cdot)}(E)\| = \left\||f|^r \middle| L_{\frac{p(\cdot)}{r}}(E)\right\|^{\frac{1}{r}}, \qquad r \in (0,\infty),$$

and

$$\|f+g \mid L_{p(\cdot)}(E)\|^r \leq \|f \mid L_{p(\cdot)}(E)\|^r + \|g \mid L_{p(\cdot)}(E)\|^r, \quad r \in \big(0, \min\{1,p^-\}\big).$$

In the setting of variable exponent function spaces it is needed to require some regularity conditions to the exponents. We recall now the standard conditions used.

We say that a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous, abbreviated $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, if there exists a positive constant $c_{\log}(g)$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \quad \text{for all } \ x, y \in \mathbb{R}^n. \tag{1}$$

We say that g is globally log-Hölder continuous, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and there exists $g_\infty \in \mathbb{R}$ and a positive constant c_∞ such that

$$|g(x) - g_\infty| \leq \frac{c_\infty}{\log(e + |x|)} \quad \text{for all } \ x \in \mathbb{R}^n.$$

Note that all functions in $C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ are bounded and if $g \in C^{\log}(\mathbb{R}^n)$ then $g_\infty = \lim_{|x|\to\infty} g(x)$. Moreover, for $g \in \mathcal{P}(\mathbb{R}^n)$, we have that $g \in C^{\log}(\mathbb{R}^n)$ if and only if $1/g \in C^{\log}(\mathbb{R}^n)$. The notation $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for those variable exponents $p \in \mathcal{P}(\mathbb{R}^n)$ with $p \in C^{\log}(\mathbb{R}^n)$.

Let $\mathcal{G}(\mathbb{R}^{n+1}_+)$ be the set of all measurable functions $\phi : \mathbb{R}^n \times [0,\infty) \to (0,\infty)$ having the following properties: there exist positive constants $c_1(\phi)$ and $\tilde{c_1}(\phi)$ such that

$$\frac{1}{\tilde{c_1}(\phi)} \leq \frac{\phi(x,r)}{\phi(x,2r)} \leq c_1(\phi) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r \in (0,\infty), \tag{2}$$

and there exists a positive constant $c_2(\phi)$ such that, for all $x, y \in \mathbb{R}^n$ and $r \in (0,\infty)$ with $|x-y| \le r$,

$$\frac{1}{c_2(\phi)} \le \frac{\phi(x,r)}{\phi(y,r)} \le c_2(\phi). \tag{3}$$

The conditions (2) and (3) are called doubling condition and compatibility condition, respectively, and have been used by Nakai [17, 18] and Nakai and Sawano [19]. Examples of functions in $\mathcal{G}(\mathbb{R}^{n+1}_+)$ are provided in [36, Remark 1.3].

In what follows, for $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$ and all cubes $Q := Q(x,r)$ with center $x \in \mathbb{R}^n$ and radius $r \in (0,\infty)$, we define $\phi(Q) := \phi(Q(x,r)) := \phi(x,r)$.

The following is our convention for dyadic cubes: For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, denote by Q_{jk} the dyadic cube $2^{-j}([0,1)^n + k)$ and $x_{Q_{jk}}$ its lower left corner. Let $\mathcal{Q} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$, $\mathcal{Q}^* := \{Q \in \mathcal{Q} : \ell(Q) \le 1\}$ and $j_Q := -\log_2 \ell(Q)$ for all $Q \in \mathcal{Q}$. When the dyadic cube Q appears as an index, such as $\sum_{Q\in\mathcal{Q}}$ and $(\cdots)_{Q\in\mathcal{Q}}$, it is understood that Q runs over all dyadic cubes in $\mathbb{R}^n$.

For the function spaces under consideration in this paper, the following modified mixed-Lebesgue sequence spaces are of special importance. Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. We denote by $L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})$ the set of all sequences $(g_j)_{j\in\mathbb{N}_0}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|(g_j)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\| := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \Big\| \Big(\sum_{j=j_P\vee 0}^{\infty} |g_j(\cdot)|^{q(\cdot)} \Big)^{\frac{1}{q(\cdot)}} \mid L_{p(\cdot)}(P) \Big\| < \infty,$$

where the supremum is taken over all dyadic cubes P in $\mathbb{R}^n$. We remark that $L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})$ is a quasi-normed space that coincides with the mixed Lebesgue-sequence space $L_{p(\cdot)}(\ell_{q(\cdot)})$ when $\phi \equiv 1$. The case of $q(\cdot) = q$ constant and $\phi(Q) := |Q|^{\tau}$ for all cubes Q and $\tau \in [0,\infty)$, has also been considered in [15].

We say that a pair (φ, φ_0) of functions in $\mathscr{S}(\mathbb{R}^n)$ is admissible if

$$\operatorname{supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\} \ \text{ and } \ |\widehat{\varphi}(\xi)| > 0 \ \text{ when } \ \frac{3}{5} \le |\xi| \le \frac{5}{3} \tag{4}$$

and

$$\operatorname{supp} \widehat{\varphi}_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\} \quad \text{and} \ \ |\widehat{\varphi}_0(\xi)| > 0 \ \text{ when } \ |\xi| \le \frac{5}{3}. \tag{5}$$

Throughout the article, for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, we set $\varphi_j(x) := 2^{jn}\varphi(2^j x)$.

Definition 2.1. *Let (φ, φ_0) be a pair of admissible functions on $\mathbb{R}^n$. Let $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. Then the Triebel-Lizorkin-type space with variable exponents $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that*

$$\|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| := \left\| \left(2^{js(\cdot)}(\varphi_j * f)(\cdot)\right)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\right\| < \infty.$$

Remarks 2.2. (i) These spaces were introduced by Yang et al. in [36], where the authors have proved the independence of the spaces on the admissible pair.

(ii) When $\phi \equiv 1$, then $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the Triebel-Lizorkin space with variable smoothness and integrability introduced and investigated by Diening et al. in [6]. There s is assumed to be non-negative, which was later generalised to the case $s \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ by Kempka in [10].

(iii) When $p(\cdot) = p$, $q(\cdot) = q$, and $s(\cdot) = s$ are constant exponents and $\phi(Q) := |Q|^\tau$ for all cubes Q and $\tau \in [0, \infty)$, then $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s,\tau}_{p,q}(\mathbb{R}^n)$ are the Triebel-Lizorkin-type spaces introduced by Yuan et al. in [34], which in turn coincide with the classical Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ when $\tau = 0$ and $p < \infty$. Moreover, also the spaces $F^s_{\infty,q}(\mathbb{R}^n)$ and the Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$ are included, as $F^s_{\infty,q}(\mathbb{R}^n) = F^{s,1/p}_{p,q}(\mathbb{R}^n)$ for $p \in (0, \infty)$, and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^n) = F^{s,1/p-1/u}_{p,q}(\mathbb{R}^n)$ if $0 < p \le u < \infty$, cf. [23].

(iv) When q, s are constant exponents and $\phi(Q) := |Q|^\tau$ for all cubes Q and $\tau \in [0, \infty)$, then $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s,\tau}_{p(\cdot),q}(\mathbb{R}^n)$ which was investigated in [15].

(v) The condition $p^+ < \infty$, included in $p \in \mathcal{P}(\mathbb{R}^n)$, is quite natural since it also exists in the case of constant exponents, cf. [34]. This is not the case for the assumption $q^+ < \infty$ as the case $q = \infty$ is included is the case of constant exponents. This restriction comes from technical reasons as explained in [36, Remark 1.5(iv)].

3. Maximal functions and local means characterization

Let $(\psi_j)_{j\in\mathbb{N}_0}$ be a sequence in $\mathscr{S}(\mathbb{R}^n)$. For each $f \in \mathscr{S}'(\mathbb{R}^n)$ and $a > 0$, the Peetre's maximal functions were defined by Peetre in [21] by

$$(\psi^*_j f)_a(x) := \sup_{y\in\mathbb{R}^n} \frac{|\psi_j * f(y)|}{(1 + |2^j(x - y)|)^a}, \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0.$$

The following result coincides with [36, Theorem 3.11].

Theorem 3.1. *Let $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. Let*

$$a > \frac{n}{\min\{p^-, q^-\}} + \log_2 \tilde{c}_1(\phi) + c_{\log}(s),$$

where $\tilde{c}_1(\phi)$ is the constant in (2) *and $c_{\log}(s)$ comes from condition* (1) *on s. Then $f \in F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathscr{S}'(\mathbb{R}^n)$ and*

$$\|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^* := \left\| \left(2^{js(\cdot)}(\varphi^*_j f)_a\right)_{j\in\mathbb{N}_0} \middle| L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\right\| < \infty.$$

Moreover, $\|\cdot \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|^*$ and $\|\cdot \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$ are equivalent quasi-norms in $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

In the above theorem the sequence $(\varphi_j)_{j\in\mathbb{N}_0}$ is the same as in Definition 2.1, which is built upon an admissible pair (φ, φ_0). One of the aims of this section is to prove that such a characterization still holds if we consider more general pairs of functions. Moreover we also show that the same happens for the definition of $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, i.e. the admissible pairs used to define the spaces can be indeed replaced by more general ones. The main theorem of this section reads then as follows.

Theorem 3.2. *Let $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. Let $R \in \mathbb{N}_0$ with $R > s^+ + \log_2 \tilde{c}_1(\phi)$, where $\tilde{c}_1(\phi)$ is the constant in* (2), *and let $\psi_0, \psi \in \mathscr{S}(\mathbb{R}^n)$ be such that*

$$(D^\beta \widehat{\psi})(0) = 0 \quad \text{for} \quad 0 \le |\beta| < R \tag{6}$$

and

$$|\widehat{\psi_0}(\xi)| > 0 \quad \text{on} \quad \{\xi \in \mathbb{R}^n : |\xi| \le k\varepsilon\}, \tag{7}$$

$$|\widehat{\psi}(\xi)| > 0 \quad \text{on} \quad \big\{\xi \in \mathbb{R}^n : \frac{\varepsilon}{2} \le |\xi| \le k\varepsilon\big\}, \tag{8}$$

for some $\varepsilon > 0$ and $k \in]1, 2]$. For

$$a > \frac{n}{\min\{p^-, q^-\}} + \log_2 \tilde{c}_1(\phi) + c_{\log}(s),$$

we have

$$\begin{aligned} \|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| &\sim \left\| \left(2^{js(\cdot)}(\psi^*_j f)_a\right)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\right\| \\ &\sim \left\| \left(2^{js(\cdot)}(\psi_j * f)\right)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\right\| \end{aligned}$$

for all $f \in \mathscr{S}'(\mathbb{R}^n)$.

Remarks 3.3. (i) The conditions (6) are usually called moment conditions while (7) and (8) are the so-called Tauberian conditions. If $R = 0$ then no moment conditions (6) on ψ are required. This is possible when $s^+ < -\log_2 \tilde{c_1}(\phi)$.

(ii) The case $\phi \equiv 1$ is covered by [1, Theorem 3.1(ii)], where we can find also a discussion on the importance of having a k in the conditions (7) and (8) in contract with the case $k = 2$ usually found in the literature in such type of result, cf. e.g. [9, 11, 22].

(iii) When $p(\cdot) = p$, $q(\cdot) = q$, $s(\cdot) = s$ are constant exponents and $\phi(Q) := |Q|^\tau$ for all cubes Q and $\tau \in [0,\infty)$, then $\log_2 \tilde{c_1}(\phi) = n\tau$ and such a characterization with $k = 2$ has been already established in the homogeneous case by Yang and Yuan, cf. [35, Theorem 2.1(ii)].

(iv) In [36] the authors proved the independence of the spaces from the admissible pair as a consequence of the φ-transform characterization. The above theorem provides an alternative proof, since an admissible pair satisfies conditions (7) and (8) with $\varepsilon = \frac{6}{5}$ and $k = \frac{25}{18}$. Moreover, it becomes clear that $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ can be defined using more general pairs than the admissible ones (in the sense of equivalent quasi-norms). We refer in particular to the case stated in Corollary 3.8 below.

The proof of Theorem 3.2 relies in the proof done by Rychkov [22] in the classical case, and will be given after some auxiliar results, for what we follow the same structure as in [9, 11] in the context of 2-microlocal spaces with variable exponents.

The first theorem provides an inequality between two different sequences of Peetre maximal functions.

Theorem 3.4. *Let $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{\rm loc}(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. Let $a > 0$ and $R \in \mathbb{N}_0$ with $R > s^+ + \log_2 \tilde{c_1}(\phi)$. Further, let $\mu_0, \mu, \psi_0, \psi \in \mathscr{S}(\mathbb{R}^n)$ be such that*

$$(D^\beta \widehat{\mu})(0) = 0 \quad \textit{if} \quad 0 \le |\beta| < R$$

and

$$|\widehat{\psi_0}(x)| > 0 \quad \textit{for} \quad |x| \le k\varepsilon \tag{9}$$

$$|\widehat{\psi}(x)| > 0 \quad \textit{for} \quad \frac{\varepsilon}{2} \le |x| \le k\varepsilon, \tag{10}$$

for some $\varepsilon > 0$ and $k \in]1,2]$. Then there exists $c > 0$ such that

$$\left\| \left(2^{js(\cdot)} (\mu_j^* f)_a\right)_{j\in\mathbb{N}_0} \mid L^\phi_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \le c \left\| \left(2^{js(\cdot)} (\psi_j^* f)_a\right)_{j\in\mathbb{N}_0} \mid L^\phi_{p(\cdot)}(\ell_{q(\cdot)}) \right\|$$

holds for every $f \in \mathscr{S}'(\mathbb{R}^n)$.

Proof: We only give a sketch of the proof as it is based in [22]. There k is restricted to $k = 2$, but the extension to $k \in]1, 2]$ follows easily. Indeed, by (9) and (10), there exist $\lambda_0, \lambda \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\widehat{\lambda}_0(x)\widehat{\psi_0}(x) + \sum_{j=1}^{\infty} \widehat{\lambda}(2^{-j+1}x)\widehat{\psi}(2^{-j}x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Therefore we can carry on parallel pointwise estimates as those in [22] to arrive at

$$2^{js(x)}(\mu_j^* f)_a(x) \lesssim \sum_{\nu=0}^{j} 2^{-(j-\nu)(R-s^+)} 2^{\nu s(x)} (\psi_\nu^* f)_a(x) + \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)} 2^{\nu s(x)} (\psi_\nu^* f)_a(x),$$

where we also used the fact that

$$2^{js(x)} \leq 2^{\nu s(x)} \times \begin{cases} 2^{(j-\nu)s^+}, & j \geq \nu \\ 2^{(j-\nu)s^-}, & j \leq \nu. \end{cases} \tag{11}$$

See also the proof of Theorem 3.8 in [9]. The proof then is completed by applying Lemma 3.5 below. ■

Lemma 3.5. *Let* $p, q \in \mathcal{P}(\mathbb{R}^n)$ *and* $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. *Let* $D_1, D_2 \in (0, \infty)$ *with* $D_2 > \log_2 \tilde{c}_1(\phi)$. *For any sequence* $(g_j)_{j\in\mathbb{N}_0}$ *of measurable functions on* $\mathbb{R}^n$, *consider*

$$G_j(x) := \sum_{\nu=0}^{j} 2^{-(j-\nu)D_2} g_\nu(x) + \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)D_1} g_\nu(x), \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}_0.$$

Then

$$\|(G_j)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\| \lesssim \|(g_j)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\|.$$

Proof: *Step 1.* Assume that $p, q \geq 1$. Let $P \in \mathcal{Q}$ be arbitrarily chosen and

$$I(P) := \frac{1}{\phi(P)} \left\| \Big(\sum_{j=j_P \vee 0}^{\infty} |G_j(\cdot)|^{q(\cdot)} \Big)^{\frac{1}{q(\cdot)}} \Big| L_{p(\cdot)}(P) \right\|.$$

We need to show that

$$I(P) \lesssim \|(g_j)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\|$$

with implicit constant independent of P and $(g_j)_{j\in\mathbb{N}_0}$.

By the triangle inequality, we have

$$I(P) \le \frac{1}{\phi(P)}\left\|\left(\sum_{j=j_P\vee 0}^{\infty}\Big(\sum_{\nu=0}^{j}2^{-(j-\nu)D_2}|g_\nu(\cdot)|\Big)^{q(\cdot)}\right)^{\frac{1}{q(\cdot)}}\middle| L_{p(\cdot)}(P)\right\|$$
$$+\frac{1}{\phi(P)}\left\|\left(\sum_{j=j_P\vee 0}^{\infty}\Big(\sum_{\nu=j+1}^{\infty}2^{-(\nu-j)D_1}|g_\nu(\cdot)|\Big)^{q(\cdot)}\right)^{\frac{1}{q(\cdot)}}\middle| L_{p(\cdot)}(P)\right\|$$
$$=: I_1(P)+I_2(P).$$

Denoting by $q'(\cdot)$ the conjugate exponent of $q(\cdot)$, that is $\frac{1}{q(x)}+\frac{1}{q'(x)}=1$, for $x\in\mathbb{R}^n$, we use Hölder's inequality and, with ε such that $0<\varepsilon<\min\{D_2, D_2-\log_2\tilde{c_1}(\phi)\}$, we get

$$I_1(P) \le \frac{1}{\phi(P)}\left\|\left(\sum_{j=j_P\vee 0}^{\infty}\Big(\sum_{\nu=0}^{j}2^{-(j-\nu)(D_2-\varepsilon)q(\cdot)}|g_\nu(\cdot)|^{q(\cdot)}\Big)\right.\right.$$
$$\left.\left.\cdot\Big(\sum_{\nu=0}^{j}2^{-(j-\nu)\varepsilon q'(\cdot)}\Big)^{\frac{q(\cdot)}{q'(\cdot)}}\right)^{\frac{1}{q(\cdot)}}\middle| L_{p(\cdot)}(P)\right\|$$
$$\lesssim \frac{1}{\phi(P)}\left\|\left(\sum_{j=j_P\vee 0}^{\infty}\sum_{\nu=0}^{j}2^{-(j-\nu)(D_2-\varepsilon)q(\cdot)}|g_\nu(\cdot)|^{q(\cdot)}\right)^{\frac{1}{q(\cdot)}}\middle| L_{p(\cdot)}(P)\right\|$$
$$= \frac{1}{\phi(P)}\left\|\left(\sum_{\nu=0}^{\infty}\sum_{j=j_P\vee 0\vee\nu}^{\infty}2^{-(j-\nu)(D_2-\varepsilon)q(\cdot)}|g_\nu(\cdot)|^{q(\cdot)}\right)^{\frac{1}{q(\cdot)}}\middle| L_{p(\cdot)}(P)\right\|.$$

Then we split the sum in ν and, since we are assuming $p,q\ge 1$, it follows

$$I_1(P) \lesssim \frac{1}{\phi(P)}\left\|\left(\sum_{\nu=0}^{j_P\vee 0}\sum_{j=j_P\vee 0}^{\infty}2^{-(j-\nu)(D_2-\varepsilon)q(\cdot)}|g_\nu(\cdot)|^{q(\cdot)}\right)^{\frac{1}{q(\cdot)}}\middle| L_{p(\cdot)}(P)\right\|$$
$$+\frac{1}{\phi(P)}\left\|\left(\sum_{\nu=j_P\vee 0}^{\infty}\sum_{j=\nu}^{\infty}2^{-(j-\nu)(D_2-\varepsilon)q(\cdot)}|g_\nu(\cdot)|^{q(\cdot)}\right)^{\frac{1}{q(\cdot)}}\middle| L_{p(\cdot)}(P)\right\|$$

$$\begin{aligned}
&\lesssim \frac{1}{\phi(P)} \Big\| \sum_{\nu=0}^{j_P\vee 0} \sum_{j=j_P\vee 0}^{\infty} 2^{-(j-\nu)(D_2-\varepsilon)} |g_\nu(\cdot)| \Big| L_{p(\cdot)}(P) \Big\| \\
&\quad + \frac{1}{\phi(P)} \Big\| \Big(\sum_{\nu=j_P\vee 0}^{\infty} |g_\nu(\cdot)|^{q(\cdot)} \Big)^{\frac{1}{q(\cdot)}} \Big| L_{p(\cdot)}(P) \Big\| \\
&\lesssim \frac{1}{\phi(P)} \sum_{\nu=0}^{j_P\vee 0} 2^{(\nu-j_P\vee 0)(D_2-\varepsilon)} \| g_\nu(\cdot) | L_{p(\cdot)}(P) \| \\
&\quad + \| (g_j)_{j\in\mathbb{N}_0} \mid L^\phi_{p(\cdot)}(\ell_{q(\cdot)}) \|.
\end{aligned}$$

Now condition (2) allows us to get the desired estimate as follows:

$$\begin{aligned}
I_1(P) &\lesssim \sum_{\nu=0}^{j_P\vee 0} 2^{(\nu-j_P\vee 0)(D_2-\varepsilon)} \tilde{c_1}(\phi)^{j_P\vee 0-\nu} \frac{1}{\phi(2^{j_P\vee 0-\nu}P)} \| g_\nu(\cdot) | L_{p(\cdot)}(2^{j_P\vee 0-\nu}P) \| \\
&\quad + \| (g_j)_{j\in\mathbb{N}_0} \mid L^\phi_{p(\cdot)}(\ell_{q(\cdot)}) \| \\
&\lesssim \sum_{\nu=0}^{j_P\vee 0} 2^{-(j_P\vee 0-\nu)(D_2-\varepsilon-\log_2 \tilde{c_1}(\phi))} \| (g_j)_{j\in\mathbb{N}_0} \mid L^\phi_{p(\cdot)}(\ell_{q(\cdot)}) \| \\
&\quad + \| (g_j)_{j\in\mathbb{N}_0} \mid L^\phi_{p(\cdot)}(\ell_{q(\cdot)}) \| \\
&\lesssim \| (g_j)_{j\in\mathbb{N}_0} \mid L^\phi_{p(\cdot)}(\ell_{q(\cdot)}) \|.
\end{aligned}$$

Now we estimate $I_2(P)$, using again Hölder's inequality and choosing $\varepsilon \in (0, D_1)$:

$$\begin{aligned}
I_2(P) &\le \frac{1}{\phi(P)} \Big\| \Big(\sum_{j=j_P\vee 0}^{\infty} \Big(\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)(D_1-\varepsilon)q(\cdot)} |g_\nu(\cdot)|^{q(\cdot)} \Big) \\
&\qquad \cdot \Big(\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)\varepsilon q'(\cdot)} \Big)^{\frac{q(\cdot)}{q'(\cdot)}} \Big)^{\frac{1}{q(\cdot)}} \Big| L_{p(\cdot)}(P) \Big\| \\
&\lesssim \frac{1}{\phi(P)} \Big\| \Big(\sum_{j=j_P\vee 0}^{\infty} \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)(D_1-\varepsilon)q(\cdot)} |g_\nu(\cdot)|^{q(\cdot)} \Big)^{\frac{1}{q(\cdot)}} \Big| L_{p(\cdot)}(P) \Big\| \\
&\le \frac{1}{\phi(P)} \Big\| \Big(\sum_{\nu=j_P\vee 0}^{\infty} \sum_{j=j_P\vee 0}^{\nu} 2^{-(\nu-j)(D_1-\varepsilon)q(\cdot)} |g_\nu(\cdot)|^{q(\cdot)} \Big)^{\frac{1}{q(\cdot)}} \Big| L_{p(\cdot)}(P) \Big\|
\end{aligned}$$

$$\lesssim \frac{1}{\phi(P)} \left\| \Big(\sum_{\nu=j_P\vee 0}^{\infty} |g_\nu(\cdot)|^{q(\cdot)} \Big)^{\frac{1}{q(\cdot)}} \Big| L_{p(\cdot)}(P) \right\|$$

$$\leq \|(g_j)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\|.$$

Step 2. We consider now the case when $p, q \in \mathcal{P}(\mathbb{R}^n)$ have no more restrictions. Let $r \in \big(0, \min\{1, p^-, q^-\}\big)$. We have

$$\begin{aligned}\|(G_j)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\| &= \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \Big(\sum_{j=j_P\vee 0}^{\infty} |G_j(\cdot)|^{r\frac{q(\cdot)}{r}} \Big)^{\frac{r}{q(\cdot)}} \mid L_{\frac{p(\cdot)}{r}}(P) \right\|^{\frac{1}{r}} \\ &\leq \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \Big(\sum_{j=j_P\vee 0}^{\infty} |F_j(\cdot)|^{\frac{q(\cdot)}{r}} \Big)^{\frac{r}{q(\cdot)}} \mid L_{\frac{p(\cdot)}{r}}(P) \right\|^{\frac{1}{r}} \\ &= \left\|(F_j)_{j\in\mathbb{N}_0} \mid L^{\phi^r}_{\frac{p(\cdot)}{r}}(\ell_{\frac{q(\cdot)}{r}})\right\|^{\frac{1}{r}} \qquad (12)\end{aligned}$$

where

$$F_j(x) := \sum_{\nu=0}^{j} 2^{-(j-\nu)D_2 r} |g_\nu(x)|^r + \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)D_1 r} |g_\nu(x)|^r, \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0.$$

Note that $\tilde{c_1}(\phi^r) = \tilde{c_1}(\phi)^r$ and by the hypothesis on D_2, we then have $D_2 r > \log_2 \tilde{c_1}(\phi^r)$. Since $\frac{p}{r}, \frac{q}{r} \geq 1$ we can use the result from Step 1 in (12) to obtain

$$\|(G_j)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\| \lesssim \left\|(|g_j|^r)_{j\in\mathbb{N}_0} \mid L^{\phi^r}_{\frac{p(\cdot)}{r}}(\ell_{\frac{q(\cdot)}{r}})\right\|^{\frac{1}{r}} = \|(g_j)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\|,$$

which completes the proof. ■

Remark 3.6. The case where $q(\cdot) = q$ is constant and $\phi(Q) := |Q|^\tau$ for all cubes Q and $\tau \in [0, \infty)$ is covered by [15, Lemma 2.9]. If $p(\cdot) = p$ is also constant we refer to [35, Lemma 2.3] and [14, Lemma 2.1].

The next theorem generalizes Theorem 3.4.

Theorem 3.7. *Let $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. Further, let $\psi_0, \psi \in \mathscr{S}(\mathbb{R}^n)$ be such that*

$$|\widehat{\psi_0}(x)| > 0 \quad \textit{for} \quad |x| \leq k\varepsilon$$

$$|\widehat{\psi}(x)| > 0 \quad \textit{for} \quad \frac{\varepsilon}{2} \leq |x| \leq k\varepsilon,$$

or some $\varepsilon > 0$ *and* $k \in]1,2]$. *For*

$$a > \frac{n}{\min\{p^-, q^-\}} + \log_2 \tilde{c}_1(\phi) + c_{\log}(s),$$

there exists $c > 0$ *such that*

$$\left\| \left(2^{js(\cdot)}(\psi_j^* f)_a\right)_{j\in\mathbb{N}_0} \middle| L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \le c \left\| \left(2^{js(\cdot)}(\psi_j * f)\right)_{j\in\mathbb{N}_0} \middle| L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)}) \right\|$$

holds for all $f \in \mathscr{S}'(\mathbb{R}^n)$.

Proof: We are in conditions to apply formula (2.66) from [32], in particular, for $f \in \mathscr{S}'(\mathbb{R}^n)$, $N \in \mathbb{N}$, $a \in (0, N]$, $j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, we have

$$\left[(\psi_j^* f)_a(x)\right]^r \le c \sum_{\nu=0}^{\infty} 2^{-\nu N r} 2^{(\nu+j)n} \int_{\mathbb{R}^n} \frac{|(\psi_{\nu+j} * f)(y)|^r}{(1+2^j|x-y|)^{ar}}\, dy,$$

where r is an arbitrarily fixed positive number and c is a positive constant independent of ψ_0, ψ, f, x and j. Using this inequality, for what is not necessary that (ψ_0, ψ) is an admissible pair, then the proof is the same as the one of Theorem 3.11 in [36]. ■

Proof of Theorem 3.2: Observe that $|(\alpha_j * f)(x)| \le (\alpha_j^* f)_a(x)$ for any $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $\alpha_j \in \mathscr{S}(\mathbb{R}^n)$. Using this observation together with the lattice property of the $L_{p(\cdot)}$ spaces, and applying Theorems 3.4, 3.7, and 3.1, we obtain the following chain of inequalities

$$\begin{aligned}
\|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| &= \left\| \left(2^{js(\cdot)}(\varphi_j * f)\right)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \\
&\le \left\| \left(2^{js(\cdot)}(\varphi_j^* f)_a\right)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \lesssim \left\| \left(2^{js(\cdot)}(\psi_j^* f)_a\right)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \\
&\lesssim \left\| \left(2^{js(\cdot)}(\psi_j * f)\right)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \le \left\| \left(2^{js(\cdot)}(\psi_j^* f)_a\right)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \\
&\lesssim \left\| \left(2^{js(\cdot)}(\varphi_j^* f)_a\right)_{j\in\mathbb{N}_0} \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)}) \right\| \lesssim \|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|.
\end{aligned}$$

■

We finish this section by presenting an important application of Theorem 3.2, that is when ψ_0 and ψ, satisfying (6)-(8), are local means. The name comes from the compact support of $\psi_0 := k_0$ and $\psi := k$, which is admitted in the following statement.

Corollary 3.8. *Let* $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{\rm loc}(\mathbb{R}^n)$ *and* $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. *For given* $N \in \mathbb{N}_0$ *and* $d > 0$, *let* $k_0, k \in \mathscr{S}(\mathbb{R}^n)$ *with* supp k_0, supp $k \subset dQ_{0,0}$,

$$(D^\beta \widehat{k})(0) = 0 \quad \textit{if} \quad 0 \le |\beta| < N, \tag{13}$$

$\widehat{k_0}(0) \neq 0$ *and* $\widehat{k}(x) \neq 0$ *if* $0 < |x| < \varepsilon$, *for some* $\varepsilon > 0$. *If* $N > s^+ + \log_2 \tilde{c}_1(\phi)$, *then*

$$\|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \left\| \left(2^{js(\cdot)}(k_j * f)\right)_{j\in\mathbb{N}_0} \mid L^\phi_{p(\cdot)}(\ell_{q(\cdot)}) \right\|$$

for all $f \in \mathscr{S}'(\mathbb{R}^n)$.

Proof: It is clear the existence of $k_0, k^0 \in \mathscr{S}(\mathbb{R}^n)$ with supp k_0, supp $k \subset dQ_{0,0}$, $\widehat{k_0}(0) \neq 0$ and $\widehat{k^0}(0) \neq 0$. Then, following [30, 11.2] and taking $M \in \mathbb{N}_0$ with $2M \ge N$ define $k := \Delta^M k^0$. Since $\widehat{k}(x) = (-\sum_{i=1}^n |x_i|^2)^M \widehat{k^0}(x)$, we immediately have (13) and $\widehat{k}(x) \neq 0$ if $0 < |x| < \varepsilon$, for a small enough $\varepsilon > 0$. The rest is a direct consequence of Theorem 3.2. ■

Remark 3.9. The Triebel-Lizorkin spaces with variable exponents $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, which coincide with $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ when $\phi \equiv 1$ (cf. Remark 4.11(ii)), are also included in the class of 2-microlocal Triebel-Lizorkin spaces $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ defined in [9]. There $\boldsymbol{w} = (w_j(x))_{j\in\mathbb{N}_0}$ is an admissible weight sequence, i.e. a sequence of non-negative measurable functions in $\mathbb{R}^n$ belonging to a class $\mathcal{W}^\alpha_{\alpha_1,\alpha_2}(\mathbb{R}^n)$, where $\alpha \ge 0$, $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \le \alpha_2$, satisfying the conditions:

(i) there exists a constant $c > 0$ such that

$$0 < w_j(x) \le c\, w_j(y)\,(1 + 2^j|x-y|)^\alpha \quad \text{for all } j \in \mathbb{N}_0 \text{ and all } x, y \in \mathbb{R}^n;$$

(ii) for all $j \in \mathbb{N}_0$ it holds

$$2^{\alpha_1}\, w_j(x) \le w_{j+1}(x) \le 2^{\alpha_2}\, w_j(x) \quad \text{for all } x \in \mathbb{R}^n.$$

In fact, it holds $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ when $w_j(x) := 2^{js(x)}$, $j \in \mathbb{N}_0$, and in this case $\alpha_1 = s^-$, $\alpha_2 = s^+$ and $\alpha = c_{\log}(s)$. Replacing the weight sequence $(2^{js(\cdot)})_{j\in\mathbb{N}_0}$ in the quasi-norm from Definition 2.1 by a general admissible weight sequence $\boldsymbol{w} = (w_j(x))_{j\in\mathbb{N}_0}$ leads to the definition of a more general scale of functions spaces $F^{\boldsymbol{w},\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. These cover all the particular cases described in Remark 4.11 as well as the 2-microlocal Triebel-Lizorkin spaces studied in [9].

It can be easily seen that Theorem 3.2 as well as Corollary 3.8 hold more generally for $F^{\boldsymbol{w},\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with the conditions $R > s^+ + \log_2 \tilde{c_1}(\phi)$ and $a > \frac{n}{\min\{p^-,q^-\}} + \log_2 \tilde{c_1}(\phi) + c_{\log}(s)$ replaced by

$$R > \alpha_2 + \log_2 \tilde{c_1}(\phi) \quad \text{and} \quad a > \frac{n}{\min\{p^-, q^-\}} + \log_2 \tilde{c_1}(\phi) + \alpha.$$

Indeed, for a general weight we have, in replacement of (3.9),

$$w_j(x) \leq w_\nu(x) \times \begin{cases} 2^{(j-\nu)\alpha_2}, & j \geq \nu \\ 2^{(j-\nu)\alpha_1}. & j \leq \nu, \end{cases}$$

Furthermore, the counterpart of [11, Lemma 19] for a general weight sequence also holds and is a variant of [16, Lemma 2.2].

4. Non-smooth atomic decomposition

In [36] the authors obtained a characterization of the spaces $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ by smooth atomic decompositions, generalizing previous results obtained in [7, 10, 34] for the particular cases described in Remark 4.11. We recall the result from [36] and start by defining smooth atoms and appropriate sequence spaces, where we opted here for a different normalization.

Definition 4.1. *Let $K, L \in \mathbb{N}_0$. A function $a_Q \in C^K(\mathbb{R}^n)$ is called a $[K, L]$-smooth atom centered at $Q := Q_{\nu k} \in \mathcal{Q}$, where $\nu \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$, if*

$$\operatorname{supp} a_Q \subset 3\,Q,$$

$$\|a_Q(2^{-\nu}\cdot) \mid C^K(\mathbb{R}^n)\| \leq 1, \tag{14}$$

and, when $\nu \in \mathbb{N}$,

$$\int_{\mathbb{R}^n} x^\gamma a_Q(x)dx = 0, \tag{15}$$

for all multi-indices $\gamma \in \mathbb{N}_0^n$ with $|\gamma| < L$.

Remark 4.2. As usual when $L = 0$ no moment conditions are required by (15).

Definition 4.3. *Let p, s, and ϕ as in Definition 2.1. Let q be either as in Definition 2.1 or $q(\cdot) \equiv \infty$. Then the sequence space $f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all sequences $t := \{t_Q\}_{Q\in\mathcal{Q}^*} \subset \mathbb{C}$ such that*

$$\|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \Big\| \Big(\sum_{Q\subset P, Q\in\mathcal{Q}^*} \big[|Q|^{-\frac{s(\cdot)}{n}}|t_Q|\chi_Q\big]^{q(\cdot)} \Big)^{\frac{1}{q(\cdot)}} \mid L_{p(\cdot)}(P) \Big\|$$

is finite (with the usual modification if $q(\cdot) \equiv \infty$).

The following atomic decomposition characterization was obtained in [36, Theorem 3.18].

Theorem 4.4. *Let p, q, s, ϕ as in Definition 2.1.*

(i) *Let $K, L \in \mathbb{N}_0$ with*

$$K > s^+ + \max\{0, \log_2 \tilde{c}_1(\phi)\} \quad \text{and} \quad L > \frac{n}{\min\{1, p^-, q^-\}} - n - s^-.$$

Suppose that $\{a_Q\}_{Q\in\mathcal{Q}^*}$ is a family of $[K, L]$-smooth atoms and that $\{t_Q\}_{Q\in\mathcal{Q}^*} \in f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Then $f := \sum_{Q\in\mathcal{Q}^*} t_Q\, a_Q$ converges in $\mathscr{S}'(\mathbb{R}^n)$ and

$$\|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \le c\,\|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$$

with c being a positive constant independent of t.

(ii) *Conversely, if $f \in F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then, for any given $K, L \in \mathbb{N}_0$, there exists a sequence $\{t_Q\}_{Q\in\mathcal{Q}^*} \in f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and a sequence $\{a_Q\}_{Q\in\mathcal{Q}^*}$ of $[K, L]$-smooth atoms such that $f = \sum_{Q\in\mathcal{Q}^*} t_Q\, a_Q$ in $\mathscr{S}'(\mathbb{R}^n)$ and*

$$\|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \le c\,\|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$$

with c being a positive constant independent of f.

Next we present the notion of non-smooth atoms already used in [8] in the context of 2-microlocal spaces with variable exponents and which were slightly adapted from [24]. Note that the usual parameters K and L are now non-negative real numbers instead of non-negative integer numbers.

Definition 4.5. *Let $K, L \ge 0$. A function $a_Q : \mathbb{R}^n \to \mathbb{C}$ is called a $[K, L]$-non-smooth atom centered at $Q := Q_{\nu k} \in \mathcal{Q}$, with $\nu \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$, if*

$$\operatorname{supp} a_Q \subset 3\,Q, \tag{16}$$

$$\|a_Q(2^{-\nu}\cdot) \mid \mathscr{C}^K(\mathbb{R}^n)\| \le 1, \tag{17}$$

and for every $\psi \in \mathscr{C}^L(\mathbb{R}^n)$ it holds

$$\Big|\int_{\mathbb{R}^n} \psi(x) a_Q(x) dx\Big| \le c\, 2^{-\nu(L+n)} \|\psi \mid \mathscr{C}^L(\mathbb{R}^n)\|. \tag{18}$$

Remark 4.6. Since $C^k(\mathbb{R}^n) \hookrightarrow \mathscr{C}^k(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$, it is clear that condition (17) follows from (14). Moreover, using a Taylor expansion, (18) can be derived from (15) when $L \in \mathbb{N}$, cf. [24, Remark 3.4]. Therefore, when $K, L \in \mathbb{N}_0$ any $[K, L]$-smooth atom is a $[K, L]$-non-smooth atom. Moreover, both conditions (17) and (18) are ordered in K and L, i.e. the conditions are stricter for increasing K and L, see [24, Remark. 3.4].

For the next two auxiliary results we refer to [8, Lemmas 3.6,3.7].

Lemma 4.7. *Let k_j be the local means according to Corollary 3.8 with $d = 3$. Then $c2^{-jn}\, k_j$ is a non-smooth $[K, L]$-atom centered at Q_{j0}, for some constant $c > 0$ independently of j and for arbitrary large $K > 0$ and $L \leq N + 1$.*

Lemma 4.8. *Let k_j be the local means according to Corollary 3.8 with $d = 3$. Let also $(a_Q)_{Q\in\mathcal{Q}^*}$ be non-smooth $[K, L]$-atoms. Then, with $Q = Q_{\nu,k}$, $\nu \in \mathbb{N}_0$, $k \in \mathbb{Z}^n$, it holds*

$$\left| \int_{\mathbb{R}^n} k_j(y) a_Q(x-y)\, dy \right| \leq c\, 2^{-(j-\nu)K} \chi(c\, Q)(x), \qquad \textit{for } j \geq \nu$$

and

$$\left| \int_{\mathbb{R}^n} k_j(x-y) a_Q(y)\, dy \right| \leq c\, 2^{-(\nu-j)(L+n)} \chi(c\, 2^{\nu-j} Q)(x), \qquad \textit{for } j < \nu.$$

Remark 4.9. We would like to highlight that the results in [8] are not properly written. There the authors used a sequence $(\psi_j)_{j\in\mathbb{N}_0}$ built upon a pair of functions defined as in Theorem 3.2, with $k = 2$. However, these functions are not required to have compact support, crucial condition in the proof of these two results. Therefore, the correct formulation reads as we present here, where we use the local means from Corollary 3.8.

Theorem 4.10. *Let p, q, s, ϕ as in Definition 2.1.*

(i) *Let $K, L \geq 0$ with*

$$K > s^+ + \max\{0, \log_2 \tilde{c}_1(\phi)\} \qquad \textit{and} \qquad L > \frac{n}{\min\{1, p^-, q^-\}} - n - s^-. \tag{19}$$

Suppose that $\{a_Q\}_{Q\in\mathcal{Q}^*}$ is a family of $[K, L]$-non-smooth atoms and that $\{t_Q\}_{Q\in\mathcal{Q}^*} \in f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Then $f := \sum_{Q\in\mathcal{Q}^*} t_Q\, a_Q$ converges in $\mathscr{S}'(\mathbb{R}^n)$ and

$$\|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \leq c\, \|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$$

with c being a positive constant independent of t.

(ii) *Conversely, if* $f \in F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*, then, for any given* $K, L \geq 0$*, there exists a sequence* $\{t_Q\}_{Q\in\mathcal{Q}^*} \in f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and a sequence* $\{a_Q\}_{Q\in\mathcal{Q}^*}$ *of* $[K, L]$*-non-smooth atoms such that* $f = \sum_{Q\in\mathcal{Q}^*} t_Q\, a_Q$ *in* $\mathscr{S}'(\mathbb{R}^n)$ *and*

$$\|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \leq c\, \|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$$

with c *being a positive constant independent of* f*.*

Proof: *Step 1.* We start by proving (ii) for what we fix $K, L \geq 0$ and assume that $f \in F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is given. Then, by Theorem 4.4(ii), we know that f can be written as an atomic decomposition with $[K_1, L_1]$-smooth-atoms with $K_1, L_1 \in \mathbb{N}_0$ chosen so that $K_1 \geq K$ and $L_1 \geq L$. Since those atoms are $[K, L]$-non-smooth-atoms, cf. Remark 4.6, part (ii) is proved.

Step 2. In this step we show that $f = \sum_{Q\in\mathcal{Q}^*} t_Q\, a_Q$ converges in $\mathscr{S}'(\mathbb{R}^n)$ if $\{t_Q\}_{Q\in\mathcal{Q}^*} \in f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\{a_Q\}_{Q\in\mathcal{Q}^*}$ is a family of $[K, L]$-non-smooth atoms with $K, L \geq 0$ such that (19) holds. To this end, it suffices to show that

$$\lim_{N\to\infty,\Lambda\to\infty} \sum_{\nu=0}^{N} \sum_{k\in\mathbb{Z}^n, |k|\leq\Lambda} t_{Q_{\nu k}} a_{Q_{\nu k}} \tag{20}$$

exists in $\mathscr{S}'(\mathbb{R}^n)$, and we rely on the proof of Step 1 of [36, Theorem 3.8]. For convenience of the reader we repeat here the arguments. For $r \in (0, \min\{1, p^-, q^-\})$, we have the following sequence of embeddings

$$f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow f^{\tilde{s}(\cdot),\phi}_{\tilde{p}(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow f^{\tilde{s}(\cdot),\phi}_{\tilde{p}(\cdot),\infty}(\mathbb{R}^n),$$

with $\tilde{s}(x) := s(x) + \frac{n}{p(x)}(r-1)$ and $\tilde{p}(x) := \frac{p(x)}{r}$ for all $x \in \mathbb{R}^n$, cf. [36, Proposition 3.1]. Therefore we can assume that $\{t_Q\}_{Q\in\mathcal{Q}^*} \in f^{\tilde{s}(\cdot),\phi}_{\tilde{p}(\cdot),\infty}(\mathbb{R}^n)$.

Let $h \in \mathscr{S}(\mathbb{R}^n)$. From conditions (16) and (17) of Definition 4.5, we obtain

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n, |k|\leq\Lambda} t_{Q_{\nu k}} a_{Q_{\nu k}}(x) h(x)\, dx \Big| \\
&\qquad \lesssim 2^{-\nu(L+n)} \sum_{k\in\mathbb{Z}^n, |k|\leq\Lambda} |t_{Q_{\nu k}}|\, \|h(\cdot) H(2^\nu \cdot - k) \mid \mathscr{C}^L(\mathbb{R}^n)\| \\
&\qquad = 2^{-\nu L} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n, |k|\leq\Lambda} |t_{Q_{\nu k}}|\, \|h(\cdot) H(2^\nu \cdot - k) \mid \mathscr{C}^L(\mathbb{R}^n)\|\, \chi_{Q_{\nu k}}(y)\, dy
\end{aligned} \tag{21}$$

where $H \in C^\infty(\mathbb{R}^n)$ with $H(x) = 1$ for $x \in 3\,Q_{00}$ and supp $H \subset 4\,Q_{00}$. Since $h \in \mathscr{S}(\mathbb{R}^n)$, we can estimate the norm in (21) from above by

$$\|h(\cdot)H(2^\nu \cdot -k) \mid \mathscr{C}^L(\mathbb{R}^n)\| \lesssim (1+|x_{Q_{\nu k}}|)^{-\delta} \sim (1+|y|)^{-\delta}, \quad \text{for } y \in Q_{\nu k},$$

with $\delta > 0$ at our disposal and the underlying constants do not depend neither on ν nor on k. Then, (21) becomes

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n, |k|\le\Lambda} t_{Q_{\nu k}} a_{Q_{\nu k}}(x)\phi(x)\,dx \Big| \\
&\quad\lesssim 2^{-\nu L} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n, |k|\le\Lambda} |t_{Q_{\nu k}}|(1+|y|)^{-\delta}\chi_{Q_{\nu k}}(y)\,dy \\
&\quad\lesssim 2^{-\nu L} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n, |k|\le\Lambda} |t_{Q_{\nu k}}| \frac{(1+|y|)^{-\delta}}{(1+2^\nu|y-x_{Q_{\nu k}}|)^R}\,dy \\
&\quad\lesssim \sum_{j=0}^{\infty} 2^{-\nu L} \int_{D_j} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}| \frac{(1+|y|)^{-\delta}}{(1+2^\nu|y-x_{Q_{\nu k}}|)^R}\,dy, \qquad (22)
\end{aligned}$$

where $R > 0$ is as big as we want, $D_0 := \{x \in \mathbb{R}^n : |x| \le 1\}$ and, for all $j \in \mathbb{N}$,

$$D_j := \{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^j\}.$$

For all $j, \nu \in \mathbb{N}_0$ and $y \in D_j$, let $W_0^{y,\nu} := \{k \in \mathbb{Z}^n : 2^\nu|y - x_{Q_{\nu k}}| \le 1\}$ and, for $i \in \mathbb{N}$,

$$W_i^{y,\nu} := \{k \in \mathbb{Z}^n : 2^{i-1} < 2^\nu|y - x_{Q_{\nu k}}| \le 2^i\}.$$

Then, we have

$$\begin{aligned}
H(j,\nu,y) &:= \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|(1+2^\nu|y-x_{Q_{\nu k}}|)^{-R} \sim \sum_{i=0}^{\infty} \sum_{k\in W_i^{y,\nu}} |t_{Q_{\nu k}}| 2^{-iR} 2^{\nu n} 2^{-\nu n} \\
&\sim \sum_{i=0}^{\infty} 2^{-iR} \int_{\bigcup_{\bar{k}\in W_i^{y,\nu}} Q_{\nu\bar{k}}} 2^{\nu n} \Big[\sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}| \chi_{Q_{\nu k}}(z) \Big] dz.
\end{aligned}$$

Note that, if $z \in \bigcup_{\bar{k}\in W_i^{y,\nu}} Q_{\nu\bar{k}}$, then $z \in Q_{\nu\bar{k}_0}$ for some $\bar{k}_0 \in W_i^{y,\nu}$ and, for $y \in D_j$, $1 + 2^\nu|y - z| \sim 1 + 2^i$; moreover,

$$|z| \le |z - x_{Q_{\nu\bar{k}_0}}| + |x_{Q_{\nu\bar{k}_0}} - y| + |y| \lesssim 2^{-\nu} + 2^{-\nu+i} + 2^j \lesssim 2^{i+j},$$

which imples that

$$\bigcup_{\bar{k}\in W_i^{y,\nu}} Q_{\nu\bar{k}} \subset Q(0,2^{i+j+c_0})$$

with some positive constant $c_0 \in \mathbb{N}$. From this, we deduce that, for all $a \in (n,\infty)$,

$$\begin{aligned} H(j,\nu,y) &\sim \sum_{i=0}^{\infty} 2^{-i(R-a)} \int_{\bigcup_{\bar{k}\in W_i^{y,\nu}} Q_{\nu\bar{k}}} \frac{2^{\nu n}}{(1+2^{\nu}|y-z|)^a} \\ &\qquad \cdot \left[\sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|\chi_{Q_{\nu k}}(z)\chi_{Q(0,2^{i+j+c_0})}(z)\right] dz \\ &\lesssim \sum_{i=0}^{\infty} 2^{-i(R-a)} \eta_{\nu,a} * \left(\sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|\chi_{Q_{\nu k}}\chi_{Q(0,2^{i+j+c_0})}\right)(y), \end{aligned}$$

where $\eta_{\nu,a}(x) := 2^{\nu n}(1+2^{\nu}|x|)^{-a}$, $x \in \mathbb{R}^n$. We go back to (22) and get

$$\begin{aligned} \Bigg|\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n,|k|\le\Lambda} t_{Q_{\nu k}} a_{Q_{\nu k}}(x)\phi(x)\,dx\Bigg| &\lesssim 2^{-\nu L} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} 2^{-i(R-a)} \\ &\quad \cdot \int_{D_j} \eta_{\nu,a} * \left(\sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|\chi_{Q_{\nu k}}\chi_{Q(0,2^{i+j+c_0})}\right)(y)\,(1+|y|)^{-\delta}\,dy \\ &\lesssim 2^{-\nu L} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} 2^{-i(R-a)}(1+2^j)^{-\delta_0} \\ &\quad \cdot \int_{D_j} \eta_{\nu,a} * \left(\sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|\chi_{Q_{\nu k}}\chi_{Q(0,2^{i+j+c_0})}\right)(y)\,(1+|y|)^{-\delta+\delta_0}\,dy \\ &\lesssim 2^{-\nu(L+\tilde{s}^-)} \sum_{j=0}^{\infty} 2^{-j\delta_0} \sum_{i=0}^{\infty} 2^{-i(R-a)} \left\|(1+|\cdot|)^{-\delta+\delta_0} \mid L_{\tilde{p}'(\cdot)}(\mathbb{R}^n)\right\| \\ &\quad \cdot \left\|\eta_{\nu,a} * \left(\sum_{k\in\mathbb{Z}^n} 2^{\nu\tilde{s}(\cdot)}|t_{Q_{\nu k}}|\chi_{Q_{\nu k}}\chi_{Q(0,2^{i+j+c_0})}\right) \Bigg| L_{\tilde{p}(\cdot)}(\mathbb{R}^n)\right\| \end{aligned}$$

$$\lesssim 2^{-\nu(L+\tilde{s}^-)} \sum_{j=0}^{\infty} 2^{-j\delta_0} \sum_{i=0}^{\infty} 2^{-i(R-a)}$$
$$\cdot \Big\| \Big| \sum_{k\in\mathbb{Z}^n} 2^{\nu\tilde{s}(\cdot)} |t_{Q_{\nu k}}| \chi_{Q_{\nu k}} \chi_{Q(0,2^{i+j+c_0})} \Big| \, L_{\tilde{p}(\cdot)}(\mathbb{R}^n) \Big\|,$$

with $\delta > \delta_0 + n(1 - \frac{r}{p^+})$, applying the Hölder inequality for $L_{p(\cdot)}$-spaces with $\tilde{p}'(x) := \frac{p(x)-r}{p(x)}$ for all $x \in \mathbb{R}^n$, as well as [6, Theorem 3.2] with $a > n$. Then, using (2), we obtain

$$\Big| \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n, |k|\le\Lambda} t_{Q_{\nu k}} a_{Q_{\nu k}}(x)\phi(x)\, dx \Big| \lesssim 2^{-\nu(L+\tilde{s}^-)} \sum_{j=0}^{\infty} 2^{-j\delta_0} \sum_{i=0}^{\infty} 2^{-i(R-a)}$$
$$\cdot \Big\| \Big| \sum_{k\in\mathbb{Z}^n} 2^{\nu\tilde{s}(\cdot)} |t_{Q_{\nu k}}| \chi_{Q_{\nu k}} \Big| \, L_{\tilde{p}(\cdot)}(Q(0,2^{i+j+c_0})) \Big\|$$
$$\lesssim 2^{-\nu(L+\tilde{s}^-)} \sum_{j=0}^{\infty} 2^{-j\delta_0} \sum_{i=0}^{\infty} 2^{-i(R-a)} \phi(Q(0,2^{i+j+c_0})) \big\| t \mid f^{\tilde{s}(\cdot),\phi}_{\tilde{p}(\cdot),\infty}(\mathbb{R}^n) \big\|$$
$$\lesssim 2^{-\nu(L+\tilde{s}^-)} \sum_{j=0}^{\infty} 2^{-j(\delta_0 - \log_2 \tilde{c}_1(\phi)} \sum_{i=0}^{\infty} 2^{-i(R-a-\log_2 \tilde{c}_1(\phi))} \big\| t \mid f^{\tilde{s}(\cdot),\phi}_{\tilde{p}(\cdot),\infty}(\mathbb{R}^n) \big\|$$
$$\lesssim 2^{-\nu(L+\tilde{s}^-)} \big\| t \mid f^{\tilde{s}(\cdot),\phi}_{\tilde{p}(\cdot),\infty}(\mathbb{R}^n) \big\|, \tag{23}$$

considering $\delta_0 > \max\{0, \log_2 \tilde{c}_1(\phi)\}$ and $R > a + \log_2 \tilde{c}_1(\phi)$. By (19), we find that there exists $r \in (0, \min\{1, p^-, q^-\})$ such that $s^- + \frac{n}{p^-}(r-1) > -L$. Therefore,

$$\tilde{s}^- \ge \inf_{x\in\mathbb{R}^n} s(x) + \inf_{x\in\mathbb{R}^n} \frac{n(r-1)}{p(x)} = s^- + \frac{n}{p^-}(r-1) > -L,$$

which, together with (23), implies that (20) exists in $\mathscr{S}'(\mathbb{R}^n)$.

Step 3. We deal with part (i). Assume that $\{t_Q\}_{Q\in\mathcal{Q}^*} \in f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and that $\{a_Q\}_{Q\in\mathcal{Q}^*}$ is a family of $[K,L]$-non-smooth atoms with $K, L \ge 0$ such that (19) holds. We have shown in Step 2 that $f = \sum_{Q\in\mathcal{Q}^*} t_Q\, a_Q$ converges in $\mathscr{S}'(\mathbb{R}^n)$. We shall prove now that

$$\|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \lesssim \|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|.$$

Let k_j, $j \in \mathbb{N}_0$, be local means as in Corollary 3.8. For a given dyadic cube $P \in \mathcal{Q}$ and $j \in \mathbb{N}_0$, it holds

$$k_j * f = \sum_{\nu=0}^{(j_P\vee 0)-1} \sum_{k\in\mathbb{Z}^n} t_{Q_{\nu k}}\, k_j * a_{Q_{\nu k}} + \sum_{\nu=(j_P\vee 0)}^{j} \sum_{k\in\mathbb{Z}^n} t_{Q_{\nu k}}\, k_j * a_{Q_{\nu k}} + \sum_{\nu=j+1}^{\infty} \sum_{k\in\mathbb{Z}^n} t_{Q_{\nu k}}\, k_j * a_{Q_{\nu k}}$$

where $\sum_{\nu=0}^{(j_P\vee 0)-1} \dots = 0 \quad \text{if} \quad j_P \le 0.$

Let $r \in (0, \min\{1, p^-, q^-\})$ be such that $L > \frac{n}{r} - n - s^-$. Thus, we find that

$$\begin{aligned}
&\frac{1}{\phi(P)}\left\| \Big(\sum_{j=(j_P\vee 0)}^{\infty} 2^{js(\cdot)q(\cdot)} |k_j * f|^{q(\cdot)} \Big)^{\frac{1}{q(\cdot)}} \mid L_{p(\cdot)}(P) \right\| \\
&\lesssim \frac{1}{\phi(P)}\left\| \left\{ \sum_{j=(j_P\vee 0)}^{\infty} \Big(2^{js(\cdot)r} \sum_{\nu=0}^{(j_P\vee 0)-1} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r\, |k_j * a_{Q_{\nu k}}|^r \Big)^{\frac{q(\cdot)}{r}} \right\}^{\frac{r}{q(\cdot)}} \Bigg| \, L_{\frac{p(\cdot)}{r}}(P) \right\|^{\frac{1}{r}} \\
&\quad + \frac{1}{\phi(P)}\left\| \left\{ \sum_{j=(j_P\vee 0)}^{\infty} \Big(2^{js(\cdot)r} \sum_{\nu=(j_P\vee 0)}^{j} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r\, |k_j * a_{Q_{\nu k}}|^r \Big)^{\frac{q(\cdot)}{r}} \right\}^{\frac{r}{q(\cdot)}} \Bigg| \, L_{\frac{p(\cdot)}{r}}(P) \right\|^{\frac{1}{r}} \\
&\quad + \frac{1}{\phi(P)}\left\| \left\{ \sum_{j=(j_P\vee 0)}^{\infty} \Big(2^{js(\cdot)r} \sum_{\nu=j+1}^{\infty} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r\, |k_j * a_{Q_{\nu k}}|^r \Big)^{\frac{q(\cdot)}{r}} \right\}^{\frac{r}{q(\cdot)}} \Bigg| \, L_{\frac{p(\cdot)}{r}}(P) \right\|^{\frac{1}{r}} \\
&=: I + II + III. \qquad (24)
\end{aligned}$$

Observe that $I = 0$ if $j_P \le 0$. Thus, to estimate I we need to assume that $j_P > 0$. By Lemma 4.8, we have

$$\begin{aligned}
I \lesssim \frac{1}{\phi(P)}\Bigg\| \Bigg\{ \sum_{j=j_P}^{\infty} \Big(2^{js(\cdot)r} \sum_{\nu=0}^{j_P-1} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r 2^{-(j-\nu)Kr} \\
\cdot (1 + 2^{\nu}|\cdot - x_{Q_{\nu k}}|)^{-Mr} \Big)^{\frac{q(\cdot)}{r}} \Bigg\}^{\frac{r}{q(\cdot)}} \Bigg| \, L_{\frac{p(\cdot)}{r}}(P) \Bigg\|^{\frac{1}{r}}
\end{aligned}$$

$$\leq \frac{1}{\phi(P)} \Bigg\| \sum_{j=j_P}^{\infty} \sum_{\nu=0}^{j_P-1} 2^{js(\cdot)r} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r 2^{-(j-\nu)Kr} \cdot (1+2^\nu|\cdot - x_{Q_{\nu k}}|)^{-Mr} \Bigg| L_{\frac{p(\cdot)}{r}}(P) \Bigg\|^{\frac{1}{r}},$$

where $M \in (0,\infty)$ is as large as we want.

Proceeding as in Step 2 of the proof of Theorem 3.8 in [36] to estimate I_1 therein, and using the fact that $K > s^+ + \max\{0, \log_2 \tilde{c}_1(\phi)\}$ and by choosing $M > a + \frac{\varepsilon}{r} + \log_2 \tilde{c}_1(\phi)$, we arrive at

$$I \lesssim \big\| t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)} \big\|. \tag{25}$$

Now we estimate II. By Lemma 4.8, we have

$$\begin{aligned} II &\lesssim \frac{1}{\phi(P)} \Bigg\| \Bigg\{ \sum_{j=(j_P\vee 0)}^{\infty} \Bigg(2^{js(\cdot)r} \sum_{\nu=(j_P\vee 0)}^{j} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r 2^{-(j-\nu)Kr} \\ &\qquad \cdot (1+2^\nu|\cdot - x_{Q_{\nu k}}|)^{-Mr} \Bigg)^{\frac{q(\cdot)}{r}} \Bigg\}^{\frac{r}{q(\cdot)}} \Bigg| L_{\frac{p(\cdot)}{r}}(P) \Bigg\|^{\frac{1}{r}} \\ &\leq \frac{1}{\phi(P)} \Bigg\| \sum_{j=(j_P\vee 0)}^{\infty} \sum_{\nu=(j_P\vee 0)}^{j} 2^{js(\cdot)r} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r 2^{-(j-\nu)Kr} \\ &\qquad \cdot (1+2^\nu|\cdot - x_{Q_{\nu k}}|)^{-Mr} \Bigg| L_{\frac{p(\cdot)}{r}}(P) \Bigg\|^{\frac{1}{r}} \end{aligned}$$

where $M \in (0,\infty)$ is as large as we want. We claim that, for fixed $x \in P$, $j \geq (j_P \vee 0)$ and ν with $(j_P \vee 0) \leq \nu \leq j$,

$$\begin{aligned} J(\nu,j,x,P) &:= 2^{js(x)r} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r \, 2^{-(j-\nu)Kr} (1+2^\nu|x - x_{Q_{\nu k}}|)^{-Mr} \\ &\lesssim 2^{-(j-\nu)(K-s^+)r} \sum_{i=0}^{\infty} 2^{-(j+i)(M-a-\frac{\varepsilon}{r})r} \\ &\qquad \cdot \eta_{\nu,ar} * \Bigg(\Bigg[\sum_{k\in\Omega_i^{x,\nu,j}} |t_{Q_{\nu k}}| 2^{\nu s(\cdot)} \chi_{Q_{\nu k}} \chi_{Q(c_P, 2^{-i+j-j_P+c_0})} \Bigg]^r \Bigg)(x), \end{aligned}$$

where $a > \frac{n}{r}, \varepsilon > c_{\log}(s)$, c_P is the center of P, $c_0 \in \mathbb{N}$ is a positive constant independent of x, P, i, ν, k, j,

$$\Omega_0^{x,\nu,j} := \{k \in \mathbb{Z}^n : 2^\nu |x - x_{Q_{\nu k}}| \le 2^j\}$$

and, for all $i \in \mathbb{N}$,

$$\Omega_i^{x,\nu,j} := \{k \in \mathbb{Z}^n : 2^{j+i-1} < 2^\nu |x - x_{Q_{\nu k}}| \le 2^{j+i}\}.$$

Indeed, we can see that

$$\begin{aligned} J(\nu, j, x, P) &= 2^{js(x)r} \sum_{i=0}^{\infty} \sum_{k \in \Omega_i^{x,\nu,j}} |t_{Q_{\nu k}}|^r \, 2^{-(j-\nu)Kr} (1 + 2^\nu |x - x_{Q_{\nu k}}|)^{-Mr} \\ &\lesssim 2^{js(x)r} \sum_{i=0}^{\infty} \sum_{k \in \Omega_i^{x,\nu,j}} |t_{Q_{\nu k}}|^r \, 2^{-(j-\nu)Kr} \, 2^{-(j+i)Mr} \, 2^{\nu n} \int_{\bigcup_{\bar{k} \in \Omega_i^{x,\nu,j}} Q_{\nu \bar{k}}} \chi_{Q_{\nu k}}(y) \, dy. \end{aligned}$$

Observe that, if $y \in \bigcup_{\bar{k} \in \Omega_i^{x,\nu,j}} Q_{\nu \bar{k}}$, then there exists a $\bar{k}_0 \in \Omega_i^{x,\nu,j}$ such that $y \in Q_{\nu \bar{k}_0}$ and $2^\nu |x - x_{Q_{\nu \bar{k}_0}}| \le 2^{j+i}$. Then,

$$2^\nu |x - y| \le 2^\nu |x - x_{Q_{\nu \bar{k}_0}}| + 2^\nu |x_{Q_{\nu \bar{k}_0}} - y| \lesssim 2^{j+i} + 2^\nu 2^{-\nu}.$$

Moreover, since $\nu \ge j_P$, it follows that

$$|y - c_P| \le |y - x_{Q_{\nu \bar{k}_0}}| + |x - x_{Q_{\nu \bar{k}_0}}| + |x - c_P| \lesssim 2^{-\nu} + 2^{j+i-\nu} + 2^{-j_P} \lesssim 2^{i+j-j_P},$$

which implies that

$$\bigcup_{\bar{k} \in \Omega_i^{x,\nu,j}} Q_{\nu \bar{k}} \subset Q(c_P, 2^{i+j-j_P+c_0}),$$

for some constant $c_0 \in \mathbb{N}$. From this, we get

$$\begin{aligned} J(\nu, j, x, P) &\lesssim 2^{-(j-\nu)(K-s^+)r} \sum_{i=0}^{\infty} 2^{-(j+i)Mr + \nu n} \\ &\quad \cdot \int_{\bigcup_{\bar{k} \in \Omega_i^{x,\nu,j}} Q_{\nu \bar{k}}} 2^{\nu s(x) r} \sum_{k \in \Omega_i^{x,\nu,j}} |t_{Q_{\nu k}}|^r \chi_{Q_{\nu k}}(y) \chi_{Q(c_P, 2^{i+j-j_P+c_0})}(y) \, dy \end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{-(j-\nu)(K-s^+)r} \sum_{i=0}^{\infty} 2^{-(j+i)(M-a-\frac{\varepsilon}{r})r} \int_{\bigcup_{\bar{k}\in\Omega_i^{x,\nu,j}} Q_{\nu\bar{k}}} 2^{\nu s(x)r} \eta_{\nu,ar+\varepsilon}(x-y) \\
&\qquad \cdot \Big[\sum_{k\in\Omega_i^{x,\nu,j}} |t_{Q_{\nu k}}| \chi_{Q_{\nu k}}(y) \chi_{Q(c_P,2^{i+j-j_P+c_0})} \Big]^r (y)\, dy \\
&\lesssim 2^{-(j-\nu)(K-s^+)r} \sum_{i=0}^{\infty} 2^{-(j+i)(M-a-\frac{\varepsilon}{r})r} \\
&\qquad \cdot \eta_{\nu,ar} * \bigg(\Big[\sum_{k\in\Omega_i^{x,\nu,j}} |t_{Q_{\nu k}}| 2^{\nu s(\cdot)} \chi_{Q_{\nu k}} \chi_{Q(c_P,2^{i+j-j_P+c_0})} \Big]^r \bigg)(x),
\end{aligned}
$$

where the last step follows by Lemma 19 of [11] with $\varepsilon > c_{\log}(s)$. This implies that the claim holds true.

By this claim, we go back to the norm and conclude that, by Theorem 3.2 of [6] with $a > \frac{n}{r}$ and using (2),

$$
\begin{aligned}
II &\lesssim \frac{1}{\phi(P)} \Bigg\{ \sum_{j=(j_P\vee 0)}^{\infty} 2^{-j(K-s^+ +M-a-\frac{\varepsilon}{r})r} \sum_{\nu=(j_P\vee 0)}^{j} 2^{\nu(K-s^+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r})r} \\
&\qquad \cdot \bigg\| \eta_{\nu,ar} * \bigg(\Big[\sum_{k\in\Omega_i^{x,\nu,j}} |t_{Q_{\nu k}}| 2^{\nu s(\cdot)} \chi_{Q_{\nu k}} \chi_{Q(c_P,2^{i+j-j_P+c_0})} \Big]^r \bigg) \bigg| L_{\frac{p(\cdot)}{r}}(P) \bigg\| \Bigg\}^{\frac{1}{r}} \\
&\lesssim \frac{1}{\phi(P)} \Bigg\{ \sum_{j=(j_P\vee 0)}^{\infty} 2^{-j(K-s^+ +M-a-\frac{\varepsilon}{r})r} \sum_{\nu=(j_P\vee 0)}^{j} 2^{\nu(K-s^+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r})r} \\
&\qquad \cdot \bigg\| \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}| 2^{\nu s(\cdot)} \chi_{Q_{\nu k}} \bigg| L_{p(\cdot)}(Q(c_P, 2^{i+j-j_P+c_0})) \bigg\|^r \Bigg\}^{\frac{1}{r}} \\
&\lesssim \big\| t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \big\| \Bigg\{ \sum_{j=(j_P\vee 0)}^{\infty} 2^{-j(K-s^+ +M-a-\frac{\varepsilon}{r})r} \\
&\qquad \cdot \sum_{\nu=(j_P\vee 0)}^{j} 2^{\nu(K-s^+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r})r} \frac{\big[\phi(Q(c_P, 2^{i+j-j_P+c_0}))\big]^r}{\phi(P)^r} \Bigg\}^{\frac{1}{r}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \left\| t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| \Bigg\{ \sum_{j=(j_P\vee 0)}^{\infty} 2^{-j(K-s^+ +M-a-\frac{\varepsilon}{r})r} \\
&\qquad \cdot \sum_{\nu=(j_P\vee 0)}^{j} 2^{\nu(K-s^+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r})r} 2^{r(i+j)\log_2 \tilde{c}_1(\phi)} \Bigg\}^{\frac{1}{r}} \\
&= \left\| t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| \Bigg\{ \sum_{j=(j_P\vee 0)}^{\infty} 2^{-j(K-s^+ +M-a-\frac{\varepsilon}{r}-\log_2 \tilde{c}_1(\phi))r} \\
&\qquad \cdot \sum_{\nu=(j_P\vee 0)}^{j} 2^{\nu(K-s^+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r}-\log_2 \tilde{c}_1(\phi))r} \Bigg\}^{\frac{1}{r}} \\
&\lesssim \left\| t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|, \qquad (26)
\end{aligned}
$$

provided that $K > s^+$ and $M > a + \frac{\varepsilon}{r} + \log_2 \tilde{c}_1(\phi)$.

Now we deal with III. By Lemma 4.8, we have

$$
\begin{aligned}
III &\lesssim \frac{1}{\phi(P)} \Bigg\| \Bigg\{ \sum_{j=j_P}^{\infty} \Bigg(2^{js(\cdot)r} \sum_{\nu=j+1}^{\infty} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r 2^{-(\nu-j)(L+n)r} \\
&\qquad \cdot (1+2^j|\cdot - x_{Q_{\nu k}}|)^{-Mr} \Bigg)^{\frac{q(\cdot)}{r}} \Bigg\}^{\frac{r}{q(\cdot)}} \Bigg| L_{\frac{p(\cdot)}{r}}(P) \Bigg\|^{\frac{1}{r}},
\end{aligned}
$$

where $M \in (0,\infty)$ is as large as we want. We claim that, for fixed $x \in P$, $\nu, j \in \mathbb{N}_0$ with $\nu > j$,

$$
\begin{aligned}
J(\nu,j,x,P) &:= 2^{js(x)r} \sum_{k\in\mathbb{Z}^n} |t_{Q_{\nu k}}|^r 2^{-(\nu-j)(L+n)r} (1+2^j|x-x_{Q_{\nu k}}|)^{-Mr} \\
&\lesssim 2^{-(\nu-j)(L+n+s^- -\frac{n}{r})r} \sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r})r} \\
&\qquad \cdot \eta_{j,ar} * \Bigg(\Bigg[\sum_{k\in\Lambda_i^{x,\nu,j}} |t_{Q_{\nu k}}| 2^{\nu s(\cdot)} \chi_{Q_{\nu k}} \chi_{Q(c_P, 2^{i-j_P+c_0})} \Bigg]^r \Bigg)(x),
\end{aligned}
$$

where $a > \frac{n}{r}, \varepsilon > c_{\log}(s)$, c_P is the center of P, $c_0 \in \mathbb{N}$ is a positive constant independent of x, P, i, ν, k,

$$
\Lambda_0^{x,\nu} := \{k \in \mathbb{Z}^n : 2^j|x - x_{Q_{\nu k}}| \le 1\}
$$

and, for all $i \in \mathbb{N}$,

$$\Lambda_i^{x,\nu,j} := \{k \in \mathbb{Z}^n : 2^{i-1} < 2^j|x - x_{Q_{\nu k}}| \leq 2^i\}.$$

Indeed, we can see that

$$\begin{aligned} J(\nu,j,x,P) &= 2^{js(x)r} \sum_{i=0}^{\infty} \sum_{k\in\Lambda_i^{x,\nu,j}} |t_{Q_{\nu k}}|^r 2^{-(\nu-j)(L+n)r}(1+2^j|x-x_{Q_{\nu k}}|)^{-Mr} \\ &\leq 2^{js(x)r} \sum_{i=0}^{\infty} \sum_{k\in\Lambda_i^{x,\nu,j}} |t_{Q_{\nu k}}|^r 2^{-(\nu-j)(L+n)r} 2^{-iMr} 2^{\nu n} \int_{\bigcup_{\bar{k}\in\Lambda_i^{x,\nu}} Q_{\nu \bar{k}}} \chi_{Q_{\nu k}}(y)\, dy. \end{aligned}$$

Observe that, if $y \in \bigcup_{\bar{k}\in\Lambda_i^{x,\nu,j}} Q_{\nu\bar{k}}$, then there exists a $\bar{k}_0 \in \Lambda_i^{x,\nu,j}$ such that $y \in Q_{\nu\bar{k}_0}$ and $2^j|x - x_{Q_{\nu\bar{k}_0}}| \leq 2^i$. Then, since $\nu \geq j$, it follows that

$$2^j|x-y| \leq 2^j|x - x_{Q_{\nu\bar{k}_0}}| + 2^j|x_{Q_{\nu\bar{k}_0}} - y| \lesssim 2^i + 2^{j-\nu} \lesssim 2^i$$

and hence

$$|y - c_P| \leq |y - x_{Q_{\nu\bar{k}_0}}| + |x - x_{Q_{\nu\bar{k}_0}}| + |x - c_P| \lesssim 2^{-\nu} + 2^{i-j} + 2^{-j_P} \lesssim 2^{i-j_P},$$

which implies that

$$\bigcup_{\bar{k}\in\Lambda_i^{x,\nu,j}} Q_{\nu\bar{k}_0} \subset Q(c_P, 2^{i-j_P+c_0}),$$

for some constant $c_0 \in \mathbb{N}$. From this, we get

$$\begin{aligned} J(\nu,j,x,P) &\lesssim 2^{-(\nu-j)(L+n+s^-)r} \sum_{i=0}^{\infty} 2^{-iMr+\nu n} \\ &\quad \cdot \int_{\bigcup_{\bar{k}\in\Lambda_i^{x,\nu,j}} Q_{\nu\bar{k}}} 2^{\nu s(x)r} \sum_{k\in\Lambda_i^{x,\nu,j}} |t_{Q_{\nu k}}|^r \chi_{Q_{\nu k}}(y) \chi_{Q(c_P,2^{i-j_P+c_0})}(y)\, dy \end{aligned}$$

$$\lesssim 2^{-(\nu-j)(L+n+s^-)r}\sum_{i=0}^{\infty}2^{-i(M-a-\frac{\varepsilon}{r})r}2^{(\nu-j)n}\int_{\bigcup_{\bar{k}\in\Lambda_i^{x,\nu,j}}Q_{\nu\bar{k}}}2^{\nu s(x)r}$$

$$\cdot\,\eta_{j,ar+\varepsilon}(x-y)\Big[\sum_{k\in\Lambda_i^{x,\nu,j}}|t_{Q_{\nu k}}|\chi_{Q_{\nu k}}(y)\chi_{Q(c_P,2^{i-j_P+c_0})}\Big]^r(y)\,dy$$

$$\lesssim 2^{-(\nu-j)(L+n-\frac{n}{r}+s^-)r}\sum_{i=0}^{\infty}2^{-i(M-a-\frac{\varepsilon}{r})r}$$

$$\cdot\,\eta_{j,ar}*\left(\Big[\sum_{k\in\Lambda_i^{x,\nu,j}}|t_{Q_{\nu k}}|2^{\nu s(\cdot)}\chi_{Q_{\nu k}}\chi_{Q(c_P,2^{i-j_P+c_0})}\Big]^r\right)(x),$$

where the last step follows by Lemma 19 of [11] with $\varepsilon > c_{\log}(s)$. This implies that the claim holds true. Using this claim, we go back to the norm and, by Minkowski's inequality and Theorem 3.2 of [6] with $a > \frac{n}{r}$, we conclude that

$$III \lesssim \frac{1}{\phi(P)}\Bigg\|\sum_{i=0}^{\infty}\Bigg\{\sum_{j=(j_P\vee 0)}^{\infty}\Bigg(2^{-i(M-a-\frac{\varepsilon}{r})r}\eta_{\nu,ar}*\Big[\sum_{\nu=j+1}^{\infty}2^{-(\nu-j)(L+n-\frac{n}{r}+s^-)r}$$

$$\cdot\sum_{k\in\Lambda_i^{x,\nu,j}}|t_{Q_{\nu k}}|^r 2^{\nu s(\cdot)r}\chi_{Q_{\nu k}}\chi_{Q(c_P,2^{i-j_P+c_0})}\Big]\Bigg)^{\frac{q(\cdot)}{r}}\Bigg\}^{\frac{r}{q(\cdot)}}\Bigg|L_{\frac{p(\cdot)}{r}}(P)\Bigg\|^{\frac{1}{r}}$$

$$\lesssim \frac{1}{\phi(P)}\Bigg\{\sum_{i=0}^{\infty}2^{-i(M-a-\frac{\varepsilon}{r})r}\Bigg\|\Bigg\{\sum_{j=(j_P\vee 0)}^{\infty}\Bigg(\sum_{\nu=j+1}^{\infty}2^{-(\nu-j)(L+n-\frac{n}{r}+s^-)r}$$

$$\cdot\sum_{k\in\Lambda_i^{x,\nu,j}}|t_{Q_{\nu k}}|^r 2^{\nu s(\cdot)r}\chi_{Q_{\nu k}}\Bigg)^{\frac{q(\cdot)}{r}}\Bigg\}^{\frac{r}{q(\cdot)}}\Bigg|L_{\frac{p(\cdot)}{r}}(Q(c_P,2^{i-j_P+c_0}))\Bigg\|\Bigg\}^{\frac{1}{r}}.$$

We apply now Lemma 3.10 in [8] with $L > \frac{n}{r} - n - s^-$, and get

$$III \lesssim \frac{1}{\phi(P)}\Bigg\{\sum_{i=0}^{\infty}2^{-i(M-a-\frac{\varepsilon}{r})r}$$

$$\cdot\Bigg\|\Bigg[\sum_{\nu=(j_P\vee 0)}^{\infty}\Big(\sum_{k\in\mathbb{Z}^n}|t_{Q_{\nu k}}|^r 2^{\nu s(\cdot)r}\chi_{Q_{\nu k}}\Big)^{\frac{q(\cdot)}{r}}\Bigg]^{\frac{r}{q(\cdot)}}\Bigg|L_{\frac{p(\cdot)}{r}}(Q(c_P,2^{i-j_P+c_0}))\Bigg\|\Bigg\}^{\frac{1}{r}}$$

$$
\begin{aligned}
&\le \frac{1}{\phi(P)}\Bigg\{\sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r})r} \\
&\quad \cdot \Bigg\|\Bigg[\sum_{\nu=(j_Q\vee 0)}^{\infty}\Big(\sum_{k\in\mathbb{Z}^n}|t_{Q_{\nu k}}|^r 2^{\nu s(\cdot)r}\chi_{Q_{\nu k}}\Big)^{\frac{q(\cdot)}{r}}\Bigg]^{\frac{r}{q(\cdot)}} \Bigg| L_{\frac{p(\cdot)}{r}}(Q(c_P,2^{i-j_P+c_0}))\Bigg\|\Bigg\}^{\frac{1}{r}} \\
&\lesssim \big\|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\big\| \left\{\sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r})r}\frac{\big[\phi(Q(c_P,2^{i-j_P+c_0}))\big]^r}{\phi(P)^r}\right\}^{\frac{1}{r}} \\
&\lesssim \big\|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\big\| \left\{\sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r}-\log_2\tilde{c}_1(\phi))r}\right\}^{\frac{1}{r}} \\
&\lesssim \big\|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\big\| \qquad (27)
\end{aligned}
$$

provided that $M > a + \frac{\varepsilon}{r} + \log_2 \tilde{c}_1(\phi)$.

By (24), (25), (26) and (27), and applying Corollary 3.8, we finally conclude that

$$
\begin{aligned}
\big\|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\big\| &\lesssim \big\|\big(2^{js(\cdot)}(k_j * f)\big)_j \mid L^{\phi}_{p(\cdot)}(\ell_{q(\cdot)})\big\| \\
&\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)}(I+II+III) \\
&\lesssim \big\|t \mid f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\big\|,
\end{aligned}
$$

which completes the proof. ■

Remarks 4.11. (i) In the case of classical Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ the above result has been proved in [24]. Regarding the Triebel-Lizorkin spaces with variable exponents $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, it is covered by [8, Theorem 3.14].

(ii) Our proof relies mainly on the proof of the smooth atomic decomposition from [37], although we were able to avoid the use of the maximal operator.

5. Pointwise multipliers

Let φ be a bounded function on $\mathbb{R}^n$. The question is under which conditions the mapping $f \mapsto \varphi \cdot f$ makes sense and generates a bounded operator in a given space $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

For the classical spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, Triebel studied this problem in Section 4.2 of [29], where two different approaches were followed. The first idea used a smooth atomic decomposition of f, but requiring the non-existence of moment conditions like (15), since the moment conditions are in general destroyed by multiplication with φ. A more general result was then obtained by Triebel with the help of local means. Recently, Scharf has shown in [24] that it is possible to get a very general result on pointwise multipliers using atomic decomposition but now with the non-smooth atoms.

Regarding the scale of Triebel-Lizorkin-type spaces, in [34, Theorem 6.1] the authors proved a corresponding result for the spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$, based in the same techniques as Triebel in [29]. Our aim is to extend this result for the scale of variable exponents $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. In this direction, we follow [8] and [24], and use the non-smooth atoms to get the desired result. Initially we refer two helpful results proved in [24]. The first lemma shows that the product of two functions in $\mathscr{C}^s(\mathbb{R}^n)$ is still a function in this space, as in Lemma 4.2 in [24].

Lemma 5.1. *Let $s \geq 0$. There exists a constant $c > 0$ such that for all $f, g \in \mathscr{C}^s(\mathbb{R}^n)$, the product $f \cdot g$ belongs to $\mathscr{C}^s(\mathbb{R}^n)$ and it holds*

$$\|f \cdot g \mid \mathscr{C}^s(\mathbb{R}^n)\| \leq c\,\|f \mid \mathscr{C}^s(\mathbb{R}^n)\| \cdot \|g \mid \mathscr{C}^s(\mathbb{R}^n)\|.$$

The next result states that the product of a non-smooth $[K, L]$-atom with a function $\varphi \in \mathscr{C}^\rho(\mathbb{R}^n)$ is still a non-smooth $[K, L]$-atom, and represents a slight normalization of Lemma 4.3 in [24].

Lemma 5.2. *There exists a constant c with the following property: for all $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, all non-smooth $[K, L]$-atoms $a_{\nu,m}$ with support in $3\,Q_{\nu,m}$ and all $\varphi \in \mathscr{C}^\rho(\mathbb{R}^n)$ with $\rho \geq \max(K, L)$, the product*

$$c\,\|\varphi \mid \mathscr{C}^\rho(\mathbb{R}^n)\|^{-1} \cdot \varphi \cdot a_{\nu,m}$$

is a non-smooth $[K, L]$-atom with support in $3\,Q_{\nu,m}$.

Now we have all the tools we need to prove the main theorem of this section. Since the proof follows exactly as in Theorem 4.3 in [8], we do not present it here.

Theorem 5.3. *Let p, q, s, ϕ as in Definition 2.1. Let*

$$\rho > \max\{s^+, s^+ + \log_2 \tilde{c}_1(\phi), \frac{n}{\min\{1, p^-, q^-\}} - n - s^-\}.$$

Then there exists a positive number c such that

$$\|\varphi f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \leq c\, \|\varphi \mid \mathscr{C}^{\rho}(\mathbb{R}^n)\| \cdot \|f \mid F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|$$

for all $\varphi \in \mathscr{C}^{\rho}(\mathbb{R}^n)$ *and all* $f \in F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Remark 5.4. In the particular case of the classical Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ this result is well-known and coincides with [28, Corollary 2.8.2]; see also to [24].

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