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ON ZEROS OF POLYNOMIALS IN BEST L^p-APPROXIMATION AND INSERTING MASS POINTS

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ABSTRACT: The purpose of this note is to revive in L^p spaces the original A. Markov ideas to study monotonicity of zeros of orthogonal polynomials. This allows us to prove and improve in a simple and unified way our previous result [Electron. Trans. Numer. Anal., 44 (2015), pp. 271–280] concerning the discrete version of A. Markov's theorem on monotonicity of zeros.

KEYWORDS: Polynomials; minimal L^p norm; monotonicity; zeros.
MATH. SUBJECT CLASSIFICATION (2000): 30C15.

1. Introduction and main results

Let μ be a positive and nontrivial Radon measure on a compact set $A \subset \mathbb{R}$. For $1 < p < \infty$, the space $L^p(\mu)$ denotes the set of all equivalent classes of μ-measurable functions f such that $|f|^p$ is μ-summable, endowed with the usual vector operations and with the norm

$$\|f\|_p := \left(\int |f(x)|^p d\mu(x) \right)^{1/p}. \tag{1}$$

Set $X := L^p(\mu)$. By a well known result by Clarkson [4, Corollary, p. 403], X is uniformly convex. Following Bourbaki [1, Definition I, p. 166], define $\mathbb{N} := \{0, 1, \dots\}$. Fix $n \in \mathbb{N}$ and set $K := \mathcal{P}_n$, $\mathcal{P}_n$ being the set of all real polynomials of degree at most n regarded as a subspace of X. Since K is finite dimensional, K is a closed convex subspace of X. Following Singer [12, p. 15], $\mathfrak{L}_K(f)$ denotes the set of all elements of best approximation of $f \in X$ by elements of K. It is known that for any point $f \in X$, there is a unique point $g_0 \in \mathfrak{L}_K(f)$ (cf. [10, Theorem 8, p. 45]). The preceding affirmation thus guarantees the existence and uniqueness of $g_0 \in \mathfrak{L}_K(x^{n+1})$.

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By the characterization of elements of best approximation (cf. [12, Theorem 1.11]) $g_0 \in \mathfrak{L}_K(x^{n+1})$ if and only if

$$\int g(x)|x^{n+1} - g_0(x)|^{p-1}\mathrm{sgn}(x^{n+1} - g_0(x))d\mu(x) = 0 \quad (g \in K). \tag{2}$$

Consider the (monic) polynomial $P_{n+1,p}(x) := x^{n+1} - g_0(x)$. As a consequence of (2), the minimum of the norm (1) taken over all (monic) real polynomials P_{n+1} of degree $n+1$ is attained when $P_{n+1} := P_{n+1,p}$. By Fejér's convex hull theorem (cf. [5, Theorem 10.2.2]), the zeros of $P_{n+1,p}$ all lie in the closure of the convex hull of $\mathrm{supp}(\mu)$. Furthermore, all the zeros of $P_{n+1,p}$ are simple [1].

The central concern of this work is the following

Question (Q). *Let μ be a positive and nontrivial Radon measure on a compact set $A \subset \mathbb{R}$. Assume that* $\mathrm{d}\mu(x,t)$ *has the form* [2]

$$\mathrm{d}\alpha(x,t) + \jmath(t)\delta_{y(t)}, \tag{3}$$

where $\mathrm{d}\alpha(x,t) := \omega(x,t)\mathrm{d}\nu(x)$ *and,* $\jmath(t) \in \mathbb{R}_+$ *and* $y(t) \in \mathbb{R}$ *are continuous differentiable function of* $t \in U$*,* U *being an open interval on* $\mathbb{R}$*. Determine sufficient conditions in order for the zeros of the polynomial* $P_{n+1,p}(x,t)$ $(2 \leq p < \infty)$ *to be strictly increasing functions of* t*.*

For reasons of economy of exposition, we intentionally avoided the case $1 < p < 2$. Even though the reader has to proceed with caution in this case, under natural additional assumptions, Theorem 1.1 below remains true, *mutantis mutandis*. When (3) has the form $\omega(x,t)\mathrm{d}x$ and $p = 2$, Question (Q) was studied as early as 1886 by A. Markov [11, p. 178], in a work with many lights and some shadows (see, for instance, [2, Section 1] for some historical remarks). When (3) has the form $\omega(x,t)\mathrm{d}\nu(x)$ and $p = 2$, Question (Q) was posed as an exercise in Freud's book [6, Problem 16, p. 133] (a proof of such result can be found in the more recent book by Ismail [8, Theorem 7.1.1]). When (3) has the form $\omega(x,t)\mathrm{d}x$, $A := [-1,1]$, and $1 \leq p \leq \infty$, Question (Q) was studied by Kroó and Peherstorfer [9]. When (3) has the form $\omega(x)\mathrm{d}x + \jmath\delta_{y(t)}$ and $p = 2$, Question (Q) was considered in [3, Theorem 2.2] through a combination of elementary facts. We recall that

[1] Suppose, contrary to our claim, that x_0 is a multiple zero. From (2) we have

$$\int \frac{P_{n+1,p}(x)}{(x-x_0)^2}|P_{n+1,p}(x)|^{p-1}\mathrm{sgn}(P_{n+1,p}(x))d\mu(x) = \int \frac{|P_{n+1,p}(x)|^p}{|x-x_0|^2}d\mu(x) = 0,$$

a contradiction.

[2] The Dirac measure δ_y is a positive Radon measure whose support is the set $\{y\}$.

this partially solves an open problem posed by Ismail at the end of the 1980's within the framework of orthogonal polynomials (cf. [7, Problem 1] and [8, Problem 24.9.1]). It is, therefore, natural that this last result be broadened to L^p spaces— as it was done in [9] with the standard version of A. Markov's theorem. Not surprisingly, this can be easily achieved by using A. Markov's original ideas [3]. Our main result reads as follows:

THEOREM 1.1. *Assume the notation and conditions of Question (Q). Assume further the existence and continuity for each $x \in A$ and $t \in U$ of $(\partial\omega/\partial t)(x,t)$. Denote by $x_0(t), \dots, x_n(t)$ the zeros of $P_{n+1,p}(x,t)$. Fix $k \in \{0, \dots, n\}$ and set*

$$d_k(t) := \begin{cases} y(t) - x_k(t) & \text{if } y(t) \neq x_k(t), \\ 1 & \text{if } y(t) = x_k(t). \end{cases}$$

Define the rational function

$$R(t) := \sum_{j=0}^{n}{}' \frac{p - \delta_{j,k}}{y(t) - x_j(t)},$$

where the prime means that the sum is over all values j and t for which $y(t) \neq x_j(t)$. Then $x_k(t)$ is a strictly increasing function for those values of t such that

$$\frac{1}{d_k(t)} \left\{ \frac{\jmath'(t)}{\jmath(t)} + y'(t)R(t) - \frac{1}{\omega(x_k(t),t)} \frac{\partial\omega}{\partial t}(x_k(t),t) \right\} \geq 0, \tag{4}$$

and

$$\frac{1}{\omega(x,t)} \frac{\partial\omega}{\partial t}(x,t) \tag{5}$$

is an increasing function of $x \in A$, provided that at least the inequality (4) *be strict or the function* (5) *be nonconstant on A.*

[3]In his classical book [14, Footnote 31, p. 116], Szegő refers his proof of A. Markov's theorem in the following terms: *"This proof does not differ essentially from the original one by A. Markov, although the present arrangement is somewhat clearer."*. Probably this assertion has avoided the attention of some mathematicians to A. Markov's work. While it is true that in the framework of orthogonal polynomials Szegő's argument becomes especially elegant, A. Markov's approach works in a more general framework. Szegő's approach is based on Gauss mechanical quadrature, which was an approach that Stieltjes suggested to handle the problem, see [13, Section 5, p. 391].

The next observations concern the cases studied in the literature for $p = 2$. As far as we know, these are the only ones that have been studied up to now. It is worth highlighting that such cases are the simplest consequences that can be derived from Theorem 1.1.

OBSERVATION 1. [4] *Assume the notation and conditions of Theorem 1.1 under the constraint that* $\mathrm{d}\mu(x,t) = \mathrm{d}\alpha(x) + \jmath\delta_{y(t)}$. *Define the sets*

$$B_- := \{t \in U \mid y(t) \in A^c \cap \mathbb{R} \wedge y'(t) < 0\},$$
$$B_+ := \{t \in U \mid y(t) \in A^c \cap \mathbb{R} \wedge y'(t) > 0\}.$$

Then all the zeros of $P_{n+1,p}(x,t)$ are strictly decreasing (respectively, increasing) functions of t on B_- (respectively, on B_+).

OBSERVATION 2. [5] *Assume the notation and conditions of Theorem 1.1 under the constraint that* $\mathrm{d}\mu(x,t) = \mathrm{d}\alpha(x) + \jmath(t)\delta_y$. *Define the sets*

$$C_- := \{t \in U \mid \jmath'(t) < 0\}, \quad C_+ := \{t \in U \mid \jmath'(t) > 0\}.$$

If $x_k(t) < y$ (respectively, $x_k(t) > y$) for each $t \in U$, then $x_k(t)$ is a strictly increasing (respectively, decreasing) function of t on C_- (respectively, on C_+).

The proof of Theorem 1.1 rests on two pillars: one is the characterization of elements of best approximation (2) and the other one is the implicit function theorem. A. Markov used the orthogonality relation that yields (2) when $p = 2$ (cf. [11, Equation 2]) together with the chain rule (cf. [11, Equation 5], assuming that the zeros are implicitly defined as differentiable functions of the parameter. In any case, as we have already mentioned, we follow the reasoning by A. Markov. In some steps of our proof, the reader will be addressed to the corresponding step in A. Markov's work.

2. Proof of Theorem 1.1

Differentiability of the zeros: Let $P_{n+1}(x) := (x - x_0)\cdots(x - x_n)$, $x_j \in \mathbb{R}$ $(j = 0, \ldots, n)$. (Note that the x_j's do not depend on t.) Define the map $\mathrm{f} := (f_0, \ldots, f_n) : U \subset \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1}$, where we have set $\mathrm{x} := (x_0, \ldots, x_n)$

[4]Observation 1 for $p = 2$ was proved for the first time in [3, Theorem 2.2]. In order to have monotonicity of zero the location of the mass point outside A is quite natural. In this regard, the statements of Theorem 2 and Corollary 3 in `arXiv:1501.07235 [math.CA]` appear to be incorrect.

[5]The case $p = 2$, often considered in the literature, can be easily handled by using very elementary results.

and

$$f_k(\mathrm{x},t) := \int \frac{|P_{n+1}(x)|^p}{x-x_k} \mathrm{d}\mu(x,t). \tag{6}$$

For $j \neq k$ one has

$$\frac{\partial f_k}{\partial x_j}(\mathrm{x},t) = p\int \frac{1}{x-x_k}\frac{\partial P_{n+1}}{\partial x_j}(x)|P_{n+1}(x)|^{p-1}\mathrm{sgn}(P_{n+1}(x))\mathrm{d}\mu(x,t); \tag{7}$$

otherwise [6]

$$\begin{aligned}\frac{\partial f_k}{\partial x_k}(\mathrm{x},t) &= \int \left|\frac{P_{n+1}(x)}{(x-x_k)}\right|^p \frac{\partial}{\partial x_k}\left(\frac{|x-x_k|^p}{x-x_k}\right)\mathrm{d}\mu(x,t)\\ &= (1-p)\int \frac{|P_{n+1}(x)|^p}{(x-x_k)^2}\mathrm{d}\mu(x,t) < 0. \end{aligned}\tag{8}$$

Set $\mathrm{x}(t) := (x_0(t), \ldots, x_n(t))$. From (6), (7) and (8), and using (2) we obtain

$$\mathrm{f}(\mathrm{x}(0),0) = 0, \quad \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(\mathrm{x}(0),0) = \det\begin{pmatrix} \frac{\partial f_0}{\partial x_0}(\mathrm{x}(0),0) & & \\ & \ddots & \\ & & \frac{\partial f_n}{\partial x_n}(\mathrm{x}(0),0)\end{pmatrix} \neq 0.$$

According to the implicit function theorem, under these conditions the equation $\mathrm{f}(\mathrm{s},t) = 0$ has a solution $\mathrm{s} = \mathrm{x}(t)$ in a neighborhood of $(\mathrm{x}(0),0)$ that depends differentiable on t.

Expression for the derivative of the zeros: In view of the above result [7],

$$\frac{\mathrm{d}x_k}{\mathrm{d}t}(t) = -\frac{\frac{\partial f_k}{\partial t}(\mathrm{x}(t),t)}{\frac{\partial f_k}{\partial x_k}(\mathrm{x}(t),t)}.$$

We see at once that

$$\begin{aligned}\frac{\partial f_k}{\partial t}(\mathrm{x}(t),t) = &\int \frac{|P_{n+1,p}(x,t)|^p}{x-x_k(t)}\frac{\partial \omega}{\partial t}(x,t)\mathrm{d}\nu(x)\\ &+ \big(\jmath'(t) + \jmath(t)y'(t)R(t)\big)\frac{|P_{n+1,p}(y(t),t)|^p}{y(t)-x_k(t)}. \end{aligned}\tag{9}$$

[6] Cf. the denominator on the right-hand side of [11, Equation 5].

[7] Cf. the left-hand side of [11, Equation 5].

Clearly [8]

$$\frac{1}{\omega(x_k(t),t)}\frac{\partial\omega}{\partial t}(x_k(t),t)\int\frac{|P_{n+1,p}(x,t)|^p}{x-x_k(t)}\mathrm{d}\mu(x,t)=0.$$

Subtracting this from the left-hand side of (9) yields [9]

$$\begin{aligned}&\frac{\partial f_k}{\partial t}(\mathrm{x}(t),t) \qquad (10)\\ &=\int\frac{|P_{n+1,p}(x,s)|^p}{x-x_k(t)}\left(\frac{1}{\omega(x,t)}\frac{\partial\omega}{\partial t}(x,t)-\frac{1}{\omega(x_k(t),t)}\frac{\partial\omega}{\partial t}(x_k(t),t)\right)\omega(x,t)\mathrm{d}\nu(x)\\ &\quad+\left(\jmath'(t)+\jmath(t)y'(t)R(t)-\frac{\jmath(t)}{\omega(x_k(t),t)}\frac{\partial\omega}{\partial t}(x_k(t),t)\right)\frac{|P_{n+1,p}(y(t),t)|^p}{y(t)-x_k(t)}.\end{aligned}$$

It only remains to note that [10]

$$\frac{1}{x-x_k(t)}\left(\frac{1}{\omega(x,t)}\frac{\partial\omega}{\partial t}(x,t)-\frac{1}{\omega(x_k(t),t)}\frac{\partial\omega}{\partial t}(x_k(t),t)\right)\geq 0.$$

Thus

$$\operatorname{sgn}\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}(t)\right)=\operatorname{sgn}\left(\frac{\partial f_k}{\partial t}(\mathrm{x}(t),t)\right),$$

and the desired result follows from (10).

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[9] Cf. the numerator on the right-hand side of [11, Equation 5].

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