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SUMS OF ORBITS OF INTEGRAL MATRICES

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ABSTRACT: For square matrices A and B over an elementary divisor domain, we study the possible invariant factors of $A+B$ in terms of the invariant factors of A and B. In particular, we find the exact range of $\det(A+B)$ in that situation.

KEYWORDS: Invariant factors, elementary divisor domains, determinants.
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1. Introduction

In this note we work with $n \times n$ matrices over principal ideal domains or, more generally, elementary divisor domains R, defined by the condition that every matrix is equivalent to its Smith normal form (for a recent treatment of these rings, see [3]). We shall be concerned with the action on $R^{n\times n}$ defined by $A \mapsto UAV$, where U, V are invertible matrices over R. The orbits for this action are characterized by n-tuples of elements forming divisibility chains, the invariant factors.

For such an n-tuple a, we denote by $\mathcal{O}_a$ the corresponding orbit. A problem that attracted attention for a long time was the description of $\mathcal{O}_a\mathcal{O}_b$ for given a and b. An answer in the principal ideal domain case was provided by Klein in 1968 [5], working with modules and localizing over a prime, in terms of Littlewood-Richardson sequences. An explicit solution involving divisibility relations was found much later (see *e.g.* [9] for the relation between the module and matrix problems and [4] for an account of the solution). Recently, the same list of divisibility relations was shown to be necessary for the product problem in the larger class of elementary divisor domains [1]. In all this work, the invariant factor product problem was seen to be a deep question, with relations to various important areas of mathematics.

In the 1980s, the analogous additive problem, that of describing $\mathcal{O}_a + \mathcal{O}_b$ for given chains of invariant factors a and b, also received some attention, in

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particular by R. C. Thompson, one of the main protagonists in the study of the product question. But after contributions by some authors (see below), the sum problem remains open. So, given three divisibility chains with the same length, it is not known when the third chain is the invariant factor sequence of a sum of two matrices having the elements of the other two chains as invariant factors.

Here we make some progress on this problem, in two ways. First, we shall be concerned with a natural function of the third sequence, namely the product of its elements. In other words, we aim at describing the possible values of the determinant of the sum of two matrices with given invariant factors. We present a complete solution to this problem for matrices over elementary divisor domains. Second, we relate this to previous work and present a new conjecture for the solution of the sum problem.

For matrices of the form $\lambda I - A$ over the polynomial ring $\mathbb{K}[\lambda]$, where $\mathbb{K}$ is a field, the question is known as the additive Deligne-Simpson problem and it is considered in [2].

2. Invariant factors of sums

For completeness, we briefly describe what is known about invariant factors of sums of matrices.

Our notation is standard. Let $n \geq 2$. Given $n \times n$ matrices A and B over an elementary divisor domain R, we denote their invariant factors by $a_1 \mid \cdots \mid a_n$ and $b_1 \mid \cdots \mid b_n$, respectively. As is well-known, we have

$$a_k = \frac{d_k(A)}{d_{k-1}(A)} \ , \ \ k = 1, \ldots, \mathrm{rank}(A) \ , \ \ a_k = 0 \ \text{ for } k > \mathrm{rank}(A),$$

where, for each k, $d_k(A)$ is the gcd of all $k \times k$ minors of A, $d_0 := 1$. This means that the invariant factors of a matrix are well defined up to associates, *i.e.* apart from units in the ring. Also, we clearly have $d_k(A) = a_1 \cdots a_k$, apart from unit factors, for all k.

In [10], R. C. Thompson proved that, if $c_1 \mid \cdots \mid c_n$ are the invariant factors of $A + B$, then

$$\gcd\{a_i, b_j\} \mid c_{i+j-1} \tag{1}$$

for all indices i, j such that $i + j - 1 \leq n$. His proof uses localization at a prime, requiring R to be a principal ideal domain, but the result is valid for matrices over an elementary divisor domain (see the argument in [7]).

Write $A + B = C$. From the equalities $B = -A + C$ and $A = -B + C$, we see that we have not one but three families of relations

$$\gcd\{a_i, b_j\} \mid c_{i+j-1} \quad , \quad \gcd\{a_i, c_j\} \mid b_{i+j-1} \quad \text{and} \quad \gcd\{b_i, c_j\} \mid a_{i+j-1}\,. \tag{2}$$

Also, trivially, taking determinants (with invariant factors chosen so that their product equals the determinant) we have

$$a_1 \cdots a_n \equiv c_1 \cdots c_n \pmod{b_1},$$

$$b_1 \cdots b_n \equiv c_1 \cdots c_n \pmod{a_1},$$

$$a_1 \cdots a_n \equiv (-1)^n b_1 \cdots b_n \pmod{c_1}. \tag{3}$$

In [11], Thompson conjectured that, for $n \geq 2$, the six conditions (2) and (3) are the complete solution for the invariant factor sum problem, *i.e.* that, if they hold (with a_1, b_1, c_1 relatively prime, which entails no loss of generality), then matrices $A \in \mathcal{O}_a$, $B \in \mathcal{O}_b$ exist such that $A + B \in \mathcal{O}_c$.

In [8], E. Marques de Sá showed that Thompson's conjecture is false in the general case, by finding additional necessary conditions (see the final section).

3. Determinants of sums

In this section we are interested in the possible values of $\det(A + B)$ when A and B have prescribed invariant factors.

A source of inspiration when looking for invariant factor relations is the corresponding situation for singular values of complex matrices. The analogy is well-known (see *e.g.* [7], [4]). So it is of interest that, in [6], the following inequality was proved:

$$\prod_{i=1}^{n} (\alpha_i + \beta_{n-i+1}) \geq \det(A + B)$$

where A and B are $n \times n$ complex matrices with singular values $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_n$.

In keeping with the analogy, we should replace sum by gcd on the left-side, as in relation (1) above (without that, the result is easily seen to be false). Indeed, we have:

Theorem 1. Let A and B be $n \times n$ matrices over R with invariant factors $a_1 \mid \cdots \mid a_n$ and $b_1 \mid \cdots \mid b_n$. Then

$$\prod_{i=1}^{n} \gcd\{a_i, b_{n-i+1}\} \mid \det(A+B)\,. \tag{4}$$

Proof. We have the well-known formula for determinants of sums:

$$\det(A+B) = \sum_{k=0}^{n} \sum_{\mu,\nu \in Q_{k,n}} (-1)^{\sum \mu + \sum \nu} \det A[\mu|\nu].\det B[\mu'|\nu'] \tag{5}$$

where $Q_{k,n}$ is the set of strictly increasing sequences with k elements taken from $\{1, 2, \ldots, n\}$, $A[\mu|\nu]$ is the submatrix of A with rows and columns indexed by μ and ν, and μ', ν' are the complementary sequences to μ, ν.

Apart from the sign, each summand is the product of a $k \times k$ minor of A, which is a multiple of $a_1 \cdots a_k$, by a $(n-k) \times (n-k)$ minor of B, which is a multiple of $b_1 \cdots b_{n-k}$. The whole sum clearly must be a multiple of $\prod_{i=1}^{n} \gcd\{a_i, b_{n-i+1}\}$. ∎

Condition (4) is simple and elegant but it is not the complete solution to the determinant question. An example showing this is the following:

Example. Consider $R = \mathbb{Z}$ and $n = 2$. For any matrices A and B with invariant factors $2 \mid 4$ and $3 \mid 6$, $\det(A+B)$ is never a multiple of 6. So, not every multiple of $\gcd\{2,6\}\gcd\{4,3\} = 2$ is attainable as the determinant of the sum of two matrices with the prescribed invariant factors.

We must take formula (5) a bit further. In the following statement, we choose the invariant factors of a matrix so that their product equals the determinant of the matrix.

Theorem 2. Let A and B be $n\times n$ matrices over R with invariant factors $a_1 \mid \cdots \mid a_n$ and $b_1 \mid \cdots \mid b_n$. Put $\delta = \gcd\{a_1 \cdots a_k b_1 \cdots b_{n-k} \colon k=1,\ldots,n\text{–}1\}$. Then

$$\det(A+B) \equiv a_1 \cdots a_n + b_1 \cdots b_n \pmod{\delta}\,. \tag{6}$$

Proof. This is immediate from formula (5). For $k \in \{1,\ldots,n\text{–}1\}$ and $\mu,\nu \in Q_{k,n}$, we have $\det A[\mu|\nu] \equiv 0 \pmod{a_1 \cdots a_k}$ and $\det B[\mu'|\nu'] \equiv 0 \pmod{b_1 \cdots b_{n-k}}$, so

$$\det A[\mu|\nu].\det B[\mu'|\nu'] \equiv 0 \pmod{a_1 \cdots a_k b_1 \cdots b_{n-k}}\,.$$

Summing for all k, we get from (5) that

$$\det(A+B) \equiv \det(A) + \det(B) \pmod{\delta}$$

which is the same as

$$\det(A+B) \equiv a_1 \cdots a_n + b_1 \cdots b_n \pmod{\delta},$$

as required. ∎

We proceed to show that condition (6) is also sufficient.

First, a technical lemma.

Lemma 1. For $n \geq 2$ and $\alpha_1,\ldots,\alpha_{n-1},\beta_1,\ldots,\beta_{n-1},\gamma_1,\ldots,\gamma_n \in R$, we have

$$\det \begin{bmatrix} \alpha_1 & & & & & \gamma_1 \\ \beta_1 & \alpha_2 & & 0 & & \gamma_2 \\ & \beta_2 & \alpha_3 & & & \gamma_3 \\ & & \ddots & \ddots & & \vdots \\ & 0 & & \beta_{n-2} & \alpha_{n-1} & \gamma_{n-1} \\ & & & & \beta_{n-1} & \gamma_n \end{bmatrix} = \sum_{i=1}^{n} (-1)^{n+i}\,\gamma_i \prod_{j=1}^{i-1} \alpha_j \prod_{j=i}^{n-1} \beta_j\,.$$

Proof. The case $n=2$ is trivial. For $n \geq 3$, we get, using Laplace's Theorem,

$$\det\begin{bmatrix} \alpha_1 & & & & \gamma_1 \\ \beta_1 & \alpha_2 & & 0 & \gamma_2 \\ & \ddots & \ddots & & \vdots \\ & & \beta_{n-2} & \alpha_{n-1} & \gamma_{n-1} \\ & 0 & & \beta_{n-1} & \gamma_n \end{bmatrix} =$$

$$-\beta_{n-1}\det\begin{bmatrix} \alpha_1 & & & & \gamma_1 \\ \beta_1 & \alpha_2 & 0 & & \gamma_2 \\ & \ddots & \ddots & & \vdots \\ & & \beta_{n-3} & \alpha_{n-2} & \gamma_{n-2} \\ & 0 & & \beta_{n-2} & \gamma_{n-1} \end{bmatrix} + \gamma_n \det\begin{bmatrix} \alpha_1 & & & & \\ \beta_1 & \alpha_2 & & 0 & \\ & \beta_2 & \alpha_3 & & \\ & & \ddots & \ddots & \\ & 0 & & \beta_{n-2} & \alpha_{n-1} \end{bmatrix},$$

and the result follows by induction on n. ■

Theorem 3. Take sequences $a_1 \mid \cdots \mid a_n$ and $b_1 \mid \cdots \mid b_n$ of elements of R. Define δ as in Theorem 2. Suppose that $x \equiv a_1 \cdots a_n + b_1 \cdots b_n \pmod{\delta}$. Then there exist $n \times n$ matrices A and B with invariant factors $a_1 \mid \cdots \mid a_n$ and $b_1 \mid \cdots \mid b_n$ such that $\det(A+B) = x$.

Proof. For $n = 2$, let $x = a_1a_2 + b_1b_2 + qa_1b_1$. Then the matrices $A = \begin{bmatrix} a_1 & 0 \\ qa_1 & a_2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -b_1 \\ b_2 & 0 \end{bmatrix}$ satisfy the required.

Suppose $n \geq 3$ and let $x \in R$ be such that $x \equiv a_1 \cdots a_n + b_1 \cdots b_n \pmod{\delta}$. Since every elementary divisor domain is also a Bézout domain, there exist $q_1, \ldots, q_{n-1} \in R$ such that $x = a_1 \cdots a_n + b_1 \cdots b_n + \sum_{k=1}^{n-1} q_k a_1 \cdots a_k b_1 \cdots b_{n-k}$.

Consider the matrices $A = \begin{bmatrix} a_1 & 0 \\ q_1a_1 & a_2 \end{bmatrix} \oplus \operatorname{diag}(a_3, \ldots, a_n)$ and

$$B = \left[\begin{array}{c|c} 0 & (-1)^{n-1}b_1 \\ \hline & 0 \\ & (-1)^{n-3}q_2b_1 \\ \operatorname{diag}(b_n, b_{n-1}, \ldots, b_2) & \vdots \\ & -q_{n-2}b_1 \\ & q_{n-1}b_1 \end{array}\right].$$

By means of elementary row and column operations we see that A and B are equivalent to $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ and $\operatorname{diag}(b_1, b_2, \ldots, b_n)$, respectively. Hence $a_1 \mid \cdots \mid a_n$ and $b_1 \mid \cdots \mid b_n$ are the invariant factors of A and B.

From the previous Lemma we have

$$
\begin{aligned}
\det(A+B) &= \det \begin{bmatrix} a_1 & & & & & (-1)^{n-1}b_1 \\ q_1a_1+b_n & a_2 & & 0 & & 0 \\ & b_{n-1} & a_3 & & & (-1)^{n-3}q_2b_1 \\ & & \ddots & \ddots & & \vdots \\ & 0 & & b_3 & a_{n-1} & -q_{n-2}b_1 \\ & & & & b_2 & a_n + q_{n-1}b_1 \end{bmatrix} \\
&= b_1(q_1a_1+b_n)b_2\cdots b_{n-1} + \sum_{i=3}^{n-1} q_{i-1}a_1\cdots a_{i-1}b_1\cdots b_{n-i+1} + \\
&\quad (a_n + q_{n-1}b_1)a_1\cdots a_{n-1} \\
&= x\,. \ \blacksquare
\end{aligned}
$$

So condition (6) is necessary and sufficient for the existence of two matrices with the given invariant factors and such that the determinant of their sum has the prescribed value.

4. Further notes on the sum problem

At first sight, Thompson's conjecture is quite a bold one, since it simply joins, as possible sufficient conditions for the sum problem, the divisibility relations (2) to the trivial congruences (3). But, in fact, the relations (2) are highly restrictive on the three n-tuples of invariant factors, so the conjecture was not unnatural. Sá showed it is not true for $n \geq 3$, but the conditions he added in [8] are just modifications of (3).

Let's illustrate how restrictive conditions (2) are. We might think of using formula (5) as in Theorem 2 to obtain some additional necessary conditions. Let $m \in \{1, 2, \ldots, n\}$. For every $\xi, \zeta \in Q_{m,n}$ we have

$$
\det(A+B)[\xi|\zeta] = \sum_{k=0}^{m} \sum_{\mu,\nu \in Q_{k,m}} (-1)^{\sum\mu + \sum\nu} \det A[\xi|\zeta][\mu|\nu] . \det B[\xi|\zeta][\mu'|\nu'] \,,
$$

and so every $m \times m$ minor of $A+B$ is a multiple of

$$\varepsilon_m = \gcd\{a_1 \cdots a_m, a_1 \cdots a_{m-1}b_1, a_1 \cdots a_{m-2}b_1b_2, \ldots, a_1b_1 \cdots b_{m-1},\ b_1 \cdots b_m\}.$$

Hence, if $c_1 \mid \cdots \mid c_n$ are the invariant factors of $A+B$, we have

$$c_1c_2 \cdots c_m \equiv 0 \pmod{\varepsilon_m}, \qquad m = 1, \ldots, n\,.$$

But these n congruences actually are not new necessary conditions, since they are a consequence of (2):

Theorem 4. If $a_1 \mid \cdots \mid a_n$, $b_1 \mid \cdots \mid b_n$ and $c_1 \mid \cdots \mid c_n$ are sequences of elements in R that satisfy (2) then $c_1c_2 \cdots c_m \equiv 0 \pmod{\varepsilon_m}$, for all $m = 1, \ldots, n$.

Proof. For each $m \in \{2, \ldots, n\}$,

$$\begin{aligned}\varepsilon_m = \gcd\{\, & a_1 \cdots a_{m-1} \gcd\{a_m, b_1\}\,,\ a_1 \cdots a_{m-2}b_1 \gcd\{a_{m-1}, b_2\}\,, \ldots, \\ & a_1b_1 \cdots b_{m-2} \gcd\{a_2, b_{m-1}\}\,,\ b_1 \cdots b_{m-1} \gcd\{a_1, b_m\}\,\}\,.\end{aligned}$$

From (2) we have that $\gcd\{a_{m-k+1}, b_k\} \mid c_m$, for $k = 1, \ldots, m$ and so $\varepsilon_m \mid \varepsilon_{m-1}c_m$. Since $\varepsilon_1 = \gcd\{a_1, b_1\} \mid c_1$ it follows that $\varepsilon_m \mid c_1c_2 \cdots c_m$. ■

Given Theorem 2, we can add condition (6) to the list of necessary conditions on invariant factors of sums of matrices with given invariant factors. It is natural to ask how it relates to Thompson's conjecture and Sá's later work. To do this we present all the known necessary conditions, renaming them for convenience. For symmetry purposes we follow [8] and consider matrices A, B and C such that $A+B+C=0$. So, if A, B and C are $n \times n$ matrices with invariant factors $a_1 \,|\, a_2 \,|\, \cdots \,|\, a_n$, $b_1 \,|\, b_2 \,|\, \cdots \,|\, b_n$ and $c_1 \,|\, c_2 \,|\, \cdots \,|\, c_n$, chosen so that the determinant of each matrix is equal to the product of its invariant factors, and $A+B+C=0$, then the following conditions hold:

$$\text{(I)} \qquad \left\{\begin{array}{l} \gcd\{a_i, b_j\} \mid c_{i+j-1} \\ \gcd\{a_i, c_j\} \mid b_{i+j-1} \\ \gcd\{c_i, b_j\} \mid a_{i+j-1} \end{array}\right. , \quad 1 \le i, j \le n,\ \ i+j-1 \le n\,;$$

$$\text{(II)} \qquad \begin{cases} \prod_{i=1}^{n} a_i \equiv (-1)^n \prod_{i=1}^{n} b_i \pmod{c_1\delta_1^{n-1}} \\ \prod_{i=1}^{n} b_i \equiv (-1)^n \prod_{i=1}^{n} c_i \pmod{a_1\delta_1^{n-1}} \\ \prod_{i=1}^{n} c_i \equiv (-1)^n \prod_{i=1}^{n} a_i \pmod{b_1\delta_1^{n-1}} \end{cases},$$

where $\delta_1 = \gcd\{a_1, b_1, c_1\}$;

$$\text{(III)} \qquad \begin{cases} \prod_{i=1}^{n} a_i \equiv (-1)^n \prod_{i=1}^{n} b_i \pmod{\gamma} \\ \prod_{i=1}^{n} b_i \equiv (-1)^n \prod_{i=1}^{n} c_i \pmod{\alpha} \\ \prod_{i=1}^{n} c_i \equiv (-1)^n \prod_{i=1}^{n} a_i \pmod{\beta} \end{cases},$$

where

$$\alpha = a_1 \prod_{i=1}^{n-1} \gcd\{a_1, b_i\}, \quad \beta = b_1 \prod_{i=1}^{n-1} \gcd\{b_1, c_i\}, \quad \gamma = c_1 \prod_{i=1}^{n-1} \gcd\{c_1, a_i\};$$

and

$$\text{(IV)} \qquad \begin{cases} (-1)^n \prod_{i=1}^{n} c_i \equiv \prod_{i=1}^{n} a_i + \prod_{i=1}^{n} b_i \pmod{\delta} \\ (-1)^n \prod_{i=1}^{n} a_i \equiv \prod_{i=1}^{n} b_i + \prod_{i=1}^{n} c_i \pmod{\theta} \\ (-1)^n \prod_{i=1}^{n} b_i \equiv \prod_{i=1}^{n} a_i + \prod_{i=1}^{n} c_i \pmod{\eta} \end{cases},$$

where

$$\begin{aligned} \delta &= \gcd\{a_1 \cdots a_k b_1 \cdots b_{n-k} \colon k = 1, \ldots, n-1\}, \\ \theta &= \gcd\{c_1 \cdots c_k b_1 \cdots b_{n-k} \colon k = 1, \ldots, n-1\}, \\ \eta &= \gcd\{a_1 \cdots a_k c_1 \cdots c_{n-k} \colon k = 1, \ldots, n-1\}. \end{aligned}$$

(II) are Thompson's conditions from [11], in the version where we do not assume that a_1, b_1, c_1 are relatively prime. (III) are Sá's conditions from [8]. (IV) are our new necessary conditions obtained as in Theorem 2 from $C = -A - B$, $A = -B - C$ and $B = -A - C$.

Concerning conditions (I) to (IV) we have the following list of remarks.

Remark 1. Conditions (I) together with (III) imply (II) but conditions (I) together with (II) do not imply (III) [8].

Remark 2. Conditions (I) together with (II) do not imply (IV). Take, for instance, $n = 3$, $a : 1 \mid 2 \mid 2$, $b : 1 \mid 6 \mid 30$ and $c : 8 \mid 16 \mid 96$. This example also shows that the conditions in Theorem 4 do not imply (IV).

Remark 3. For n odd, it is easy to show that (IV) are equivalent to

$$\prod_{i=1}^{n} a_i + \prod_{i=1}^{n} b_i + \prod_{i=1}^{n} c_i \equiv 0 \ (\mathrm{mod}\ \mathrm{lcm}\{\delta, \theta, \eta\}),$$

and (III) are equivalent to

$$\prod_{i=1}^{n} a_i + \prod_{i=1}^{n} b_i + \prod_{i=1}^{n} c_i \equiv 0 \ (\mathrm{mod}\ \mathrm{lcm}\{\alpha, \beta, \gamma\}),$$

since $\alpha \mid \prod_{i=1}^{n} a_i$, $\beta \mid \prod_{i=1}^{n} b_i$ and $\gamma \mid \prod_{i=1}^{n} c_i$.

Remark 4. Elementary calculations show that, for $n = 2$, the three sets of conditions (I)$\wedge$(II), (I)$\wedge$(III) and (I)$\wedge$(IV) are equivalent. They are the complete answer to the sum problem for 2×2 matrices [11].

Next we prove that (IV) implies (III).

Lemma 2. If $a_1 \mid \cdots \mid a_n$, $b_1 \mid \cdots \mid b_n$ and $c_1 \mid \cdots \mid c_n$ are sequences of elements in R that satisfy (I) then

$$\mathrm{lcm}\,\{\alpha, \beta\} \mid \delta\,, \quad \mathrm{lcm}\,\{\beta, \gamma\} \mid \theta \quad \text{and} \quad \mathrm{lcm}\,\{\alpha, \gamma\} \mid \eta\,.$$

Proof. We will prove that $\alpha \mid \delta$ and $\beta \mid \delta$. The other relations are proved similarly. For each $k \in \{1, \ldots, n\}$, $a_1 \cdots a_k b_1 \cdots b_{n-k}$ is a multiple of $a_1^k b_1 \cdots b_{n-k}$ and hence a multiple of α. So $\alpha \mid \delta$.

Let $k \in \{1, \ldots, n\}$. From (I), $\gcd\{b_1, c_i\} \mid a_i$, for $i = 1, \ldots, k$. Then

$$\beta = b_1 \prod_{i=1}^{n-1} \gcd\{b_1, c_i\} \mid a_1 \cdots a_k b_1^{n-k} \mid a_1 \cdots a_k b_1 \cdots b_{n-k} \,.$$

Therefore $\beta \mid \delta$. ■

This argument shows that, even for sequences that do not satisfy (I), $\alpha \mid \delta$, $\beta \mid \theta$ and $\gamma \mid \eta$ will always hold.

Theorem 5. If $a_1 \mid \cdots \mid a_n$, $b_1 \mid \cdots \mid b_n$ and $c_1 \mid \cdots \mid c_n$ are sequences of elements in R that satisfy (IV) then they also satisfy (III).

Proof. The result follows promptly from $\alpha \mid \delta$, $\beta \mid \theta$, $\gamma \mid \eta$, $\alpha \mid \prod_{i=1}^n a_i$, $\beta \mid \prod_{i=1}^n b_i$ and $\gamma \mid \prod_{i=1}^n c_i$. ■

It is natural to ask whether (III) implies (IV).

All of the above, together with some computational evidence, leads us to the following conjecture on the solution to the invariant factor sum problem:

Conjecture. Given chains a, b and c, we have $\mathcal{O}_c \subset \mathcal{O}_a + \mathcal{O}_b$ if and only if (I) holds and $\prod_{i=1}^n c_i$ is congruent modulo δ to the sum of an associate of $\prod_{i=1}^n a_i$ with an associate of $\prod_{i=1}^n b_i$, together with similar conditions to $\prod_{i=1}^n a_i$ modulo θ and $\prod_{i=1}^n b_i$ modulo η.

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