Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 17–34

TWO-STEP ESTIMATION PROCEDURES FOR COMPOUND POISSON INARCH PROCESSES

ESMERALDA GONÇALVES, NAZARÉ MENDES LOPES AND FILIPA SILVA

Abstract: Considering the wide class of CP-INARCH models, introduced in [6], the main goal of this paper is to develop and compare parametric estimation procedures for order one models, applicable without specifying the conditional distribution of the process. Therefore, two-step estimation procedures, combining either the conditional least square (CLS) or the Poisson quasi-maximum likelihood (PQML) methods with that of the moment's estimation, are introduced and discussed. Specifying the process conditional distribution, we develop also within this class of models the conditional maximum likelihood (CML) methodology. A simulation study illustrates, particularly, the competitive performance of the two-step approaches regarding the more classical CML one which requires the conditional distribution knowledge. A final real-data example shows the relevance of this wide class of models, as it will be clear the better performance in the data fitting of some new models emerging in such class.

Keywords: integer-valued time series, CP-INGARCH model, estimation.
Math. Subject Classification (2000): 62M10.

1. Introduction

The family of discrete compound Poisson distributions, which includes as particular cases the Poisson, the Neyman type-A or the geometric Poisson laws, was recently used to define a new class of integer-valued GARCH models, the compound Poisson INGARCH ones [6], specified through the characteristic function of the conditional law of the process given its past. Namely, $X = (X_t, t \in \mathbb{Z})$ follows a CP-INGARCH process if the characteristic function of X_t conditioned on $\underline{X}_{t-1}$ is such that

$$\begin{cases} \Phi_{X_t|\underline{X}_{t-1}}(u) = \exp\left\{ i\frac{\lambda_t}{\varphi_t'(0)}\left[\varphi_t(u) - 1\right]\right\},\ u \in \mathbb{R}, \\ E(X_t|\underline{X}_{t-1}) = \lambda_t = \alpha_0 + \displaystyle\sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k}, \end{cases}$$

where $\alpha_0 > 0$, $\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q \geq 0$, $\underline{X}_{t-1}$ represents the σ-field generated by $\{X_{t-s}, s \geq 1\}$ and $(\varphi_t, t \in \mathbb{Z})$ is a family of characteristic functions on $\mathbb{R}$,

Received July 4, 2017.

$\underline{X}_{t-1}$-measurables, associated to a family of discrete laws with support in $\mathbb{N}_0$ and finite mean. The functional form of the conditional characteristic function $\Phi_{X_t|\underline{X}_{t-1}}$ allows a wide flexibility of the class of CP-INGARCH models. In fact, as it is assumed that the family of discrete characteristic functions $(\varphi_t, t \in \mathbb{Z})$ is $\underline{X}_{t-1}$-measurable it means that its elements may be random functions or deterministic ones. Thus, this general model unifies and enlarges substantially the family of conditionally heteroscedastic integer-valued processes. In fact, it is possible to present new specific models with conditional distributions with interest in practical applications as, for instance, the geometric Poisson INGARCH ([6]) or the Neyman type-A INGARCH ([5]) ones, and also recover recent contributions such as the Poisson INGARCH ([4]), the generalized Poisson INGARCH ([12]), the negative binomial INGARCH ([11]) and the negative binomial DINARCH ([10]) processes (correspon- ding to random or deterministic functions φ_t, respectively). In addition to having the ability to describe different distributional behaviors and consequently different kinds of conditional heteroscedasticity, the CP-INGARCH model is able to incorporate simultaneously the overdispersion characteristic that has been recorded in real count data.

In this paper, we focus on the case where φ_t is deterministic and constant in time which still includes many of the particular cases referred above. For that reason, from now on we will refer these functions simply as φ. In this subclass of models, there exists a strictly stationary and ergodic solution with finite first and second order moments under $\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1$ ([6]). For $p = q = 1$, Gonçalves, Mendes-Lopes and Silva [7] stated that this simple coefficient condition is also necessary and sufficient to establish the existence of all the moments of X_t.

The remainder of the paper proceeds as follows. In Section 2 we consider the subclass of CP-INARCH models of order one, with $\varphi_t = \varphi$ deterministic, and deduce its moments, central moments and cumulants up to the order 4. These results are particularly important in Section 3, devoted to estimation procedures, to deduce explicit expressions for the asymptotic distribution of the Conditional Least Squares (CLS) estimators of the conditional mean parameters, α_0 and α_1. In a second step, the method of moments is used to estimate the additional parameter associated to the function φ. Another two-step estimation procedure, combining the Poisson Quasi Maximum Likelihood (PQML) and the moment methods, is also proposed in this section, followed by the Conditional Maximum Likelihood (CML) estimation for the NTA-INARCH(1) and GEOMP2-INARCH(1) models. Section 4 presents some simulation studies that illustrate and compare the performance of these three methodologies of estimation. In Section 5 an integer-valued time series related to the prices of electricity in Portugal and Spain during 2013 is considered. The data is fitted by several CP-INARCH(1) models estimated by the CML method and the quality of the fitting is discussed using, in particular, the values of the log likelihood function, Akaike information criterion and Bayesian information criterion. Detailed calculations are included in the Appendices.

2. The CP-INARCH(1) process

Let us consider the subclass of CP-INGARCH(1) models for which $p = q = 1$ and $\beta_1 = 0$. Supposing $\varphi_t = \varphi$ constant in time and deterministic we recall that $\alpha_1 < 1$ is a necessary and sufficient condition to assure the existence of a strictly stationary and ergodic solution of the model. Moreover the process has moments of all the orders.

Setting $X = (X_t, t \in \mathbb{Z})$ a CP-INARCH(1) process we derive in the following closed-form expressions for the joint (central) moments and cumulants of the CP-INARCH(1) up to order 4. In fact, setting the notations below (used, for instance, by Weiß in [9]),

$$f_k = \frac{\alpha_0}{\prod_{j=1}^{k}(1-\alpha_1^j)}, \ k \in \mathbb{N},$$

$$\begin{aligned} \mu(s_1, ..., s_{r-1}) &= E\left(X_t X_{t+s_1} ... X_{t+s_{r-1}}\right), \\ \widetilde{\mu}(s_1, ..., s_{r-1}) &= E\left((X_t - \mu)(X_{t+s_1} - \mu)...(X_{t+s_{r-1}} - \mu)\right), \\ \kappa(s_1, ..., s_{r-1}) &= Cum\left[X_t, X_{t+s_1}, ..., X_{t+s_{r-1}}\right], \end{aligned} \tag{1}$$

with $r = 2, 3, 4$ and $0 \leq s_1 \leq ... \leq s_{r-1}$, and

$$v_0 = -i\frac{\varphi''(0)}{\varphi'(0)}, \ d_0 = -\frac{\varphi'''(0)}{\varphi'(0)}, \ c_0 = i\frac{\varphi^{(iv)}(0)}{\varphi'(0)},$$

we establish the following results whose proofs may be found in Appendices 1 and 2, respectively.

Theorem 2.1 (Moments of a CP-INARCH(**1**) process).
We have:

(a): *For any* $k \geq 0$, $\mu(k) = f_2(v_0\alpha_1^k + \alpha_0(1+\alpha_1))$.

(b): *For any* $l \geq k \geq 0$,

$$\mu(k,l) = [d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]f_3\alpha_1^{l+k} + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1}f_2\alpha_1^l + v_0 f_1 f_2 \alpha_1^{l-k} + f_1\mu(k).$$

(c): *For any* $m \geq l \geq k \geq 0$,

$$\begin{aligned}\mu(k,l,m) \;=\; & \alpha_1^{m-l}\left[\left\{(c_0 - 4v_0d_0 + 3v_0^3) + 3v_0(v_0^2-d_0)\alpha_1 + (3v_0d_0 - c_0)\alpha_1^2\right.\right.\\ & \left. + (7v_0d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0-2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0d_0 + c_0)\alpha_1^5\right\} f_4\alpha_1^{2l+k}\\ & + \frac{2v_0+\alpha_0}{1-\alpha_1} f_3\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2l}\\ & + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2\left[2v_0\alpha_0 + d_0(1-\alpha_1) + v_0^2(2\alpha_1-1)\right]\alpha_1^{2l-k}\\ & + \frac{\alpha_0 f_3}{1-\alpha_1}\left\{d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right\}\alpha_1^{2(l-k)} + \frac{v_0+\alpha_0}{1-\alpha_1}\mu(k,l)\\ & \left. - f_2\mu(k)[\alpha_0 + (v_0+\alpha_0)\alpha_1]\right] + f_1\mu(k,l).\end{aligned}$$

Corollary 2.1 (Central Moments and Cumulants of a CP-INARCH**(1)** process)**.**
We have:

(a): *For any* $s \geq 0$, $\widetilde{\mu}(s) = \kappa(s) \;=\; v_0\alpha_1^s f_2$.
(b): *For any* $l \geq s \geq 0$,

$$\widetilde{\mu}(s,l) \;=\; \kappa(s,l) \;=\; f_3\alpha_1^l[v_0^2(1+\alpha_1+\alpha_1^2) - \{v_0^2(1+\alpha_1-2\alpha_1^2) - d_0(1-\alpha_1^2)\}\alpha_1^s].$$

(c): *For any* $m \geq l \geq s \geq 0$,

$$\begin{aligned}\kappa(s,l,m) \;=\; & \alpha_1^m f_4\left[\left\{c_0 + 3v_0^3 - 4v_0d_0 + 3v_0(v_0^2-d_0)\alpha_1 + (3\alpha_0d_0 - c_0)\alpha_1^2\right.\right.\\ & \left. +(7v_0d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0-2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0d_0 + c_0)\alpha_1^5\right\}\alpha_1^{l+s}\\ & +v_0(1+\alpha_1+\alpha_1^2+\alpha_1^3)[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)](2\alpha_1^l + \alpha_1^s)\\ & \left. +v_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)[(1+\alpha_1)v_0^2 + (d_0(1-\alpha_1) + v_0^2(2\alpha_1-1))\alpha_1^{l-s}]\right],\\ \widetilde{\mu}(s,l,m) \;=\; & \kappa(s,l,m) + v_0^2 f_2^2(\alpha_1^{m-l+s} + 2\alpha_1^{m+l-s}).\end{aligned}$$

From Theorem 2.1 we deduce, for instance,

$$E(X_t^2) \;=\; \mu(0) \;=\; \frac{\alpha_0(v_0+\alpha_0(1+\alpha_1))}{(1-\alpha_1)(1-\alpha_1^2)}, \tag{2}$$

$$E(X_t^3) \;=\; \mu(0,0) \;=\; \frac{\alpha_0}{(1-\alpha_1)^3}\left[\frac{d_0 + (3v_0^2-d_0)\alpha_1^2}{(1+\alpha_1)(1+\alpha_1+\alpha_1^2)} + \frac{3v_0\alpha_0}{1+\alpha_1} + \alpha_0^2\right].$$

These results generalize those of Weiß [9] for the INARCH(1) model. They are particularly important to deduce explicit expressions for the asymptotic distribution of the CLS estimators of the parameters α_0 and α_1 provided in the next section. As we will take in our study some important particular cases concerning the process conditional law, we conclude this section recalling such cases and deducing the corresponding values of v_0, d_0 and c_0, previously introduced.

a) The INARCH(1) model ([4]) corresponds to a CP-INARCH model considering φ the characteristic function of the Dirac's law concentrated in $\{1\}$. So, we deduce that $v_0 = d_0 = c_0 = 1$.

b) When φ is the characteristic function of the Poisson distribution with mean $\phi > 0$, $X_t|\underline{X}_{t-1}$ follows a Neyman type-A law with parameter $(\lambda_t/\phi, \phi)$, and we have the NTA-INARCH(1) model introduced in [5]. For this case, $v_0 = 1 + \phi$, $d_0 = 1 + 3\phi + \phi^2$ and $c_0 = 1 + 7\phi + 6\phi^2 + \phi^3$.

c) Considering in the above expressions $v_0 = (2 - p^*)/p^*$, $d_0 = (6 - 6p^* + (p^*)^2)/(p^*)^2$ and $c_0 = ((2-p^*)(12-12p^* + (p^*)^2))/(p^*)^3$, we obtain the expressions for the GEOMP2-INARCH(1) model [6]. In fact, this process is defined considering φ the characteristic function of the geometric distribution with parameter $p^* \in]0, 1[$ and $X_t|\underline{X}_{t-1}$ following a geometric Poisson $(p^*\lambda_t, p^*)$ law.

d) Another particular case of the CP-INARCH model is the NB2-INARCH (that is identical to the NB-DINARCH model proposed by Xu et al. [10]), where $X_t|\underline{X}_{t-1}$ follows a negative binomial law with parameter $(\lambda_t/(\beta - 1), 1/\beta)$ and $\beta > 0$. This process is stated when φ is the characteristic function of the logarithmic distribution with parameter $(\beta-1)/\beta$ and then, we deduce $v_0 = \beta$, $d_0 = 2\beta^2 - \beta$ and $c_0 = 6\beta^2(\beta - 1) + \beta$.

e) When φ is the characteristic function of the Borel law with parameter $\kappa \in]0, 1[$, $X_t|\underline{X}_{t-1}$ follows a generalized Poisson distribution with parameter $((1 - \kappa)\lambda_t, \kappa)$ and we recover the GP-INARCH model ([12]). So, $v_0 = (1-\kappa)^{-2}$, $d_0 = (2\kappa + 1)(1 - \kappa)^{-4}$ and $c_0 = (6\kappa^2 + 8\kappa + 1)(1 - \kappa)^{-6}$.

3. Estimation Procedures

In this section, we will focus on the estimation of the vector $\theta = (\alpha_0, \alpha_1, v_0)^\top$, where v_0 includes the additional parameter associated to the conditional distribution of the CP-INARCH(1) model (for example, $v_0 = 1 + \phi$ in the NTA-INARCH(1) model and $v_0 = (2 - p^*)/p^*$ in the GEOMP2-INARCH(1)). To estimate the true value of θ, we start by discussing a two-step approach using the conditional least squares and moment estimation methods; after we consider the combination of the Poisson Quasi-Maximum Likelihood and moments estimation methods and finally develop the conditional maximum likelihood estimation. For this purpose, let $(x_1, ..., x_n)$ be n particular values, arbitrarily fixed, of the process X.

3.1. Two-step estimation procedures.

3.1.1. *Conditional Least Squares and Moments estimation methods.*

In the first step, we discuss the conditional least squares (CLS) approach for the estimation of the conditional mean parameters α_0 and α_1 and, for parameter v_0 associated to the CP-INARCH(1) conditional distribution, an approach based on the moment estimation method is developed.

The CLS estimator of $\alpha = (\alpha_0, \alpha_1)$ is obtained by minimizing the sum of squares

$$Q_n(\alpha) = \sum_{t=2}^{n} \left[x_t - E\left(X_t | X_{t-1} = x_{t-1}\right)\right]^2 = \sum_{t=2}^{n} \left[x_t - \alpha_0 - \alpha_1 x_{t-1}\right]^2,$$

with respect to α. Solving the least squares equations

$$\begin{cases} \dfrac{\partial Q_n(\alpha)}{\partial \alpha_0} = -2 \displaystyle\sum_{t=2}^{n} \left(x_t - \alpha_0 - \alpha_1 x_{t-1}\right) = 0 \\ \dfrac{\partial Q_n(\alpha)}{\partial \alpha_1} = -2 \displaystyle\sum_{t=2}^{n} x_{t-1}\left(x_t - \alpha_0 - \alpha_1 x_{t-1}\right) = 0, \end{cases}$$

we obtain the following explicit expressions for the CLS estimator $\widehat{\alpha}_n = (\widehat{\alpha}_{0,n}, \widehat{\alpha}_{1,n})$:

$$\widehat{\alpha}_{1,n} = \frac{\sum_{t=2}^{n} X_t X_{t-1} - \frac{1}{n-1} \cdot \sum_{t=2}^{n} X_t \cdot \sum_{s=2}^{n} X_{s-1}}{\sum_{t=2}^{n} X_{t-1}^2 - \frac{1}{n-1} \left(\sum_{t=2}^{n} X_{t-1}\right)^2},$$

$$\widehat{\alpha}_{0,n} = \frac{\sum_{t=2}^{n} X_t - \widehat{\alpha}_{1,n} \sum_{t=2}^{n} X_{t-1}}{n-1}. \tag{3}$$

The consistency and the asymptotic distribution of these estimators are stated in the next theorem.

Theorem 3.1. *Let $\widehat{\alpha}_n = (\widehat{\alpha}_{0,n}, \widehat{\alpha}_{1,n})$ be the CLS estimator of $\alpha = (\alpha_0, \alpha_1)$ given in* (3)*. Then $\widehat{\alpha}_n$ converges almost surely to α and*

$$\sqrt{n}(\widehat{\alpha}_n - \alpha) \xrightarrow{d} N(\boldsymbol{0}_{2\times 1},\, \boldsymbol{V}^{-1} \boldsymbol{W} \boldsymbol{V}^{-1}),$$

as $n \to \infty$, where the entries of the matrix $\boldsymbol{V}^{-1}\boldsymbol{W}\boldsymbol{V}^{-1} = (b_{ij})$, $i, j = 1, 2$, are given by

$$
\begin{aligned}
b_{11} &= \frac{\alpha_0}{1-\alpha_1}\left(\alpha_0(1+\alpha_1) + \frac{v_0^2 + (d_0 - v_0^2)\alpha_1(1+\alpha_1-\alpha_1^2) + (3v_0^2 - d_0)\alpha_1^4}{v_0(1+\alpha_1+\alpha_1^2)}\right),\\
b_{12} &= b_{21} = v_0\alpha_1 - \alpha_0(1+\alpha_1) - \frac{\alpha_1(1+\alpha_1)(d_0 + (3v_0^2 - d_0)\alpha_1^2)}{v_0(1+\alpha_1+\alpha_1^2)},\\
b_{22} &= (1-\alpha_1^2)\left(1 + \frac{\alpha_1(d_0 + (3v_0^2 - d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right),
\end{aligned}
$$

and $\xrightarrow{d}$ means convergence in distribution.

Proof. The results announced are proved using those of Klimko and Nelson [13, Section 3]. In fact, it is easily checked that the regularity conditions (i) to (iii) defined on [13, p. 634] are satisfied taking into account that $g(\alpha; X_{t-1}) = E\left(X_t | \underline{X}_{t-1}\right) = \alpha_0 + \alpha_1 X_{t-1}$, and thus, by their Theorem 3.1, it follows that the CLS estimators are strongly consistent. Furthermore, the matrix $\mathbf{V}$ is invertible as it is given by

$$
\mathbf{V} = \begin{bmatrix} E\left(\frac{\partial g}{\partial\alpha_0}\frac{\partial g}{\partial\alpha_0}\right) & E\left(\frac{\partial g}{\partial\alpha_0}\frac{\partial g}{\partial\alpha_1}\right) \\ E\left(\frac{\partial g}{\partial\alpha_1}\frac{\partial g}{\partial\alpha_0}\right) & E\left(\frac{\partial g}{\partial\alpha_1}\frac{\partial g}{\partial\alpha_1}\right) \end{bmatrix} = \begin{bmatrix} E(1) & E(X_{t-1}) \\ E(X_{t-1}) & E\left(X_{t-1}^2\right) \end{bmatrix} = \begin{bmatrix} 1 & \frac{\alpha_0}{1-\alpha_1} \\ \frac{\alpha_0}{1-\alpha_1} & \frac{\alpha_0(v_0+\alpha_0(1+\alpha_1))}{(1-\alpha_1)(1-\alpha_1^2)} \end{bmatrix},
$$

considering the expressions stated in Theorem 2.1. Thus, Theorem 3.2 of [13] is satisfied implying the asymptotic normality of the CLS estimators. The entries of the covariance matrix of the asymptotic distribution $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$ are derived in Appendix 3. $\square$

To estimate the parameter v_0 we propose to use the moments estimation method. Taking into consideration the expression (2) of the second order moment of the CP-INARCH(1) model, an estimator for v_0, whose strong consistence is a consequence from the strict stationarity and ergodicity of the process X, is given by solving the equation

$$
\frac{\widehat{\alpha}_{0,n}(v_0 + \widehat{\alpha}_{0,n}(1+\widehat{\alpha}_{1,n}))}{(1-\widehat{\alpha}_{1,n})(1-\widehat{\alpha}_{1,n}^2)} = \frac{1}{n}\sum_{t=1}^{n} X_t^2
$$

in order to v_0; in this way we get the two-step CLS+M estimator for $(\alpha_0, \alpha_1, v_0)$.

We deduce, for instance, that the corresponding estimator of ϕ for the NTA-INARCH(1) model is given by

$$\widehat{\phi}_n = -1 - \widehat{\alpha}_{0,n}(1+\widehat{\alpha}_{1,n}) + \frac{(1-\widehat{\alpha}_{1,n})(1-\widehat{\alpha}_{1,n}^2)}{n\widehat{\alpha}_{0,n}} \sum_{t=1}^{n} X_t^2,$$

and for the GEOMP2-INARCH(1) process an estimator of the p^* parameter is

$$\widehat{p^*} = 2\left[1 - \widehat{\alpha}_{0,n}(1+\widehat{\alpha}_{1,n}) + \frac{(1-\widehat{\alpha}_{1,n})(1-\widehat{\alpha}_{1,n}^2)}{n\widehat{\alpha}_{0,n}} \sum_{t=1}^{n} X_t^2\right]^{-1}.$$

3.1.2. *Poisson Quasi-Maximum Likelihood and Moments estimation methods.*

One of the advantages of using the above CLS+M approach is the fact that we do not need to specify entirely the conditional distribution of the CP-INARCH(1) model to estimate its para- meters. We refer now another two-step approach where it is used the Poisson quasi-conditional maximum likelihood estimator (PQMLE) to estimate the conditional mean parameters α_0 and α_1 and, as previously, the moment estimation method for parameter v_0. The resulting estimator is denoted PQML+M.

The PQMLE provides a general approach for estimating the conditional mean parameters of the CP-INARCH(1) models by maximizing a pseudo-likelihood function considering the conditional distribution the Poisson one, that is, the function

$$\widetilde{L}_n(\theta|\mathbf{x}) = \sum_{t=2}^{n} (x_t \log(\lambda_t) - \lambda_t).$$

Ahmad and Francq [1] found some regularity conditions to establish the consistency and asymptotic normality of the Poisson quasi-maximum likelihood estimator of the conditional mean parameters of a count time series. These regularity conditions are easily satisfied by a CP-INARCH (1) process with $\alpha_1 < 1$, and so the Poisson QML estimator of (α_0, α_1) is consistent and asymptotically Gaussian. The almost sure convergence of the v_0 estimator follows as previously.

3.2. Conditional Maximum Likelihood Estimation.

When the distribution of $X_t|\underline{X}_{t-1}$ is known, we can estimate its parameters using the conditional maximum likelihood estimation (CMLE) method. In this section, we discuss this procedure by considering NTA-INARCH(1) and GEOMP2-INARCH(1) models.

Starting by a NTA-INARCH(1) process, we have the conditional probability mass function of X_t ([8]) given by

$$P\left[X_t = x_t|\underline{X}_{t-1}\right] = \frac{e^{-\frac{\lambda_t}{\phi}}\phi^{x_t}}{x_t!}Z(\lambda_t, x_t, \phi),\ \ Z(\lambda_t, X_t, \phi) = \sum_{j=0}^{\infty}\frac{\left(\lambda_t e^{-\phi}\right)^j j^{X_t}}{\phi^j j!},$$

for $x_t = 0, 1, ...$ The conditional likelihood function is then

$$L_n(\theta|\mathbf{x}) = \prod_{t=2}^{n}\frac{e^{-\frac{\lambda_t}{\phi}}\phi^{x_t}}{x_t!}Z(\lambda_t, x_t, \phi),$$

where for convenience $\theta = (\alpha_0, \alpha_1, \phi)$ as $v_0 = 1 + \phi$. So the log-likelihood function has the form

$$\log L_n(\theta|\mathbf{x}) = \sum_{t=2}^{n} l_t(\theta) = \sum_{t=2}^{n}\left\{-\frac{\lambda_t}{\phi} + x_t\log(\phi) - \log\left(x_t!\right) + \log\left(Z(\lambda_t, x_t, \phi)\right)\right\}.$$

The first derivatives of l_t are given as

$$\frac{\partial l_t(\theta)}{\partial\phi} = \frac{\lambda_t}{\phi^2} + \frac{x_t}{\phi} - \left(\frac{\phi+1}{\phi}\right)\frac{Z(\lambda_t, x_t+1, \phi)}{Z(\lambda_t, x_t, \phi)},$$

$$\frac{\partial l_t(\theta)}{\partial\alpha_j} = \left[-\frac{1}{\phi} + \frac{1}{\lambda_t}\frac{Z(\lambda_t, x_t+1, \phi)}{Z(\lambda_t, x_t, \phi)}\right]\frac{\partial\lambda_t}{\partial\alpha_j},\ \ j = 0, 1,$$

and the second derivatives of l_t are

$$\frac{\partial^2 l_t(\theta)}{\partial\phi^2} = -2\frac{\lambda_t}{\phi^3} - \frac{x_t}{\phi^2} + \frac{1}{\phi^2}\frac{Z(\lambda_t, x_t+1, \phi)}{Z(\lambda_t, x_t, \phi)} + \left(\frac{\phi+1}{\phi}\right)^2\left[\frac{Z(\lambda_t, x_t+2, \phi)}{Z(\lambda_t, x_t, \phi)} - \frac{Z^2(\lambda_t, x_t+1, \phi)}{Z^2(\lambda_t, x_t, \phi)}\right],$$

$$\frac{\partial^2 l_t(\theta)}{\partial\phi\partial\alpha_j} = \left[\frac{1}{\phi^2} - \frac{\phi+1}{\phi\lambda_t}\left\{\frac{Z(\lambda_t, x_t+2, \phi)}{Z(\lambda_t, x_t, \phi)} - \frac{Z^2(\lambda_t, x_t+1, \phi)}{Z^2(\lambda_t, x_t, \phi)}\right\}\right]\frac{\partial\lambda_t}{\partial\alpha_j},$$

$$\begin{aligned}\frac{\partial^2 l_t(\theta)}{\partial\alpha_j\partial\alpha_k} &= \frac{1}{\lambda_t^2}\left[-\frac{Z(\lambda_t, x_t+1, \phi)}{Z(\lambda_t, x_t, \phi)} + \frac{Z(\lambda_t, x_t+2, \phi)}{Z(\lambda_t, x_t, \phi)} - \frac{Z^2(\lambda_t, x_t+1, \phi)}{Z^2(\lambda_t, x_t, \phi)}\right]\frac{\partial\lambda_t}{\partial\alpha_j}\frac{\partial\lambda_t}{\partial\alpha_k}\\ &+ \left[\frac{1}{\lambda_t}\frac{Z(\lambda_t, x_t+1, \phi)}{Z(\lambda_t, x_t, \phi)} - \frac{1}{\phi}\right]\frac{\partial^2\lambda_t}{\partial\alpha_j\partial\alpha_j},\ \ 0 \le j, k \le 1,\end{aligned}$$

where the expressions for $\partial\lambda_t/\partial\alpha_j$ and $\partial^2\lambda_t/\partial\alpha_j\partial\alpha_k$ are easily deduced.

Analogously, for the GEOMP2-INARCH(1) process we obtain the following expression

$$\log L_n(\theta|\mathbf{x}) = \sum_{t=2}^{n} l_t(\theta)$$

$$= \sum_{t=2}^{n} \left\{ -\lambda_t + \log \left(1_{x_t=0} + \left[\sum_{n=1}^{x_t} \frac{\lambda_t^n}{n!} \begin{pmatrix} x_t - 1 \\ n-1 \end{pmatrix} (p^*)^n (1-p^*)^{x_t - n} \right] 1_{x_t \neq 0}, \right) \right\},$$

where $\theta = (\alpha_0, \alpha_1, p^*)$, as $v_0 = (2-p^*)/p^*$ and taking into consideration that the conditional probability mass function of X_t is given by

$$P\left[X_t = 0 | \underline{X}_{t-1}\right] = e^{-\lambda_t},$$

$$P\left[X_t = x_t | \underline{X}_{t-1}\right] = \sum_{n=1}^{x_t} e^{-\lambda_t} \frac{\lambda_t^n}{n!} \begin{pmatrix} x_t - 1 \\ n-1 \end{pmatrix} (p^*)^n (1-p^*)^{x_t-n}, \; x_t = 1, 2, ...$$

Similarly to the previous case, the first and second derivatives of l_t in order to α_0, α_1 and p^* are deduced.

4. A simulation study

Some simulation studies were developed to examine and compare the performance of the different estimators considered in Section 3 for the model parameters.

We begin by illustrating the two-step approach based on CLS and moments estimation methods by computing the estimates and analyzing its performance. In the sequel, the several estimation procedures are discussed and compared. All the study is developed considering the NTA-INARCH(1) and the GEOMP2-INARCH(1) models.

4.1. CLS estimators performance.

4.1.1. *NTA-INARCH(1) model.*

To illustrate the CLS method, we focus on the NTA-INARCH(1) model with true parameters $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$ and, for different sample sizes $n = 100$, 250, 500, 750, 1000, we present in Table 1 the expected values, variances and covariance of $\widehat{\alpha}_{0,n}$, $\widehat{\alpha}_{1,n}$ and $\widehat{\phi}_n$. In the last column of this table we present the true values of α_0, α_1 and ϕ as well as the entries of the asymptotic matrix $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$, respectively b_{11}, b_{22} and b_{12} given in Theorem 3.1. We verify that the asymptotic and the sample values are quite similar for large values of n.

Figure 1 displays the boxplot and histogram of the 1000 values of the CLS estimator (centered and reduced) of $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$ for samples of length $n = 2000$ of a NTA-INARCH(1) model with $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$. In agreement with Theorem 3.1, the plots indicates the adequacy of the normal for the empirical marginal distributions of the estimator $\widehat{\alpha}$. In addition, the Kolmogorov-Smirnov test for the sampling laws of the standardized CLS estimation gives large p-values, 0.9454 and 0.4051, for testing against the standard normal distribution.

TABLE 1. Means, variances and covariances for the CLS+M estimates of the NTA-INARCH(1) model with coefficients $\alpha_0 = 2$, $\alpha_1 = 0.2$, $\phi = 2$ and for different sample sizes n.

	$n=$ 100	250	500	750	1000	
$E_{est}(\widehat{\alpha}_{0,n})$	2.0221	2.0108	2.0071	2.0076	2.0014	2
$E_{est}(\widehat{\alpha}_{1,n})$	0.1859	0.1937	0.1971	0.1975	0.1988	0.2
$E_{est}(\widehat{\phi}_n)$	1.9490	1.9666	1.9876	1.9836	1.9874	2
$n \cdot V_{est}(\widehat{\alpha}_{0,n})$	12.1454	11.6264	11.7927	12.7023	12.0195	12.3774
$n \cdot V_{est}(\widehat{\alpha}_{1,n})$	1.1320	1.2236	1.2530	1.2381	1.2368	1.2604
$n \cdot V_{est}(\widehat{\phi}_n)$	22.1794	22.0772	23.1773	20.5149	22.2516	
$n \cdot Cov_{est}(\widehat{\alpha}_{0,n}, \widehat{\alpha}_{1,n})$	-2.0654	-2.4147	-2.5108	-2.4944	-2.3405	-2.5510

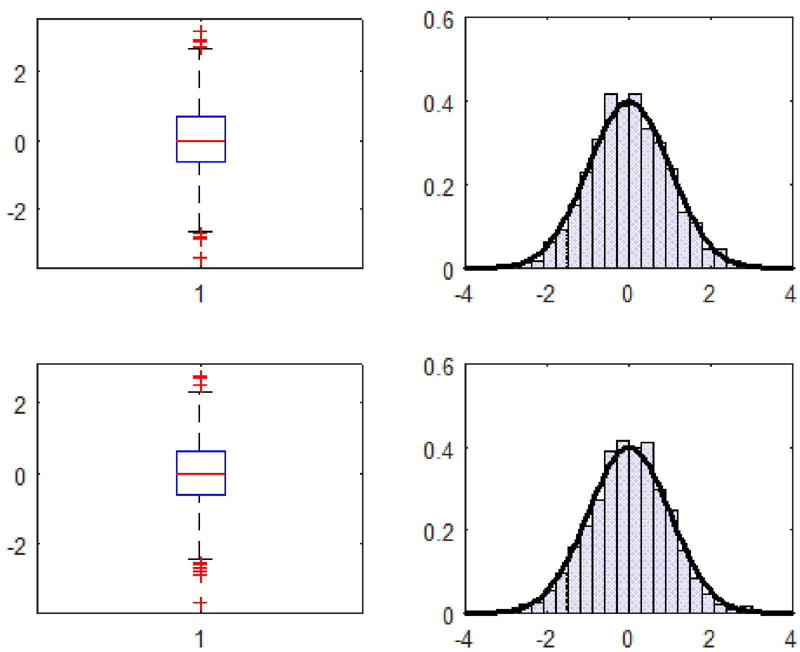


FIGURE 1. Boxplot and histogram of the empirical distribution of $\widehat{\alpha}_0$ (on top) and $\widehat{\alpha}_1$ (bellow) for a NTA-INARCH(1) process when $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$. Superimposed is the standard normal density function.

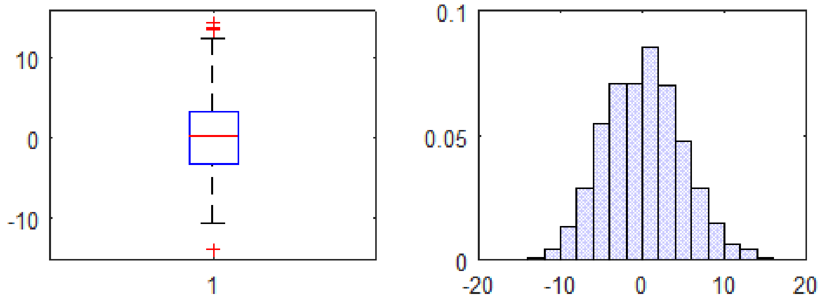


FIGURE 2. Boxplot and histogram of the empirical distribution of $\sqrt{n}\left(\widehat{\phi}_n - \phi\right)$ when $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$ for a NTA-INARCH(1).

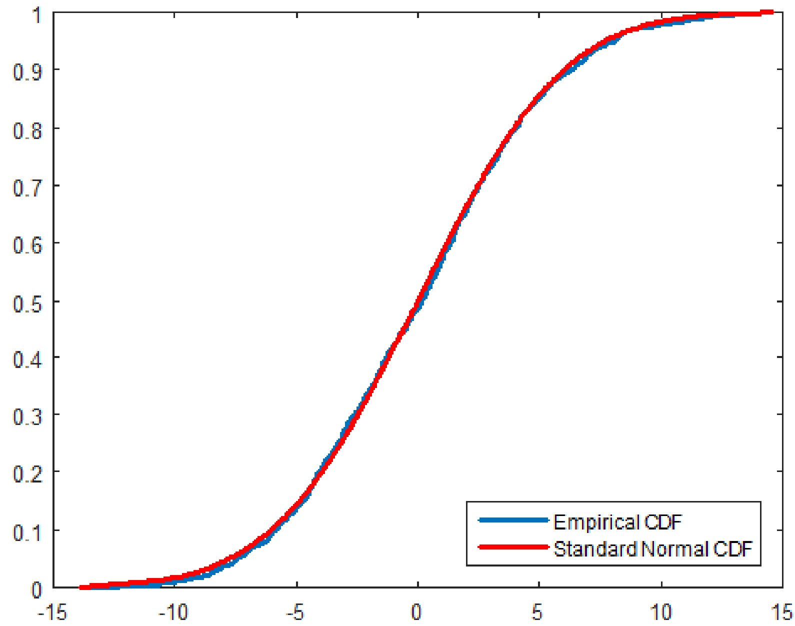


FIGURE 3. Empirical CDF of the distribution of $\sqrt{n}\left(\widehat{\phi}_n - \phi\right)$ when $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$ for a NTA-INARCH(1) (in blue) and the CDF of the normal(0, 4.7) distribution (in red).

In Figure 2 we present the boxplot and the histogram of the distribution of $\sqrt{n}(\widehat{\phi}_n - \phi)$. Figure 3 shows the similarity between the empirical cumulative distribution function of $\sqrt{n}(\widehat{\phi}_n - \phi)$ (represented in blue) and the cumulative distribution function of the normal(0, 4.7) law (in red), whose parameters are the sample mean and variance of $\sqrt{n}(\widehat{\phi}_n - \phi)$. Once again, the p-value of the

Kolmogorov-Smirnov test, namely 0.8231, indicates the adequacy of the normal for the empirical distribution of $\sqrt{n}(\widehat{\phi}_n - \phi)$.

From the empirical results presented in the two last lines of Table 2, we can presume that the estimators of α_0, (resp., α_1) and ϕ are asymptotically uncorrelated. In fact, for the NTA-INARCH(1) model in study, the empirical correlations $\rho_{est}(\widehat{\alpha}_{0,n}, \widehat{\phi}_n)$ and $\rho_{est}(\widehat{\alpha}_{1,n}, \widehat{\phi}_n)$ are significantly low. To support this statement we use the Monte Carlo method to determine confidence intervals for the mean of $\rho_{est}(\widehat{\alpha}_{0,n}, \widehat{\phi}_n)$ and for the mean of $\rho_{est}(\widehat{\alpha}_{1,n}, \widehat{\phi}_n)$ which we denote by $m_{0,n,\widetilde{n}}$ and $m_{1,n,\widetilde{n}}$, respectively. The confidence intervals are obtained considering $\widetilde{n} = 35$ and $\widetilde{n} = 50$ replications of n-dimensional samples ($n = 500$ and $n = 1000$) of a NTA INARCH(1) model with $\alpha_0 = 2$, $\alpha_1 = 0.2$ and $\phi = 2$. Such intervals with confidence level 0.99 are presented in Table 3, where we stress the lower values when n or $\widetilde{n}$ increase. So we have estimated (α_0, α_1) and ϕ separately, likely without loss of efficiency.

Table 2. Empirical correlations for the CLS+M estimates of the NTA-INARCH(1) model with coefficients $\alpha_0 = 2$, $\alpha_1 = 0.2$, $\phi = 2$ and for different sample sizes n.

	$n =$ 250	750	1000	5000	10000
$\rho_{est}(\widehat{\alpha}_{0,n}, \widehat{\alpha}_{1,n})$	-0.6277	-0.6455	-0.6264	-0.6379	-0.6273
$\rho_{est}(\widehat{\alpha}_{0,n}, \widehat{\phi}_n)$	0.1242	0.1163	0.0828	0.0683	0.0636
$\rho_{est}(\widehat{\alpha}_{1,n}, \widehat{\phi}_n)$	0.0061	0.0265	0.0043	0.0703	0.0462

Table 3. Confidence intervals for the mean of $\rho_{est}(\widehat{\alpha}_{0,n}, \widehat{\phi}_n)$ and for the mean of $\rho_{est}(\widehat{\alpha}_{1,n}, \widehat{\phi}_n)$, with confidence level $\gamma = 0.99$ and for different sample sizes n and $\widetilde{n}$.

	$\widetilde{n} = 35$		$\widetilde{n} = 50$	
	$n = 500$	$n = 1000$	$n = 500$	$n = 1000$
$m_{0,n,\widetilde{n}}$	$[0.0917, 0.1180]$	$[0.0883, 0.1162]$	$[0.0940, 0.1160]$	$[0.0814, 0.1064]$
$m_{1,n,\widetilde{n}}$	$[0.0113, 0.0412]$	$[0.0165, 0.0412]$	$[0.0137, 0.0354]$	$[0.0132, 0.0397]$

4.1.2. *GEOMP2-INARCH(1) model.*

Let us consider now the GEOMP2-INARCH(1) model with true parameters $\alpha_0 = 2$, $\alpha_1 = 0.4$ and $p^* = 0.1$. As in the previous section, for different sample sizes n, we compute the expected values, variances and covariance of $\widehat{\alpha}_{0,n}$, $\widehat{\alpha}_{1,n}$ and $\widehat{p^*}_n$ (see Table 4) and for samples of length $n = 2000$ we plot boxplots and histograms for 1000 values of the CLS+M estimators (in Figure 4). To show the adequacy of the normal for the empirical distribution of $\sqrt{n}(\widehat{p^*}_n - p^*)$, in Figure 5 we present the empirical cumulative distribution function of $\sqrt{n}(\widehat{p^*}_n - p^*)$ (represented in blue) and the cumulative distribution function of the normal(0, 0.3) law (in red).

TABLE 4. Expected values, variances and covariances for the CLS+M estimates of the GEOMP2-INARCH(1) model with $\alpha_0 = 2$, $\alpha_1 = 0.4$, $p^* = 0.1$ and different sample sizes n.

	$n =$ 100	250	500	750	1000	
$E_{est}(\widehat{\alpha}_0)$	2.0527	2.0758	2.0400	2.0177	2.0182	2
$E_{est}(\widehat{\alpha}_1)$	0.3367	0.3637	0.3817	0.3890	0.3892	0.4
$E_{est}(\widehat{p^*})$	0.1170	0.1071	0.1040	0.1026	0.1012	0.1
$n \cdot V_{est}(\widehat{\alpha}_0)$	50.6433	53.6845	61.9295	56.6796	59.6529	61.5325
$n \cdot V_{est}(\widehat{\alpha}_1)$	2.9292	3.3704	3.5536	4.2723	3.8424	4.3979
$n \cdot V_{est}(\widehat{p^*})$	0.0961	0.0881	0.0841	0.0856	0.0800	
$n \cdot Cov_{est}(\widehat{\alpha}_0, \widehat{\alpha}_1)$	−1.5730	−3.2463	−5.9278	−5.5029	−5.7433	−7.0598

4.2. Comparative analysis of the estimation procedures.

To examine and compare the finite sample performances of the CLS+M, Poisson QMLE+M and CMLE methods, we consider two different NTA-INARCH(1) models with parameter values $\alpha_0 = 2, \alpha_1 = 0.2, \phi = 2$ and $\alpha_0 = 5, \alpha_1 = 0.3, \phi = 1$, and two different GEOMP2-INARCH(1) models with parameter values $\alpha_0 = 2, \alpha_1 = 0.2, p^* = 0.1$ and $\alpha_0 = 5, \alpha_1 = 0.3, p^* = 0.6$. The sample sizes considered are $n = 500$ and 1000 and the number of replications $m = 1000$.

For the maximization of the log-likelihood functions, we use the MATLAB function *fmincon* where the estimates obtained using the CLS+M method were used as the initial values and the constrained conditions are $\alpha_0 > 0$, $0 < \alpha_1 < 1$, $\phi > 0$ (for the NTA) and $0 < p^* < 1$ (for the GEOMP2). The performance of the estimators is evaluated by the mean square error, i.e.,

$$\frac{1}{m}\sum_{k=1}^{m}\left(\widehat{\theta}_{j,k} - \theta_j\right)^2, \ j = 1, 2, 3.$$

The results of the simulation experiments are presented in Tables 5 and 6.

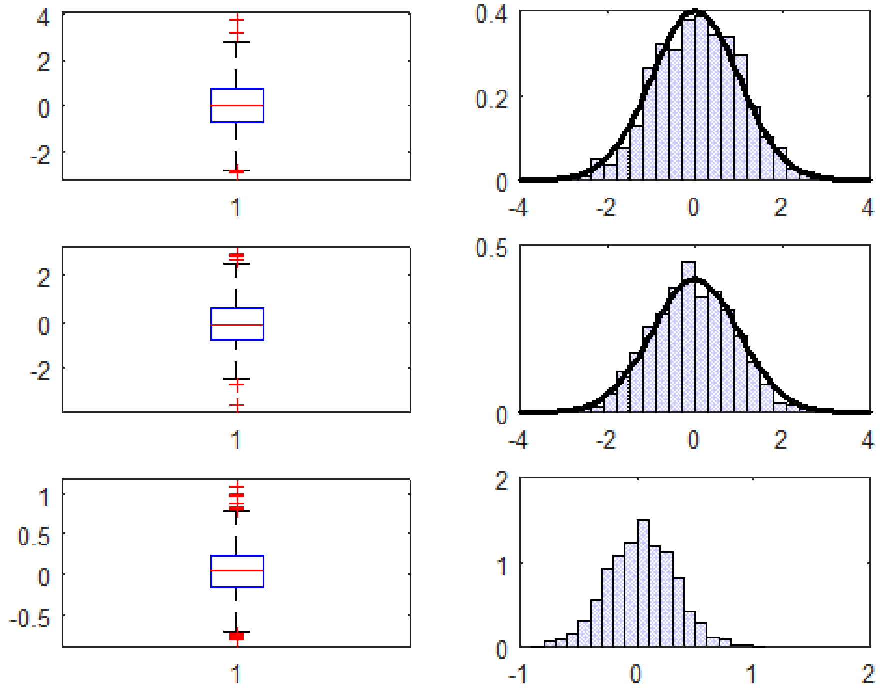


FIGURE 4. Boxplot and histogram of the law of $\widehat{\alpha}_0$ (on top), $\widehat{\alpha}_1$ (in the middle) and $\widehat{p^*}$ (bellow) when $\alpha_0 = 2$, $\alpha_1 = 0.4$ and $p^* = 0.1$ for a GEOMP2-INARCH(1). Superimposed is the standard normal density function. The results are based on 1000 simulations and samples of size $n = 2000$.

From this study we may conclude that the three methods seem to perform quite well, although the CMLE gives slightly smaller mean square errors in most cases.

5. Real data example - Counts of differences in the prices of electricity in Portugal and Spain

OMIE (http://www.omie.es) is the company that manages the wholesale electricity market on the Iberian Peninsula. Electricity prices in Europe are set on a daily basis (every day of the year) at 12 noon, for the twenty-four hours of the following day, known as daily market. The market splitting is the mechanism used for setting the price of electricity on the daily market. When the price of electricity is the same in Portugal and Spain, which corresponds to the desired situation, it means that the integration of the Iberian market is working properly.

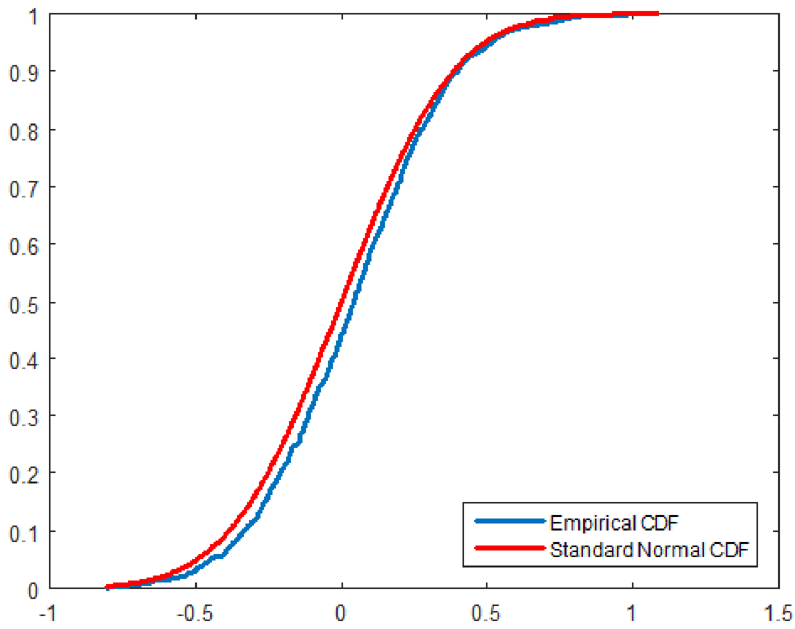


FIGURE 5. Empirical CDF of the law of $\sqrt{n}\left(\widehat{p^*}_n - p^*\right)$ when $\alpha_0 = 2$, $\alpha_1 = 0.4$ and $p^* = 0.1$ for a GEOMP2-INARCH(1) model (in blue) and the CDF of the normal(0, 0.3) law (in red).

TABLE 5. Mean estimates (in bold) and mean square errors (within parentheses) for the NTA-INARCH(1) model with different sample sizes n.

n	Method	$\alpha_0 = 2$	$\alpha_1 = 0.2$	$\phi = 2$	$\alpha_0 = 5$	$\alpha_1 = 0.3$	$\phi = 1$
500	CLS+M	**2.0023** (0.0224)	**0.1974** (0.0023)	**1.9921** (0.0431)	**5.0293** (0.1204)	**0.2952** (0.0022)	**0.9948** (0.0178)
	PQMLE+M	**2.0013** (0.0214)	**0.1978** (0.0022)	**1.9921** (0.0432)	**5.0288** (0.1159)	**0.2953** (0.0021)	**0.9951** (0.0181)
	MLE	**1.9996** (0.0206)	**0.1985** (0.0021)	**1.9969** (0.0172)	**5.0291** (0.1140)	**0.2953** (0.0021)	**0.9923** (0.0139)
1000	CLS+M	**2.0027** (0.0123)	**0.1973** (0.0013)	**1.9887** (0.0225)	**5.0044** (0.0571)	**0.2986** (0.0010)	**0.9956** (0.0089)
	PQMLE+M	**2.0040** (0.0118)	**0.1968** (0.0012)	**1.9897** (0.0227)	**5.0042** (0.0538)	**0.2986** (0.0010)	**0.9957** (0.0089)
	MLE	**2.0023** (0.0115)	**0.1975** (0.0011)	**1.9956** (0.0082)	**5.0050** (0.0531)	**0.2985** (0.0010)	**0.9961** (0.0070)

In the following, we consider the time series that represents the number of hours in a day in which the prices of electricity for Portugal and Spain are

TABLE 6. Mean estimates (in bold) and mean square errors (within parentheses) for the GEOMP2-INARCH(1) model with different sample sizes n.

n	Method	$\alpha_0 = 2$	$\alpha_1 = 0.2$	$p^* = 0.1$	$\alpha_0 = 5$	$\alpha_1 = 0.3$	$p^* = 0.6$
500	CLS+M	**2.0432** (0.0998)	**0.1887** (0.0055)	**0.1033** (0.0002)	**5.0307** (0.1191)	**0.2944** (0.0020)	**0.6025** (0.0008)
	PQMLE+M	**2.0351** (0.0949)	**0.1918** (0.0050)	**0.1034** (0.0002)	**5.0291** (0.1141)	**0.2946** (0.0020)	**0.6026** (0.0008)
	MLE	**2.0228** (0.0840)	**0.1967** (0.0035)	**0.1005** (0.0001)	**5.0290** (0.1114)	**0.2946** (0.0019)	**0.6025** (0.0007)
1000	CLS+M	**2.0020** (0.0464)	**0.1960** (0.0031)	**0.1016** (0.0001)	**5.0121** (0.0578)	**0.2985** (0.0010)	**0.6011** (0.0004)
	PQMLE+M	**1.9980** (0.0438)	**0.1976** (0.0027)	**0.1016** (0.0001)	**5.0084** (0.0552)	**0.2990** (0.0010)	**0.6012** (0.0004)
	MLE	**1.9947** (0.0393)	**0.1991** (0.0019)	**0.1005** (0.0000)	**5.0091** (0.0528)	**0.2989** (0.0009)	**0.6008** (0.0004)

different. The data presented in Figure 6 consists of 365 observations, starting from January 2013 and ending in December 2013.

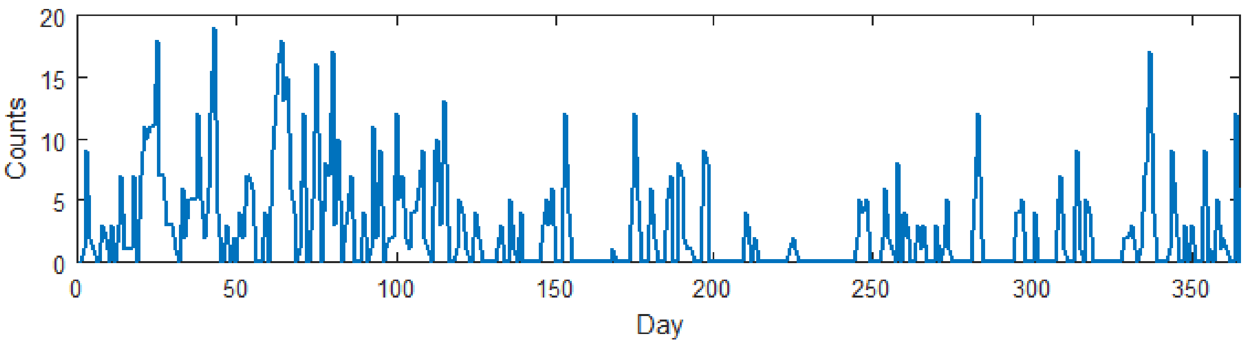
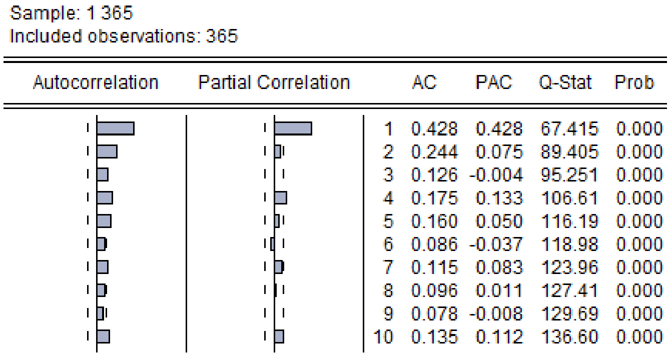


FIGURE 6. Daily number of hours in which the price of electricity of Portugal and Spain are different, starting from January 2013 and ending in December 2013.

Empirical mean and variance of the data are 2.5863 and 15.0894, respectively, indicating that the true marginal distribution is overdispersed. Let us observe that this time series exhibits also volatility clusters suggesting characteristics of conditional heteroscedasticity. The partial autocorrelation function presented in Figure 7, suggests an order 1 dependence and so a CP-INARCH(1) model may be a reasonable choice to fit the data within the CP-INGARCH class.



Sample: 1 365
Included observations: 365

	AC	PAC	Q-Stat	Prob
1	0.428	0.428	67.415	0.000
2	0.244	0.075	89.405	0.000
3	0.126	-0.004	95.251	0.000
4	0.175	0.133	106.61	0.000
5	0.160	0.050	116.19	0.000
6	0.086	-0.037	118.98	0.000
7	0.115	0.083	123.96	0.000
8	0.096	0.011	127.41	0.000
9	0.078	-0.008	129.69	0.000
10	0.135	0.112	136.60	0.000

FIGURE 7. Sample autocorrelations and partial autocorrelations.

Trying to obtain a suitable model for this count time series, we present a comparative study between CP-INARCH(1) processes associated to the Poisson ([4]), the generalized Poisson ([12]), the Neyman type-A, the geometric Poisson and the negative binomial ([10]) laws. For the parameter estimation we use the conditional maximum likelihood method and the results, obtained with the help of MATLAB software, are displayed in Table 7. Based on the values of the log likelihood function, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC), we conclude that the GEOMP2-INARCH(1) model gives better fit than the other CP-INARCH(1) models considered and the NTA-INARCH(1) fit is the second one. The mean, variance and the first-order autocorrelation coefficient (FOAC) for the fitted CP-INARCH(1) models are summarized in Table 8. The results are in accordance with the previous conclusions as, although the similarity of the mean values, the variance values point clearly to a NTA-INARCH(1) or GEOMP2-INARCH(1) fitting.

TABLE 7. Conditional ML parameters estimates for several CP-INARCH(1) models.

	Model	$\widehat{\alpha}_{0,365}$	$\widehat{\alpha}_{1,365}$	Additional parameter	-Log L	AIC	BIC
INARCH(1)	P	1.4189	0.4529		1047.8	2099.6	2107.4
	GP	1.2689	0.5107	$\widehat{\kappa}_{365} = 0.6429$	695.5	1396.9	1408.6
	NTA	1.3891	0.4644	$\widehat{\phi}_{365} = 2.9439$	683.7	1373.6	1385.3
	GEOMP2	1.3518	0.4787	$\widehat{p^*}_{365} = 0.3111$	676.5	1359.0	1370.7
	NB2	1.3030	0.4976	$\widehat{\beta}_{365} = 6.6531$	685.9	1377.8	1389.5

TABLE 8. Sample and estimated means, variances and FOACs under CP-INARCH(1) models.

Model	Sample	P	GP	NTA	GEOMP2	NB2
Mean	2.5863	2.5935	2.5933	2.5935	2.5931	2.5936
Variance	15.0894	3.2627	27.5118	13.0412	18.2625	22.9337
FOAC	0.428	0.4529	0.5107	0.4644	0.4787	0.4976

6. Conclusion

The class of integer-valued GARCH models, specified through the characteristic function of the compound Poisson law and denoted CP-INGARCH, [6], unifies and enlarges substantially the family of conditionally heteroscedastic integer-valued processes. With this new class, we may capture simultaneously different kinds of conditional volatility and the overdispersion characteristic often recorded in real count data. The probabilistic analysis of this kind of models, concerning stationarity and ergodicity properties as well as moments studies, was the goal of previous works among which we may refer those established in [5] and [6]. The aim of this paper is to develop some statistical studies, regarding the process parametric estimation, that allow the use of this general class with real data and show its true practical usefulness. We concentrate our study on the CP-INARCH models of order one, and a two-step estimation methodology, involving the conditional least square or the Poisson quasi-maximum likelihood methods in a first-step, and the moment's estimation method in the second one, has been introduced and developed. We point out the great advantage of this procedure regarding the more classical conditional maximum likelihood one, as its application is independent from the specific conditional distribution of the process. In fact, the simulation study presented allows concluding that the two-step methodology performance is strongly competitive with that of the conditional maximum likelihood estimation. We should also stress that the practical relevance of this wide class is clearly shown with the real-data example presented which illustrates the better quality of the fitting performed by new models emerged from that class.

Future developments of the present study should concern, particularly, the parametric estimation of a general CP-INGARCH model. In which regards the CP-INARCH process of any order p, these procedures will be easily generalized with a natural increase of complexity.

Acknowledgements. This work was supported by the Centre for Mathematics of the University of Coimbra - UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

References

[1] Ahmad, A., Francq, C.: Poisson QMLE of count time series models. *Journal of Time Series Analysis*, DOI: 10.1111/jtsa.12167 (2015).

[2] Bakouch, H.: Higher-order moments, cumulants and spectral densities of the NGINAR(1) process. *Statistical Methodology*, **7** 1–21 (2010).

[3] Cramér, H.: Mathematical Methods of Statistics. Princeton University Press, Princeton (1946).

[4] Ferland, R., Latour, A., Oraichi, D.: Integer-valued GARCH process. *Journal of Time Series Analysis* **27**, 923–942 (2006).

[5] Gonçalves, E., Mendes Lopes, N., Silva, F.: A new approach to integer-valued time series modeling: The Neyman Type-A INGARCH model, *Lithuanian Mathematical Journal*, **55(2)**, 231–242 (2015a).

[6] Gonçalves, E., Mendes Lopes, N., Silva, F.: Infinitely divisible distributions in integer valued GARCH models, *Journal of Time Series Analysis*, **36**, 503–527 (2015b).

[7] Gonçalves, E., Mendes Lopes, N., Silva, F.: Zero-inflated compound Poisson distributions in integer-valued GARCH models, *Statistics*, **50**, 558–578 (2016).

[8] Johnson, N.L., Kotz, S., Kemp, A.W.: Univariate discrete distributions. Wiley, New York, 3rd ed. (2005).

[9] Weiß, C. H.: INARCH(1) Processes: Higher-order Moments and Jumps. *Stat. Probab. Lett.* **80**, 1771–1780 (2010).

[10] Xu, H.-Y., Xie, M., Goh, T. N., and Fu, X.: A model for integer-valued time series with conditional overdispersion. *Computational Statistics and Data Analysis*, **56**, 4229–4242 (2012).

[11] Zhu, Fk.: A negative binomial integer-valued GARCH model. *Journal of Time Series Analysis* **32**, 54–67 (2011).

[12] Zhu, Fk.: Modelling overdispersed or underdispersed count data with generalized Poisson integer-valued GARCH models. *Journal of Mathematical Analysis and Applications* **389 1**, 58–71 (2012).

[13] Klimko, L.A. and Nelson, P.I.: On conditional least squares estimation for stochastic processes. *Ann. Stat.* **6(3)**, 629–642 (1978).

Appendix 1. Proof of Theorem 2.1

To establish the results present in Theorem 2.1 let us begin by recalling the expression of the following conditional moments:

$$\begin{aligned}
E(X_t|\underline{X}_{t-1}) &= \lambda_t \;=\; \alpha_0+\alpha_1 X_{t-1}, \\
E(X_t^2|\underline{X}_{t-1}) &= v_0\lambda_t+\lambda_t^2 \;=\; \alpha_1^2X_{t-1}^2+\alpha_1(2\alpha_0+v_0)X_{t-1}+\alpha_0(\alpha_0+v_0), \qquad (4)\\
E(X_t^3|\underline{X}_{t-1}) &= i\,\Phi'''_{X_t|\underline{X}_{t-1}}(0) \;=\; d_0\lambda_t+3v_0\lambda_t^2+\lambda_t^3 \\
&= \alpha_1^3X_{t-1}^3+3\alpha_1^2(v_0+\alpha_0)X_{t-1}^2+\alpha_1(3\alpha_0^2+6v_0\alpha_0+d_0)X_{t-1} \\
&\quad +\alpha_0(d_0+3v_0\alpha_0+\alpha_0^2). \qquad (5)
\end{aligned}$$

(a): Using the fact that for $k \geq 0$, $\Gamma(k) = \alpha_1^k f_2$ we get

$$\begin{aligned}\mu(k) &= E(X_t X_{t+k}) = Cov(X_t, X_{t+k}) + E(X_t)^2 \\ &= f_2(v_0\alpha_1^k + \alpha_0(1+\alpha_1)). \end{aligned} \tag{6}$$

(b): To derive $\mu(k,l)$, $0 \leq k \leq l$, we distinguish the following three cases:

Case 1: $l > k$**:** We have

$$\begin{aligned}\mu(k,l) &= E(X_t X_{t+k} X_{t+l}) \ = \ E[X_t X_{t+k} E(X_{t+l} | \underline{X}_{t+l-1})] \\ &= \alpha_0 E(X_t X_{t+k}) + \alpha_1 E(X_t X_{t+k} X_{t+l-1}) \\ &= \alpha_0 \mu(k) + \alpha_1 \mu(k, l-1) \\ &= \alpha_0 \mu(k) + \alpha_1[\alpha_0 \mu(k) + \alpha_1 \mu(k, l-2)] \\ &= \ \ldots \ = \ \alpha_1^{l-k}[\mu(k,k) - f_1\mu(k)] + f_1\mu(k). \end{aligned}$$

Case 2: $l = k > 0$**:** We have

$$\begin{aligned}\mu(k,k) &= E[X_t E(X_{t+k}^2 | \underline{X}_{t+k-1})] \\ &= \alpha_1^2 E(X_t X_{t+k-1}^2) + \alpha_1(2\alpha_0 + v_0) E(X_t X_{t+k-1}) + \alpha_0(\alpha_0+v_0)E(X_t) \\ &= \alpha_1^2 \mu(k-1,k-1) + \alpha_1(2\alpha_0+v_0)\mu(k-1) + \alpha_0(\alpha_0+v_0) f_1 \\ &= \ \ldots = \alpha_1^{2k}\left[\mu(0,0) - \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1} f_2 - f_1\mu(0)\right] + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1} f_2 \alpha_1^k + f_1\mu(0). \end{aligned}$$

Case 3: $l = k = 0$**:** According to the relations between the moments and the cumulants (e.g., formula (15.10.4) in [3, p. 186]) and Theorem 4.2 of [7], we have

$$\begin{aligned}\mu(0,0) &= E(X_t^3) = \kappa_3 + 3\kappa_2\mu + \mu^3 = f_3[d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2] + 3v_0 f_2 f_1 + f_1^3 \\ &= [d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2] f_3 + \frac{2\alpha_0 v_0}{1-\alpha_1} f_2 + f_1\mu(0), \end{aligned}$$

since $f_1 = (1-\alpha_1^2) f_2$.

So the above formula for $\mu(k,k)$ simplifies to

$$\begin{aligned}\mu(k,k) &= \alpha_1^{2k}\left[[d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2] f_3 - \frac{v_0^2}{1-\alpha_1} f_2\right] + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1} f_2\alpha_1^k + f_1\mu(0) \\ &= \alpha_1^{2k} f_3\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2)\right] + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1} f_2 \alpha_1^k + f_1\mu(0), \end{aligned}$$

which also holds for $k = 0$. Replacing this expression in $\mu(k,l)$ above, it follows that

$$\begin{aligned}\mu(k,l) &= \alpha_1^{l-k}\Bigg[[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)] f_3 \alpha_1^{2k} + \frac{v_0(2\alpha_0+v_0)}{1-\alpha_1} f_2\alpha_1^k \\ &\quad + f_1\mu(0) - f_1\mu(k)\Bigg] + f_1\mu(k). \end{aligned}$$

As

$$f_1\mu(0) - f_1\mu(k) = v_0 f_1 f_2 - \frac{v_0\alpha_0}{1-\alpha_1} f_2 \alpha_1^k,$$

we finally obtain, for any $0 \le k \le l$,

$$\mu(k,l) = [d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)] f_3 \alpha_1^{l+k} + \frac{v_0(\alpha_0+v_0)}{1-\alpha_1} f_2 \alpha_1^l + v_0 f_1 f_2 \alpha_1^{l-k} + f_1\mu(k).$$

(c): In what concerns the fourth-order moments $\mu(k,l,m)$ with $0 \le k \le l \le m$, we proceed in a similar way as above and distinguish the following four cases:

Case 1: $m > l$**:** As above we have

$$\begin{aligned} \mu(k,l,m) &= E(X_t X_{t+k} X_{t+l} X_{t+m}) \\ &= \alpha_1^{m-l}[\mu(k,l,l) - f_1\mu(k,l)] + f_1\mu(k,l). \end{aligned}$$

Case 2: $m = l > k$**:** For this case, using formula (4), we obtain

$$\begin{aligned} \mu(k,l,l) &= E[X_t X_{t+k} E(X_{t+l}^2 | \underline{X}_{t+l-1})] \\ &= \alpha_1^2 \mu(k,l-1,l-1) + \alpha_1(v_0+2\alpha_0)\mu(k,l-1) + \alpha_0(v_0+\alpha_0)\mu(k). \end{aligned}$$

Replacing $\mu(k,l-1)$, using $\mu(0) = (v_0+\alpha_0(1+\alpha_1))f_2$ and replacing $\mu(k)$, we obtain

$$\begin{aligned} \mu(k,l,l) &= \alpha_1^{2(l-k)}\mu(k,k,k) + \mu(k)\mu(0) \\ &\quad - f_2 v_0 \left[f_2(v_0+\alpha_0(1+\alpha_1)) + \frac{(v_0+2\alpha_0)(v_0+\alpha_0)}{(1-\alpha_1)^2} \right] \alpha_1^{2l-k} \\ &\quad - f_1 \left[f_1\mu(0) + \frac{v_0(v_0+2\alpha_0)}{1-\alpha_1} f_2 \right] \alpha_1^{2(l-k)} + \frac{v_0+2\alpha_0}{1-\alpha_1}[\mu(k,l) - f_1\mu(k)] \\ &\quad - \frac{v_0+2\alpha_0}{1-\alpha_1}\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2) \right] f_3 \alpha_1^{2l}. \end{aligned}$$

So, replacing $\mu(0)$, recalling $\mu(0,0)$ and taking into account that $\frac{f_1}{1-\alpha_1} = (1+\alpha_1)f_2$, we get

$$\begin{aligned} \mu(k,l,l) &= \alpha_1^{2(l-k)}\mu(k,k,k) - \mu(k) f_2[\alpha_0 + (v_0+\alpha_0)\alpha_1] \\ &\quad - \frac{f_2 v_0}{(1-\alpha_1)(1-\alpha_1^2)} \left[v_0^2(1+\alpha_1) + v_0\alpha_0(4+3\alpha_1) + 3\alpha_0^2(1+\alpha_1) \right] \alpha_1^{2l-k} \\ &\quad - f_1 \left\{ \mu(0,0) - [d_0(1-\alpha_1^2) + 3v_0^2\alpha_1^2] f_3 + \frac{v_0^2 f_2}{1-\alpha_1} \right\} \alpha_1^{2(l-k)} \\ &\quad + \frac{v_0+2\alpha_0}{1-\alpha_1}\mu(k,l) - \frac{v_0+2\alpha_0}{1-\alpha_1}\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2) \right] f_3 \alpha_1^{2l}. \end{aligned} \tag{7}$$

Case 3: $m = l = k > 0$**:** From formula (5) we have

$$\begin{aligned}\mu(k,k,k) &= E[X_t E(X_{t+k}^3 | \underline{X}_{t+k-1})] \\ &= \alpha_1^3 \mu(k-1,k-1,k-1) + 3\alpha_1^2(v_0+\alpha_0)\mu(k-1,k-1) \\ &\quad + \alpha_1(d_0 + 6v_0\alpha_0 + 3\alpha_0^2)\mu(k-1) + \alpha_0(d_0 + 3v_0\alpha_0 + \alpha_0^2)\mu.\end{aligned}$$

Replacing $\mu(k-1,k-1)$ and thereafter $\mu(k-1)$, we deduce

$$\begin{aligned}\mu(k,k,k) &= \alpha_1^3\mu(k-1,k-1,k-1) \\ &\quad +3(v_0+\alpha_0)\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right] f_3\alpha_1^{2k} \\ &\quad +\frac{v_0 f_2}{1-\alpha_1}\left[3\alpha_1(v_0+\alpha_0)^2 + 3\alpha_1(v_0+\alpha_0)\alpha_0 + (d_0+6v_0\alpha_0+3\alpha_0^2)(1-\alpha_1)\right]\alpha_1^k \\ &\quad +f_1 f_2\left\{3\alpha_1^2(v_0+\alpha_0)(v_0+\alpha_0(1+\alpha_1)) + (d_0+6v_0\alpha_0+3\alpha_0^2)\alpha_1(1-\alpha_1)(1+\alpha_1)\right. \\ &\quad \left. +(d_0+3v_0\alpha_0+\alpha_0^2)(1-\alpha_1)(1-\alpha_1^2)\right\}.\end{aligned}$$

Making some calculations and then recalling the expression of $\mu(0,0)$, we obtain

$$\begin{aligned}\mu(k,k,k) &= \alpha_1^3\mu(k-1,k-1,k-1) \\ &\quad +3(v_0+\alpha_0)\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right] f_3\alpha_1^{2k} \\ &\quad +\frac{v_0 f_2}{1-\alpha_1}\left[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) + d_0(1-\alpha_1) + 3v_0^2\alpha_1\right]\alpha_1^k \\ &\quad +f_1(1-\alpha_1^3)\mu(0,0).\end{aligned}$$

Replacing successively the expression of $\mu(k-j,k-j,k-j)$, $j = 1,...,k-1$, it remains

$$\begin{aligned}\mu(k,k,k) &= \alpha_1^{3k}\left\{\mu(0,0,0) - 3(v_0+\alpha_0)\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right]\frac{f_3}{1-\alpha_1}\right. \\ &\quad -\frac{v_0 f_2}{(1-\alpha_1)(1-\alpha_1^2)}\left[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) + d_0(1-\alpha_1) + 3v_0^2\alpha_1\right] \\ &\quad \left. -f_1\mu(0,0)\right\} + \frac{3(v_0+\alpha_0)f_3\alpha_1^{2k}}{1-\alpha_1}\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right] \\ &\quad +\frac{v_0 f_2\alpha_1^k}{(1-\alpha_1)(1-\alpha_1^2)}\left[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) + d_0(1-\alpha_1) + 3v_0^2\alpha_1\right] \\ &\quad +f_1\mu(0,0). \qquad (8)\end{aligned}$$

Replacing $\mu(0,0)$, highlighting $\frac{f_3}{1-\alpha_1^2}$, noting that $f_2 = (1-\alpha_1^3)f_3$ and $\frac{f_3}{1-\alpha_1^2} = f_4(1+\alpha_1^2)$ and developing the calculations, we finally get

$$\begin{aligned}
\mu(k,k,k) &= \big\{\mu(0,0,0) - f_4\big[4v_0d_0 - 3v_0^3 + 3v_0(d_0-v_0^2)\alpha_1 + v_0(3v_0^2+d_0)\alpha_1^2 \\
&\quad + v_0(6v_0^2-d_0)\alpha_1^3 + 3v_0(2v_0^2-d_0)\alpha_1^4 + v_0(9v_0^2-4d_0)\alpha_1^5 \\
&\quad + \alpha_0(1+\alpha_1^2)\left[3v_0^2+4d_0+(3v_0^2+4d_0)\alpha_1+(15v_0^2-4d_0)\alpha_1^2+(12v_0^2-4d_0)\alpha_1^3\right] \\
&\quad + 6v_0\alpha_0^2(1+\alpha_1^2)(1+\alpha_1)(1+\alpha_1+\alpha_1^2) \\
&\quad + \alpha_0^3(1+\alpha_1^2)(1+\alpha_1)^2(1+\alpha_1+\alpha_1^2)\big]\big\}\,\alpha_1^{3k} \\
&\quad + 3\frac{v_0+\alpha_0}{1-\alpha_1} f_3\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2k} + f_1\mu(0,0) \\
&\quad + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2\big[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) \\
&\quad + d_0(1-\alpha_1) + 3v_0^2\alpha_1\big]\,\alpha_1^k. \qquad (9)
\end{aligned}$$

Case 4: $m=l=k=0$: Once again, according to the relations between the moments and the cumulants, we obtain

$$\begin{aligned}
\mu(0,0,0) &= E(X_t^4) = \kappa_4 + 3\kappa_2^2 + 6\kappa_2\mu^2 + 4\kappa_3\mu + \mu^4 \\
&= f_4\big\{c_0 + (3v_0^3+4v_0d_0-c_0)\alpha_1^2 + (6v_0d_0-c_0)\alpha_1^3 + (15v_0^3-10v_0d_0+c_0)\alpha_1^5 \\
&\quad + \alpha_0(1+\alpha_1^2)\left[3v_0^2+4d_0+(3v_0^2+4d_0)\alpha_1+(15v_0^2-4d_0)\alpha_1^2+(12v_0^2-4d_0)\alpha_1^3\right] \\
&\quad + 6v_0\alpha_0^2(1+\alpha_1)(1+\alpha_1^2)(1+\alpha_1+\alpha_1^2) + \alpha_0^3(1+\alpha_1)^2(1+\alpha_1^2)(1+\alpha_1+\alpha_1^2)\big\}.
\end{aligned}$$

So the formula (9) for $\mu(k,k,k)$ studied in case 3 simplifies to

$$\begin{aligned}
\mu(k,k,k) &= f_4\big\{c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2-d_0)\alpha_1 + (3v_0d_0-c_0)\alpha_1^2 \\
&\quad + (7v_0d_0-6v_0^3-c_0)\alpha_1^3 + 3v_0(d_0-2v_0^2)\alpha_1^4 + (6v_0^3-6v_0d_0+c_0)\alpha_1^5\big\}\,\alpha_1^{3k} \\
&\quad + 3\frac{v_0+\alpha_0}{1-\alpha_1} f_3\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2k} + f_1\mu(0,0) \\
&\quad + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2\left[3\alpha_0^2(1+\alpha_1) + 3v_0\alpha_0(2+\alpha_1) + d_0(1-\alpha_1) + 3v_0^2\alpha_1\right]\alpha_1^k.
\end{aligned}$$

Inserting into the formula (7) for $\mu(k,l,l)$ stated in case 2, we obtain

$$\begin{aligned}
\mu(k,l,l) &= f_4\big\{c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2-d_0)\alpha_1 + (3v_0d_0-c_0)\alpha_1^2 \\
&\quad + (7v_0d_0-6v_0^3-c_0)\alpha_1^3 + 3v_0(d_0-2v_0^2)\alpha_1^4 + (6v_0^3-6v_0d_0+c_0)\alpha_1^5\big\}\,\alpha_1^{2l+k} \\
&\quad + \frac{2v_0+\alpha_0}{1-\alpha_1} f_3\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)\right]\alpha_1^{2l} \\
&\quad + \left\{\frac{\alpha_0 f_3}{1-\alpha_1}[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]\right\}\alpha_1^{2(l-k)} \\
&\quad + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2\left[2v_0\alpha_0 + d_0(1-\alpha_1) + v_0^2(2\alpha_1-1)\right]\alpha_1^{2l-k} \\
&\quad + \frac{v_0+2\alpha_0}{1-\alpha_1}\mu(k,l) - f_2\mu(k)[\alpha_0 + (v_0+\alpha_0)\alpha_1].
\end{aligned}$$

So it follows that we have

$$
\begin{aligned}
\mu(k,l,m) &= \alpha_1^{m-l}[\mu(k,l,l) - f_1\mu(k,l)] + f_1\mu(k,l) \\
&= \alpha_1^{m-l}\Big[f_4\left\{c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2 - d_0)\alpha_1 + (3v_0d_0 - c_0)\alpha_1^2\right. \\
&\quad \left. +(7v_0d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0d_0 + c_0)\alpha_1^5\right\}\alpha_1^{2l+k} \\
&\quad + \frac{2v_0 + \alpha_0}{1-\alpha_1} f_3\left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2)\right]\alpha_1^{2l} \\
&\quad + \left\{\frac{\alpha_0 f_3}{1-\alpha_1}[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)]\right\}\alpha_1^{2(l-k)} \\
&\quad + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2\left[2v_0\alpha_0 + d_0(1-\alpha_1) + v_0^2(2\alpha_1 - 1)\right]\alpha_1^{2l-k} \\
&\quad \left. + \frac{v_0+\alpha_0}{1-\alpha_1}\mu(k,l) - f_2\mu(k)[\alpha_0 + (v_0+\alpha_0)\alpha_1]\right] + f_1\mu(k,l),
\end{aligned}
$$

which holds for all $0 \le k \le l \le m$.

Appendix 2. Proof of Corollary 2.1

To establish the results present in Corollary 2.1 we use the general relations between joint moments and joint cumulants (see [2], p. 5),

(a): the second-order central moments and cumulants of X, for any $s \ge 0$, are given by

$$\widetilde{\mu}(s) = \kappa(s) = Cov(X_t, X_{t+s}) = v_0\alpha_1^s f_2.$$

(b): the third-order central moments and cumulants, for any $l \ge s \ge 0$, are given by

$$
\begin{aligned}
\widetilde{\mu}(s,l) &= \kappa(s,l) \\
&= f_3\alpha_1^l[v_0^2(1+\alpha_1+\alpha_1^2) - \{v_0^2(1+\alpha_1-2\alpha_1^2) - d_0(1-\alpha_1^2)\}\alpha_1^s].
\end{aligned}
$$

(c): In what concerns the fourth-order cumulants we have, for $m \ge l \ge s \ge 0$,

$$
\begin{aligned}
\kappa(s,l,m) \;=\; & \boldsymbol{\alpha_1^{m-l}} \Big[\; \alpha_1^{2l+s} f_4 \big\{ c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2-d_0)\alpha_1 + (3v_0d_0-c_0)\alpha_1^2 \\
& + (7v_0d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0-2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0d_0 + c_0)\alpha_1^5 \big\} \\
& + \frac{2v_0+\alpha_0}{1-\alpha_1} f_3 \left[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2) \right] \alpha_1^{2l} \\
& + \boldsymbol{\left\{ \frac{\alpha_0 f_3}{1-\alpha_1} [d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)] \right\} \alpha_1^{2(l-s)}} \\
& + \frac{v_0}{(1-\alpha_1)(1-\alpha_1^2)} f_2 \left[2v_0\alpha_0 + d_0(1-\alpha_1) + v_0^2(2\alpha_1 - 1) \right] \alpha_1^{2l-s} \\
& + \frac{v_0+\alpha_0}{1-\alpha_1} \mu(s,l) - f_2\mu(s)[\alpha_0 + (v_0+\alpha_0)\alpha_1] \;\Big] \boldsymbol{+ f_1\mu(s,l) - f_1\mu(s,l)} \\
& \boldsymbol{- f_1} \Big(\; \boldsymbol{[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1 - 2\alpha_1^2)] f_3 \alpha_1^{m+l-2s}} + \frac{v_0(v_0+\alpha_0)}{1-\alpha_1} f_2 \alpha_1^{m-s} \\
& + v_0 f_1 f_2 \alpha_1^{m-l} \boldsymbol{+ f_1\mu(l-s) - f_1 f_2 (v_0 \alpha_1^{l-s} + \alpha_0(1+\alpha_1))} \\
& + [d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)] f_3 \alpha_1^{m+l} \\
& + \frac{v_0(v_0+\alpha_0)}{1-\alpha_1} f_2 \alpha_1^m + f_1 f_2 v_0 \alpha_1^{m-l} \boldsymbol{+ f_1\mu(l) - f_1\mu(l)} + \frac{v_0(v_0+\alpha_0)}{1-\alpha_1} f_2 \alpha_1^m \\
& + [d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)] f_3 \alpha_1^{m+s} + v_0 f_1 f_2 \alpha_1^{m-s} \boldsymbol{+ f_1\mu(s) - f_1\mu(s)} \;\Big) \\
& - (f_2[v_0\alpha_1^s + \alpha_0(1+\alpha_1)] - f_1^2)(f_2[v_0\alpha_1^{m-l} + \alpha_0(1+\alpha_1)] - f_1^2) \\
& - (f_2[v_0\alpha_1^l + \alpha_0(1+\alpha_1)] - f_1^2)(f_2[v_0\alpha_1^{m-s} + \alpha_0(1+\alpha_1)] - f_1^2) \\
& - (f_2[v_0\alpha_1^m + \alpha_0(1+\alpha_1)] - f_1^2)(f_2[v_0\alpha_1^{l-s} + \alpha_0(1+\alpha_1)] - f_1^2) \\
& + f_1^2 \; \big(\; f_2[v_0\alpha_1^m + \alpha_0(1+\alpha_1)] + f_2[v_0\alpha_1^{m-s} + \alpha_0(1+\alpha_1)] \\
& + f_2[v_0\alpha_1^{m-l} + \alpha_0(1+\alpha_1)] - 3f_1^2 \; \big),
\end{aligned}
$$

where we highlight, using bold, expressions whose sum equals zero.

So, taking into account that

$$
-f_2\mu(s)[\alpha_0 + (v_0+\alpha_0)\alpha_1]\alpha_1^{m-l} = \left[-f_1 \frac{\alpha_0+v_0}{1-\alpha_1}\mu(s) + v_0 f_2 \mu(s) \right] \alpha_1^{m-l}
$$

and

$$
\begin{aligned}
& -(f_2[v_0\alpha_1^s + \alpha_0(1+\alpha_1)] - f_1^2)(f_2[v_0\alpha_1^{m-l} + \alpha_0(1+\alpha_1)] - f_1^2) \\
& -(f_2[v_0\alpha_1^l + \alpha_0(1+\alpha_1)] - f_1^2)(f_2[v_0\alpha_1^{m-s} + \alpha_0(1+\alpha_1)] - f_1^2) \\
& -(f_2[v_0\alpha_1^m + \alpha_0(1+\alpha_1)] - f_1^2)(f_2[v_0\alpha_1^{l-s} + \alpha_0(1+\alpha_1)] - f_1^2) \\
& + f_1^2 \left(f_2[v_0\alpha_1^m + \alpha_0(1+\alpha_1)] + f_2[v_0\alpha_1^{m-s} + \alpha_0(1+\alpha_1)] \right. \\
& \left. + f_2[v_0\alpha_1^{m-l} + \alpha_0(1+\alpha_1)] - 3f_1^2 \right) \\
= \; & -v_0^2 f_2^2 [\alpha_1^{m-l+s} + 2\alpha_1^{m+l-s}] + v_0 f_1^2 f_2 [\alpha_1^{m-l} + \alpha_1^{m-s} + \alpha_1^m]
\end{aligned}
$$

we obtain by replacing $\mu(s,l)$,

$$\begin{aligned}\kappa(s,l,m) &= \alpha_1^m f_4 \left[\left\{c_0 - 4v_0d_0 + 3v_0^3 + 3v_0(v_0^2 - d_0)\alpha_1 + (3\alpha_0 d_0 - c_0)\alpha_1^2\right.\right.\\ &\quad \left. +(7v_0d_0 - 6v_0^3 - c_0)\alpha_1^3 + 3v_0(d_0 - 2v_0^2)\alpha_1^4 + (6v_0^3 - 6v_0d_0 + c_0)\alpha_1^5\right\}\alpha_1^{l+s}\\ &\quad +v_0(1+\alpha_1+\alpha_1^2+\alpha_1^3)[d_0(1-\alpha_1^2) - v_0^2(1+\alpha_1-2\alpha_1^2)](2\alpha_1^l + \alpha_1^s)\\ &\quad \left. +v_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)[(1+\alpha_1)v_0^2 + (d_0(1-\alpha_1) + v_0^2(2\alpha_1 - 1))\alpha_1^{l-s}]\right],\end{aligned}$$

for any $m \geq l \geq s \geq 0$.

Finally, the fourth-order central moments of X are given by

$$\begin{aligned}\widetilde{\mu}(s,l,m) &= \kappa(s,l,m) + v_0\alpha_1^s f_2 v_0 \alpha_1^{m-l} f_2 + v_0\alpha_1^l f_2 v_0\alpha_1^{m-s} f_2 + v_0\alpha_1^{l-s} f_2 v_0 \alpha_1^m f_2\\ &= \kappa(s,l,m) + v_0^2 f_2^2 \alpha_1^{m-l+s} + 2v_0^2 f_2^2 \alpha_1^{m+l-s}.\end{aligned}$$

Appendix 3. Covariance matrix of the asymptotic distribution of CLS estimators in CP-INARCH(1) model

To obtain the entries of the covariance matrix $\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1}$, let us begin by deducing the inverse of V.

$$\mathbf{V}^{-1} = \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\begin{bmatrix} \frac{\alpha_0(v_0+\alpha_0(1+\alpha_1))}{(1-\alpha_1)(1-\alpha_1^2)} & -\frac{\alpha_0}{1-\alpha_1}\\ -\frac{\alpha_0}{1-\alpha_1} & 1\end{bmatrix} = \begin{bmatrix} 1+\frac{\alpha_0}{v_0}(1+\alpha_1) & -\frac{1}{v_0}(1-\alpha_1^2)\\ -\frac{1}{v_0}(1-\alpha_1^2) & \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\end{bmatrix}.$$

Furthermore, considering $u_t(\alpha) = X_t - g(\alpha, X_{t-1})$,

$$\begin{aligned}E\left[f(X_{t-1})\cdot u_t^2(\alpha)\right] &= E\left[f(X_{t-1})\cdot E\left[(X_t - \alpha_0 - \alpha_1 X_{t-1})^2|X_{t-1}\right]\right]\\ &= E\left[f(X_{t-1})\cdot V\left[X_t - \alpha_0 - \alpha_1 X_{t-1}|X_{t-1}\right] + 0\right]\\ &= E\left[f(X_{t-1})\cdot V\left[X_t|X_{t-1}\right]\right] = E\left[f(X_{t-1})\cdot v_0(\alpha_0 + \alpha_1 X_{t-1})\right],\end{aligned}$$

because of the conditional compound Poisson distribution, and then

$$\mathbf{W} = \begin{bmatrix} E\left(u_t^2 \frac{\partial g}{\partial\alpha_0}\frac{\partial g}{\partial\alpha_0}\right) & E\left(u_t^2 \frac{\partial g}{\partial\alpha_0}\frac{\partial g}{\partial\alpha_1}\right)\\ E\left(u_t^2 \frac{\partial g}{\partial\alpha_1}\frac{\partial g}{\partial\alpha_0}\right) & E\left(u_t^2 \frac{\partial g}{\partial\alpha_1}\frac{\partial g}{\partial\alpha_1}\right)\end{bmatrix}$$

$$= \begin{bmatrix} E[1\cdot v_0(\alpha_0 + \alpha_1 X_{t-1})] & E[X_{t-1}\cdot v_0(\alpha_0+\alpha_1 X_{t-1})]\\ E[X_{t-1}\cdot v_0(\alpha_0+\alpha_1X_{t-1})] & E[X_{t-1}^2 \cdot v_0(\alpha_0 + \alpha_1 X_{t-1})]\end{bmatrix}$$

$$= \frac{v_0\alpha_0}{1-\alpha_1}\begin{bmatrix} 1 & \frac{v_0\alpha_1 + \alpha_0(1+\alpha_1)}{1-\alpha_1^2}\\ \frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} & \frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)} + \frac{\alpha_0^2}{(1-\alpha_1)^2} + \frac{\alpha_1(d_0 + (3v_0^2 - d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\end{bmatrix},$$

since

$$E[v_0(\alpha_0 + \alpha_1 X_{t-1})] = v_0\left[\alpha_0 + \alpha_1\frac{\alpha_0}{1-\alpha_1}\right] = \frac{v_0\alpha_0}{1-\alpha_1},$$

$$E[X_{t-1}\cdot v_0\,(\alpha_0+\alpha_1 X_{t-1})] = v_0\left[\frac{\alpha_0^2}{1-\alpha_1}+\frac{\alpha_1\alpha_0(v_0+\alpha_0(1+\alpha_1))}{(1-\alpha_1)(1-\alpha_1^2)}\right] = \frac{v_0\alpha_0}{1-\alpha_1}\cdot\frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2},$$

$$E[X_{t-1}^2\cdot v_0\,(\alpha_0+\alpha_1 X_{t-1})] = v_0\left[\frac{\alpha_0^2(v_0+\alpha_0(1+\alpha_1))}{(1-\alpha_1)(1-\alpha_1^2)}\right.$$

$$\left.+\frac{\alpha_1\alpha_0}{(1-\alpha_1)^3}\left(\frac{d_0+(3v_0^2-d_0)\alpha_1^2}{(1+\alpha_1)(1+\alpha_1+\alpha_1^2)}+\frac{3v_0\alpha_0}{1+\alpha_1}+\alpha_0^2\right)\right]$$

$$=\frac{v_0\alpha_0}{1-\alpha_1}\left[\frac{v_0\alpha_0(1-\alpha_1)+3v_0\alpha_0\alpha_1}{(1-\alpha_1)^2(1+\alpha_1)}+\frac{\alpha_0^2(1-\alpha_1)+\alpha_0^2\alpha_1}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right]$$

$$=\frac{v_0\alpha_0}{1-\alpha_1}\left[\frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right],$$

using again the expressions stated in Theorem 2.1.

Now, the product of $\mathbf{V}^{-1}\mathbf{W}$ is given by

$$\begin{bmatrix} 1+\frac{\alpha_0}{v_0}(1+\alpha_1) & -\frac{1}{v_0}(1-\alpha_1^2) \\ -\frac{1}{v_0}(1-\alpha_1^2) & \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\end{bmatrix}\begin{bmatrix} 1 & \frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} \\ \frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} & \frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\end{bmatrix}$$

$$=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} = \begin{bmatrix} 1-\alpha_1 & \frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)} \\ \frac{\alpha_1(1-\alpha_1)}{\alpha_0} & 1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\end{bmatrix},$$

since

$$\begin{aligned}
a_{11} &= 1+\frac{\alpha_0(1+\alpha_1)}{v_0}-\frac{1-\alpha_1^2}{v_0}\,\frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2} \;=\; 1-\alpha_1,\\
a_{12} &= \left(1+\frac{\alpha_0}{v_0}(1+\alpha_1)\right)\frac{v_0\alpha_1+\alpha_0(1+\alpha_1)}{1-\alpha_1^2}\\
&\quad -\frac{1-\alpha_1^2}{v_0}\left[\frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}+\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right]\\
&= \frac{v_0\alpha_1}{1-\alpha_1^2}+\frac{\alpha_0}{1-\alpha_1}+\frac{\alpha_0\alpha_1}{1-\alpha_1}+\frac{\alpha_0^2(1+\alpha_1)}{v_0(1-\alpha_1)}-\frac{\alpha_0(1+2\alpha_1)}{1-\alpha_1}\\
&\quad -\frac{\alpha_0^2(1+\alpha_1)}{v_0(1-\alpha_1)}-\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1-\alpha_1^3)}\\
&= \frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)},
\end{aligned}$$

$$
\begin{aligned}
a_{21} &= -\frac{(1-\alpha_1^2)}{v_0}+\frac{(1-\alpha_1)(1-\alpha_1^2)(v_0\alpha_1+\alpha_0(1+\alpha_1))}{v_0\alpha_0(1-\alpha_1^2)} \\
&= -\frac{(1-\alpha_1^2)}{v_0}+\frac{\alpha_1(1-\alpha_1)}{\alpha_0}+\frac{(1-\alpha_1^2)}{v_0} \;=\; \frac{\alpha_1(1-\alpha_1)}{\alpha_0}, \\
a_{22} &= -\frac{(1-\alpha_1^2)(v_0\alpha_1+\alpha_0(1+\alpha_1))}{v_0(1-\alpha_1^2)}+\frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\left[\frac{v_0\alpha_0(1+2\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2)}\right. \\
&\quad \left. +\frac{\alpha_0^2}{(1-\alpha_1)^2}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_1^3)}\right] \\
&= -\alpha_1-\frac{\alpha_0(1+\alpha_1)}{v_0}+1+2\alpha_1+\frac{\alpha_0(1+\alpha_1)}{v_0}+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)} \\
&= 1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}.
\end{aligned}
$$

So, the asymptotic covariance matrix is such that

$$
\mathbf{V}^{-1}\mathbf{W}\mathbf{V}^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}
$$

$$
= \frac{v_0\alpha_0}{1-\alpha_1}\begin{bmatrix} 1-\alpha_1 & \frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)} \\ \frac{\alpha_1(1-\alpha_1)}{\alpha_0} & 1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)} \end{bmatrix}\begin{bmatrix} 1+\frac{\alpha_0}{v_0}(1+\alpha_1) & -\frac{1}{v_0}(1-\alpha_1^2) \\ -\frac{1}{v_0}(1-\alpha_1^2) & \frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0} \end{bmatrix}
$$

where

$$
\begin{aligned}
b_{11} &= \frac{\alpha_0}{1-\alpha_1}\left(\alpha_0(1+\alpha_1)+\frac{v_0^2+(d_0-v_0^2)\alpha_1(1+\alpha_1-\alpha_1^2)+(3v_0^2-d_0)\alpha_1^4}{v_0(1+\alpha_1+\alpha_1^2)}\right), \\
b_{12} &= b_{21} \;=\; v_0\alpha_1-\alpha_0(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1+\alpha_1+\alpha_1^2)}, \\
b_{22} &= (1-\alpha_1^2)\left(1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right).
\end{aligned}
$$

In fact, we have

$$
\begin{aligned}
b_{11} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[(1-\alpha_1)\left(1+\frac{\alpha_0}{v_0}(1+\alpha_1)\right)\right.\\
&\left.-\frac{1}{v_0}(1-\alpha_1^2)\left(\frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)}\right)\right]\\
&= \frac{\alpha_0}{1-\alpha_1}\left[v_0(1-\alpha_1)+\alpha_0(1-\alpha_1^2)-v_0\alpha_1+\alpha_0\alpha_1(1+\alpha_1)\right.\\
&\left.+\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)(1+\alpha_1)}{v_0(1+\alpha_1+\alpha_1^2)}\right]\\
&= \frac{\alpha_0}{1-\alpha_1}\left[\alpha_0(1+\alpha_1)+\frac{v_0^2(1-2\alpha_1)(1+\alpha_1+\alpha_1^2)+\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)(1+\alpha_1)}{v_0(1+\alpha_1+\alpha_1^2)}\right]\\
&= \frac{\alpha_0}{1-\alpha_1}\left(\alpha_0(1+\alpha_1)+\frac{v_0^2+(d_0-v_0^2)\alpha_1(1+\alpha_1-\alpha_1^2)+(3v_0^2-d_0)\alpha_1^4}{v_0(1+\alpha_1+\alpha_1^2)}\right),\\
b_{12} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[-\frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0}\right.\\
&\left.+\frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\left(\frac{v_0\alpha_1}{1-\alpha_1^2}-\frac{\alpha_0\alpha_1}{1-\alpha_1}-\frac{\alpha_1\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1-\alpha_1^3)}\right)\right]\\
&= -\alpha_0(1-\alpha_1^2)+v_0\alpha_1-\alpha_0\alpha_1(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)\left(d_0+(3v_0^2-d_0)\alpha_1^2\right)}{v_0(1+\alpha_1+\alpha_1^2)}\\
&= v_0\alpha_1-\alpha_0(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1+\alpha_1+\alpha_1^2)},\\
b_{21} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[\frac{\alpha_1(1-\alpha_1)}{\alpha_0}\left(1+\frac{\alpha_0(1+\alpha_1)}{v_0}\right)-\frac{1-\alpha_1^2}{v_0}\left(1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right)\right]\\
&= v_0\alpha_1+\alpha_0\alpha_1(1+\alpha_1)-\alpha_0(1+\alpha_1)-\alpha_0\alpha_1(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1+\alpha_1+\alpha_1^2)}\\
&= v_0\alpha_1-\alpha_0(1+\alpha_1)-\frac{\alpha_1(1+\alpha_1)(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0(1+\alpha_1+\alpha_1^2)},\\
b_{22} &= \frac{v_0\alpha_0}{1-\alpha_1}\left[-\frac{\alpha_1(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}+\frac{(1-\alpha_1)(1-\alpha_1^2)}{v_0\alpha_0}\left(1+\alpha_1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right)\right]\\
&= -\alpha_1(1-\alpha_1^2)+\alpha_1(1-\alpha_1^2)+(1-\alpha_1^2)\left(1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right)\\
&= (1-\alpha_1^2)\left(1+\frac{\alpha_1(d_0+(3v_0^2-d_0)\alpha_1^2)}{v_0\alpha_0(1+\alpha_1+\alpha_1^2)}\right).
\end{aligned}
$$

Esmeralda Gonçalves
CMUC, Department of Mathematics,University of Coimbra, Apartado 3008, EC Santa Cruz, 3001-501 Coimbra, Portugal

E-mail address: esmerald@mat.uc.pt

Nazaré Mendes Lopes
CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz, 3001-501 Coimbra, Portugal
E-mail address: nazare@mat.uc.pt

Filipa Silva
CMUC
E-mail address: mat0504@mat.uc.pt