Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 17–35

ON q-PERMANENT EXPANSIONS AND A THEOREM ON CYCLE SURGERY

EDUARDO MARQUES DE SÁ

Abstract: The q-permanent linear preservers are described, and several expansion formulas for the q-permanent of a square matrix are given. Some of these formulas are valid for all matrices, but others are not; for each such formula Φ we determine all digraphs D such that Φ holds for all matrices with digraph D. The proof technique is based on a combinatorial result where we accurately evaluate what happens to the the number of inversions of a permutation π when one of its cycles if excised from π. In the last section some structural issues are raised concerning the q-permanent expansions previously studied, and some open problems are presented.

Keywords: q-permanent, determinant, polynomial identities, digraphs.
Math. Subject Classification (2010): 15A15, 15A86, 05C20, 05C50.

1. Introduction

This paper is a continuation of [19], on the q-permanent of an n-square matrix $A = (a_{ij})$, a polynomial given by

$$\operatorname{per}_q A = \sum_{\sigma\in\mathscr{S}_n} q^{\ell(\sigma)} \prod_{i=1}^n a_{i\sigma_i}\,,$$

where $\mathscr{S}_n$ is the symmetric group of order n, and $\ell(\sigma)$ is the number of inversions of the permutation σ, here called the *length* of σ. In [14, 5, 21, 24] the reader will find the genesis and uses of this function in the areas of mathematical physics, and quantum groups and algebras. Further developments may be found in [1, 11, 2, 12].

Section 3 describes the q-permanent linear preservers. In [22, 23] the q-permanent is generalized to multivariable quantum parameters and, in this context, some expansions are obtained for the q-permanent which are reminiscent of the archetypal expansions of Laplace along a set of rows or columns. The expansions considered below (sections 4, 6, 7) are of a different nature in that we collect the q-permanent terms according to the cycle structure of the digraph of the matrix A, as has been done for the determinant in

Received July 5, 2017.
This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

[16, 18, 17]. In section 5 we get a result on 'cycle surgery' that may have an independent life: the variation in length is determined when we excise from a permutation one of its cycles. This paves the way to section 6 where the main q-permanent expansion formula is modified in several ways. Each modified formula Φ is *combinatorially solved*, *i.e.*, all digraphs D are found such that Φ holds for all matrices with digraph D. In the last section we discuss *in abstracto* other possibilities of expanding the q-permanent using the cycle structure, and leave some open problems.

2. Preliminaries

On digraphs, graphs and matrices we follow the traditional concepts as may be seen in, *e.g.*, [6, 7], with minor changes. Here, the set $V(D)$ of the vertices of a digraph D is a subset of $[n] = \{1, \dots, n\}$. Notations like $(i, j) \in D \subseteq D'$ mean that (i, j) is an arc of D, and D is a subdigraph of D'; an arc is also denoted $i{\to}j$. We write $[r, s]$, $]r, s]$, etc. to refer *integer intervals.* A *k-cycle* c, or *cycle of order k*, is a digraph on k vertices, say $v_1, \dots, v_k$, ordered so that $v_i{\to}v_{i+1}$ are the arcs of c, with i read modulo k; the *short notation* $c = (v_1 v_2 \cdots v_k)$ will be used, and we say c is a cycle *through* each one of its vertices. With due care, we may identify c with the set $V(c)$ of its vertices. The set of all cycles through a given vertex v is denoted by $\mathscr{C}_v$, or $\mathscr{C}_v(n)$ if needed. The sole 1-cycle of $\mathscr{C}_v$ is the loop (v). Given $\sigma \in \mathscr{S}_n$, we define $\mathrm{Mov}(\sigma) = \{i \in [n] : \sigma(i) \neq i\}$; the *permutation digraph* Δ_σ has arcs $i{\to}\sigma(i)$, for $i \in [n]$. Each cycle $c = (v_1 v_2 \cdots v_k)$ determines a permutation of $\mathscr{S}_n$, also denoted c and called *cyclic permutation*, or just *cycle*, given by $c(v_i) = v_{i+1}$, for $i \in [k]$, and $c(j) = j$, for j a non-vertex of c; in this way, all loops are mapped into the identity $e \in \mathscr{S}_n$; this mapping is a bijection between *non-loop cycles* and *cyclic permutations* $\neq e$, a fact that allows an identification of the two concepts. A *cycle of a permutation* π is a permutation corresponding to a cycle of Δ_π.

Of course all this framework may be naturally extended to an arbitrary $K = \{k_1, \dots, k_r\} \subseteq [n]$, $k_1 < \cdots < k_r$. For example, we may consider the group $\mathscr{S}_K$ of all permutations of K, and permutation digraphs on K. A permutation of K, say $\omega : K \to K$, may be represented in the traditional *complete*, or *one-line* notation $\omega = \omega(k_1)\omega(k_2)\cdots\omega(k_r)$, and in short notation if ω is a cyclic permutation.

$\mathfrak{M}_n$ denotes the set of n-square matrices over a field $\mathbb{F}$ with more than n elements. So an $\mathbb{F}$-polynomial $f(x_1, \dots, x_N)$, of degree $\leqslant n$, satisfying $f(a) =$

0 for all $a \in \mathbb{F}^N$, is necessarily the zero polynomial (*e.g.*, [9, p. 235 *ff*]). For $A \in \mathfrak{M}_n$ and $S \subseteq [n]$, $A(S)$ or A_S denote the principal submatrix obtained by eliminating the rows and columns of A indexed by elements of S; the notations $A_{\{i\}}, A_{\{i,j\}}, A_{V(c)}$, where c is a cycle, will be simplified to A_i, A_{ij}, A_c. The *digraph of* A, denoted $D(A)$, has (i,j) as arc iff $a_{ij} \neq 0$. A matrix A is said to be *generic* if its nonzero entries are independent commuting variables over the base field. For a digraph D and a permutation σ, the *weight of D in A* and the *total weight of σ in A* are defined by

$$\mathrm{wt}_D(A) = \textstyle\prod_{(i,j)\in D} a_{ij} \quad \text{and} \quad \mathrm{twt}_\sigma(A) = \prod_{i\in[n]} a_{i\sigma_i}.$$

Clearly $\mathrm{wt}_{\Delta_\sigma}(A) = \mathrm{twt}_\sigma(A)$. We define P_σ as the permutation matrix with ij-entry $\delta_{\sigma(i),j}$; this is designed so that $\mathrm{per}_q\, P_\sigma = q^{\ell(\sigma)}$. The transpose of A is denoted A^T, and A^R denotes the *reverse* of A, obtained by reversing the order of the rows and the columns of A. Hence $A^R = P_{w_\circ} A P_{w_\circ}$, where $w_\circ$ is the so-called *reversal permutation*, given by $w_\circ(i) = n+1-i$. This $w_\circ$ is the top element of $\mathscr{S}_n$ in the Bruhat order, the unique permutation of maximum length, $\ell(w_\circ) = \binom{n}{2}$; and the identity e is the only permutation of length 0 (e.g., [3]).

As with digraphs, the graphs G that will be considered are on a vertex subset of $[n]$. An *edge* is a 2-set $\{i,j\}$ $(i \neq j)$. The *underlying digraph* of G, denoted D_G, has (i,j) as arc iff $i = j$ or $\{i,j\}$ is an edge of G. So the underlying digraph is symmetric and has all loops. The *graph of* A, denoted $G(A)$, has $\{i,j\}$ as edge iff $i \neq j$ and either $a_{ij} \neq 0$ or $a_{ji} \neq 0$. A is said to be a *generic matrix with graph* G if A is generic and $a_{ij} \neq 0$ if and only if: $i = j$ or $\{i,j\}$ is an edge of G.

Recall from [10, 20] that $R, S \subseteq [n]$ are said to be *crossing sets*, or that they *cross*, if there exist $r, r' \in R$ and $s, s' \in S$ such that $r < s < r' < s'$ or $s < r < s' < r'$.

3. q-Permanent linear preservers

This section is a small token to the theory of linear preservers (see, *e.g.*, [15, 4, 13, 8]). A linear mapping $\mathscr{L} : \mathfrak{M}_n \to \mathfrak{M}_n$ is said to *preserve the q-permanent*, if $\mathrm{per}_q \mathscr{L}(X) = \mathrm{per}_q X$ for all $X \in \mathfrak{M}_n$.

Proposition 3.1. *The set of linear mappings $\mathscr{L}$ that preserve the q-permanent is the group generated by the transformations of the following three kinds:* (*a*) $X \rightsquigarrow DXE$*, where D and E are diagonal matrices such that* $\mathrm{per}(DE) = 1$*;* (*b*) $X \rightsquigarrow X^T$*; and* (*c*) $X \rightsquigarrow X^R$*.*

Proof: By [15, 4] the set of the permanent linear preservers is the group generated by (a), (b) and the transformations $X \rightsquigarrow PXQ$, where P, Q are permutation matrices. Clearly (a) and (b) preserve the q-permanent. To handle $X \rightsquigarrow PXQ$ note that

$$\operatorname{per}_q PXQ = \textstyle\sum_{\sigma\in\mathscr{S}_n} q^{\ell(\alpha\sigma\beta)} \prod_{i=1}^n x_{i\sigma(i)}, \tag{1}$$

where α, β are uniquely determined by $(P_\alpha, P_\beta) = (Q, P)$. So we are faced with an ℓ-preserver problem: *which $\alpha, \beta \in \mathscr{S}_n$ satisfy the equation $\ell(\alpha\sigma\beta) = \ell(\sigma)$ for all σ.* Plugging $\sigma = e$ in this equation we get $\ell(\alpha\beta) = \ell(e)$, and so $\alpha\beta = e$. Therefore $\sigma \rightsquigarrow \alpha\sigma\beta$ must be an inner automorphism of $\mathscr{S}_n$; it is well-known [3, p. 38 *ff*], and elementary, that the only inner automorphisms preserving the Bruhat order of $\mathscr{S}_n$ are the identity and the one afforded by $\alpha = w_\circ$, the reversal permutation. So only two linear mappings of the kind $X \rightsquigarrow PXQ$ preserve the q-permanent, namely the identity and the reversion. The remaining proof details are left to the reader. ■

4. Some expansions of the q-permanent

Following the strategy of [16, 18, 17], we get generalized versions of known determinantal expansions based on cycle decompositions (*cf.* theorems 2 and 3 of [17]).

Fix a set $I \subseteq [n]$. Let $\mathfrak{F}$ be the set of all digraphs $\mathfrak{f}$ that are joins of disjoint cycles and $V(\mathfrak{f}) \supseteq I$. For $\mathfrak{f} \in \mathfrak{F}$, define $\mathfrak{f} \wedge I$ as the join of the cycles of $\mathfrak{f}$ that intersect I. Let $\mathcal{M}_I$ be the set of those $\mathfrak{f}$ such that $\mathfrak{f} \wedge I = \mathfrak{f}$. For $\mathfrak{f} \in \mathcal{M}_I$ let $\mathcal{E}_\mathfrak{f}$ be the set of all $\sigma \in \mathscr{S}_n$ such that $\mathfrak{f} \subseteq \Delta_\sigma$. It is easily seen that $\sigma \in \mathcal{E}_\mathfrak{f}$ if and only if $\Delta_\sigma \wedge I = \mathfrak{f}$. Therefore $\{\mathcal{E}_\mathfrak{f}\}_{\mathfrak{f}\in\mathcal{M}_I}$ is a partition of $\mathscr{S}_n$.

Theorem 4.1. *For any set $I \subseteq [n]$ and matrix A, we have*

$$\operatorname{per}_q A = \textstyle\sum_{\mathfrak{f}\in\mathcal{M}_I} \operatorname{wt}_\mathfrak{f}(A)\ \operatorname{per}_q \mathcal{Z}(A, \mathfrak{f}), \tag{2}$$

where $\mathcal{Z}(A, \mathfrak{f})$ denotes the matrix obtained from A by zeroing out all rows and all columns indexed by the vertices of $\mathfrak{f}$, except the entries in the positions $(i, j) \in \mathfrak{f}$, which are replaced by 1's.

Proof: For each $\mathfrak{f} \in \mathcal{M}_I$, we transform A into $\mathcal{Z}(A, \mathfrak{f})$ in two steps: first we get a matrix $\mathcal{Z}^*(A, \mathfrak{f})$ by zeroing out all rows and all columns indexed by the vertices of $\mathfrak{f}$, except the a_{uw} for $(u, w) \in \mathfrak{f}$ which keep their values; in a second step we replace these a_{uw} with 1's. For $\sigma \in \mathcal{E}_\mathfrak{f}$, $\operatorname{twt}_\sigma(A) = \operatorname{twt}_\sigma(\mathcal{Z}^*(A, \mathfrak{f})) = \operatorname{wt}_\mathfrak{f}(A) \operatorname{twt}_\sigma(\mathcal{Z}(A, \mathfrak{f}))$. For $\sigma \notin \mathcal{E}_\mathfrak{f}$, there is $u \to w$ in $\mathfrak{f}$ such that $\sigma(u) \neq w$;

by definition, in row u of $\mathscr{Z}^*(A,\mathfrak{f})$ the only eventually nonzero entry is a_{uw}; therefore $\mathrm{twt}_\sigma(\mathscr{Z}^*(A,\mathfrak{f}))=0$. So we get

$$\begin{aligned}\operatorname{per}_q A &= \textstyle\sum_{\mathfrak{f}\in\mathcal{M}_I}\sum_{\sigma\in\mathcal{E}_\mathfrak{f}} q^{\ell(\sigma)}\,\mathrm{twt}_\sigma(A) = \sum_{\mathfrak{f}\in\mathcal{M}_I}\sum_{\sigma\in\mathscr{S}_n} q^{\ell(\sigma)}\,\mathrm{twt}_\sigma(\mathscr{Z}^*(A,\mathfrak{f}))\\ &= \textstyle\sum_{\mathfrak{f}\in\mathcal{M}_I}\mathrm{wt}_\mathfrak{f}(A)\sum_{\sigma\in\mathscr{S}_n} q^{\ell(\sigma)}\,\mathrm{twt}_\sigma(\mathscr{Z}(A,\mathfrak{f})),\end{aligned}$$

and the theorem is proved. ∎

For $\mathfrak{f}\in\mathfrak{F}$, let $r=\sharp V(\mathfrak{f})$, write $V(\mathfrak{f})=\{i_1,\ldots,i_r\}$ and $[n]\smallsetminus V(\mathfrak{f})=\{j_1,\ldots,j_{n-r}\}$, with $i_1<\cdots<i_r$ and $j_1<\cdots<j_{n-r}$, and let κ be the permutation (in one-line notation) $i_1\cdots i_r j_1\cdots j_{n-r}$. Clearly, we may view $\mathfrak{f}$ as a permutation (the product of the cycles of $\mathfrak{f}$); then $\kappa^{-1}\mathfrak{f}\kappa$ fixes all points of $]r,n]$, and so it fixes the set $[r]$. Denote by $\mathfrak{f}'\in\mathscr{S}_r$ the restriction of $\kappa^{-1}\mathfrak{f}\kappa$ to $[r]$. Then we have

$$\mathscr{Z}(A,\mathfrak{f})=P_\kappa^T\big[P_{\mathfrak{f}'}\oplus A(V(\mathfrak{f}))\big]P_\kappa. \tag{3}$$

(In case $V(\mathfrak{f})=[r]$, one has $\kappa=e$ and $\mathfrak{f}'$ is the restriction of $\mathfrak{f}$ to $[r]$.) Let $\mathcal{M}_I^\circ$ be the set of those $\mathfrak{f}$ such that $V(\mathfrak{f})=I$. For $\mathfrak{f}\in\mathcal{M}_I^\circ$, the κ of (3) does not depend on $\mathfrak{f}$. We have $\mathcal{M}_I^\circ=\kappa\mathcal{M}_{[r]}^\circ\kappa^{-1}$, and the mapping $\mathcal{M}_I^\circ\to\mathscr{S}_r$, $\mathfrak{f}\rightsquigarrow\mathfrak{f}'$, is a group isomorphism. Detaching from the right hand side of (2) the sum over $\mathcal{M}_I^\circ$ we get

$$\textstyle\sum_{\mathfrak{f}\in\mathcal{M}_I^\circ}\mathrm{wt}_\mathfrak{f}(A)\,\operatorname{per}_q\mathscr{Z}(A,\mathfrak{f})=\sum_{\mathfrak{f}\in\mathcal{M}_I^\circ}\mathrm{wt}_\mathfrak{f}(A)\operatorname{per}_q P_\kappa^T\big[P_{\mathfrak{f}'}\oplus A(I)\big]P_\kappa. \tag{4}$$

In case $I=[r]$, this expression transforms into

$$\operatorname{per}_q A[I]\operatorname{per}_q A(I),$$

where $A[I]$ is the principal submatrix of A corresponding to the rows and columns indexed by I. So, for $I=[r]$, theorem 4.1 yields a neat generalization of the determinantal formula [17, Th. 3]. However, for a general I, the permutation similarity by P_κ produces in (4) a seemingly hopeless situation upon application of per_q; so formula (2) was kept with no further ado.

In case I is a singleton, say $I=\{v\}$, then $\mathcal{M}_I=\mathscr{C}_v$, and we get

Corollary 4.2. *Let A be any matrix. For any vertex $v\in[n]$, we have*

$$\operatorname{per}_q A=\textstyle\sum_{c\in\mathscr{C}_v}\mathrm{wt}_c(A)\,\operatorname{per}_q\mathscr{Z}(A,c). \tag{5}$$

In the case $q=-1$, the similarity by P_κ in (3) is of no effect under the determinant. So we retrieve [17, Th. 2] from (5) as follows:

$$\begin{aligned}\det A &= \textstyle\sum_{c\in\mathscr{C}_v} \mathrm{wt}_c(A)\, \det \mathcal{Z}(A,c) = \sum_{c\in\mathscr{C}_v} \mathrm{wt}_c(A)\, \det(P_{c'}\oplus A_c)\\ &= \textstyle\sum_{c\in\mathscr{C}_v} (-1)^{\ell(c)}\, \mathrm{wt}_c(A)\, \det A_c. \end{aligned}\tag{6}$$

This method goes equally well for $q=1$, and (6) suggests to accept the expansion

$$\mathrm{per}_q\, A = \textstyle\sum_{c\in\mathscr{C}_v} q^{\ell(c)}\, \mathrm{wt}_c(A)\, \mathrm{per}_q\, A_c \tag{7}$$

as true for any matrix and any $v\in[n]$, as has been done in p. 227 of [C. Fonseca, *Lin. Mult. Alg.*, 53(2005), pp. 225-230]. Although (7) has simple counterexamples, it leaves in between our hands the interesting problem of giving a combinatorial solution to equation (7), *i.e.*, to find the set of all digraphs D such that (7) holds for a generic A with digraph D. We solve this and other related problems in section 6, but to do so we need the combinatorial result of the next section.

5. A theorem on cycle surgery

Fix a permutation $\pi\in\mathscr{S}_n$, and a cycle c of the digraph Δ_π, maybe a loop of Δ_π, and denote by θ the permutation πc^{-1}. The surgery we have in mind consists in extracting the whole c (arcs and vertices) from Δ_π. We get a digraph on $[n]\smallsetminus V(c)$ which is itself a permutation digraph corresponding to a permutation of the set $\mathscr{S}_{[n]\smallsetminus V(c)}$, denoted by $\pi\smallsetminus c$, and called *excision of c from π*. If $\theta|_S$ denotes the restriction of θ to a set S, we have $\pi\smallsetminus c=\theta|_{[n]\smallsetminus V(c)}$.

The *multiplicity of θ over a vertex k* is the number

$$\mathrm{mult}(\theta,k)=\sharp\{i : i<k<\theta(i) \text{ or } \theta(i)<k<i\}.$$

The *multiplicity of θ over a set $F\subseteq[n]$*, denoted $\mathrm{mult}(\theta,F)$, is the sum of the multiplicities of θ over the vertices of F. If F is a set of fixed points of θ, it is easy to see that $\mathrm{mult}(\theta,F)$ is an even number, and [19, Lemma 4.2] implies

$$\begin{aligned}&\ell(\theta)=\ell(\theta|_{[n]\smallsetminus F})+\mathrm{mult}(\theta,F). \quad \text{In particular}\\ &\ell(\theta)=\ell(\pi\smallsetminus c)+\mathrm{mult}(\theta,V(c)).\end{aligned}\tag{8}$$

Given two arcs $i{\to}j$ and $r{\to}s$, we say that $i{\to}j$ *lies under* $r{\to}s$ (and $r{\to}s$ *lies above* $i{\to}j$) if $r<i\leqslant j<s$ or $r>i\geqslant j>s$. If the cycle c is *not a loop*, we define

$$\mathcal{A}_c^\theta=\sharp\{(\alpha,\beta) : \alpha,\beta \text{ are arcs, } \alpha\in\theta,\ \beta\in c, \text{ and } \alpha \text{ lies above } \beta\}.$$

If $c = (v)$, we define $\mathcal{A}^{\theta}_{(v)} = \frac{1}{2}\,\mathrm{mult}(\theta, v)$.

Theorem 5.1. *With the above notation we have* $\ell(\pi) = \ell(c) + \ell(\pi \setminus c) + 2\mathcal{A}^{\theta}_{c}$.

Before the proof we give an example to show the geometrical nature of these concepts. Figure 1 depicts the digraph Δ_π, in case $\pi = (173)(264)$, in 'linear style' [20], *i.e.*, with all vertices uniformly and orderly disposed on a line, and non-loop arcs represented by similar circular arrows. The arcs of cycle $c = (264)$ are dotted and those of $\theta = (173)$ are in solid black. The

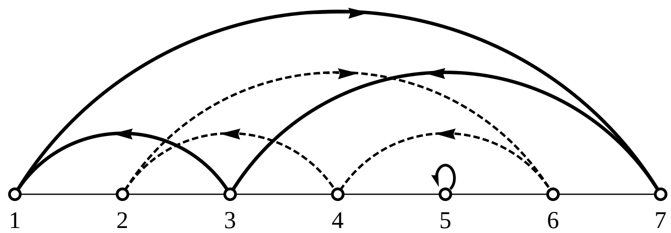

FIGURE 1. The case $c = (264)$, $\theta = (173)$.

loop at 5 lies under two arcs of θ, while the arc $6 \to 4$ lies under a sole arc of θ (note that $1 \to 7$ is not above $6 \to 4$ due to 'wrong' orientation). Excision of c from π transforms π into a permutation of $\{1, 3, 5, 7\}$, which is, in complete notation: $\pi \setminus c = 7153$; therefore $\ell(\pi \setminus c) = 4$. A glimpse at the picture shows that $\mathcal{A}^{\theta}_{c} = 2$, and a simple inversions count leads to $\ell(\pi) = 14$ and $\ell(c) = 6$. So the theorem holds in this example.

Proof: If c is a loop, the theorem follows from (8). We now treat the case of a transposition $c = (rs)$, $r < s$. From [19, Lemma 3.1] we have

$$\ell(\pi) = \ell(\theta) + 2\sharp\{i \in]r, s[: \theta_i \in]r, s[\} + 1. \tag{9}$$

Define $\mathcal{T}_1 = [1, r[$, $\mathcal{T}_2 =]r, s[$, $\mathcal{T}_3 =]s, n]$, and let T_i be the cardinality of $\mathcal{T}_i$. Denote by t_{ij} the number of indices of $\mathcal{T}_i$ that are mapped by θ into $\mathcal{T}_j$. Obviously the i-th row [column] of the matrix (t_{ij}) has sum T_i. Using these notations we have

$$\begin{aligned}
\ell(\pi) &= \ell(\theta) + 2t_{22} + 1 \quad \text{(from (9))}\\
\ell(c) &= 2(s - r) - 1 = 2T_2 + 1\\
\mathcal{A}^{\theta}_{(rs)} &= t_{13} + t_{31}\\
\mathrm{mult}(\theta, \{r, s\}) &= t_{12} + t_{13} + t_{21} + t_{31} + t_{23} + t_{13} + t_{32} + t_{31}\\
&= 2(T_2 - t_{22} + \mathcal{A}^{\theta}_{(rs)}).
\end{aligned}$$

By (8), $\ell(\theta) = \ell(\pi \setminus (rs)) + \text{mult}(\theta, \{r, s\})$. Combining all these pieces proves the theorem in case $c = (rs)$.

We now proceed by induction, assuming the theorem holds for cycles c of a given order k, $k \geqslant 2$. Let $\overline{\pi}$ be a permutation having a cycle $\overline{c}$ of order $k+1$, and define θ by $\theta\overline{c} = \overline{\pi}$. Let m be the vertex of $\overline{c}$ such that $f := \overline{c}(m)$ is the maximum of all the vertices of $\overline{c}$. In short notation, we may write $\overline{c} = (ab \ldots mf)$. The transposition (mf) transforms $\overline{c}$ into $c := \overline{c}(mf)$. Define $\pi = \theta c$. Clearly, c is the k-cycle $(ab \ldots m)$, and f is a fixed point of π. The surgery done is the replacement of the path $m \to f \to a$ of $\overline{c}$ by the arc $m \to a$ of the shorter cycle c; in complete notation we write $c = c_1 c_2 \ldots c_n$ (so a is renamed as c_m). The situation is depicted in figure 2, where we assume that $c_m < m$. This assumption is made with no loss of generality, because the roles of m and c_m may be reversed by inverting π, and it is easy to see (from (8) and the definition of $\mathcal{A}_c^\theta$) that the theorem's formula is invariant under inversion of π (and of c and θ). Next, consider $K_{\pi,m,f}$ defined by

$$K_{\pi,m,f} = \sharp\{i \in]m, f[\colon \pi_i \in]c_m, f[\}. \tag{10}$$

As $\overline{\pi}$ and $\overline{c}$ both have an inversion at (m, f), we get from [19, Lemma 3.1]

$$\ell(\overline{\pi}) - \ell(\pi) = 2K_{\pi,m,f} + 1 \quad \text{and} \quad \ell(\overline{c}) - \ell(c) = 2K_{c,m,f} + 1. \tag{11}$$

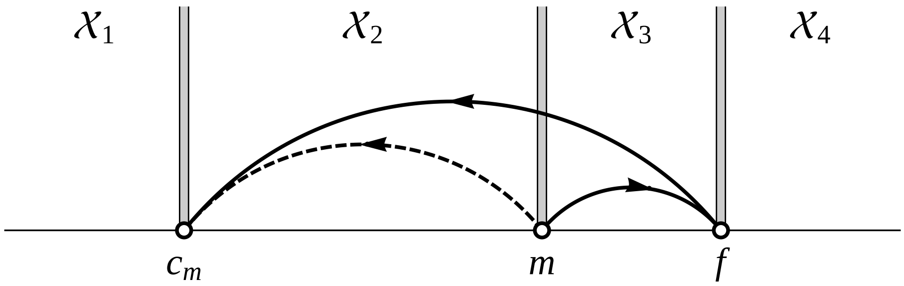

FIGURE 2. Replacing $m \to f \to c_m$ by $m \to c_m$.

In figure 2, the symbols $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$ denote the intervals $[1, c_m[$, $]c_m, m[$, $]m, f[$, $]f, n]$, respectively. We let $X_i = \sharp\mathcal{X}_i$, and let x_{ij} be the number of indices of $\mathcal{X}_i$ that are mapped by θ into $\mathcal{X}_j$. For example, in figure 2, $x_{i,i+1}$ is the number of arcs of θ that cross from left to right the vertical grey bar separating $\mathcal{X}_i$ from $\mathcal{X}_{i+1}$. In the case of the vertical grey bar over f, we have

$$\text{mult}(\theta, f) = x_{14} + x_{24} + x_{34} + x_{41} + x_{42} + x_{43} = 2(X_4 - x_{44}). \tag{12}$$

Clearly $\pi_i = c_i$ iff θ fixes i. So, to evaluate $K_{\pi,m,f} - K_{c,m,f}$ using the definitions (10) of the K's, we only have to take into account the set $\mathfrak{I}_\theta$ of those $i \in]m, f[$

that are moved by θ. As $c_i = i$ for $i \in \mathfrak{I}_\theta$, we have

$$\begin{aligned} K_{\pi,m,f} - K_{c,m,f} &= \sharp\{i \in \mathfrak{I}_\theta : \theta_i \in]c_m, f[\} - \sharp\{i \in \mathfrak{I}_\theta : c_i \in]c_m, f[\} \\ &= \sharp\{i \in \mathfrak{I}_\theta : \theta_i \in]c_m, f[\} - \sharp\mathfrak{I}_\theta \\ &= -\sharp\{i \in \mathfrak{I}_\theta : \theta_i \notin]c_m, f[\} = -\sharp\{i \in]m, f[: \theta_i \notin]c_m, f[\} \\ &= -x_{31} - x_{34}. \end{aligned} \tag{13}$$

Clearly we have $\mathcal{A}^\theta_{\overline{c}} - \mathcal{A}^\theta_c = \mathcal{A}^\theta_{m\to f} + \mathcal{A}^\theta_{f\to c_m} - \mathcal{A}^\theta_{m\to c_m}$, and from this we get

$$\mathcal{A}^\theta_{\overline{c}} - \mathcal{A}^\theta_c = (x_{14} + x_{24}) + x_{41} - (x_{31} + x_{41}) = x_{14} + x_{24} - x_{31}. \tag{14}$$

Define $\Omega(\pi, c) = \ell(\pi) - \ell(c) - \ell(\pi \smallsetminus c)$, and compute using (8), (11), (12), (13):

$$\begin{aligned} \Omega(\overline{\pi}, \overline{c}) - \Omega(\pi, c) &= [\ell(\overline{\pi}) - \ell(\pi)] - [\ell(\overline{c}) - \ell(c)] + \mathrm{mult}(\theta, V(\overline{c})) - \mathrm{mult}(\theta, V(c)) \\ &= 2K_{\pi,m,f} - 2K_{c,m,f} + \mathrm{mult}(\theta, f) \\ &= -2x_{31} - 2x_{34} + 2(X_4 - x_{44}) = 2(x_{14} + x_{24} - x_{31}) \end{aligned}$$

From (14) we obtain $\Omega(\overline{\pi}, \overline{c}) - \Omega(\pi, c) = 2\mathcal{A}^\theta_{\overline{c}} - 2\mathcal{A}^\theta_c$. The induction hypothesis is $\Omega(\pi, c) = 2\mathcal{A}^\theta_c$. Hence we get $\Omega(\overline{\pi}, \overline{c}) = 2\mathcal{A}^\theta_{\overline{c}}$, which is the desired formula. ■

6. Prospective q-permanent expansions

The combinatorial solution of (7) is now an easy matter.

Theorem 6.1. *Let $v \in [n]$. Formula (7) holds for a generic matrix with digraph D, if and only if for any permutation digraph $\Delta_\sigma \subseteq D$, the cycle c of Δ_σ passing through v satisfies one of the following equivalent conditions*

(*a*) $\ell(\sigma) = \ell(c) + \ell(\sigma \smallsetminus c)$;

(*b*) *No arc of c lies under an arc of another cycle of Δ_σ.*

Proof: The equivalence $(a) \Leftrightarrow (b)$ follows from theorem 5.1.

Let $A_{[\sigma]}$ denote the generalized permutation matrix whose ij-entry is $a_{ij}\delta_{\sigma(i),j}$. It is easy to prove that (7) holds if and only if it holds for all $A_{[\sigma]}$. So we only have to prove the theorem when $D = \Delta_\sigma$. In this case, (7) reduces to

$$q^{\ell(\sigma)} a_{1\sigma_1} \cdots a_{n\sigma_n} = q^{\ell(c)}\, \mathrm{wt}_c(A)\, \mathrm{per}_q\left[A_{[\sigma]}(c)\right],$$

where c is the cycle of Δ_σ passing through v. After canceling all $a_{i\sigma_i}$ in both members, we get $q^{\ell(\sigma)} = q^{\ell(c)}\ \mathrm{per}_q(P_\sigma)_c$; we clearly have $(P_\sigma)_c = P_{\sigma\smallsetminus c}$, and so $\mathrm{per}_q(P_\sigma)_c = q^{\ell(\sigma\smallsetminus c)}$. So, in case $A = A_{[\sigma]}$, (7) is equivalent to (a). ■

We now replace in (7) A_c by $A_c^{\checkmark}$, a matrix already considered in [19]. $A_c^{\checkmark}$ is obtained from A by zeroing out all entries of A in rows and columns indexed by the vertices of c, except the diagonal entries a_{ii} for $i \in c$ that are replaced by 1's.

Theorem 6.2. *A generic matrix A with digraph D satisfies*

$$\mathrm{per}_q A = \textstyle\sum_{c \in \mathscr{C}_v} q^{\ell(c)} \,\mathrm{wt}_c(A)\, \mathrm{per}_q A_c^{\checkmark}, \tag{15}$$

if and only if for any permutation digraph $\Delta_\sigma \subseteq D$ the orbit of σ containing v does not cross any other orbit of σ.

Proof: We adopt the strategy and the notation of the pervious proof. So we only need to prove the theorem when $D = \Delta_\sigma$. After replacing A with $A_{[\sigma]}$ in (15) and after canceling the $a_{i\sigma_i}$ in both members, we get $q^{\ell(\sigma)} = q^{\ell(c)} \mathrm{per}_q (P_\sigma)_c^{\checkmark}$, where c is the cycle of Δ_σ through v. We have $(P_\sigma)_c^{\checkmark} = P_\theta$, where $\theta := \sigma c^{-1}$. Therefore (15), for $A = A_{[\sigma]}$, is equivalent to the equation $\ell(c\theta) = \ell(c) + \ell(\theta)$; according to [19, Th. 4.3], this equation holds if and only if c crosses no orbit of θ. ■

Theorems 6.1 and 6.2 are easily remade for graphs G, by adequate handling of the underlying digraph.

Corollary 6.3. *Let G be a graph. All matrices having G as graph satisfy* (7)*, with $\mathscr{C}_v$ denoting the set of cycles through v in the underlying digraph D_G, if and only if the following two conditions hold:* (i) *there is no edge $\{r, s\}$ of G such that $r < v < s$;* (ii) *if γ is a cycle of G through v, no edge of G disjoint from all edges of γ lies above an edge of γ.*

Proof: We have to show that (b) of theorem 6.1 is equivalent to (i) $\wedge$ (ii). Assume (b) holds for any $\Delta_\sigma \subseteq D_G$. If there is an edge $\{r, s\}$ of G such that $r < v < s$, then $\Delta_{(rs)}$ is a subdigraph of D_G, because D_G has all loops; in $\Delta_{(rs)}$, the cycle through v is the loop $c = (v)$, which lies under $r{\to}s$, contradicting (b); so (i) holds. We now get a contradiction from the assumption that γ is a cycle of G through v, and G has an edge $\{r, s\}$, disjoint from all edges of γ, which lies above an edge of γ. We choose an orientation for γ and call c the oriented γ; then $\Delta_{c(rs)}$ is a subdigraph of D_G; clearly (b) fails for $\Delta_{c(rs)}$. So (ii) holds.

Now assume (i)$\wedge$(ii) holds; we seek for a contradiction from the assumption that (b) fails for some $\Delta_\sigma \subseteq D_G$. Let $c = (v_1 \dots v_k)$ be the cycle of Δ_σ through v; for some j, $v_j{\to}v_{j+1}$ lies under an arc $u{\to}w$ of another cycle of Δ_σ. As D_G

is symmetric we may assume $u < v_j \leqslant v_{j+1} < w$. If $k = 1, 2$ then $u < v < w$, contradicting (i). For $k \geqslant 3$, $\{v_j, v_{j+1}\}$ and $\{u, w\}$ are edges of G, and the former lies under the latter; this goes against (ii). ∎

Corollary 6.4. *All matrices with a given graph G satisfy* (15)*, with $\mathscr{C}_v$ denoting the set of cycles through v in the underlying digraph D_G, if and only if the following two conditions hold:* (α) *no edge of G with endpoint v is crossed by another edge of G, and* (β) *if γ is a cycle of G through v, no edge of G disjoint from all edges of γ crosses an edge of γ.*

Proof: This corollary is similar to the previous one, with the 'crossing' relation replacing the 'lie above' relation of corollary 6.3. The proof may follow the same pattern with adequate small changes which are left to the reader. ∎

In case of an acyclic graph G, the equations (7) and (15) may be written as

$$\operatorname{per}_q A = a_{vv} \operatorname{per}_q A_v + \textstyle\sum_{i \neq v} q^{\ell((vi))} a_{vi} a_{iv} \operatorname{per}_q A_{vi}$$
$$\operatorname{per}_q A = a_{vv} \operatorname{per}_q A_v^{\checkmark} + \textstyle\sum_{i \neq v} q^{\ell((vi))} a_{vi} a_{iv} \operatorname{per}_q A_{vi}^{\checkmark}.$$

Of course the combinatorial solutions to these two equations are obtained by eliminating conditions (ii) and (β) from corollaries 6.3 and 6.4, respectively.

7. Aligned matrices and q-permanent expansions

In this section A denotes a generic matrix with no zero entries. We saw that (7) and (15) do not hold for such A. It is natural to ask if the submatrix A_c may be replaced by some other matrix $\mathcal{S}(A, c)$, depending on A and c, such that

$$\operatorname{per}_q A = \textstyle\sum_{c \in \mathscr{C}_v} q^{\ell(c)} \operatorname{wt}_c(A) \operatorname{per}_q \mathcal{S}(A, c). \tag{16}$$

We shall assume that $\operatorname{per}_q \mathcal{S}(A, c)$ *is a polynomial in the entries of A that does not depend on the a_{rs} for $(r, s) \in c$.* Clearly $\operatorname{per}_q \mathcal{Z}(A, c)$ is a sum of monomials in the a_{ij}, $(i, j) \notin c$, each monomial multiplied by some $q^{\ell(\pi)}$, where π has c as cycle; theorem 5.1 implies $\ell(\pi) \geqslant \ell(c)$, therefore $\operatorname{per}_q \mathcal{Z}(A, c)$ is a multiple of $q^{\ell(c)}$; with the help of theorem 4.2 it is easy to prove that (16) is equivalent to

$$\operatorname{per}_q \mathcal{S}(A, c) = q^{-\ell(c)} \operatorname{per}_q \mathcal{Z}(A, c), \tag{17}$$

for all $c \in \mathscr{C}_v$. For some cycles c it is not difficult to find matrices $\mathcal{S}(A, c)$ satisfying (17). We collect in a lemma, with proof left to the reader, two classes of examples which follow from theorems 6.1 and 6.2 applied to the complete digraph on $[n]$.

Lemma 7.1. *A cycle c satisfies* (17) *for $\mathcal{S}(A,c) = A_c$ if and only if any arc of c has a vertex outside the interval generated by $[n] \setminus c$. A cycle c satisfies* (17) *for $\mathcal{S}(A,c) = A_c^{\checkmark}$ if and only if one of the sets c or $[n] \setminus c$ is an interval.* ■

Lemma 7.2. *We have $\mathcal{Z}(A,c)^T = \mathcal{Z}(A^T, c^{-1})$ and $\mathcal{Z}(A,c)^R = \mathcal{Z}(A^R, w_\circ c w_\circ)$. Moreover, if $\mathcal{S}(A,c)$ satisfies* (17)*, then*

$$\mathcal{S}(A^T, c) = q^{-\ell(c^{-1})} \operatorname{per}_q \mathcal{Z}(A, c^{-1}), \quad \mathcal{S}(A^R, c) = q^{-\ell(w_\circ c w_\circ)} \operatorname{per}_q \mathcal{Z}(A, w_\circ c w_\circ).$$

Proof: We only give a sketchy proof to the case of A^R. For a cycle $c = (v_1 \dots v_k)$, we have $w_\circ c w_\circ = (w_\circ(v_1) \dots w_\circ(v_k))$. Let $a_{ij}^R := a_{w_\circ(i), w_\circ(j)}$ be the ij-entry of A^R. To zero out the row [column] k in A will zero out row [column] $w_\circ(k)$ of A^R; to replace $a_{v_i v_{i+1}}$ by 1 in A, is equivalent to replacing $a^R_{w_\circ(v_i), w_\circ(v_{i+1})}$ by 1. Then the identity $\mathcal{Z}(A,c)^R = \mathcal{Z}(A^R, w_\circ c w_\circ)$ follows at once. Next, from (17) we get $\operatorname{per}_q \mathcal{S}(A^R, c) = q^{-\ell(c)} \operatorname{per}_q \mathcal{Z}(A^R, c)$ and the rest is a simple manipulation using theorem 3.1. ■

Two entries of a matrix are said to be *aligned* whenever they lie in the same row or in the same column of the matrix. We say that a matrix S is *aligned with* A if each entry of S is either a constant, or an entry of A and, whenever two entries of A occur in S and are aligned in A, they are aligned in S as well.

Note that the matrices A_c and $A_c^{\checkmark}$ are aligned with A, and other matrices aligned with A may be defined in a natural manner. For example, if $\alpha \subseteq c$, we let A_c^α be the submatrix of $A_c^{\checkmark}$ obtained by deleting the rows and columns indexed by α (A_c^α is the matrix $A(c,\alpha)$ of [19]). These matrices are shown in the following table where, for $n = 3, 4, 5$, in each line, on the right of a matrix A_c^α, we list cycles c for which (17) holds with $\mathcal{S}(A,c) = A_c^\alpha$:

$n=3$	$A_c^{\checkmark}$	(2)
$n=4$	$A_c^{\checkmark}$	(2), (3), (23)
"	A_c	(13), (24)
$n=5$	$A_c^{\checkmark}$	(2), (3), (4), (23), (34), (234), (243)
"	A_c	(13), (14), (25), (35), (124), (142), (135), (153), (245), (254)
"	$A_c^{\{3\}}$	(134), (143), (235), (253)
"	?	(24)

(18)

For all the cycles c that are not in the table, both matrices A_c and $A_c^{\checkmark}$ satisfy (17). The table is build with the help of lemmas 7.1 and 7.2. Lemma 7.1 tells that in case $n=5$, $c=(134)$, the matrices A_c and $A_c^{\checkmark}$ are of no use. Once a cycle is placed in the table, lemma 7.2 gives as bonus the location in the table of c^{-1}, $w_\circ c w_\circ$ and $w_\circ c^{-1} w_\circ$. The question mark upon $n=5$, $c=(2\,4)$ is the object of the next lemma.

Lemma 7.3. *For $n=5$, $c=(24)$, no $\mathcal{S}(A,c)$ aligned with A satisfies* (17).

Proof: Assume that $B=(b_{ij}) \in \mathfrak{M}_m$ is aligned with A, and (17) holds with $\mathcal{S}(A,c)=B$. We seek for a contradiction. Expanding $\mathcal{S}(A,c)$ in (17) we get

$$\begin{aligned}\operatorname{per}_q B = a_{11}a_{33}a_{55} + (a_{11}a_{35}a_{53} + a_{13}a_{31}a_{55})q+ \\ + (a_{13}a_{35}a_{51} + a_{15}a_{31}a_{53})q^4 + a_{15}a_{33}a_{51}q^7.\end{aligned} \tag{19}$$

The term $a_{11}a_{33}a_{55}q^0$ in the right hand side of (19) indicates that

$$b_{11}b_{22}\cdots b_{mm} = a_{11}a_{33}a_{55}. \tag{20}$$

So a_{11}, a_{33}, a_{55} are diagonal entries of B; let $\{a_{11}, a_{33}, a_{55}\} = \{b_{rr}, b_{ss}, b_{tt}\}$, with $r<s<t$, and call C the submatrix of B of rows and columns r, s, t. The variables a_{ij}, $i,j \in \{1,3,5\}$ all occur in B, so they are the entries of C because B is aligned with A. So, up to transposition and permutation similarity, C has the form

$$\begin{bmatrix} a_{11} & a_{13} & a_{15} \\ a_{31} & a_{33} & a_{35} \\ a_{51} & a_{53} & a_{55} \end{bmatrix}, \tag{21}$$

and the entries of C occur nowhere else in B. The term $(a_{11}a_{35}a_{53}+a_{13}a_{31}a_{55})q$ in $\operatorname{per}_q B$ means that $a_{11}a_{35}a_{53}+a_{13}a_{31}a_{55}$ is the sum of the total weights in B of the transpositions of consecutive indices; this implies that, for some $i,j \in$

$[m]$, $\{a_{13}, a_{31}\} = \{b_{i,i+1}, b_{i+1,i}\}$ and $\{a_{35}, a_{53}\} = \{b_{j,j+1}, b_{j+1,j}\}$. Therefore r, s, t are consecutive integers and $b_{ss} = a_{33}$.

Let F be the set of $\sigma \in \mathscr{S}_m$ that fix r, s and t. The sum of all terms of $\mathrm{per}_q B$ that are multiple of $a_{11}a_{33}a_{55}$ is $a_{11}a_{33}a_{55} \,\mathrm{per}_q B^{\checkmark}_{(rst)}$. From (19) $\mathrm{per}_q B^{\checkmark}_{(rst)} = 1$. Let ω be one of the cycles (rst) or (tsr); for any $\sigma \in F$, $\ell(\omega\sigma) = 2 + \ell(\sigma)$. By (20) $\mathrm{twt}_\omega(B) = \mathrm{wt}_\omega(B)$, and so $\mathrm{twt}_\omega(B)$ is one of the monomials $a_{13}a_{35}a_{51}$ or $a_{15}a_{31}a_{53}$. The sum of all terms of $\mathrm{per}_q B$ that are multiple of $\mathrm{twt}_\omega(B)$ is

$$\begin{aligned}\textstyle\sum_{\sigma\in F} q^{\ell(\omega\sigma)} \,\mathrm{twt}_{\omega\sigma}(B) &= q^2 \,\mathrm{twt}_\omega(B) \textstyle\sum_{\sigma\in F} q^{\ell(\sigma)} \,\mathrm{twt}_\sigma(B^{\checkmark}_{(rst)}) \\ &= q^2 \,\mathrm{twt}_\omega(B) \,\mathrm{per}_q B^{\checkmark}_{(rst)} = q^2 \,\mathrm{twt}_\omega(B). \end{aligned} \tag{22}$$

So $a_{13}a_{35}a_{51}$ and $a_{15}a_{31}a_{53}$ should occur in $\mathrm{per}_q B$ multiplied by q^2; but (19) has the factor q^4 instead. This contradiction finishes the proof. ∎

Lemma 7.4. *If $n = 6$ and c is one of the cycles* (246)*,* (264)*,* (135) *or* (153)*, no $\mathcal{S}(A, c)$ aligned with A satisfies* (17)*.*

Proof: *The case $c = (246)$.* The proof starts as that of lemma 7.3, with a matrix $\mathcal{S}(A, (246)) =: B$ satisfying (17). We seek for a contradiction. This time we get

$$\begin{aligned}\mathrm{per}_q B = a_{11}a_{33}a_{55} + (a_{11}a_{35}a_{53} + a_{13}a_{31}a_{55})q+ \\ a_{13}a_{35}a_{51}q^2 + a_{15}a_{31}a_{53}q^4 + a_{15}a_{33}a_{51}q^5.\end{aligned} \tag{23}$$

As before, we get a principal submatrix C which, up to transposition and permutational similarity, is (21); and C lies in consecutive rows and columns r, s, t. The proof continues with the arguments of the previous proof leading to (22). Now the desired contradiction is that $a_{15}a_{31}a_{53}$ occurs in (23) multiplied by q^4, instead of q^2 as (22) determines.

The other three cycles, (264), (135), (153), result from (246) by a combination of inversion with conjugation by $w_\circ$. So we may apply lemma 7.2. ∎

Theorem 7.5. *If all $\mathcal{S}(A, c)$ are required to be aligned with A, then* (16) *fails for $n = 5$ and $v \in \{2, 4\}$, and fails for $n \geqslant 6$ and all $v \in [n]$.*

Proof: The cases $n = 5$ and $n = 6$ were handled in lemmas 7.3 and 7.4.

Now fix an arbitrary (n, v), with $n \geqslant 6$, $v \in [n]$. Assume that (16) holds, where all $\mathcal{S}(A, c)$ are aligned with A, and seek for a contradiction. Choose an integer p such that $1 \leqslant p+1 \leqslant v \leqslant p+6 \leqslant n$. For each $\omega \in \mathscr{S}_6$, let $\omega^* \in \mathscr{S}_n$ be defined by

$$\omega^*(i) = \begin{cases} \omega(i-p)+p, & \text{for } i \in [p+1, p+6] \\ i, & \text{otherwise.} \end{cases}$$

This is a group isomorphism onto the subgroup $\mathscr{S}_n^*$ of the permutations that fix all indices outside $[p+1, p+6]$; clearly $\ell(\omega^*) = \ell(\omega)$. This isomorphism maps $\mathscr{C}_{v-p}(6)$ onto the set $\mathscr{C}_v^* := \mathscr{C}_v \cap \mathscr{S}_n^*$. We now let X be a generic 6×6 matrix with no zero entry, and use (16) to compute

$$\begin{aligned} \operatorname{per}_q X &= \operatorname{per}_q(I_p \oplus X \oplus I_{n-6-p}) \\ &= \textstyle\sum_{c^* \in \mathscr{C}_v^*} q^{\ell(c^*)} \operatorname{twt}_{c^*}(I_p \oplus X \oplus I_{n-6-p}) \operatorname{per}_q \mathcal{S}(I_p \oplus X \oplus I_{n-6-p}, c^*) \\ &= \textstyle\sum_{c \in \mathscr{C}_{6,v-6}} q^{\ell(c)} \operatorname{twt}_c(X) \operatorname{per}_q \mathcal{S}(I_p \oplus X \oplus I_{n-6-p}, c^*). \end{aligned}$$

We just got (16) for the generic matrix X, where the $\mathcal{S}(I_p \oplus X \oplus I_{n-6-p}, c^*)$ are obviously aligned with X. This contradicts lemma 7.4. ∎

Open Problems

As alignment proved to be too restrictive to satisfy (16)-(17), we suggest

Problem 1. *The quest for matrices $\mathcal{S}(A, c)$ satisfying* (16)*, whose entries are either constant or entries of A.*

In such less demanding setting, the case $n = 5$, $c = (24)$ is solved with the following 7×7 matrix

$$\mathcal{S}(A, (24)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{15} & a_{11} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & a_{31} & a_{33} & a_{35} & 0 & 0 \\ 0 & 0 & 0 & a_{53} & a_{55} & a_{51} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The proof that this matrix satisfies (17) is an easy exercise with the help of its digraph to exploit sparsity. This example and the table (18) give a positive answer to problem 1 for $n \leqslant 5$. An alternative to problem 1 is to relax the condition that $\mathcal{S}$ is a function of only A and c:

Problem 2. *Find matrices $\mathcal{S}(A, c, q)$, with entries depending also on q, to satisfy formula* (16)*.*

This is suggested by the definition given by H. Tagawa in [22] of a multivariable q-permanent. His paper gives a very interesting sort of Laplace expansion, that do not really look like the traditional expansion for the determinant and the permanent in that the algebraic complements depend heavily on the quantum parameter(s).

References

[1] R. Bapat, Interpolating the Determinantal and Permanental Hadamard Inequality, *Linear and Multilinear Algebra*, 32(1992), pp. 335-337.

[2] R. Bapat and A. Lal, Inequalities for the q-permanent, Second Conference of the International Linear Algebra Society (Lisbon, 1992), *Linear Algebra and its Applications*, 197/198(1994), pp. 397-409.

[3] A. Bjorner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics 231, Springer, 2005.

[4] E. Botta, Linear transformations that preserve the permanent, *Proceedings of AMS*, 18(1967), pp. 566-569.

[5] M. Bozejko and R. Speicher, An example of a generalized Brownian motion, *Communications in Mathematical Physics*, 137(1991), pp. 519-531.

[6] R. Brualdi and H. Ryser, *Combinatorial matrix theory*, Encyclopedia of mathematics and its applications, Cambridge University Press, 1991.

[7] R. Diestel, *Graph theory*, Springer Verlag, New York, 1997.

[8] M. A. Duffner, Linear transformations that preserve immanants, *Linear Algebra and its Applications*, 197/198(1994), pp. 567-588.

[9] S. Jukna, *Extremal combinatorics, with applications in computer science*, Texts in Theoretical Computer Science, Springer-Verlag, 2011 (Second Edition).

[10] G. Kreweras, Sur les partitions noncroisées d'un cycle, *Discrete Mathematics*, 1(1972), pp. 333-350.

[11] A. Lal, *Coxeter Groups and Positive Matrices*, [Ph.D. Thesis], Indian Statistical Institute, 1992.

[12] A. Lal, Inequalities for the q-Permanent, II, *Linear Algebra and its Applications*, 274(1998), pp. 1-16.

[13] C-K. Li and N-K. Tsing, Linear preserver problems: A brief introduction and some special techniques, *Linear Algebra and its Applications*, 162-164(1992), pp. 217-235.

[14] Yu I. Manin, *Quantum Groups and Non-Commutative Geometry*, Publications du C.R.M., Universite de Montreal, 1988.

[15] M. Marcus and F. May, The permanent function, *Canad. J. Math.*, 14(1962), pp. 177-189.

[16] J. Maybee, *Mathematical structure of the OMS model*, Office of Statistical Standards, Energy Information Administration, Washington DC, 1986. Cited in [17].

[17] J. Maybee, D. Olesky, P. van den Driessche and J. Wiener, Matrices, digraphs and determinants, *SIAM Journal on Matrix Analysis and Applications*, 10(1989), p. 500-519.

[18] J. Maybee and J. Quirk, Qualitative problems in matrix theory, *SIAM Review*, 11(1969), pp. 3051.

[19] E. Sá, Noncrossing partitions, noncrossing graphs and q-permanental formulas, CMUC working paper 17-22, 2017.

[20] R. Simion, Noncrossing partitions, *Discrete Mathematics* 217(2000), pp. 367-409.

[21] E. Taft and J. Towber, Quantum Deformation of Flag Schemes and Grassmann Schemes, I. A q-Deformation of the Shape-Algebra for GL(n), *Journal of Algebra*, 142(1991), pp. l-36

[22] H. Tagawa, A multivariable quantum determinant over a commutative ring, *RIMS Kōkyūroku*, 765(1991), pp. 91-103.

[23] H. Tagawa, q-Analogues of Determinants and Symmetric Chain Decompositions, *Tokyo Journal of Mathematics*, 16(1993), pp. 311-320.

[24] K-W. Yang, q-determinants and permutations, *The Fibonacci Quarterly*, 29(1991), pp. 160-163.

Eduardo Marques de Sá

CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz 3001-501 Coimbra, Portugal

E-mail address: emsa@mat.uc.pt