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ON SEMICLASSICAL ORTHOGONAL POLYNOMIALS VIA POLYNOMIAL MAPPINGS II: SIEVED ULTRASPHERICAL POLYNOMIALS REVISITED

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Abstract: In a companion paper [On semiclassical orthogonal polynomials via polynomial mappings, J. Math. Anal. Appl. (2017)] we proved that the semiclassical class of orthogonal polynomials is stable under polynomial transformations. In this work we use this fact to derive in an unified way old and new properties concerning the sieved ultraspherical polynomials of the first and second kind introduced by W. Al-Salam, W. R. Allaway, and R. Askey, and subsequently studied by several authors. Our results are stated in the more general framework of orthogonality with respect to a quasi-definite (or regular) moment linear functional, not necessarily represented by a weight function or positive Borel measure. This allow us to derive infinitely many examples of semiclassical functionals such that the pair of polynomials appearing in each corresponding canonical Pearson-type distributional differential equation is non-admissible.

Keywords: Orthogonal polynomials; semiclassical orthogonal polynomials; polynomial mappings; sieved orthogonal polynomials; differential equations.
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1. Introduction

This is the second of two papers intended to develop the theory of polynomial mappings in the framework of the semiclassical orthogonal polynomial sequences. Throughout this paper we will use the abbreviations OP and OPS for orthogonal polynomial(s) and orthogonal polynomial(s) sequence(s), respectively. In our first article [3] we obtained basic properties fulfilled by monic OPS $\{p_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ linked by a polynomial mapping, in the sense that there exist two polynomials π_k and θ_m, of (fixed) degrees k and

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m, respectively, where $0 \le m \le k-1$, such that

$$p_{nk+m}(x) = \theta_m(x)\, q_n(\pi_k(x))\,, \quad n = 0, 1, 2, \dots\ ,$$

under the assumption that one of the sequences $\{p_n\}_{n\ge0}$ or $\{q_n\}_{n\ge0}$ is a semiclassical OPS. In particular, we proved that if at least one of the sequences $\{p_n\}_{n\ge0}$ or $\{q_n\}_{n\ge0}$ is semiclassical then so is the other one, and we gave relations between their classes [3, Theorem 3.1].

Our present goal is to apply the results stated in [3] to the *sieved* OPS, introduced by Al-Salam, Allaway, and Askey [1], and subsequently studied by several authors (see e.g. [8, 4, 9, 11, 20, 2, 5, 6, 14, 10]). The connection between sieved OPS and polynomials mappings has been observed by Geronimo and Van Assche [11], who showed how many results involving sieved OPS follow by taking particular polynomial transformations. For instance, take π_k the monic Chebyshev polynomial of the first kind of degree k. Then (up to normalization) taking for q_n the monic ultraspherical polynomial of degree n of parameter $\lambda+1$ and choosing $m = 0$ and $\theta_m \equiv 1$, $\{p_n\}_{n\ge0}$ becomes the monic sieved ultraspherical OPS of the first kind. Similarly, taking for q_n the monic ultraspherical polynomial of degree n of parameter λ and choosing $m = k-1$ and θ_m the monic Chebyshev polynomial of the second kind of degree $k-1$, $\{p_n\}_{n\ge0}$ becomes the monic sieved ultraspherical OPS of the second kind.

The structure of the paper is as follows. In Section 2 we introduce some background, including the definitions of sieved ultraspherical polynomials and some basic facts concerning semiclassical OPS, complementing the information appearing in [3]. In Sections 3 and 4 we consider the sieved ultraspherical OPS of the first and of the second kind, respectively. Among other results, we prove that both families are semiclassical of class exactly $k-1$ except for one choice of the parameter λ (being classical in such a case). Using this fact and the theory of semiclassical OP presented by Maroni [15], we give the structure relation that such sieved OPS satisfy, and then we use these relations (together with general facts of the theory of semiclassical OPS) to derive the linear homogeneous second order ordinary differential equation (ODE) that sieved orthogonal polynomial fulfills. This ODE was obtained (by a different process) for the sieved OP of the second kind by Bustoz, Ismail, and Wimp [2]. As far as we know, the ODE for the sieved OPS of the first kind did not appeared before in the literature. The interest on such ODE comes at once from the original paper by Al-Salam, Allaway, and Askey, where in a final section devoted to some open problems they wrote that "A potentially very important result would be the second order differential equation these polynomials satisfy." The results presented here based on polynomial mappings in the framework of semiclassical OPS allow to obtain similar results for other sieved OPS, *mutatis mutandis*.

2. Background

For reasons of economy of exposition, we assume familiarity with most of the results and notation appearing in Sections 2 and 3 of our previous article [3]. Let $\{p_n\}_{n\geq 0}$ be a monic OPS, so that, according to Favard's theorem it is characterized by a three-term recurrence such as

$$p_{n+1}(x) = (x-\beta_n)p_n(x) - \gamma_n p_{n-1}(x)\;, \quad n = 0,1,2,\dots\;, \tag{2.1}$$

with $p_{-1}(x) := 0$ and $p_0(x) := 1$, where $\beta_n \in \mathbb{C}$ and $\gamma_{n+1} \in \mathbb{C}\setminus\{0\}$ for each $n \in \mathbb{N}_0$. In the framework of polynomial mappings, it is useful to write the recurrence relation in terms of blocks of recurrence relations as

$$\begin{aligned} (x-b_n^{(j)})p_{nk+j}(x) = p_{nk+j+1}(x) + a_n^{(j)} p_{nk+j-1}(x)\;, \\ j = 0,1,\dots,k-1\;; \quad n = 0,1,2,\dots\;. \end{aligned} \tag{2.2}$$

Without loss of generality, we assume $a_0^{(0)} := 1$. In general, the $a_n^{(j)}$'s and $b_n^{(j)}$'s are complex numbers with $a_n^{(j)} \neq 0$ for all n and j. With these numbers we may construct the determinants $\Delta_n(i,j;x)$ introduced by Charris, Ismail, and Monsalve [5, 6], so that

$$\Delta_n(i,j;x) := \begin{cases} 0 & \text{if } j < i-2 \\ 1 & \text{if } j = i-2 \\ x - b_n^{(i-1)} & \text{if } j = i-1 \end{cases} \tag{2.3}$$

and, if $j \geq i \geq 1$,

$$\Delta_n(i,j;x) := \begin{vmatrix} x-b_n^{(i-1)} & 1 & 0 & \dots & 0 & 0 \\ a_n^{(i)} & x-b_n^{(i)} & 1 & \dots & 0 & 0 \\ 0 & a_n^{(i+1)} & x-b_n^{(i+1)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x-b_n^{(j-1)} & 1 \\ 0 & 0 & 0 & \dots & a_n^{(j)} & x-b_n^{(j)} \end{vmatrix}\,, \tag{2.4}$$

for every $n \in \mathbb{N}_0$. Taking into account that $\Delta_n(i,j;\cdot)$ is a polynomial whose degree may exceed k, and since in (2.2) the $a_n^{(j)}$'s and $b_n^{(j)}$'s were defined only for $0 \leq j \leq k-1$, we adopt the convention

$$b_n^{(k+j)} := b_{n+1}^{(j)} \; , \quad a_n^{(k+j)} := a_{n+1}^{(j)} \quad i,j,n \in \mathbb{N}_0 \; , \tag{2.5}$$

and so the following useful equality holds:

$$\Delta_n(k+i,k+j;x) = \Delta_{n+1}(i,j;x) \; . \tag{2.6}$$

Theorem 2.1. [10, Theorem 2.1] *Let $\{p_n\}_{n\geq 0}$ be a monic OPS characterized by the general blocks of recurrence relations (2.2). Fix $r_0 \in \mathbb{C}$, $k \in \mathbb{N}$, and $m \in \mathbb{N}_0$, with $0 \leq m \leq k-1$ and $k \geq 3$. Then, there exist polynomials π_k and θ_m of degrees k and m (respectively) and a monic OPS $\{q_n\}_{n\geq 0}$ such that $q_1(0) = -r_0$ and*

$$p_{kn+m}(x) = \theta_m(x)\, q_n(\pi_k(x)) \; , \quad n = 0,1,2,\ldots \tag{2.7}$$

if and only if the following four conditions hold:

(i) $b_n^{(m)}$ *is independent of n for $n \geq 0$;*

(ii) $\Delta_n(m+2,m+k-1;x)$ *is independent of n for $n \geq 0$ and for every x;*

(iii) $\Delta_0(m+2,m+k-1;\cdot)$ *is divisible by θ_m, i.e., there exists a polynomial η_{k-1-m} with degree $k-1-m$ such that*

$$\Delta_0(m+2,m+k-1;x) = \theta_m(x)\, \eta_{k-1-m}(x) \, ;$$

(iv) $r_n(x)$ *is independent of x for every $n \geq 1$, where*

$$\begin{aligned} r_n(x) := {} & a_n^{(m+1)}\Delta_n(m+3,m+k-1;x) - a_0^{(m+1)}\Delta_0(m+3,m+k-1;x) \\ & + a_n^{(m)}\Delta_{n-1}(m+2,m+k-2;x) - a_0^{(m)}\Delta_0(1,m-2;x)\, \eta_{k-1-m}(x) \; . \end{aligned}$$

Under such conditions, the polynomials θ_m and π_k are explicitly given by

$$\begin{aligned} \pi_k(x) &= \Delta_0(1,m;x)\, \eta_{k-1-m}(x) - a_0^{(m+1)}\, \Delta_0(m+3,m+k-1;x) + r_0 \; , \\ \theta_m(x) &:= \Delta_0(1,m-1;x) \equiv p_m(x) \; , \end{aligned} \tag{2.8}$$

and the monic OPS $\{q_n\}_{n\geq 0}$ is generated by the three-recurrence relation

$$q_{n+1}(x) = (x - r_n)\, q_n(x) - s_n q_{n-1}(x) \, , \quad n = 0,1,2,\ldots \tag{2.9}$$

with initial conditions $q_{-1}(x) = 0$ and $q_0(x) = 1$, where

$$r_n := r_0 + r_n(0) \; , \quad s_n := a_n^{(m)} a_{n-1}^{(m+1)} \cdots a_{n-1}^{(m+k-1)} \; , \quad n = 1,2,\ldots \; . \tag{2.10}$$

Moreover, for each $j = 0, 1, 2, \ldots, k-1$ *and all* $n = 0, 1, 2, \ldots$,

$$\begin{aligned} p_{kn+m+j+1}(x) = \frac{1}{\eta_{k-1-m}(x)} \Big\{ \Delta_n(m+2, m+j; x)\, q_{n+1}(\pi_k(x)) \\ + \left(\textstyle\prod_{i=1}^{j+1} a_n^{(m+i)} \right) \Delta_n(m+j+3, m+k-1; x)\, q_n(\pi_k(x)) \Big\} . \end{aligned} \tag{2.11}$$

Remarks 2.1. *Notice that for* $j = k-1$, *(2.11) reduces to (2.7).*

Theorem 2.2. [10, Theorem 3.4] *Under the conditions of Theorem 2.1, choose* $r_0 = 0$ *and assume that* $\{p_n\}_{n\geq 0}$ *is a monic OPS in the positive-definite sense with respect to some positive measure* $\mathrm{d}\mu$ *. Then* $\{q_n\}_{n\geq 0}$ *is also a monic OPS in the positive-definite sense, orthogonal with respect to a measure* $\mathrm{d}\tau$ *. Further, assume that the following conditions hold:*

(i) $[\xi, \eta] := \mathrm{co}\,(\mathrm{supp}(\mathrm{d}\tau))$ *is a compact set;*

(ii) *if* $m \geq 1$,

$$\int_\xi^\eta \frac{\mathrm{d}\tau(x)}{|x - \pi_k(z_i)|} < \infty \quad (i = 1, 2, \ldots, m)\,,$$

where $z_1 < z_2 < \cdots < z_m$ *are the zeros of* θ_m *;*

(iii) *either* $\pi_k(y_{2i-1}) \geq \eta$ *and* $\pi_k(y_{2i}) \leq \xi$ *(for all possible* i*) if* k *is odd, or* $\pi_k(y_{2i-1}) \leq \xi$ *and* $\pi_k(y_{2i}) \geq \eta$ *if* k *is even, where* $y_1 < \cdots < y_{k-1}$ *denote the zeros of* π_k' *;*

(iv) $\theta_m \eta_{k-1-m}$ *and* π_k' *have the same sign at each point of the set* $\pi_k^{-1}([\xi, \eta])$.

Then the Stieltjes transforms $F(\cdot; \mathrm{d}\mu)$ *and* $F(\cdot; \mathrm{d}\tau)$ *are related by*

$$F(z; \mathrm{d}\mu) = \frac{-v_0\, \Delta_0(2, m-1; z) + \left(\prod_{j=1}^m a_0^{(j)} \right) \eta_{k-1-m}(z)\, F(\pi_k(z); \mathrm{d}\tau)}{\theta_m(z)}\,,$$
$$z \in \mathbb{C} \setminus \left(\pi_k^{-1}([\xi, \eta]) \cup \{z_1, \ldots, z_m\} \right)\,,$$

where the normalization condition $v_0 := \int_\xi^\eta \mathrm{d}\tau = \int_{\mathrm{supp}(\mathrm{d}\sigma)} \mathrm{d}\mu =: u_0$ *is assumed. Further, up to constant factors, the measure* $\mathrm{d}\mu$ *can be obtained from* $\mathrm{d}\tau$ *by*

$$\mathrm{d}\mu(x) = \sum_{i=1}^m M_i\, \delta(x - z_i)\, \mathrm{d}x + \left| \frac{\eta_{k-1-m}(x)}{\theta_m(x)} \right| \frac{\mathrm{d}\tau(\pi_k(x))}{\pi_k'(x)}\,, \tag{2.12}$$

where if $m \geq 1$

$$M_i := \frac{v_0\, \Delta_0(2, m-1; z_i) / \left(\prod_{j=1}^{m} a_0^{(j)} \right) - \eta_{k-1-m}(z_i)\, F(\pi_k(z_i); \mathrm{d}\tau)}{\theta_m'(z_i)} \geq 0 \tag{2.13}$$

for all $i = 1, \cdots, m$. *The support of* $\mathrm{d}\mu$ *is contained in the set*

$$\pi_k^{-1}\left([\xi, \eta] \right) \cup \{z_1, \ldots, z_m\} ,$$

an union of k *intervals and* m *possible mass points.*

Remarks 2.2. *In statement (i),* $\mathrm{co}(A)$ *means the convex hull of a set* A. *Under the conditions of Theorem 2.2, if* $\mathrm{d}\tau$ *is an absolutely continuous measure with density* w_τ, *then the absolutely continuous part of* $\mathrm{d}\mu$ *has density*

$$w_\mu(x) := \left| \frac{\eta_{k-1-m}(x)}{\theta_m(x)} \right| w_\tau(\pi_k(x))$$

with support contained in an union of at most k *closed intervals, and it may appear mass points at the zeros of* θ_m.

In [3, Section 3] we stated several results concerning OPS and polynomial mappings in the framework of the theory of semiclassical OPS. In particular, in the proof of part (ii) of [3, Theorem 3.1], we implicitly proved the following

Theorem 2.3. *Under the conditions of Theorem 2.1, let* $\mathbf{u}$ *and* $\mathbf{v}$ *be the moment regular functionals with respect to which* $\{p_n\}_{n\geq 0}$ *and* $\{q_n\}_{n\geq 0}$ *are monic OPS, respectively. Let* $S_{\mathbf{u}}(z) := -\sum_{n\geq 0} u_n/z^{n+1}$ *and* $S_{\mathbf{v}}(z) := -\sum_{n\geq 0} v_n/z^{n+1}$ *(where* $u_n := \langle \mathbf{u}, x^n \rangle$ *and* $v_n := \langle \mathbf{v}, x^n \rangle$*) be the corresponding (formal) Stieltjes series, respectively. Suppose that there exist polynomials* $\widetilde{\Phi}$, $\widetilde{C}$, *and* $\widetilde{D}$, *such that*

$$\widetilde{\Phi}(z) S_{\mathbf{v}}'(z) = \widetilde{C}(z) S_{\mathbf{v}}(z) + \widetilde{D}(z) .$$

Then $S_{\mathbf{u}}(z)$ *fulfils*

$$\Phi_1(z) S_{\mathbf{u}}'(z) = C_1(z) S_{\mathbf{u}}(z) + D_1(z) ,$$

where Φ_1*,* C_1*, and* D_1 *are polynomials given explicitly by*

$$\begin{aligned}
&\Phi_1 := v_0\theta_m\eta_{k-1-m}\sigma_{\pi_k}[\widetilde{\Phi}] \,, \\
&C_1 := v_0\left(\eta'_{k-1-m}\theta_m - v_0\theta'_m\eta_{k-1-m}\sigma_{\pi_k}[\widetilde{\Phi}] + \eta_{k-1-m}\theta_m\pi'_k\sigma_{\pi_k}[\widetilde{C}]\right) \,, \\
&D_1 := u_0v_0\left(\Delta_0(2,m-1,\cdot)\eta'_{k-1-m} - \Delta'_0(2,m-1,\cdot)\eta_{k-1-m}\right)\sigma_{\pi_k}[\widetilde{\Phi}] \\
&\qquad\quad + u_0\left(\kappa_m\eta_{k-1-m}\sigma_{\pi_k}[\widetilde{D}] + v_0\Delta_0(2,m-1,\cdot)\sigma_{\pi_k}[\widetilde{C}]\right)\eta_{k-1-m}\pi'_k \,,
\end{aligned}$$

and $\sigma_{\pi_k}[f](z) := f\big(\pi_k(z)\big)$ *for each polynomial* f*.*

Besides the basic facts concerning semiclassical OPS given in [3, Section 2], we recall that such families are characterized by a structure relation and a linear homogeneous second order ODE. Indeed, let $\{p_n\}_{n\geq 0}$ be a monic semiclassical OPS. This means that $\{p_n\}_{n\geq 0}$ is an OPS with respect to a linear functional $\mathbf{u} : \mathcal{P} \to \mathbb{C}$ ($\mathcal{P}$ being the space of all polynomials with complex coefficients) which fulfils a distributional differential equation of Pearson type

$$D(\Phi\mathbf{u}) = \Psi\mathbf{u} \,,$$

where Φ and Ψ are nonzero polynomials (i.e., they do not vanish identically), and $\deg\Psi \geq 1$. According to the theory presented by Maroni in [15], $\{p_n\}_{n\geq 0}$ fulfills the structure relation

$$\Phi(x)p'_n(x) = M_n(x)p_{n+1}(x) + N_n(x)p_n(x) \,, \quad n = 0,1,2,\dots \,, \tag{2.14}$$

where M_n and N_n are polynomials that may depend of n, but they have degrees (uniformly) bounded by a number independent of n, which can be computed successively using the relations

$$\begin{aligned}
&N_n = -C - N_{n-1} - (x-\beta_n)M_n \\
&\gamma_{n+1}M_{n+1} = -\Phi + \gamma_nM_{n-1} + (x-\beta_n)(N_{n-1} - N_n) \,,
\end{aligned} \tag{2.15}$$

with initial conditions $N_{-1} := -C$, $M_{-1} := 0$, and $M_0 := u_0^{-1}D$. Here β_n and γ_n are the parameters appearing in the three-term recurrence relation (2.1), $u_0 := \langle\mathbf{u}, 1\rangle$, and C and D are polynomials, being $C := \Psi - \Phi'$, and the definition of D may be seen in [3, Section 2.2]. The structure relation (2.14) is a characteristic property of semiclassical OPS. Another characterization of semiclassical OPS is the second order ODE

$$J_n(x)p''_n(x) + K_n(x)p'_n(x) + L_n(x)p_n(x) = 0 \,, \tag{2.16}$$

where J_n, K_n, and L_n are polynomials that may depend of n, but their degrees are (uniformly) bounded by a number independent of n. Moreover, if $\{p_n\}_{n\geq 0}$ satisfies the structure relation (2.14)–(2.15) then J_n, K_n, and L_n are given by

$$\begin{aligned} &J_n := \Phi M_n \\ &K_n := W(M_n, \Phi) + CM_n = \Psi M_n - \Phi M_n' \\ &L_n := W(N_n, M_n) + (\gamma_{n+1} M_n M_{n+1} - N_n(N_n + C))\, M_n/\Phi\,, \end{aligned} \tag{2.17}$$

where $W(f,g) := fg' - f'g$.

3. Sieved ultraspherical polynomials

Let $\{C_n^\lambda\}_{n\geq 0}$ be the ultraspherical (or Gegenbauer) OPS, defined by the recurrence relation

$$2(n+\lambda)xC_n^\lambda(x) = (n+1)C_{n+1}^\lambda(x) + (n+2\lambda-1)C_{n-1}^\lambda(x)\,, \quad n \in \mathbb{N}\,,$$

with initial conditions $C_0^\lambda(x) := 1$ and $C_1^\lambda(x) := 2\lambda x$, where $\lambda \neq 0$. This definition appears in [19, Equation (4.7.17)], where the condition $\lambda > -1/2$ is assumed, so that the polynomials are orthogonal in the positive-definite sense. If $\lambda = 0$ then a compatible definition is [19, Equation (4.7.8)]

$$C_0^0(x) := 1\,, \quad C_n^0(x) := T_n(x) = \frac{n}{2} \lim_{\substack{\lambda\to 0\\ \lambda\neq 0}} \frac{C_n^\lambda(x)}{\lambda}\,, \quad n \in \mathbb{N}\,.$$

Here we allow orthogonality with respect to a quasi-definite (or regular) functional in $\mathcal{P}$, not necessarily positive-definite. Therefore we assume that the range of values of the parameter λ is

$$\lambda \in \mathbb{C} \setminus \{-n/2 : n \in \mathbb{N}\}\,. \tag{3.1}$$

(This follows e.g. from [3, Table 1], noticing that C_n^λ is, up to normalization, a Jacobi polynomial $P_n^{(\alpha,\beta)}$ with parameters $\alpha = \beta = \lambda - 1/2$.) We recall the definition of the sieved ultraspherical polynomials, as presented in [1] and [8]. Rogers [17, 18] studied the OPS $\{C_n(\cdot;\beta|q)\}_{n\geq 0}$ defined by $C_0(x;\beta|q) := 1$, $C_1(x;\beta|q) := 2x(1-\beta)/(1-q)$, and

$$2x(1-\beta q^n)C_n(x;\beta|q) = (1-q^{n+1})C_{n+1}(x;\beta|q) + (1-\beta^2 q^{n-1})C_{n-1}(x;\beta|q)\,,$$

where β and q are real or complex parameters, and $|q| < 1$. Nowadays these polynomials are called continuous $q-$ultraspherical polynomials, since they generalize $\{C_n^\lambda\}_{n\geq 0}$ in the following sense (see [8]):

$$\lim_{q\to 1} C_n(x;q^\lambda|q) = C_n^\lambda(x)\ .$$

Let $\{c_n(\cdot;\beta|q)\}_{n\geq 0}$ be an OPS obtained renormalizing $\{C_n(\cdot;\beta|q)\}_{n\geq 0}$, so that

$$C_n(\cdot;\beta|q) = \frac{(\beta^2;q)_n}{(q;q)_n}\, c_n(\cdot;\beta|q)\ ,$$

where $(a;q)_0 := 1$ and $(a;q)_n := \prod_{j=1}^n (1-aq^{j-1})$ for each $n\in\mathbb{N}$. The sieved OP defined by Al-Salam, Allaway, and Askey [1] are limiting cases of the polynomials $C_n(\cdot;\beta|q)$ and $c_n(\cdot;\beta|q)$. Indeed, fix $k\in\mathbb{N}$ and let ω_k be an kth root of the unity, i.e.,

$$\omega_k := e^{2\pi i/k}\ .$$

Setting $\beta = s^{\lambda k}$ and $q = s\omega_k$, the OPS $\{c_n^\lambda(\cdot;k)\}_{n\geq 0}$ defined by

$$c_n^\lambda(x;k) := \lim_{s\to 1} c_n(x;s^{\lambda k}|s\omega_k)$$

is the sequence of the *sieved ultraspherical polynomials of the first kind*; and setting $\beta = s^{\lambda k+1}\omega_k$ and $q = s\omega_k$, the OPS $\{B_n^\lambda(\cdot;k)\}_{n\geq 0}$ defined by

$$B_n^\lambda(x;k) := \lim_{s\to 1} C_n(x;s^{\lambda k+1}\omega_k|s\omega_k)$$

is the sequence of the *sieved ultraspherical polynomials of the second kind.*

For $\lambda > -1/2$ the sieved ultraspherical polynomials are orthogonal in the positive-definite sense. In such a case, the orthogonality measures were given in [1, Theorems 1 and 2].

From now on (even if not stated explicitly) we assume that $k\geq 3$.

4. On sieved ultraspherical OP of the second kind

4.1. Description via a polynomial mapping. In [4], Charris and Ismail proved that $\{B_n^\lambda(\cdot;k)\}_{n\geq 0}$ satisfies

$$B_{kn+j}^\lambda(x;k) = U_j(x)C_n^{\lambda+1}(T_k(x)) + U_{k-j-2}(x)C_{n-1}^{\lambda+1}(T_k(x)) \qquad (4.1)$$

for $j = 0,1,\ldots,k-1$ and $n = 1,2,\ldots$, where $\{T_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ are the OPS of the Chebychev polynomials of the first and the second kind, respectively, defined by

$$T_n(x) := \cos(n\theta)\ , \quad U_n(x) := \frac{\sin(n+1)\theta}{\sin\theta} \quad (x=\cos\theta\ ,\ 0<\theta<\pi)\ .$$

Since $U_{-1} := 0$, then for $j = k-1$ (4.1) reduces to

$$B^{\lambda}_{kn+k-1}(x;k) = U_{k-1}(x)C^{\lambda+1}_n(T_k(x)) \;, \quad n = 0,1,2,\dots \;. \tag{4.2}$$

Relations (4.1) and (4.2) establish a connection between sieved OP of the second kind and OPS obtained via a polynomial mapping as described in [3, Section 2]. This connection was established in a different way by Geronimo and Van Assche [11], and also in [10] (see also [6]). Next we briefly describe such connection following the presentation in [10, Section 5.2]. Taking for $\{p_n\}_{n\geq 0}$ the monic OPS corresponding to $\{B^{\lambda}_n(\cdot;k)\}_{n\geq 0}$, so that

$$p_{kn+j}(x) = \frac{n!}{2^{kn+j}(\lambda+1)_n} B^{\lambda}_{kn+j}(x;k) \tag{4.3}$$

($n = 0,1,2,\dots;\; j = 0,1,\dots,k-1$), where $(\alpha)_n$ is the shifted factorial, defined by $(\alpha)_0 := 1$ and $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ whenever $n \geq 1$, and using the three-term recurrence relation for $\{B^{\lambda}_n(\cdot;k)\}_{n\geq 0}$ given in [1], we see that the coefficients appearing in the (block) three-term recurrence relation (2.2) for $\{p_n\}_{n\geq 0}$ are

$$b^{(j)}_n := 0 \quad (0 \leq j \leq k-1) \;, \quad a^{(j)}_n := \tfrac{1}{4} \quad (1 \leq j \leq k-2) \;,$$
$$a^{(0)}_{n+1} := \frac{n+1}{4(n+1+\lambda)} \;, \quad a^{(k-1)}_n := \frac{n+1+2\lambda}{4(n+1+\lambda)}$$

for each $n \in \mathbb{N}_0$. Hence, for every $n \in \mathbb{N}_0$ and $0 \leq j \leq k-1$, we compute

$$\Delta_n(1, j-1; x) = \widehat{U}_j(x) \;, \quad \Delta_n(j+2, k-2; x) = \widehat{U}_{k-j-2}(x) \;,$$

where $\widehat{T}_n$ and $\widehat{U}_n$ denote the monic polynomials corresponding to T_n and U_n,

$$\widehat{T}_n(x) := 2^{1-n}T_n(x) \;, \quad \widehat{U}_n(x) := 2^{-n}U_n(x) \;, \quad n \in \mathbb{N} \;, \tag{4.4}$$

and so one readily verifies that the hypothesis of Theorem 2.1 are fulfilled, with $m = k-1$ and being the polynomial mapping described by the polynomials

$$\pi_k(x) := \widehat{U}_k(x) - \tfrac{1}{4}\widehat{U}_{k-2}(x) = \widehat{T}_k(x) \,, \quad \theta_{k-1}(x) := \widehat{U}_{k-1}(x) \,, \quad \eta_0(x) := 1 \,. \tag{4.5}$$

Moreover, $\{q_n\}_{n\geq 0}$ is the monic OPS characterized by

$$r_0 = r_n = 0 \,, \quad s_n = 4^{2-k}\, a^{(0)}_n a^{(k-1)}_n = \frac{1}{4^k}\frac{n(n+1+2\lambda)}{(n+\lambda)(n+1+\lambda)} \quad (n \in \mathbb{N}) \;,$$

meaning that, indeed, q_n is up to an affine change of variables the ultraspherical polynomial of degree n with parameter $\lambda+1$,

$$q_n(x) = \frac{n!}{2^{kn}(\lambda+1)_n}\, C_n^{\lambda+1}\left(2^{k-1}x\right) \ . \tag{4.6}$$

Thus (4.1) and (4.2) follow immediately from Theorem 2.1. For $\lambda > -1/2$ the orthogonality measure for $\{B_n^\lambda(\cdot;k)\}_{n\geq 0}$ given in [1] may be computed easily using Theorem 2.2, being absolutely continuous with weight function

$$w(x) := (1-x^2)^{\lambda+\frac{1}{2}}\left|U_{k-1}(x)\right|^{2\lambda}, \quad -1<x<1 \ .$$

Indeed, in this situation, the masses at the zeros of $\theta_m \equiv \widehat{U}_{k-1}$ given by (2.13) all vanish, and so the measure given by (2.12) becomes absolutely continuous, with a density function given by Remark 2.2. For details, see [10, Section 5.2].

4.2. Classification. According to a result by Bustoz, Ismail, and Wimp [2], $B_n^\lambda(\cdot;k)$ is a solution of a linear second order ODE with polynomial coefficients, being the degrees of these polynomials (uniformly) bounded by a number independent of n. Therefore, $\{B_n^\lambda(\cdot;k)\}_{n\geq 0}$ is a semiclassical OPS. In the next theorem we state the semiclassical character of $\{B_n^\lambda(\cdot;k)\}_{n\geq 0}$ in an alternative way and we give its (precise) class. It is worth mentioning that usually the ODE is not the most efficient way to obtain the class of a semiclassical OPS. Often, being $\mathbf{u}$ the regular functional for the given (semiclassical) OPS, the differential equation fulfilled by the corresponding (formal) Stieltjes series $S_{\mathbf{u}}(z) := -\sum_{n\geq 0} u_n/z^{n+1}$ allow us to obtain the class in a more simpler way. In the next theorem we determine the class of $\{B_n^\lambda(\cdot;k)\}_{n\geq 0}$ using the associated Stieltjes series and the results stated in [3, Section 3].

Theorem 4.1. *Let $\{p_n\}_{n\geq 0}$ be the monic OPS corresponding to the sieved polynomials $\{B_n^\lambda(\cdot;k)\}_{n\geq 0}$ given by (4.3), being $\lambda \in \mathbb{C}\setminus\{-n/2 \,:\, n\in\mathbb{N}\}$ and and $k\geq 3$. Let $\mathbf{u}$ be the regular functional with respect to which $\{p_n\}_{n\geq 0}$ is an OPS. Then*

$$D(\Phi\mathbf{u}) = \Psi\mathbf{u} \ , \tag{4.7}$$

where Φ and Ψ are polynomials given by

$$\Phi(x) := (1-x^2)\widehat{U}_{k-1}(x) \ , \quad \Psi(x) := -\big(2x\widehat{U}_{k-1}(x) + k(2\lambda+1)\widehat{T}_k(x)\big) \ . \tag{4.8}$$

Moreover, the corresponding formal Stieltjes series $S_{\mathbf{u}}(z)$ fulfils

$$\Phi(z)S'_{\mathbf{u}}(z)=C(z)S_{\mathbf{u}}(z)+D(z)\;, \tag{4.9}$$

where C and D are polynomials given by

$$C(z):=-\big(z\widehat{U}_{k-1}(z)+2k\lambda\widehat{T}_k(z)\big)\;,\quad D(z):=-2u_0\big(\widehat{U}_{k-1}(z)+k\lambda\widehat{T}_{k-1}(z)\big)\;. \tag{4.10}$$

As a consequence, if $\lambda\in\mathbb{C}\setminus\{-n/2\,:\,n\in\mathbb{N}_0\}$ then $\{B_n^\lambda(\cdot;k)\}_{n\geq0}$ is a semiclassical OPS of class $k-1$. If $\lambda=0$ then $\{B_n^0(\cdot;k)\}_{n\geq0}$ is (up to normalization) the Chebychev OPS of the second kind, and so a classical OPS.

Proof: Let $\mathbf{v}^{\lambda+1}$ be the regular functional associated with the ultraspherical OPS $\{C_n^{\lambda+1}\}_{n\geq0}$, and let $\mathbf{v}$ be the regular functional associated with $\{q_n\}_{n\geq0}$ defined by (4.6). The relation between the corresponding formal Stieltjes series $S_{\mathbf{v}}(z):=\sum_{n\geq0}v_n/z^{n+1}$ and $S_{\mathbf{v}^{\lambda+1}}(z):=\sum_{n\geq0}v_n^{\lambda+1}/z^{n+1}$ (where $v_n:=\langle\mathbf{v},x^n\rangle$ and $v_n^{\lambda+1}:=\langle\mathbf{v}^{\lambda+1},x^n\rangle$, $n\geq0$) is

$$S_{\mathbf{v}}(z)=2^{k-1}S_{\mathbf{v}^{\lambda+1}}\left(2^{k-1}z\right)\;.$$

Therefore, using the formal ordinary differential equation fulfilled by $S_{\mathbf{v}^{\lambda+1}}$ (cf. e.g. [15], or see [3, Eq. (2.4) and Table 2]), we easily deduce

$$\widetilde{\Phi}(z)S'_{\mathbf{v}}(z)=\widetilde{C}(z)S_{\mathbf{v}}(z)+\widetilde{D}(z)\,, \tag{4.11}$$

where $\widetilde{\Phi}(x):=-x^2+4^{1-k}$, $\widetilde{C}(x):=-(2\lambda+1)x$, and $\widetilde{D}(x):=-2(\lambda+1)v_0$. Our aim is to prove that $\mathbf{u}$ is semiclassical of class $k-1$. Indeed, by Theorem 2.3,

$$\Phi_1(z)S'_{\mathbf{u}}(z)=C_1(z)S_{\mathbf{u}}(z)+D_1(z)\;, \tag{4.12}$$

with Φ_1, C_1, and D_1 given by

$$\begin{aligned}
&\Phi_1(x):=v_0\theta_{k-1}(x)\widetilde{\Phi}(\pi_k(x))\;,\\
&C_1(x):=-v_0\theta'_{k-1}(x)\widetilde{\Phi}(\pi_k(x))+v_0\theta_{k-1}(x)\pi'_k(x)\widetilde{C}(\pi_k(x))\\
&D_1(x):=-u_0v_0\Delta'_0(2,k-2,x)\widetilde{\Phi}(\pi_k(x))\\
&\qquad\qquad+u_0\pi'_k(x)\left(\big(\textstyle\prod_{j=1}^{k-1}a_0^{(j)}\big)\widetilde{D}(\pi_k(x))+v_0\Delta_0(2,k-2,x)\widetilde{C}(\pi_k(x))\right)\;.
\end{aligned}$$

Now, by (4.5) and using the elementary relations

$$\begin{aligned}
&\widehat{T}_n^2(x)+(1-x^2)\widehat{U}_{n-1}^2(x)=4^{1-n}\,, \quad \widehat{U}_n^2(x)-\widehat{U}_{n-1}(x)\widehat{U}_{n+1}(x)=4^{-n}\,,\\
&x\widehat{U}_n(x)-(1-x^2)\widehat{U}_n'(x)=(n+1)\widehat{T}_{n+1}(x)\,, \quad \widehat{T}_n'(x)=n\widehat{U}_{n-1}(x)\,,\\
&\widehat{T}_n(x)+x\widehat{U}_{n-1}(x)=2\widehat{U}_n(x)\,,
\end{aligned} \tag{4.13}$$

after straightforward computations we deduce

$$\begin{aligned}
&\Phi_1(x)=(1-x^2)\widehat{U}_{k-1}^3(x)\,, \quad C_1(x)=-\widehat{U}_{k-1}^2(x)\big(x\widehat{U}_{k-1}(x)+2k\lambda\widehat{T}_k(x)\big)\,,\\
&D_1(x)=-2u_0\widehat{U}_{k-1}^2(x)\big(\widehat{U}_{k-1}(x)+k\lambda\widehat{T}_{k-1}(x)\big)\,.
\end{aligned}$$

Therefore, canceling the common factor $\widehat{U}_{k-2}^2(x)$, we find that $S_{\mathbf{u}}$ satisfies (4.9), where Φ, C, and D given as in (4.8) and (4.10). Since $\widehat{U}_{k-1}(\pm1)=k(\pm1)^{k-1}$, $\widehat{T}_k(\pm1)=(\pm1)^k$, and taking into account that $\widehat{U}_{k-1}$ does not share zeros with $\widehat{T}_k$, we see that if $\lambda\neq0$ then the polynomials Φ, C, and D are co-prime, hence the class of $\mathbf{u}$ is equal to $s=\max\{\deg C-1,\deg D\}=k-1$. It is clear that $\mathbf{u}$ satisfies (4.7), taking into account that $\Psi(x)=C(x)+\Phi'(x)$. If $\lambda=0$, then $\widehat{U}_{k-1}(x)$ is a common factor of the polynomials Φ, C, and D in (4.10), hence canceling this factor we see that $\mathbf{u}$ is a classical functional, and so we see that $\{p_n\}_{n\geq0}$ is (up to normalization) the Chebychev OPS of the second kind. ∎

Remarks 4.1. *Some authors define semiclassical functional requiring the pair* (Φ,Ψ) *appearing in the corresponding Pearson's equation to be an admissible pair, meaning that, whenever* $\deg\Phi=1+\deg\Psi$ *the leading coefficient of* Ψ *cannot be a negative integer multiple of the leading coefficient of* Φ. *Medem* [16] *gave an example of a semiclassical functional and a corresponding pair* (Φ,Ψ) *which is not admissible. The above Theorem 4.1 shows that such a situation is not an isolated phenomenon. Indeed, choose* $n_0\in\mathbb{N}$ *such that* n_0+2 *is different from an integer multiple of* k, *and define*

$$\lambda:=-\frac{n_0+2+k}{2k}\,.$$

Then, the functional $\mathbf{u}$ *fulfilling (4.7) is semiclassical (and so* $\mathbf{u}$ *is regular), although the corresponding pair* (Φ,Ψ) *given by (4.8) is not admissible. We recall, however, that for a classical functional the admissibility condition holds necessarily, a fact known as early as the work of Geronimus* [12].

4.3. Structure relation and second order linear ODE. In this section we will give explicitly the structure relation and the second order linear ODE fulfilled by the monic sieved OPS of the second kind, given by (4.3), so that

$$p_n(x) = \nu_n B_n^{\lambda}(x;k)\,, \quad \nu_n := \lfloor n/k\rfloor!/\big\{2^n(\lambda+1)_{\lfloor n/k\rfloor}\big\}\,,$$

for each $n\in\mathbb{N}_0$, recovering in an alternative way —in the framework of the theory of semiclassical OP— the results given in [2]. In what follows next we determine explicitly M_n and N_n for the sieved OP.

Theorem 4.2. *The monic sieved OPS of the second kind $p_n(x)=\nu_n B_n^{\lambda}(x;k)$ satisfies the structure relation* (2.14)*, where*

$$\begin{aligned}
\Phi(x) &= (1-x^2)\widehat{U}_{k-1}(x)\,,\\
M_{nk+j}(x) &= -2(nk+j+1+\lambda k)\widehat{U}_{k-1}(x)\\
&\quad -\tfrac{\lambda k}{2}\big(\widehat{U}_{j-1}(x)\widehat{U}_{k-j-2}(x)-\widehat{U}_{j}(x)\widehat{U}_{k-j-3}(x)\big)\,,\\
N_{nk+j}(x) &= (nk+j+2+2\lambda k)x\widehat{U}_{k-1}(x)-\lambda k\epsilon_j\widehat{U}_{k-2}(x)\\
&\quad +\tfrac{\lambda k}{8}\big(\widehat{U}_{j-1}(x)\widehat{U}_{k-j-3}(x)-\widehat{U}_{j}(x)\widehat{U}_{k-j-4}(x)\big)
\end{aligned}\tag{4.14}$$

for every $n=0,1,2,\dots$ and $0\le j\le k-1$, being $\epsilon_{k-1}:=1$, $\epsilon_{k-2}:=0$, and $\epsilon_j:=\frac12$ for $0\le j\le k-3$.

Proof: Making $m=k-1$ in (2.11) and taking into account (4.5), we obtain

$$p_{kn+j}(x)=\widehat{U}_j(x)q_n(\widehat{T}_k(x))+4^{-j}a_n^{(0)}\widehat{U}_{k-j-2}(x)q_{n-1}(\widehat{T}_k(x)) \tag{4.15}$$

Taking derivatives in both sides of (4.15), we obtain

$$\begin{aligned}
p'_{kn+j}(x) &= \widehat{U}'_j(x)q_n(\widehat{T}_k(x))+\mathcal{A}_j(x)q'_n(\widehat{T}_k(x))\\
&\quad +4^{-j}a_n^{(0)}\widehat{U}'_{k-j-2}(x)q_{n-1}(\widehat{T}_k(x))+\mathcal{B}_j(x)q'_{n-1}(\widehat{T}_k(x))\,,
\end{aligned}\tag{4.16}$$

where $\mathcal{A}_j$ and $\mathcal{B}_j$ are polynomials defined by

$$\mathcal{A}_j(x):=\widehat{U}_j(x)\widehat{T}'_k(x)\,,\quad \mathcal{B}_j(x):=4^{-j}a_n^{(0)}\widehat{U}_{k-j-2}(x)\widehat{T}'_k(x)\,.$$

Multiplying both sides of (4.16) by $\widehat{U}_{k-j-2}(x)$ and using (4.15), we deduce

$$\begin{aligned}
&\widehat{U}_{k-j-2}(x)\Big(\mathcal{A}_j(x)q'_n(\widehat{T}_k(x))+\mathcal{B}_j(x)q'_{n-1}(\widehat{T}_k(x))\Big)\\
&\quad=\widehat{U}_{k-j-2}(x)p'_{nk+j}(x)-\widehat{U}'_{k-j-2}(x)p_{nk+j}(x)\\
&\qquad+\Big(\widehat{U}'_{k-j-2}(x)U_j(x)-\widehat{U}_{k-j-2}(x)U'_j(x)\Big)\,q_n(\widehat{T}_k(x))\,.
\end{aligned}\tag{4.17}$$

Now, since $\{q_n\}_{n\geq 0}$ is a classical OPS, it fulfills the structure relation (see e.g. [15])

$$\tilde{\Phi}(x)q_n'(x) = \tilde{M}_n(x)q_{n+1}(x) + \tilde{N}_n(x)q_n(x) \ , \tag{4.18}$$

being $\tilde{\Phi}(x) = 4^{1-k} - x^2$, $\tilde{N}_n(x) = (n+2\lambda+2)x$, and $\tilde{M}_n(x) = -2(\lambda+n+1)$. Replacing x by $\widehat{T}_k(x)$ in (4.18), and then multiplying both sides of the resulting equation by $\mathcal{A}_j(x)\widehat{U}_{k-1}(x)\widehat{U}_{k-j-2}(x)$, one obtains a first equation. Similarly, substituting x by $\widehat{T}_k(x)$ in (4.18), and then changing n into $n-1$ and multiplying both sides of the resulting equation by $\mathcal{B}_j(x)\widehat{U}_{k-1}(x)\widehat{U}_{k-j-2}(x)$, we obtain a second equation. Adding these two equations and using (4.15) and (4.17), we deduce

$$\mathscr{L}_1(x)p_{nk+j}'(x) = \mathscr{L}_2(x)p_{nk+j}(x) + \mathscr{L}_3(x)p_{nk+k-1}(x) + \mathscr{L}_4(x)p_{(n+1)k+k-1}(x) \ , \tag{4.19}$$

where $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, and $\mathscr{L}_4$ are polynomials defined by

$$\begin{aligned}
\mathscr{L}_1(x) &:= \widehat{U}_{k-j-2}(x)\widehat{U}_{k-1}(x)\tilde{\Phi}\big(\widehat{T}_k(x)\big) \ , \\
\mathscr{L}_2(x) &:= \widehat{U}_{k-j-2}'(x)\widehat{U}_{k-1}(x)\tilde{\Phi}\big(\widehat{T}_k(x)\big) + \widehat{U}_{k-j-2}(x)\widehat{U}_{k-1}(x)\widehat{T}_k'(x)\tilde{N}_{n-1}\big(\widehat{T}_k(x)\big) \ , \\
\mathscr{L}_3(x) &:= \Big(\widehat{U}_{k-j-2}(x)\widehat{U}_j'(x) - \widehat{U}_{k-j-2}'(x)\widehat{U}_j(x)\Big)\,\tilde{\Phi}\big(\widehat{T}_k(x)\big) \\
&\qquad + \widehat{U}_{k-j-2}(x)\,\Big(\mathcal{A}_j(x)\tilde{N}_n\big(\widehat{T}_k(x)\big) + \mathcal{B}_j(x)\tilde{M}_{n-1}\big(\widehat{T}_k(x)\big)\Big) \\
&\qquad - \widehat{U}_{k-j-2}(x)\widehat{U}_j(x)\widehat{T}_k'(x)\tilde{N}_{n-1}\big(\widehat{T}_k(x)\big) \ , \\
\mathscr{L}_4(x) &:= \mathcal{A}_j(x)\widehat{U}_{k-j-2}(x)\tilde{M}_n\big(\widehat{T}_k(x)\big) \ .
\end{aligned} \tag{4.20}$$

Taking into account the three-term recurrence relation for $\{p_n\}_{n\geq 0}$, we deduce

$$\begin{aligned}
p_{(n+1)k+k-1}(x) =& \big(x\widehat{U}_{k-1}(x) - a_{n+1}^{(0)}\widehat{U}_{k-2}(x)\big)p_{nk+k-1}(x) \\
&- a_n^{(k-1)}\widehat{U}_{k-1}(x)p_{nk+k-2}(x) \ , \\
p_{nk+k-i}(x) =& \widehat{U}_{k-j-i-1}(x)p_{nk+j+1}(x) - \frac{1}{4}\widehat{U}_{k-j-i-2}(x)p_{nk+j}(x)
\end{aligned} \tag{4.21}$$

for every $n \in \mathbb{N}_0$ and $0 \leq j \leq k-i-2$, $i = 1, 2$. Substituting (4.21) in (4.19), we obtain

$$\mathscr{L}_1(x)p_{nk+j}'(x) = N_{nk+j}(x)p_{nk+j}(x) + M_{nk+j}(x)p_{nk+j+1}(x)$$

for every $n=0,1,2,\dots$ and $0\le j\le k-4$, where

$$\begin{aligned}M_{nk+j}(x) &:= \mathcal{H}_1(x)\widehat{U}_{k-j-2}(x)-\mathcal{H}_2(x)\widehat{U}_{k-j-3}(x)\\ N_{nk+j}(x) &:= \mathscr{L}_2(x)-\tfrac{1}{4}\mathcal{H}_1(x)\widehat{U}_{k-j-3}(x)+\tfrac{1}{4}\mathcal{H}_2(x)\widehat{U}_{k-j-4}(x)\,,\end{aligned}\tag{4.22}$$

being

$$\begin{aligned}\mathcal{H}_1(x) &:= \mathscr{L}_3(x)+\mathscr{L}_4(x)\big(x\widehat{U}_{k-1}(x)-a_{n+1}^{(0)}\widehat{U}_{k-2}(x)\big)\,,\\ \mathcal{H}_2(x) &:= \mathscr{L}_4(x)a_n^{(k-1)}\widehat{U}_{k-1}(x)\,.\end{aligned}$$

Using some basic properties of Chebyshev polynomials we may verify that, up to the factor $\frac{1}{k}\widehat{T}_k'(x)\widehat{U}_{k-1}(x)\widehat{U}_{k-j-2}(x)$, the relations

$$\begin{aligned}\mathscr{L}_1(x) &= (1-x^2)\widehat{U}_{k-1}(x)\\ M_{nk+j}(x) &= -2(nk+j+1+\lambda k)\widehat{U}_{k-1}(x)\\ &\quad -\tfrac{\lambda k}{2}\left(\widehat{U}_{j-1}(x)\widehat{U}_{k-j-2}(x)-\widehat{U}_{j}(x)\widehat{U}_{k-j-3}(x)\right)\\ N_{nk+j}(x) &= (nk+j+2+2\lambda k)x\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{2}\widehat{U}_{k-2}(x)\\ &\quad +\tfrac{\lambda k}{8}\left(\widehat{U}_{j-1}(x)\widehat{U}_{k-j-3}(x)-\widehat{U}_{j}(x)\widehat{U}_{k-j-4}(x)\right)\end{aligned}\tag{4.23}$$

hold for every $n\in\mathbb{N}_0$ and $0\le j\le k-4$. Moreover, when $j=k-1$, using the relation $p_{nk+k-1}(x)=\widehat{U}_{k-1}(x)q_n(\widehat{T}_k(x))$ we may write

$$p_{nk+k-1}'(x)=\widehat{U}_{k-1}'(x)q_n(\widehat{T}_k(x))+\widehat{U}_{k-1}(x)\widehat{T}_k'(x)q_n'(\widehat{T}_k(x))\,.\tag{4.24}$$

Multiplying both sides of (4.18) by $\widehat{U}_{k-1}^2(x)\widehat{T}_k'(x)$ and taking into account (4.24) and (4.21), we obtain, up to the factor $\frac{1}{k}\widehat{T}_k'(x)\widehat{U}_{k-1}(x)$,

$$\mathscr{L}_1(x)p_{kn+k-1}'(x)=N_{nk+k-1}(x)p_{nk+k-1}(x)+M_{nk+k-1}(x)p_{nk+k}(x)\ ,$$

where

$$\begin{aligned}M_{nk+k-1}(x) &:= -2k(\lambda+n+1)\widehat{U}_{k-1}(x)\ ,\\ N_{nk+k-1}(x) &:= (nk+k+2\lambda k+1)x\widehat{U}_{k-1}(x)-\lambda k\widehat{U}_{k-2}(x)\,.\end{aligned}\tag{4.25}$$

Taking into account (2.15), (4.10), and (4.25), and using again some basic properties of the Chebyshev polynomials, we deduce

$$\begin{aligned}N_{nk+k-2}(x) &= -\,N_{nk+k-1}(x)-xM_{nk+k-1}(x)-C(x)\\ &= k(n+1+2\lambda)x\widehat{U}_{k-1}(x)\,.\end{aligned}\tag{4.26}$$

Combining relations (2.15) and taking into account (4.10), we deduce

$$\begin{aligned}&\big(x^2-\tfrac14\big)\,N_{nk+k-3}(x)\\&=x\Big(-\Phi(x)+\tfrac14 M_{nk+k-4}(x)+xN_{nk+k-4}(x)\Big)+\tfrac14\left(C(x)+N_{nk+k-2}(x)\right)\\&=\big(x^2-\tfrac14\big)\Big((k-1-nk)x\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{2}\big(2\widehat{U}_{k-2}(x)-4x\widehat{U}_{k-1}(x)-x\widehat{U}_{k-3}(x)\big)\Big)\,,\end{aligned}$$

so that

$$N_{nk+k-3}(x)=\big(nk+k-1+2\lambda k\big)x\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{2}\widehat{U}_{k-2}(x)+\tfrac{\lambda k}{8}\widehat{U}_{k-4}(x)\,. \quad (4.27)$$

Finally, using (2.15), (4.10), (4.26), and (4.27), we obtain

$$\begin{aligned}xM_{nk+k-2}(x)&=-N_{nk+k-3}(x)-N_{nk+k-2}(x)-C(x)\\&=-2(nk+k-1+\lambda k)x\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{2}x\widehat{U}_{k-3}(x)\\xM_{nk+k-3}(x)&=-N_{nk+k-4}(x)-N_{nk+k-3}(x)-C(x)\\&=-2(nk+k-2+\lambda k)x\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{8}x\widehat{U}_{k-5}(x)\,,\end{aligned}$$

hence

$$\begin{aligned}M_{nk+k-2}(x)&=-2(nk+k-1+\lambda k)\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{2}\widehat{U}_{k-3}(x)\\M_{nk+k-3}(x)&=-2(nk+k-2+\lambda k)\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{8}\widehat{U}_{k-5}(x)\,.\end{aligned} \quad (4.28)$$

Thus the proof is complete. ■

Remarks 4.2. *We can give alternative expressions for the polynomials M_n and N_n appearing in (4.14). Indeed, since*

$$\widehat{U}_n(x)\widehat{U}_m(x)-\widehat{U}_{n-1}(x)\widehat{U}_{m+1}(x)=\begin{cases}4^{-n}\widehat{U}_{m-n}(x) & \text{if } \ 0\le n\le m\,;\\ -4^{-m-1}\widehat{U}_{n-m-2}(x) & \text{if } \ 0\le m<n\,,\end{cases} \quad (4.29)$$

we may write

$$M_{nk+j}(x)=-2(nk+j+1+\lambda k\delta_j)\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{2}U_{k,j}(x)\,,$$

$$N_{nk+j}(x)=(nk+j+2+2\lambda k\delta_j)x\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{2}\widehat{U}_{k-2}(x)+2\lambda kV_{k,j}(x)\,,$$

where $\delta_j:=1$ if $0\le j\le k-2$, $\delta_{k-1}:=0$, and $U_{k,j}$ and $V_{k,j}$ are polynomials defined by

$$U_{k,j}(x):=\begin{cases}-4^{-j}\widehat{U}_{k-3-2j}(x) & \text{if } \ j=0,1,\dots,\lfloor\frac{k-3}{2}\rfloor\\ 4^{-k+j+2}\widehat{U}_{2j-k+1}(x) & \text{if } \ j=1+\lfloor\frac{k-3}{2}\rfloor,\dots,k-1\,,\end{cases}$$

$$V_{k,j}(x) := \begin{cases} -4^{-j-2}\widehat{U}_{k-4-2j}(x) & \text{if} \quad j = 0, 1, \dots, \lfloor \frac{k-4}{2} \rfloor \\ 4^{-k+j+1}\widehat{U}_{2j-k+2}(x) & \text{if} \quad j = 1 + \lfloor \frac{k-4}{2} \rfloor, \dots, k-1\,. \end{cases}$$

Remarks 4.3. *Theorem 4.2 allows us to recover Theorem 3.1 in* [2]*. Indeed, taking into account the three-term recurrence relation for* $\{p_n\}_{n\geq 0}$*, as well as (4.4) and the first identity in (4.13), setting* $y_n(x) := B_n^\lambda(x;k)$*, we obtain*

$$\big(1 - T_k^2(x)\big)\, y_n'(x) = g_n(x)y_{n-1}(x) + h_n(x)y_n(x)\ ,$$

where g_n *and* h_n *are polynomials defined by*

$$\begin{aligned} g_{nk+j}(x) &:= U_{k-1}(x)\left\{(nk+j+1+\lambda k)U_{k-1}(x) + \lambda k\mathscr{U}_{k,j}(x)\right\}\ , \\ h_{nk+j}(x) &:= -U_{k-1}(x)\left\{(nk+j)xU_{k-1}(x) + \lambda kU_{k-2}(x) + \lambda k\mathscr{W}_{k,j}(x)\right\} \end{aligned}$$

for every $n \in \mathbb{N}_0$ *and* $0 \leq j \leq k-1$*, being* $\mathscr{U}_{k,j}$ *and* $\mathscr{W}_{k,j}$ *polynomials defined by*

$$\mathscr{U}_{k,j}(x) := \begin{cases} -U_{k-2j-3}(x) & \text{if} \quad j = 0, 1, \dots, \lfloor \frac{k-3}{2} \rfloor \\ U_{2j-k+1}(x) & \text{if} \quad j = 1 + \lfloor \frac{k-3}{2} \rfloor, \dots, k-1\,, \end{cases}$$

$$\mathscr{W}_{k,j}(x) := \begin{cases} -U_{k-2j-2}(x) & \text{if} \quad j = 0, 1, \dots, \lfloor \frac{k-2}{2} \rfloor \\ U_{2j-k}(x) & \text{if} \quad j = 1 + \lfloor \frac{k-2}{2} \rfloor, \dots, k-1\,. \end{cases}$$

The second order linear ODE fulfilled by the sieved OPS of the second kind follows now easily.

Theorem 4.3. *The monic sieved OPS of the second kind* $p_n(x) = \nu_n B_n^\lambda(x;k)$ *satisfies the second order ODE* (2.16)*, where*

$$\begin{aligned} J_{nk+j}(x) &= \Phi(x)M_{nk+j}(x)\ , \\ K_{nk+j}(x) &= \Psi(x)M_{nk+j}(x) - \Phi(x)M_{nk+j}'(x)\ , \\ L_{nk+j}(x) &= N_{nk+j}(x)M_{nk+j}'(x) + \big(\Omega_j(x) - N_{nk+j}'(x)\big)M_{nk+j}(x) \end{aligned} \tag{4.30}$$

for all $n \geq 1$ *and* $0 \leq j \leq k-1$*, being* M_{nk+j} *and* N_{nk+j} *given by* (4.14)*, and*

$$\begin{aligned} &\Phi(x) := (1-x^2)\widehat{U}_{k-1}(x)\,, \quad \Psi(x) := -\big(2x\widehat{U}_{k-1}(x) + k(2\lambda+1)\widehat{T}_k(x)\big)\ , \\ &\Omega_j(x) = (nk+j+1)(nk+j+2+2\lambda k)\widehat{U}_{k-1}(x) - \tfrac{\lambda k}{2}\widehat{U}_j(x)\widehat{U}_{k-j-3}(x)\,. \end{aligned}$$

Proof: The first two equalities in (4.30) follow immediately from (2.17). To prove the third equality in (4.30), we only need to take into account the third equality in (2.17) and noticing that, using basic properties of the Chebyshev polynomials, as well as the relations $\widehat{U}_m^2(x) - \widehat{U}_{m+1}(x)\widehat{U}_{m-1}(x) = 4^{-m}$ ($m = 0, 1, 2, \ldots$), the equality

$$\frac{a_n^{(j+1)} M_{nk+j}(x) M_{nk+j+1}(x) - N_{nk+j}(x)\big(N_{nk+j}(x) + C(x)\big)}{\Phi(x)} = \Omega_j(x)$$

holds for every $n \in \mathbb{N}_0$ and $0 \le j \le k-1$. ■

It is worth mentioning that a misprint appeared in the ODE given in [2, Theorem 3.2], as Professor Bustoz kindly commented to the third author of the present work during a visited to the Arizona State University at the 1990's.

5. On sieved ultraspherical OP of the first kind

5.1. Description via a polynomial mapping. Taking for $\{p_n\}_{n\ge0}$ the monic OPS corresponding to $\{c_n^\lambda(\cdot;k)\}_{n\ge0}$, so that

$$p_{kn+j+1}(x) = \frac{(1+2\lambda)_n}{2^{kn+j}(\lambda+1)_n} c_{kn+j+1}^\lambda(x;k) \tag{5.1}$$

($n = 0, 1, 2, \ldots$; $j = 0, 1, \ldots, k-1$), and using the three-term recurrence relation for $\{c_n^\lambda(\cdot;k)\}_{n\ge0}$ given in [1], we see that the coefficients appearing in the (block) three-term recurrence relation (2.2) for $\{p_n\}_{n\ge0}$ are given by

$$b_n^{(j)} := 0 \quad (0 \le j \le k-1)\,, \quad a_n^{(j)} := \tfrac{1}{4} \quad (2 \le j \le k-1)\,,$$
$$a_n^{(0)} := \frac{n}{4(n+\lambda)}\,, \quad a_n^{(1)} := \frac{n+2\lambda}{4(n+\lambda)}$$

for each $n \in \mathbb{N}_0$. Hence, for every $n \in \mathbb{N}_0$ and $0 \le j \le k-1$, we compute

$$\Delta_n(2,j;x) = \widehat{U}_j(x)\,, \quad \Delta_n(j+3,k-1;x) = \widehat{U}_{k-j-2}(x)\,,$$

and so one sees that the hypothesis of Theorem 2.1 are fulfilled, with $m = 0$ and being the polynomial mapping described by the polynomials

$$\pi_k(x) := \widehat{U}_k(x) - \tfrac{1}{4}\widehat{U}_{k-2}(x) = \widehat{T}_k(x)\,, \quad \eta_{k-1}(x) := \widehat{U}_{k-1}(x)\,, \quad \theta_0(x) \equiv 1\,. \tag{5.2}$$

Moreover, $\{q_n\}_{n\ge0}$ is the monic OPS characterized by

$$r_0 = r_n = 0\,, \quad s_n = 4^{2-k} a_n^{(0)} a_{n-1}^{(1)} = \frac{1}{4^k}\frac{n(n-1+2\lambda)}{(n+\lambda)(n-1+\lambda)} \quad (n \in \mathbb{N})\,,$$

meaning that q_n is up to an affine change of variables the ultraspherical polynomial of degree n with parameter λ:

$$q_n(x) = \frac{n!}{2^{kn}(\lambda)_n} C_n^{\lambda}\left(2^{k-1}x\right) . \tag{5.3}$$

For $\lambda > -1/2$, the orthogonality measure for $\{c_n^{\lambda}(\cdot;k)\}_{n\geq 0}$—given in [1]—may be computed using Theorem 2.2, being absolutely continuous with weight function

$$w(x) := (1-x^2)^{\lambda-\frac{1}{2}}\left|U_{k-1}(x)\right|^{2\lambda}, \quad -1 \leq x \leq 1 .$$

5.2. Classification.

Theorem 5.1. *Let $\{p_n\}_{n\geq 0}$ be the monic OPS corresponding to the sieved polynomials $\{c_n^{\lambda}(\cdot;k)\}_{n\geq 0}$, given by (5.1), being $\lambda \in \mathbb{C}\setminus\{-n/2 : n \in \mathbb{N}\}$ and $k \geq 3$. Let $\mathbf{u}$ be the regular functional with respect to which $\{p_n\}_{n\geq 0}$ is an OPS. Then*

$$D(\Phi\mathbf{u}) = \Psi\mathbf{u} , \tag{5.4}$$

where Φ and Ψ are polynomials given by

$$\Phi(x) := (1-x^2)\widehat{U}_{k-1}(x) , \quad \Psi(x) := -k(2\lambda+1)\widehat{T}_k(x) . \tag{5.5}$$

Moreover, the corresponding formal Stieltjes series $S_{\mathbf{u}}(z)$ fulfils

$$\Phi(z)S'_{\mathbf{u}}(z) = C(z)S_{\mathbf{u}}(z) + D(z) , \tag{5.6}$$

where C and D are polynomials given by

$$C(z) := z\widehat{U}_{k-1}(z) - 2k\lambda\widehat{T}_k(z) , \quad D(z) := -2k\lambda u_0\widehat{U}_{k-1}(z) . \tag{5.7}$$

As a consequence, if $\lambda \in \mathbb{C}\setminus\{-n/2 : n \in \mathbb{N}_0\}$ then $\{c_n^{\lambda}(\cdot;k)\}_{n\geq 0}$ is a semiclassical OPS of class $k-1$. If $\lambda = 0$ then $\{c_n^{0}(\cdot;k)\}_{n\geq 0}$ is (up to normalization) the Chebychev OPS of the first kind, hence it is a classical OPS.

Proof: The case $\lambda = 0$ is trivial, so we will assume $\lambda \neq 0$. Let $\mathbf{v}^{\lambda}$ be the regular functional associated with the ultraspherical OPS $\{C_n^{\lambda}\}_{n\geq 0}$, and let $\mathbf{v}$ be the regular functional associated with $\{q_n\}_{n\geq 0}$ defined by (5.3). The relation between the corresponding formal Stieltjes series is

$$S_{\mathbf{v}}(z) = 2^{k-1}S_{\mathbf{v}^{\lambda}}\left(2^{k-1}z\right) .$$

Moreover,

$$\widetilde{\Phi}(z)S'_{\mathbf{v}}(z) = \widetilde{C}(z)S_{\mathbf{v}}(z) + \widetilde{D}(z) , \tag{5.8}$$

where $\widetilde{\Phi}(x) := -x^2 + 4^{1-k}$, $\widetilde{C}(x) := -(2\lambda-1)x$, and $\widetilde{D}(x) := -2\lambda v_0$. Hence, by Theorem 2.3,

$$\Phi_1(z)S'_{\mathbf{u}}(z) = C_1(z)S_{\mathbf{u}}(z) + D_1(z)\ , \tag{5.9}$$

where Φ_1, C_1, and D_1 are given by

$$\begin{aligned}
&\Phi_1(x) := v_0\eta_{k-1}(x)\widetilde{\Phi}(\pi_k(x))\ ,\\
&C_1(x) := v_0\eta'_{k-1}(x)\widetilde{\Phi}(\pi_k(x)) + v_0\eta_{k-1}(x)\pi'_k(x)\widetilde{C}(\pi_k(x))\\
&D_1(x) := u_0\eta^2_{k-1}(x)\pi'_k(x)\widetilde{D}(\pi_k(x))\ .
\end{aligned}$$

Now, taking into account (5.2), and using relations (4.13), after straightforward computations and canceling a common factor $\widehat{U}^2_{k-1}(x)$, we deduce

$$\Phi(z)S'_{\mathbf{u}}(z) = C(z)S_{\mathbf{u}}(z) + D(z)\ , \tag{5.10}$$

where Φ, C, and D are given by (5.5) and (5.7). Since $\widehat{U}_{k-1}(\pm1) = k(\pm1)^{k-1}$, $\widehat{T}_k(\pm1) = (\pm1)^k$, and taking into account that $\lambda \neq 0$ and $\widehat{U}_{k-1}$ does not share zeros with $\widehat{T}_k$, we see that the polynomials Φ, C, and D are co-prime, hence the class of $\mathbf{u}$ is equal to $s = \max\{\deg C - 1, \deg D\} = k-1$. ■

5.3. Structure relation and second order linear ODE. In this section we derive the structure relation and the second order linear ODE fulfilled by the monic sieved OPS of the first kind given by (5.1), so that

$$p_{n+1}(x) = \vartheta_n c^\lambda_{n+1}(x;k)\ ,\quad \vartheta_n := (2\lambda+1)_{\lfloor n/k\rfloor}/\big\{2^n(\lambda+1)_{\lfloor n/k\rfloor}\big\}$$

for each $n \in \mathbb{N}_0$, and $p_0(x) \equiv 1$.

Theorem 5.2. *The monic sieved OPS of the first kind $p_n(x) = \vartheta_{n-1}c^\lambda_n(x;k)$ satisfies the structure relation (2.14), where*

$$\begin{aligned}
\Phi(x) &= (1-x^2)\widehat{U}_{k-1}(x)\ ,\\
M_{nk+j}(x) &= -2(nk+j+\lambda k)\widehat{U}_{k-1}(x)\\
&\qquad -\tfrac{\lambda k}{2}\big(\widehat{U}_{j-1}(x)\widehat{U}_{k-j-2}(x) - \widehat{U}_{j-2}(x)\widehat{U}_{k-j-1}(x)\big)\ ,\\
N_{nk+j}(x) &= (nk+j+2\lambda k)x\widehat{U}_{k-1}(x) - \lambda k\epsilon_j\widehat{U}_{k-2}(x)\\
&\qquad +\tfrac{\lambda k}{8}\big(\widehat{U}_{j-1}(x)\widehat{U}_{k-j-3}(x) - \widehat{U}_{j-2}(x)\widehat{U}_{k-j-2}(x)\big)
\end{aligned} \tag{5.11}$$

for every $n = 0,1,2,\dots$ and $0 \leq j \leq k-1$, being $\epsilon_{k-1} := 1$, $\epsilon_0 := 0$, and $\epsilon_j := \frac{1}{2}$ for $1 \leq j \leq k-2$.

Proof: Making $m=0$ in (2.11) and taking into account (5.2), we obtain

$$p_{kn+j}(x)=\mathcal{A}_j(x)q_{n+1}(\widehat{T}_k(x))+4^{1-j}a_n^{(1)}\mathcal{B}_j(x)q_n(\widehat{T}_k(x)) \tag{5.12}$$

where $\mathcal{A}_j(x):=\widehat{U}_{j-1}(x)/\widehat{U}_{k-1}(x)$ and $\mathcal{B}_j(x):=\widehat{U}_{k-j-1}(x)/\widehat{U}_{k-1}(x)$. Taking derivatives in both sides of (5.12), we obtain

$$\begin{aligned} p'_{kn+j}(x)=&\,\mathcal{A}'_j(x)q_{n+1}(\widehat{T}_k(x))+\mathcal{C}_j(x)q'_{n+1}(\widehat{T}_k(x)) \\ &+4^{1-j}a_n^{(1)}\mathcal{B}'_j(x)q_n(\widehat{T}_k(x))+\mathcal{D}_j(x)q'_n(\widehat{T}_k(x))\,, \end{aligned} \tag{5.13}$$

where $\mathcal{C}_j(x):=\mathcal{A}_j(x)\widehat{T}'_k(x)$ and $\mathcal{D}_j(x):=4^{1-j}a_n^{(1)}\mathcal{B}_j(x)\widehat{T}'_k(x)$. Multiplying both sides of (5.13) by $\mathcal{B}_j(x)$ and using (5.12), we deduce

$$\begin{aligned} &\mathcal{B}_j(x)\left(\mathcal{C}_j(x)q'_{n+1}(\widehat{T}_k(x))+\mathcal{D}_j(x)q'_n(\widehat{T}_k(x))\right) \\ &\qquad=\mathcal{B}_j(x)p'_{nk+j}(x)-\mathcal{B}'_j(x)p_{nk+j}(x) \\ &\qquad\quad+\big(\mathcal{A}_j(x)\mathcal{B}'_j(x)-\mathcal{A}'_j(x)\mathcal{B}_j(x)\big)\,q_{n+1}(\widehat{T}_k(x))\,. \end{aligned} \tag{5.14}$$

Now, since $\{q_n\}_{n\geq 0}$ is a classical OPS, it fulfills the structure relation (see e.g. [15])

$$\tilde{\Phi}(x)q'_n(x)=\tilde{M}_n(x)q_{n+1}(x)+\tilde{N}_n(x)q_n(x)\,, \tag{5.15}$$

being $\tilde{\Phi}(x)=4^{1-k}-x^2$, $\tilde{N}_n(x)=(n+2\lambda)x$, and $\tilde{M}_n(x)=-2(\lambda+n)$. Substituting x by $\widehat{T}_k(x)$ in (5.15), and then multiplying both sides of the resulting equation by $\mathcal{B}_j(x)\mathcal{D}_j(x)$, one obtains a certain equation. Similarly, substituting x by $\widehat{T}_k(x)$ in (5.15), and then changing n into $n+1$ and multiplying both sides of the resulting equation by $\mathcal{B}_j(x)\mathcal{C}_j(x)$, we obtain a second equation. Adding these two equations and using (5.12) and (5.14), we deduce

$$\mathscr{S}_1(x)p'_{nk+j}(x)=\mathscr{S}_2(x)p_{nk+j}(x)+\mathscr{S}_3(x)p_{(n+1)k}(x)+\mathscr{S}_4(x)p_{(n+2)k}(x)\,, \tag{5.16}$$

where $\mathscr{S}_1$, $\mathscr{S}_2$, $\mathscr{S}_3$, and $\mathscr{S}_4$ are polynomials defined by

$$\begin{aligned} \mathscr{S}_1(x):=&\,\mathcal{B}_j(x)\tilde{\Phi}\big(\widehat{T}_k(x)\big)\,, \\ \mathscr{S}_2(x):=&\,\mathcal{B}'_j(x)\tilde{\Phi}\big(\widehat{T}_k(x)\big)+\mathcal{B}_j(x)\widehat{T}'_k(x)\tilde{N}_n\big(\widehat{T}_k(x)\big)\,, \\ \mathscr{S}_3(x):=&\,\big(\mathcal{A}'_j(x)\mathcal{B}_j(x)-\mathcal{A}_j(x)\mathcal{B}'_j(x)\big)\,\tilde{\Phi}\big(\widehat{T}_k(x)\big) \\ &+\mathcal{B}_j(x)\left(\mathcal{C}_j(x)\tilde{N}_{n+1}\big(\widehat{T}_k(x)\big)+\mathcal{D}_j(x)\tilde{M}_n\big(\widehat{T}_k(x)\big)\right) \\ &-\mathcal{A}_j(x)\mathcal{B}_j(x)\widehat{T}'_k(x)\tilde{N}_n\big(\widehat{T}_k(x)\big)\,, \\ \mathscr{S}_4(x):=&\,\mathcal{B}_j(x)\mathcal{C}_j(x)\tilde{M}_{n+1}\big(\widehat{T}_k(x)\big)\,. \end{aligned} \tag{5.17}$$

Taking into account the three-term recurrence relation for $\{p_n\}_{n\geq 0}$, we deduce

$$\begin{aligned} p_{(n+2)k}(x) &= \big(x\widehat{U}_{k-1}(x) - a_{n+1}^{(1)}\widehat{U}_{k-2}(x)\big)p_{(n+1)k}(x) \\ &\qquad -a_{n+1}^{(0)}\widehat{U}_{k-1}(x)p_{(n+1)k-1}(x)\,, \\ p_{nk+k-i}(x) &= \widehat{U}_{k-j-i-1}(x)p_{nk+j+1}(x) - \tfrac{1}{4}\widehat{U}_{k-j-i-2}(x)p_{nk+j}(x) \end{aligned} \tag{5.18}$$

for every $n \in \mathbb{N}_0$ and $1 \leq j \leq k-i-1$, $i = 0, 1$. Substituting (5.18) in (5.16), we obtain

$$\mathscr{S}_1(x)p'_{nk+j}(x) = N_{nk+j}(x)p_{nk+j}(x) + M_{nk+j}(x)p_{nk+j+1}(x)$$

for every $n = 0, 1, 2, \ldots$ and $1 \leq j \leq k-2$, where

$$\begin{aligned} &M_{nk+j}(x) := \mathcal{K}_1(x)\widehat{U}_{k-j-1}(x) - \mathcal{K}_2(x)\widehat{U}_{k-j-2}(x) \\ &N_{nk+j}(x) := \mathscr{S}_2(x) - \tfrac{1}{4}\mathcal{K}_1(x)\widehat{U}_{k-j-2}(x) + \tfrac{1}{4}\mathcal{K}_2(x)\widehat{U}_{k-j-3}(x)\,, \end{aligned} \tag{5.19}$$

being

$$\begin{aligned} &\mathcal{K}_1(x) := \mathscr{S}_3(x) + \mathscr{S}_4(x)\big(x\widehat{U}_{k-1}(x) - a_{n+1}^{(1)}\widehat{U}_{k-2}(x)\big)\,, \\ &\mathcal{K}_2(x) := \mathscr{S}_4(x)a_{n+1}^{(0)}\widehat{U}_{k-1}(x)\,. \end{aligned}$$

Using some basic properties of Chebyshev polynomials we may verify that, up to the factor $\widehat{U}_{k-j-1}(x)$, the relations

$$\begin{aligned} \mathscr{S}_1(x) &= (1-x^2)\widehat{U}_{k-1}(x) \\ M_{nk+j}(x) &= -2(nk+j+\lambda k)\widehat{U}_{k-1}(x) \\ &\qquad -\tfrac{\lambda k}{2}\Big(\widehat{U}_{j-1}(x)\widehat{U}_{k-j-2}(x) - \widehat{U}_{j-2}(x)\widehat{U}_{k-j-1}(x)\Big) \\ N_{nk+j}(x) &= (nk+j+2\lambda k)x\widehat{U}_{k-1}(x) - \tfrac{\lambda k}{2}\widehat{U}_{k-2}(x) \\ &\qquad +\tfrac{\lambda k}{8}\Big(\widehat{U}_{j-1}(x)\widehat{U}_{k-j-3}(x) - \widehat{U}_{j-2}(x)\widehat{U}_{k-j-2}(x)\Big) \end{aligned} \tag{5.20}$$

hold for every $n \in \mathbb{N}_0$ and $1 \leq j \leq k-2$. Moreover, when $j = 0$, then using the relation $p_{nk}(x) = q_n(\widehat{T}_k(x))$ we may write

$$p'_{nk}(x) = \widehat{T}'_k(x)q'_n(\widehat{T}_k(x))\,. \tag{5.21}$$

Substituting x by $\widehat{T}_k(x)$ in (5.15) and multiplying both sides of (5.15) by $\widehat{T}'_k(x)$ and taking into account (5.21) and (5.18), we obtain, up to the factor $\widehat{U}_{k-1}(x)$,

$$\mathscr{S}_1(x)p'_{kn}(x) = M_{nk}(x)p_{nk+1}(x) + N_{nk}(x)p_{nk}(x)\,,$$

where

$$M_{nk}(x) := -2k(\lambda+n)\widehat{U}_{k-1}(x)\,, \quad N_{nk}(x) := k(n+2\lambda)x\widehat{U}_{k-1}(x)\,. \tag{5.22}$$

Taking into account (2.15), (5.7), (5.20), and (5.22), and using again some basic properties of the Chebyshev polynomials, we deduce

$$\begin{aligned} N_{nk+k-1}(x) &= \tfrac{1}{x}\big(-\Phi(x)+\tfrac{1}{4}M_{nk+k-2}(x)-a_{n+1}^{(0)}M_{(n+1)k}(x)\big)+N_{nk+k-2}(x) \\ &= \big(nk+k-1+2\lambda k\big)x\widehat{U}_{k-1}(x)-\lambda k\widehat{U}_{k-2}(x)\,. \end{aligned} \tag{5.23}$$

Finally, taking into account (2.15), (5.7), (5.20), and (5.23), we obtain

$$\begin{aligned} xM_{nk+k-1}(x) &= -N_{nk+k-1}(x)-N_{nk+k-2}(x)-C(x) \\ &= -2(nk+k-1+\lambda k)x\widehat{U}_{k-1}(x)+\tfrac{\lambda k}{2}x\widehat{U}_{k-3}(x) \end{aligned}$$

hence

$$M_{nk+k-1}(x) = -2(nk+k-1+\lambda k)\widehat{U}_{k-1}(x)+\tfrac{\lambda k}{2}\widehat{U}_{k-3}(x)\,. \tag{5.24}$$

Thus the proof is complete. ■

Remarks 5.1. *We can give alternative expressions for the polynomials M_n and N_n appearing in (5.11). Indeed, taking into account (4.29), we may write*

$$M_{nk+j}(x) = -2(nk+j+\lambda k\delta_j)\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{2}U_{k,j}(x)\,,$$

$$N_{nk+j}(x) = (nk+j+2\lambda k)x\widehat{U}_{k-1}(x)-\tfrac{\lambda k}{2}\widehat{U}_{k-2}(x)+\tfrac{\lambda k}{2}V_{k,j}(x)\,,$$

where $\delta_j := 1$ if $1\leq j\leq k-1$, $\delta_0 := 0$, and $U_{k,j}$ and $V_{k,j}$ are polynomials defined by

$$U_{k,j}(x) := \begin{cases} 4^{1-j}\widehat{U}_{k-1-2j}(x) & \text{if } \ j=0,1,\dots,\lfloor\frac{k-1}{2}\rfloor \\ -4^{-k+j+1}\widehat{U}_{2j-k-1}(x) & \text{if } \ j=1+\lfloor\frac{k-1}{2}\rfloor,\dots,k-1\,, \end{cases}$$

$$V_{k,j}(x) := \begin{cases} 4^{-j}\widehat{U}_{k-2-2j}(x) & \text{if } \ j=0,1,\dots,\lfloor\frac{k-2}{2}\rfloor \\ -4^{-k+j+1}\widehat{U}_{2j-k}(x) & \text{if } \ j=1+\lfloor\frac{k-2}{2}\rfloor,\dots,k-1\,. \end{cases}$$

Theorem 5.3. *The monic sieved OPS of the first kind $p_n(x)=\vartheta_{n-1}c_n^{\lambda}(x;k)$ satisfies the second order ODE (2.16), where*

$$\begin{aligned} J_{nk+j}(x) &= \Phi(x)M_{nk+j}(x)\,, \\ K_{nk+j}(x) &= \Psi(x)M_{nk+j}(x)-\Phi(x)M'_{nk+j}(x)\,, \\ L_{nk+j}(x) &= N_{nk+j}(x)M'_{nk+j}(x)+\big(\Omega_j(x)-N'_{nk+j}(x)\big)M_{nk+j}(x) \end{aligned} \tag{5.25}$$

for all $n \geq 1$ *and* $0 \leq j \leq k-1$*, being* M_{nk+j} *and* N_{nk+j} *given by* (4.14)*, and*

$$\Phi(x) := (1-x^2)\widehat{U}_{k-1}(x)\,, \quad \Psi(x) := -k(2\lambda+1)\widehat{T}_k(x)\,,$$
$$\Omega_j(x) = (nk+j+1)(nk+j+2\lambda k)\widehat{U}_{k-1}(x) + \tfrac{\lambda k}{2}\widehat{U}_{j-1}(x)\widehat{U}_{k-j-2}(x)\,.$$

Proof: The first two equalities in (5.25) follow immediately from (2.17). To prove the third equality in (5.25), we only need to take into account the third equality in (2.17) and noticing that, using basic properties of the Chebyshev polynomials, as well as the relations $\widehat{U}_m^2(x) - \widehat{U}_{m+1}(x)\widehat{U}_{m-1}(x) = 4^{-m}$ ($m = 0,1,2,\ldots$), the equality

$$\frac{a_n^{(j+1)} M_{nk+j}(x)M_{nk+j+1}(x) - N_{nk+j}(x)\big(N_{nk+j}(x)+C(x)\big)}{\Phi(x)} = \Omega_j(x)$$

holds for every $n \in \mathbb{N}_0$ and $0 \leq j \leq k-1$. ■

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