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ON LINKS MINIMIZING THE TUNNEL NUMBER

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Abstract: We determine a large class of links, including satellite and hyperbolic links, for each of which the tunnel number is the minimum possible, the number of its components minus one, and observe that the rank versus genus conjecture is valid for this class of links.

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1. Introduction

Given a link L in S^3, an unknotting tunnel system for L is a collection of arcs, properly embedded in the exterior of L, with the exterior of a regular neighborhood of their union with L being a handlebody. The minimum cardinality of an unknotting tunnel system for L is referred to as the tunnel number of L and is denoted by $t(L)$. The purpose of this paper is to study the tunnel number of a large class of links, which we refer to as *band links*, and to prove that these links have the lowest possible tunnel number. That is, given a link with n components its tunnel number must be at least $n-1$, and we will show that for links in this class that minimum is attained.

We define *band links* as follows. Consider and embedding $H \times I$ in S^3, where $H \equiv H \times 0$ is a Heegaard surface of S^3. Let K be a link in $H \times I$ lying in $H \times 0$ except near crossings where the overstrand touches $H \times 1$. Suppose that the projection D_K of K onto H separates H into a collection of disks. The projection D_K is a graph with a 4-valent vertex in each crossing of K in H, where we introduce at least a 2-valent vertex in each arc of the projection. From D_K we construct a new graph D_L by replacing each 4-valent vertex of D_K by two pairs of parallel arcs crossing at a square, each 2-valent vertex of

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D_K by two arcs crossing in two points and each edge of D_K by two parallel (possibly twisted) arcs connecting to the corresponding endpoints of the arcs in the replacements of the vertices. A link L with regular projection D_L onto H is called an *n-band link over K*, where n is the number of connected components of L. In Figure 1 we have an illustration of this construction of L from D_K.

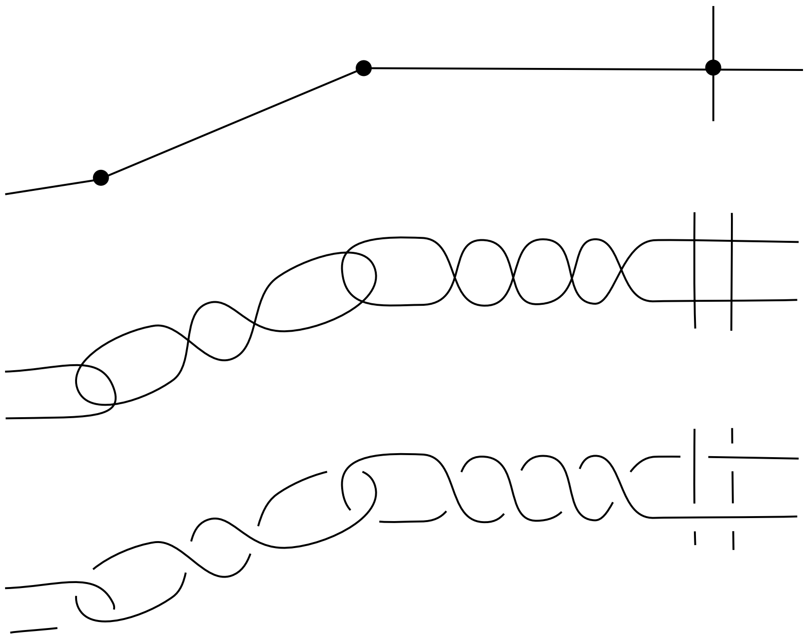

FIGURE 1. A graph D_K, a constructed graph D_L and a corresponding band link L.

Notice that all components of a band link L are unknotted. Moreover, in the projection D_L, each component intersects two other components in two crossings and at most one component in four crossings. We observe also that L can be a satellite link, with companion K, or a hyperbolic link, for instance, when D_L is alternating on the 2-sphere (from Corollary 2 of [5]).

In this paper we study the tunnel number of an n-band link L over a regular projection D_K of a generic link K, and its relation with the rank of the exterior of L, denoted $E(L)$. If the projection D_K of K is a simple circle on a sphere, it is straightforward to observe, as in Section 4, that the tunnel number of a n-band link L over K is $n-1$. In the following theorem we prove that this is also the case for every regular projection D_L of a link L.

Theorem 1.1. *The tunnel number of an n-band link exterior is $n-1$.*

If we consider the Heegaard genus $g(E(L))$ of the exterior of L, then it is well known that $g(E(L)) = t(L) + 1$. Therefore, Theorem 1.1 states that the Heegaard genus of a n-band link exterior is n.

Waldhausen [8] asked whether the rank $r(M)$ of M, that is, the minimal number of generators of $\pi_1(M)$, can be realized geometrically as the genus of a Heegaard splitting decomposing M into one handlebody and a compression body, that is if $r(M) = g(M)$, for every compact 3-manifold M. This question became to be known as the Rank versus Genus Conjecture. In [1] Boileau–Zieschang provided the first counter-examples by showing there are Seifert manifolds where the rank is strictly smaller than the Heegaard genus. Later Schultens and Weidman [7] generalized these counter-examples to graph manifolds. Very recently, Li [4] proved that the conjecture also doesn't hold true for hyperbolic 3-manifolds. As far as we know, the conjecture remains open for link exteriors in S^3. The first author [2] proved this conjecture to be true for augmented links. Theorem 1.1 shows that this is also the case for band links, as stated in the following corollary.

Corollary 1.2. *If L is an n-band link, then $r\big(E(L)\big) = g\big(E(L)\big)$.*

In fact, by the "half lives, half dies" theorem ([3], Lemma 3.5) applied to $E(L)$, we have $r\big(E(L)\big) \geq |L|$, where $|L|$ denotes the number of components of L. The corollary now follows simply from Theorem 1.1 and the observation that

$$n = |L| \leq r\big(E(L)\big) \leq g\big(E(L)\big) = n,$$

Therefore, $r\big(E(L)\big) = g\big(E(L)\big) = n$.

This paper is organized as follows. In Section 2 we describe a procedure to determine an unknotting tunnel system for links from a projection diagram. In Section 3 we present a combinatorial version of this procedure. Finally, in Section 4 we use this combinatorial procedure to find the tunnel number of band links. We use the survey [6] by Yoav Moriah as a reference for context on Heegaard decompositions of knot exteriors.

2. Minimal number of vertical tunnels

Before we proceed we introduce and review some terminology. A *stabilization* of a genus-g Heegaard surface in S^3 is a surface of genus $g + 1$ obtained by adding a trivial 1-handle, that is, a handle whose core is parallel to the surface. A *destabilization* of a Heegaard surface is a surface obtained from the reversed procedure. Note that a surface obtained by (de)stabilization of a Heegaard surface is also a Heegaard surface.

Let K be a link in S^3 with a regular projection D_K onto a Heegaard surface H of S^3, such that the complement of D_K in H is a collection of disks. Note that we always have such a property when H is a 2-sphere. We refer to these disks, together with its boundary edges and vertices, as *faces*.

We can construct an unknotting tunnel system for K from the crossings of D_K in H. In fact, for each crossing v of D_K, consider an arc in S^3 connecting the understrand to the overstrand, of the form $v \times I$, as in Figure 2. We call such an arc *vertical*.

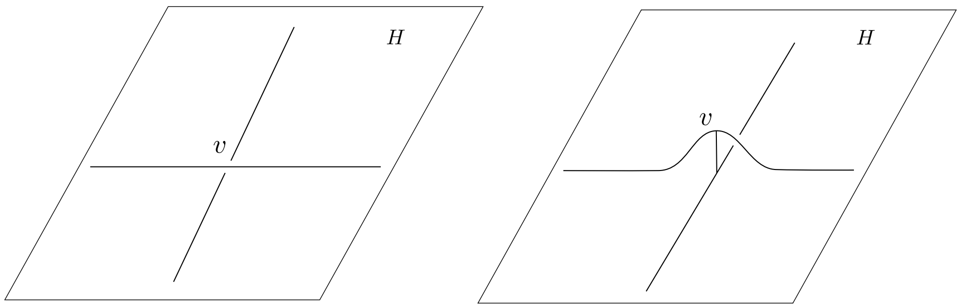

FIGURE 2. A vertical arc on a crossing v of D_K.

This collection of vertical arcs, together with K determines a 1-complex homotopically equivalent to D_K, which we also denote by D_K. As D_K separates the Heegaard surface H into disks, the exterior of D_K in S^3 is characterized by two handlebodies connected along 1-handles (with co-core the disks of $H - D_K$). Hence, the exterior of D_K is a handlebody, and the collection of vertical arcs is an unknotting tunnel system of H.

Instead of adding one vertical arc at each crossing, we want to find the minimal number of vertical arcs needed to constitute an unknotting tunnel system. To do this, we will add vertical arcs only at certain crossings and determine if the exterior of the resulting 1-complex is a handlebody, by showing that the corresponding decomposition of $E(K)$ is connected by a sequence of (de)stabilizations to the Heegaard decomposition of $E(K)$ obtained by adding one vertical arc at each crossing.

In this context, we will use the following remark to establish an upper bound for the minimal number of vertical arcs defining an unknotting tunnel system of a band link.

Remark 1. Suppose one starts to add vertical arcs and, at some point, there is a face f determined by the projection D_K such that all but one of the crossings of f has a vertical arc. Let v represent this crossing. Then, the vertical arc on the crossing v is trivial with respect to the exterior of K with the vertical arcs added up to this stage, as it is parallel to the edges and vertical arcs with respect to f. Hence, if we add the vertical arc at v, which we refer to as an *automatic arc*, we have a stabilization of the decomposition defined by K and the vertical arcs added towards the decomposition defined by D_K. Therefore, we can add or remove the vertical arc at v, without changing the decomposition defined by the collection of vertical arcs added being a Heegaard decomposition. When the last vertex v of f is colored automatically, we also color the face f for notation, as in Figure 3.

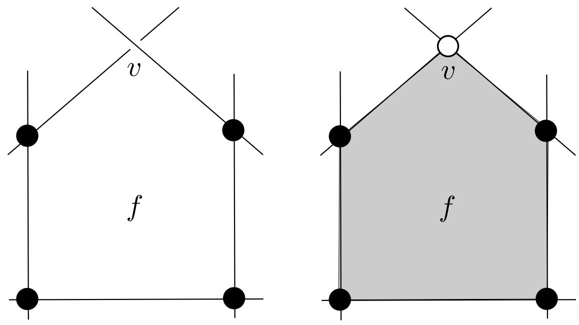

FIGURE 3. Coloring a vertex automatically

3. Percolation on link projections

Let G be an embedded graph in a Heegaard surface H with complement in H being a collection of disks, which we refer to as faces, as mentioned before. Consider the following *coloring rule (percolation rule)* on the set $V(G)$, the vertex set of G.

3.1. Coloring Rule (Percolation Rule). Vertices will either be *manually colored* or *automatically colored.* At each step $s \in \{0, 1, \dots, k, \dots\}$ some subset (possibly empty) of vertices is manually colored. A vertex v will be automatically colored at step $s + 1$ if it belongs to a face in which all other vertices have already been colored (either manually or automatically) at some previous step.

We say a subset $V' \subset V(G)$ *percolates* G if manually coloring all vertices in V' implies all remaining vertices of $V(G)$ will be automatically colored at some step. A *hull set* for G is a minimal subset $H \subset V(G)$ such that H percolates G. The *hull number* for G is the size of a hull set, denoted $h(G)$.

3.2. Relationship between hull number and tunnel number. Let K be a link with a regular projection D_K onto a Heegaard surface H of S^3, such that the complement of D_K in H is a collection of disks.

Lemma 3.1. *$t(K) \leq h(D_K)$.*

Proof: Let $\mathcal{V}$ be a hull set of D_K.
At each crossing corresponding to a vertex in $\mathcal{V}$ we add a vertical arc to K, and denote this collection of arcs by $\mathcal{V}_a$. We proceed by adding a vertical arc at each crossing corresponding to an automatically colored vertex at each step of the percolation of $\mathcal{V}$. As $\mathcal{V}$ percolates D_K, we stop this process only when all crossings of D_K have a vertical arc attached.
At each step of the percolation of $\mathcal{V}$, a vertex w is automatically colored because it belongs to a face f with all the other vertices already colored at that step. This translates into having vertical arcs at all crossings of f, except at w. From Remark 1, adding a vertical arc at w corresponds to a stabilization of the decomposition of $E(K)$ determined by the vertical arcs added up to this step. We add a vertical arc at w and proceed to the next step, untill all crossings have a vertical arc added.
This process determines a sequence of stabilizations from the decomposition of $E(K)$ determined by $\mathcal{V}_a$, to the decomposition of $E(K)$ determined by the collection of a vertical arc at each crossing. Following observations from Section 2, as the collection of vertical arcs at each crossing determines a Heegaard decomposition, this means that $\mathcal{V}_a$ is an unknotting tunnel system for K. ■

4. Tunnel number of band links

In this section we prove Theorem 1.1. We first prove for a regular projection D_L of a n-band link L, on some Heegaard surface, that we have $n-1 \leq t(L) \leq h(D_L) \leq n-1$. Then, by using the above percolation on these diagrams, we show that $h(D_L) \leq n-1$, and Theorem 1.1 immediately follows from the inequalities $n-1 \leq t(L) \leq h(D_L) \leq n-1$.

As a warm-up, we look at the case when L is a band over the unknot. Here, it is fairly easy to see that $t(L) = n-1$. In fact, if the components of L are cyclically labeled $C_1, C_2, \ldots, C_n$, adding one vertical arc between all pairs of consecutive components C_i, C_{i+1}, for $i = 1, \ldots, n-1$, yields an unknotting tunnel system for L. This procedure is illustrated in Figure 4.

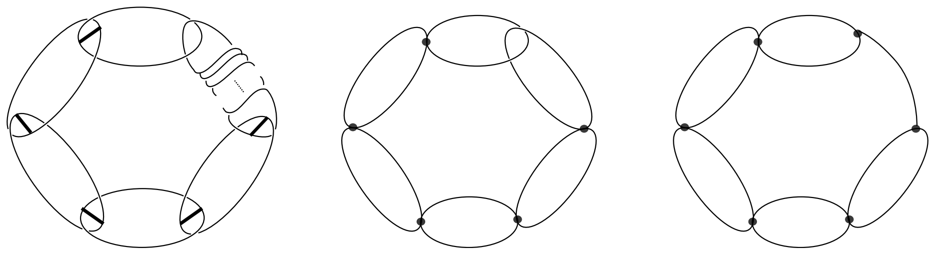

FIGURE 4. Left:1-complex K' (tunnels are the bold segments); Middle: collapse tunnels to vertices; Right: K' can be made planar after a sequence of handle slides.

If we attempt this procedure for a generic band link L, by adding one vertical arc between all pairs of consecutive components, we see that the resulting 1-complex has a similar diagram to the knot K. What this means is that, following this approach, we would need to add another $t(K)$ arcs to this 1-complex in order to obtain an unknotting tunnel system for L. Instead, we will follow a different strategy to realize that adding $n-1$ vertical arcs suffices to obtain an unknotting tunnel system for a n-band links, which is the minimum possible. This is achieved by the above percolation procedure applied to D_L. First we observe the following lemma.

Lemma 4.1. *Let D_L be a projection of an n-band link. Then $h(D_L) \geq n-1$.*

Proof: We know that the tunnel number of a link L is at least its number of components minus 1. Combining this with Lemma 3.1, we obtain $n-1 \leq t(L) \leq h(D_L)$. ■

We are now ready to present

Proof of Theorem 1.1: Let L be an n-band link with corresponding projection D_L, as in the definition of band link. We will show that $h(D_L) \leq n-1$. The theorem then follows by considering Lemma 4.1.

First we will deal with the case in which all arcs of D_L are untwisted. In this situation, the projection of each component of L is a simple closed curve and intersections on the projection are as in Figure 5. We refer to these simple closed curves by *circle components*.

There are two types of faces in the projection D_L: those which arise from faces of the projection D_K and those which are within a projection of a circle component of L.

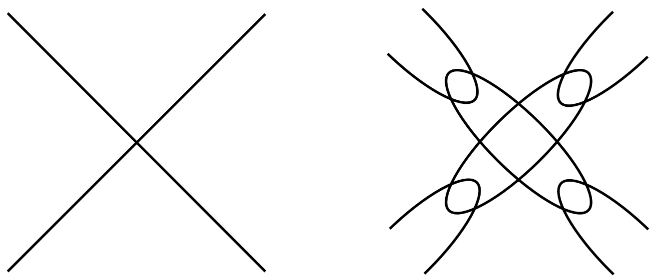

FIGURE 5. Left: a crossing of D_K; Right: portion of the band corresponding to this crossing.

Step 1. Choose a face f_0 of D_L corresponding to a face of the projection D_K.

In the face f_0, manually color all vertices except one. By the coloring rule, the remaining vertex is automatically colored. Notice that if the face f_0 has m vertices, then there are m circle components making up f_0.

Since the projection of every component of L intersects at most one other component in four crossings, then, by allowing further automatic colorings, we observe that all faces sharing a vertex with f_0 have their remaining vertices automatically colored. As mentioned in Remark 1, we also color all these faces having all of their vertices colored, and denote this region by R. These steps are illustrated in Figure 6.

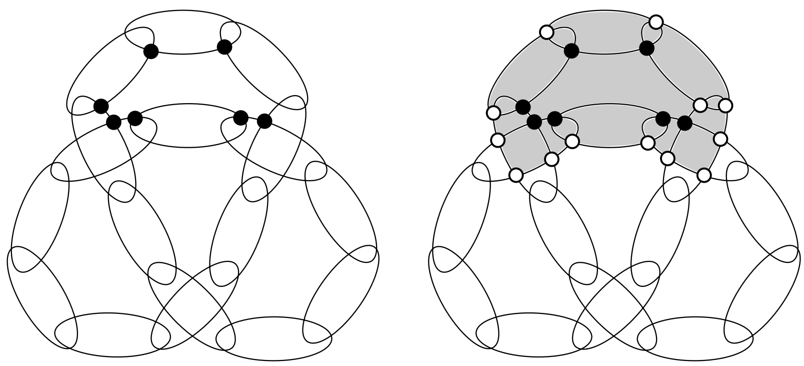

FIGURE 6. Left: black vertices are colored; Right: white vertices are colored automatically and faces sharing a vertex with f_0 are colored.

Step 2. Consider a face f_1 of D_L, corresponding to a face of D_K, adjacent to R. Assume f_1 has m_1 vertices, other than the ones in R. Again, since the projection of every component of L intersects at most one other component in four crossings, then these vertices are given by intersections of $m_1 - 1$ circle components in which vertices have not been colored yet. Manually color $m_1 - 1$ vertices of f_1. Since all vertices in R are already colored, only one vertex of f_1 remains to be colored, and thus it is automatically colored. Color the face f_1, together with all faces within a projection of a circle components and which share a vertex with f_1. We denote the new colored region also by R.

Step 3. Now we repeat Step 2 inductively, each time to a face of D_L adjacent to R, corresponding to a face of D_K.

Eventually all faces of D_L will be colored, that is all vertices of D_L will be colored, either automatically or manually. With the above steps we determined a subset of vertices of D_L, the ones which were manually colored, that percolates D_L. Observe also that we manually colored exactly $n-1$ vertices. Therefore, $h(D_L) \leq n-1$. This proves the theorem in the case where all arcs of D_L are untwisted.

For the general case, we just need to observe that all additional vertices coming from twisting will be colored automatically when we color one crossing between two different components. ■

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