Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 17–39

MATRIX BIORTHOGONAL POLYNOMIALS, ASSOCIATED POLYNOMIALS AND FUNCTIONS OF THE SECOND KIND

A. BRANQUINHO, J.C. GARCÍA-ARDILA AND F. MARCELLÁN

ABSTRACT: In this work the interplay between matrix biorthogonal polynomials with respect to a matrix of linear functionals, the k-th associated matrix polynomials and the second kind matrix functions, is studied in terms of quasideterminants. A sort of Poincaré's theorem for the ratio of two consecutive matrix functions solutions of a linear difference equation is also presented. Some new formulas connecting these families of matrix functions are given.

KEYWORDS: ratio asymptotic; quadrature formulae; Markov functions; matrix biorthogonal polynomials; generalized Chebyshev polynomials.
AMS SUBJECT CLASSIFICATION (2010): 33C45, 42C05, 47A56 (primary), 41A10 (secondary).

1. Introduction

Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of orthonormal polynomials with respect to a probability measure, μ, supported on an infinite subset of the real line. It is well known that $\{p_n\}_{n\in\mathbb{N}}$ satisfies a three term recurrence relation

$$x\,p_n(x) = a_{n+1}\,p_{n+1}(x) + b_n\,p_n(x) + a_n\,p_{n-1}(x), \quad n \geq 0\,,$$

with initial conditions $p_0(x) = 1$, $p_{-1}(x) = 0$, where $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers with $a_n \neq 0$. If we assume that $a_n \to a \neq 0$ and $b_n \to b$, then Nevai [9] proved that the convergence

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n-1}(z)} = \frac{z - b + \sqrt{(z-b)^2 - 4a^2}}{2a}\,, \quad z \in \mathbb{C} \setminus \operatorname{supp}(u)\,, \tag{1}$$

holds uniformly on compact subsets of $\mathbb{C} \setminus \operatorname{supp}(\mu)$.

Received August 4, 2017.
AB acknowledges Centro de Matemática da Universidade de Coimbra (CMUC) – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020. JCA and FM acknowledge the project MTM2015-65888-C4-2-P.

The functions of second kind, $\{q_n\}_{n\in\mathbb{N}}$, defined as

$$q_n(z) = \int \frac{p_n(x)}{z-x}\, d\mu(x),\ n\in\mathbb{N},\ \ z\in\mathbb{C}\setminus \operatorname{supp}(u),$$

satisfy the same three term recurrence relation as $\{p_n\}_{n\in\mathbb{N}}$, but with initial conditions $q_{-1}(z) = 1/a_0$, $q_0(z) = \int d\mu(x)/(z-x)$.

By the Poincaré's theorem for difference equations [11] it is possible conclude that

$$\lim_{n\to\infty} \frac{q_n(z)}{q_{n-1}(z)} = \frac{z-b-\sqrt{(z-b)^2-4a^2}}{2a},\ \ z\in\mathbb{C}\setminus \operatorname{supp}(u), \tag{2}$$

uniformly on compact subsets of $\mathbb{C}\setminus \operatorname{supp}(\mu)$. From this, Van Assche [14] (see also [12]) showed that

$$\lim_{n\to\infty} p_n(z)q_n(z) = \frac{1}{\sqrt{(z-b)^2-4a^2}},\ \ z\in\mathbb{C}\setminus \operatorname{supp}(u), \tag{3}$$

and the convergence holds uniformly on compact subsets of $\mathbb{C}\setminus \operatorname{supp}(u)$.

Now, taking into account [6], we consider a positive definite $N\times N$ matrix of measures and its corresponding sequence of matrix orthonormal polynomials $\{P_n\}_{n\in\mathbb{N}}$ satisfying a recurrence relation

$$x\,P_n(x) = A_{n+1}\,P_{n+1}(x) + B_n\,P_n(x) + A_n^{\top} P_{n-1},\ n\geq 0,$$

with initial conditions $P_0(x) = I_N$, $P_{-1}(x) = \mathbf{0}$, where, A_n, are nonsingular matrices and B_n are Hermitian matrices; then the outer ratio asymptotics of two consecutive polynomials belonging to the *matrix Nevai class*, i.e. if $A_n\to A$ and $B_n\to B$ with A a nonsingular matrix, then $\{P_{n-1}\,P_n^{-1}A_n^{-1}\}_{n\in\mathbb{N}}$ uniformly converges on compact subsets of $\mathbb{C}\setminus\Gamma^{(0)}$ to $\int dW_{A,B}(y)/(x-y)$, where $\Gamma^{(0)}$ will be defined later, cf. (9), and $W_{A,B}$ is the matrix weight for the Chebyshev matrix polynomials (cf. [5]). A more general case was studied in [4, Theorem 4] for matrix biorthogonal polynomials (cf. also [2]).

In the present contribution we are interested to analyze the analogous results in the matrix biorthogonal case (cf. [1] for a fresh introduction on matrix biorthogonal polynomials) of those given in (2) and (3).

Observe that in this matrix scenario the Poincaré's theorem is no longer valid. The answers to these problems are given as Corollaries of the main result of this paper (cf. Theorem 6). Here we present the result for matrices of linear functionals (and biorthogonal polynomials).

The structure of this manuscript is as follows. In Section 2, we exhibit the basic theory of linear difference equations on the noncommutative ring of matrices. In Section 3, we introduce the concept of matrix of linear functionals and its associated families of matrix biorthogonal polynomials. Recall that the families of biorthogonal polynomials are solutions of a second order linear difference equation but these are no the unique ones. With this background, in Section 4, we present results concerning the independence of solutions for linear difference equations with matrix coefficients, as well as an explicit representation for these solutions. In Section 5, we study the outer ratio asymptotics for the second kind matrix functions (see Theorem 6) generalizing for the matrix case the results given in (2) and (3).

2. Linear difference equations on the ring of matrices

First of all we will fix some notation. Let $\mathbb{R}$ and $\mathbb{C}$ be the set of complex and real numbers, respectively, and denote by $\mathbb{C}^{N\times N}$ (respectively, $\mathbb{R}^{N\times N}$) the linear space of $N \times N$ matrices with complex entries (respectively, the linear space of $N \times N$ matrices with real entries).

For an arbitrary finite or infinite matrix A, the matrix $A^{\top}$ is its transpose. The matrix $\mathbf{0}$ will be understand as the null matrix of size $N \times N$.

In the sequel, we will use the definition of quasideterminants coming from the last corner of the block matrix to obtain connection formulas between some families of orthogonal polynomials. They constitute a generalization of the determinants when the entries of the matrix belong to a noncommutative ring, and they share several properties with them.

Let $A \in \mathbb{C}^{M\times M}$, $B \in \mathbb{C}^{M\times N}$, $C \in \mathbb{C}^{N\times M}$ and $D \in \mathbb{C}^{N\times N}$, with A a nonsingular matrix. For the 2×2 block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the *last quasideterminants* is defined by

$$\Theta_* \begin{pmatrix} A & B \\ C & D \end{pmatrix} := D - CA^{-1}B \,.$$

Notice that the last quasideterminant is just the Schur complement of the block A.

Proposition 1. *Given the block matrix,* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* A, B, C, *and* D *are matrices of size* $N \times N$, *then:*

1. $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \begin{pmatrix} -C & -D \\ A & B \end{pmatrix}$.
2. *If* A *is nonsingular then* $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B)$.
3. *If* A *and* $D - CA^{-1}B$ *are nonsingular matrices, then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

Recall that if R is a ring, we say that a *left module over* R is a set M together with two operations

$$\oplus : M \times M \to M \text{ and } \odot : R \times M \to M,$$

such that for $m, n \in M$ and $a, b \in R$ we have:

1. $(M, \oplus)$ is an Abelian group.
2. $(a \oplus b) \odot m = (a \odot m) \oplus (b \odot m)$ and $a \odot (m \oplus n) = (a \odot m) \oplus (a \odot n)$.
3. $(a \odot b) \odot m = a \odot (b \odot m)$.

In a similar way, one defines a right module on R. If M is a left and right module over R, then M is said to be a *bimodule* (cf. [10, 13]).

The *module* M is said to be a *free left* module (respectively, *right* module) over R if M admits a basis, that is, there exists a subset S of M such that S is not empty, S generates M, i.e. $M = \operatorname{span}(S)$ and S is linearly independent.

Recall that for matrices $A_k \in \mathbb{C}^{N\times N}$, $0 \leqslant k \leqslant n$, with $\det(A_n) \neq 0$, the matrix $P(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$ is said to be a matrix polynomial of degree n. In particular, if $A_n = I_N$, i.e. A_n is the identity $N \times N$ matrix, then the polynomial is said to be monic. The set of matrix polynomials with coefficients in $\mathbb{C}^{N\times N}$ will be denoted by $\mathbb{C}^{N\times N}[x]$.

Observe that $\mathbb{C}^{N\times N}[x]$ with the usual sum and product for matrices is a free bimodule and, in particular, a left module, on the ring $\mathbb{C}^{N\times N}$ with basis $\{x^n I_N\}_{n\in\mathbb{N}}$. Important submodules of $\mathbb{C}^{N\times N}[x]$ are the sets $\mathbb{C}_n^{N\times N}[x]$ of matrix polynomials of degree less than or equal to n with the basis $\{I_N, \ldots, x^n I_N\}$ of cardinality $n+1$.

The complex number, x_0, is said to be a zero of P if $\det P(x_0) = 0$. Clearly, as a consequence, we have that P has at most $n N$ zeros.

Now, we consider a sequence of matrices $(A_n)_{n\in\mathbb{N}}$ in $\mathbb{C}^{N\times N}[x]$ and take the following k-th order difference equation

$$y_{n+k} + A_{n+k-1}\, y_{n+k-1} + \cdots + A_n\, y_n = \mathbf{0}, \quad n \in \mathbb{N}, \tag{4}$$

with initial conditions $y_0 = c_0, \ldots, y_{k-1} = c_{k-1}$, in $\mathbb{C}^{N\times N}[x]$. We denote by $(y_n(c))_{n\in\mathbb{N}}$, which belong to $\mathbb{C}^{N\times N}[x]$, the solution of (4) with the initial condition $c = (c_0, c_1, \ldots, c_{k-1})$.

Proposition 2. *The equation (4) with initial condition c has a unique solution.*

Proof: It is clear from the fact that given an initial condition c, $y_n(c)$ is completely determined. ■

Now, we introduce the operator L as follows

$$\mathrm{L}\, y_n = \sum_{i=0}^{k} A_{n+k-i}\, y_{n+k-i} \quad \text{with} \quad A_{n+k} = I_N\,.$$

Observe that if $(y_n)_{n\in\mathbb{N}}$ is a solution of (4), then $\mathrm{L}\, y_n = \mathbf{0}$. It is easy verify that the operator L is a *right linear operator*, i.e.

$$\mathrm{L}(y_n\,\alpha + z_n\,\beta) = (\mathrm{L}\, y_n)\,\alpha + (\mathrm{L}\, z_n)\,\beta\,, \quad \alpha, \beta \in \mathbb{C}^{N\times N}[x]\,.$$

We will denote by $\mathbb{S}$ the set of solutions of (4).

Notice that if $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ are solutions of (4), then since the operator L is right linear, we have $y_n\,\alpha + z_n\,\beta$ is also a solution of (4).

Moreover, it is clear that $\mathbb{S}$ is an Abelian group under addition and since

$$y_n\,(\alpha + \beta) = y_n\,\alpha + y_n\,\beta \quad \text{and} \quad (y_n\,\alpha)\,\beta = y_n\,(\alpha\,\beta)\,,$$

then we can conclude that $\mathbb{S}$ is also a right module over the noncommutative ring $C^{N\times N}[x]$.

Proposition 3. *Let $(y_n(e_0))_{n\in\mathbb{N}}, (y_n(e_1))_{n\in\mathbb{N}}, \ldots, (y_n(e_{k-1}))_{n\in\mathbb{N}}$ be the solution of (4) with the initial conditions*

$$e_0 = (I_N, 0, \ldots, 0),\; e_1 = (0, I_N, \ldots, 0), \ldots, e_{k-1} = (0, 0, \ldots, I_N)\,.$$

Given a solution of (4), $(y_n(c))_{n\in\mathbb{N}}$, with the set of initial conditions given by $c = (c_0 \cdots c_{k-1})$, then $(y_n(c))_{n\in\mathbb{N}}$ can be expressed as a linear combination of $(y_n(e_i))_{n\in\mathbb{N}}$, $i = 0, \ldots, k-1$.

Proof: Let $z_n = \sum_{i=0}^{k-1} y_n(e_i)c_i$. Since $\mathbb{S}$ is a right module on $\mathbb{C}^{N\times N}[x]$, then $\sum_{i=0}^{k-1} y_n(e_i)c_i \in \mathbb{S}$. Moreover, observe that $z_i = c_i$, for $i = 0,\dots,k-1$. The above implies that $(z_n)_{n\in\mathbb{N}}$ is a solution of (4) with initial condition c, but from Proposition 2, $z_n = y_n$ for every $n \in \mathbb{N}$. ∎

Given a set of functions $\{f_{i,n}\}_{n\in\mathbb{N}}$, $i = 0,\dots,k-1$, $f_{i,n} : \mathbb{N} \to \mathbb{C}^{N\times N}[x]$. The *set of functions* $\{f_{i,n}\}_{n\in\mathbb{N}}$, $i = 0,\dots,k-1$, are said to be *linearly independent* if for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{k-1} f_{i,n}\,\alpha_i = \mathbf{0}\,, \quad \alpha_i \in C^{N\times N}[x], \text{ implies } \alpha_i = \mathbf{0}\,. \tag{5}$$

Corollary 1. *The set of solutions $\{(y_n(e_i))_{n\in\mathbb{N}} : i = 0,\dots,k-1\}$ is a basis for $\mathbb{S}$, or equivalently, $\mathbb{S}$ is a free right module.*

Proof: This fact follows from Proposition 3. ∎

Given a set of functions $\{f_{i,n}\}_{n\in\mathbb{N}}$, $i = 0,\dots,k-1$, we define the *block Casorati matrix* as

$$W(f_{0,n},\dots,f_{k-1,n}) = \begin{pmatrix} f_{0,n} & \cdots & f_{k-1,n} \\ \vdots & \ddots & \vdots \\ f_{0,n+k-1} & \cdots & f_{k-1,n+k-1} \end{pmatrix}.$$

Theorem 1. *A sufficient condition for the set of functions $\{f_{i,n}\}_{n\in\mathbb{N}}$, $i = 0,\dots,k-1$ be linearly independent is that there exists $\hat{n} \in \mathbb{N}$ such that*

$$\det W(f_{0,\hat{n}},\dots,f_{k-1,\hat{n}}) \neq 0\,.$$

Proof: If (5) holds for some $\hat{n} \in \mathbb{N}$, then

$$\begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} f_{0,\hat{n}} & \cdots & f_{k-1,\hat{n}} \\ \vdots & \ddots & \vdots \\ f_{0,\hat{n}+k-1} & \cdots & f_{k-1,\hat{n}+k-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}.$$

The above system has a unique solution if and only if $W(f_{1,\hat{n}},\dots,f_{k,\hat{n}})$ is a nonsingular matrix (see [8]). ∎

3. Matrix biorthogonal polynomials

A *sesquilinear form* on the bimodule $\mathbb{C}^{N\times N}[x]$, with real variable, is a map

$$\langle\cdot,\cdot\rangle : \mathbb{C}^{N\times N}[x]\times\mathbb{C}^{N\times N}[x]\to\mathbb{C}^{N\times N},$$

such that for any triple $P,Q,R\in\mathbb{C}^{N\times N}[x]$ of matrix polynomials we have for all $A,B\in\mathbb{C}^{N\times N}$:

1. $\langle A\,P(x)+B\,Q(x),R(x)\rangle = A\,\langle P(x),R(x)\rangle + B\,\langle Q(x),R(x)\rangle$;
2. $\langle P(x),A Q(x)+B\,R(x)\rangle = \langle P(x),Q(x)\rangle\,A^{\top} + \langle P(x),R(x)\rangle\,B^{\top}$.

If $\langle P(t),Q(t)\rangle = \langle Q(t),P(t)\rangle^{\top}$, $\langle\cdot,\cdot\rangle$ is called a *symmetric sesquilinear form*.

Given a matrix of linear functionals, i.e.

$$u=\begin{pmatrix} u_{1,1} & \cdots & u_{1,p}\\ \vdots & \ddots & \vdots\\ u_{p,1} & \cdots & u_{p,p}\end{pmatrix},$$

where $u_{i,j}$ belong to the the dual space of $\mathbb{C}[x]$, we define its associated sesquilinear form $\langle P,Q\rangle_u$ as follows

$$\left(\langle P,Q\rangle_u\right)_{i,j} := \sum_{k,l=1}^{p}\left\langle u_{k,l}, P_{i,k}(x)\,Q_{j,l}(x)\right\rangle .$$

In this case the support is defined as $\operatorname{supp}(u) := \bigcup_{k,l=1}^{p}\operatorname{supp}(u_{k,l})$.

An important property of the sesquilinear form defined in terms of a matrix of linear functional is that $\langle x\,P(x),Q(x)\rangle = \langle P(x),x\,Q(x)\rangle$.

Let $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ be two sequences of matrix polynomials satisfying

$$\left\langle V_n(x),G_m(x)\right\rangle_u = I_N\,\delta_{n,m},\ \ n,m\in\mathbb{N}.$$

The sequences of matrix polynomials $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ are said to be *biorthogonal with respect to u*.

It is well known, cf. for instance [3], that for the sequences of matrix polynomials $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ there exist sequences of matrices $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, and $(C_n)_{n\in\mathbb{N}}$, with A_n a lower triangular matrix and C_n a upper triangular matrix, both nonsingular for $n=0,1,2,\ldots$, such that

$$x\,V_n(x) = A_n\,V_{n+1}(x) + B_n\,V_n(x) + C_n\,V_{n-1}(x), \tag{6}$$

$$x\,G_n^{\top}(x) = G_{n+1}^{\top}(x)\,C_{n+1} + G_n^{\top}(x)\,B_n + G_{n-1}^{\top}(x)\,A_{n-1}, \tag{7}$$

with initial conditions $V_0(x)=G_0^{\top}(x)=I_N$ and $V_{-1}(x)=G_{-1}^{\top}(x)=\mathbf{0}$.

Observe that from definition, $C_0 = I_N$. In the same way $A_{-1} = I_N$. Moreover, the converse is also true, i.e. if we have two sequences of matrix polynomials $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ satisfying (6) and (7), respectively, then there exists a matrix of linear functionals u, with respect to they are biorthogonal.

Now, from the sequences of matrices $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, and $(C_n)_{n\in\mathbb{N}}$ with $A_0 = I_N$, one defines for each $n \in \mathbb{N}$ the *k-th associated polynomials* $\{V_n^{(k)}\}_{n\in\mathbb{N}}$ and $\{G_n^{(k)}\}_{n\in\mathbb{N}}$ by the recurrence formula, for $n \in \mathbb{N}$,

$$\begin{aligned} x\,V_n^{(k)}(x) &= A_{n+k}\,V_{n+1}^{(k)}(x) + B_{n+k}\,V_n^{(k)}(x) + C_{n+k}\,V_{n-1}^{(k)}(x), \\ x\,G_n^{(k)\top}(x) &= G_{n+1}^{(k)\top}(x)\,C_{n+k+1} + G_n^{(k)\top}(x)\,B_{n+k} + G_{n-1}^{(k)\top}(x)A_{n+k-1}, \end{aligned} \tag{8}$$

with initial conditions $V_{-1}^{(k)}(x) = \mathbf{0}$, $V_0^{(k)}(x) = I_N$ and $G_{-1}^{(k)}(x) = \mathbf{0}$, $G_0^{(k)}(x) = I_N$.

In [4] is proved that for every $n \in \mathbb{N}$, $V_n^{(k)}(x)$ and $G_n^{(k)}(x)$ have the same zeros. Moreover, taking the N-block Jacobi matrix associated with the recurrence relation (8), i.e.

$$J^{(k)} = \begin{pmatrix} B_k & A_k & \mathbf{0} & \\ C_{k+1} & B_{k+1} & A_{k+1} & \ddots \\ \mathbf{0} & C_{k+2} & B_{k+2} & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

$J^{(0)} = J$, the zeros of $V_n^{(k)}(x)$ are the eigenvalues of $J_n^{(k)}$, where J_n is the truncated matrix of $J^{(k)}$ with size $nN \times nN$. So, denoting by $\Delta_n^{(k)}$ the set of zeros of $V_n^{(k)}(x)$ (or equivalently, of $G_n^{(k)}(x)$), we define

$$\Gamma^{(k)} = \bigcap_{n\in\mathbb{N}} M_N^{(k)} \quad \text{where} \quad M_N^{(k)} = \overline{\bigcup_{n\in\mathbb{N}} \Delta_n^{(k)}}. \tag{9}$$

If A_n, B_n, and C_n converge, then by the Gershgorin disk theorem, there exists a real number $M > 0$ such that $\overline{\bigcup_{n\in\mathbb{N}} \Delta_n} \subset \mathbf{D}_M$, where $\mathbf{D}_M = \{x : |x| \leq M\}$. Moreover, since $J^{(k)}$ is a submatrix of J, then again from the Gershgorin disk theorem, we have $\bigcup_{k\in\mathbb{N}} \Gamma^{(k)} \subset \mathbf{D}_M$.

For $y \notin \operatorname{supp}(u)$, the corresponding families of *second kind functions*, $\{Q_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$, are defined by

$$Q_n(y) = \left\langle V_n(x), \frac{I_N}{y-x} \right\rangle_u \quad \text{and} \quad R_n^{\top}(y) = \left\langle \frac{I_N}{y-x}, G_n(x) \right\rangle_u .$$

Observe that the families of second kind functions $\{Q_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ also satisfy the following three term recurrence relations

$$\begin{aligned} y\,Q_n(y) &= A_n\,Q_{n+1}(y) + B_n\,Q_n(y) + C_n\,Q_{n-1}(y)\,,\\ y\,R_n^\top(y) &= R_{n+1}^\top(y)\,C_{n+1} + R_n^\top(y)\,B_n + R_{n-1}^\top(y)\,A_{n-1}\,, \end{aligned}$$

with $Q_0(y) = \left\langle I_N, \frac{I_N}{y-x}\right\rangle_u$, $Q_{-1}(y) = C_0^{-1}$, $R_0^\top(y) = \left\langle \frac{I_N}{y-x}, I_N\right\rangle_u$, and $R_{-1}^\top(y) = A_{-1}^{-1}$.

Proposition 4 (Christoffel-Darboux type formulas)**.** *Let $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ be the sequences of biorthogonal polynomials with respect to u. Let $\{Q_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ be, respectively, the corresponding families of second kind functions, then*

$$(x-y)\sum_{m=0}^{n} G_m^\top(y)\,Q_m(x) = G_n^\top(y)\,A_n\,Q_{n+1}(x) - G_{n+1}^\top(y)\,C_{n+1}\,Q_n(x) + I_N\,, \tag{10}$$

and its confluent formula

$$\sum_{m=0}^{n} G_m^\top(x)\,Q_m(x) = (G_{n+1}^\top(x))'\,C_{n+1}\,Q_n(x) - (G_n^\top(x))'\,A_n\,Q_{n+1}(x)\,. \tag{11}$$

Moreover, we get the analogous Christoffel-Darboux and confluent formulas

$$(x-y)\sum_{m=0}^{n} R_m^\top(y)\,V_m(x) = R_n^\top(y)\,A_n\,V_{n+1}(x) - R_{n+1}^\top(y)\,C_{n+1}\,V_n(x) - I_N\,, \tag{12}$$

$$\sum_{m=0}^{n} R_m^\top(x)\,V_m(x) = R_n^\top(x)\,A_n\,V_{n+1}'(x) - R_{n+1}^\top(y)\,C_{n+1}\,V_n'(x)\,. \tag{13}$$

Proof: We only prove (10) and (11). The formulas (12) and (13) follow in a similar way. From recurrence formulas for $\{Q_m\}_{m\in\mathbb{N}}$ (respectively, $\{G_m\}_{m\in\mathbb{N}}$) multiplied on the left by $G_m^\top$ (respectively, multiplied on the right by $\{Q_m\}_{m\in\mathbb{N}}$), we have

$$\begin{aligned} &x\,G_m^\top(y)\,Q_m(x)\\ &\quad = G_m^\top(y)\,A_m\,Q_{m+1}(x) + G_m^\top(y)\,B_m\,Q_m(x) + G_m^\top(y)\,C_m\,Q_{m-1}(x)\,, \end{aligned} \tag{14}$$

$$\begin{aligned} &y\,G_m^\top(y)\,Q_m(x)\\ &\quad = G_{m+1}^\top(y)\,C_{m+1}\,Q_m(x) + G_m^\top(y)\,B_m\,Q_m(x) + G_{m-1}^\top(y)\,A_{m-1}\,Q_m(x)\,. \end{aligned} \tag{15}$$

From here, if we subtract (15) from (14)

$$
\begin{aligned}
(x-y)\,G_m^{\top}(y)Q_m(x) = \big(G_m^{\top}(y)A_m\,Q_{m+1}(x) - G_{m-1}^{\top}(y)A_{m-1}\,Q_m(x)\big)\\
-\big(G_{m+1}^{\top}(y)\,C_{m+1}\,Q_m(x) - G_m^{\top}(y)\,C_m\,Q_{m-1}(x)\big)\,.
\end{aligned}
$$

Summing the later from 0 to n and taking into account the initial conditions, the result follows. To show the confluent formula notice that

$$
G_n^{\top}(y)A_n\,Q_{n+1}(y) - G_{n+1}^{\top}(y)\,C_{n+1}\,Q_n(y) + I_N = \mathbf{0}\,.
$$

Now, observe that the Christoffel-Darboux formula can successively be rewritten as

$$
\begin{aligned}
\sum_{m=0}^{n} G_m^{\top}(y)Q_m(x) &= \frac{G_{n+1}^{\top}(y)\,C_{n+1}\,Q_n(y) - G_n^{\top}(y)A_n\,Q_{n+1}(y) + C_0^{-1}}{x-y}\\
&\quad + G_n^{\top}(y)A_n\frac{Q_{n+1}(x)-Q_{n+1}(y)}{x-y} - G_{n+1}^{\top}(y)\,C_{n+1}\frac{Q_n(x)-Q_n(y)}{x-y}\,,\\
&= G_n^{\top}(y)A_n\frac{Q_{n+1}(x)-Q_{n+1}(y)}{x-y} - G_{n+1}^{\top}(y)\,C_{n+1}\frac{Q_n(x)-Q_n(y)}{x-y}\,;
\end{aligned}
$$

and taking $x \to y$ we get the desired result. ∎

Now, we state a sort of reciprocal of Proposition 4.

Proposition 5. *Suppose that we have two sequences, $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$, of matrix polynomials, and two matrices of linear functionals u^1, u^2 such that $\langle V_n, I_N\rangle_{u^1} = \delta_{n,0}$ and $\langle I_N, G_n\rangle_{u^2} = \delta_{n,0}$, $n \in \mathbb{N}$. We define the matrix functions $Q_n(y) = \left\langle V_n(x), \frac{I_N}{y-x}\right\rangle_{u^1}$, $R_n^{\top}(y) = \left\langle \frac{I_N}{y-x}, G_n(x)\right\rangle_{u^2}$ and we assume that (10) and (12) are satisfied, then $u^1 \equiv u^2$.*

Proof: Observe that from the confluent formula

$$
\sum_{m=0}^{n} G_m^{\top}(x)Q_m(x) = \sum_{m=0}^{n-1} G_m^{\top}(x)Q_m(x) + G_n^{\top}(x)Q_n(x)\,;
$$

we get using (10)

$$
\begin{aligned}
G_n^{\top}(x)Q_n(x) = \big((G_{n+1}^{\top}(x))'\,C_{n+1} + (G_{n-1}^{\top}(x))'A_{n-1}\big)\,Q_n(x)\\
-(G_n^{\top}(x))'\,\big(A_n\,Q_{n+1}(x) + C_n\,Q_{n-1}(x)\big)\,.
\end{aligned}
$$

or, equivalently,

$$I_N = G_n^{-\top}(x)\left(G_{n+1}^{\top}(x)C_{n+1} + G_{n-1}^{\top}(x)A_{n-1}\right)' Q_n(x) - G_n^{-\top}(x)(G_n^{\top}(x))'\left(A_n Q_{n+1}(x) + C_n Q_{n-1}(x)\right) Q_n^{-1}(x). \quad (16)$$

Now, for every $n \in \mathbb{N}$,

$$I_N = G_{n+1}^{\top}(x)C_{n+1}Q_n(x) - G_n^{\top}(x)A_{n+1}Q_{n+1}(x), \quad (17)$$

and as $(G_n^{-\top}(x))' = -G_n^{-\top}(x)(G_n^{\top}(x))' G_n^{-\top}(x)$, we can rewrite (16) as follows

$$I_N = G_n^{-\top}(x)\left(G_{n+1}^{\top}(x)C_{n+1} + G_{n-1}^{\top}(x)A_{n-1}\right)' Q_n(x) + (G_n^{-\top}(x))'\left(G_{n+1}^{\top}(x)C_{n+1} + G_{n-1}^{\top}(x)A_{n-1}\right),$$

and so $I_N = \left(G_n^{-\top}(x)\left(G_{n+1}^{\top}(x)C_{n+1} + G_{n-1}^{\top}(x)A_{n-1}\right)\right)'$. Integrating the above with respect to the variable x, we get that $\{G_n\}_{n\in\mathbb{N}}$ satisfies the following recurrence relation

$$G_n^{\top}(x)(xI - B_n) = G_{n+1}^{\top}(x)C_{n+1} + G_{n-1}^{\top}(x)A_{n-1}. \quad (18)$$

A similar procedure for $\{Q_n\}_{n\in\mathbb{N}}$ yield

$$(xI - \widetilde{B}_n)Q_n(x) = A_n Q_{n+1}(x) + C_n Q_{n-1}(x). \quad (19)$$

From (17), (18) and (19), we obtain $\sum_{k=0}^{n} G^{\top}(x)(\widetilde{B}_k - B_k)Q_n(x) = \mathbf{0}$, and this implies that $\widetilde{B}_n = B_n$, for every $n \in \mathbb{N}$.

Since $\{Q_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ satisfy the same recurrence relation (with different initial conditions), from the Favard's theorem we can conclude, that there exist a matrix of linear functionals such that $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ are biorthogonal. ■

Lemma 1. *For every $n \in \mathbb{N}$, $Q_{n-1}(x)G_{n-1}^{\top}(x) - V_{n-1}(x)R_{n-1}^{\top}(x) = \mathbf{0}$.*

Proof: From the definition of second kind functions we get

$$\begin{aligned} Q_{n-1}(x)G_{n-1}^{\top}(x) &- V_{n-1}(x)R_{n-1}^{\top}(x) \\ &= \left\langle V_{n-1}(y), \frac{G_{n-1}(x)}{x-y} \right\rangle_u - \left\langle \frac{V_{n-1}(x)}{x-y}, G_{n-1}(y) \right\rangle_u \\ &= \left\langle V_{n-1}(y), \frac{G_{n-1}(x) - G_{n-1}(y)}{x-y} \right\rangle_u - \left\langle \frac{V_{n-1}(y) - V_{n-1}(x)}{x-y}, G_{n-1}(y) \right\rangle_u, \end{aligned}$$

and from the orthogonality conditions the result follows. ■

Proposition 6 (Liouville-Ostrogradski type formulas). *Let $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ be the sequences of biorthogonal polynomials with respect to a matrix of linear functionals u and let $\{Q_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ be their respective sequences of second kind functions, then*

$$Q_{n-1}(x)\,G_n^\top(x) - V_{n-1}(x)\,R_n^\top(x) = C_n^{-1}\,, \tag{20}$$

$$V_n(x)\,R_{n-1}^\top(x) - Q_n(x)\,G_{n-1}^\top(x) = A_{n-1}^{-1}\,. \tag{21}$$

Proof: We will prove (20) and (21) follows by using analogous arguments. We proceed by induction. For $n=0$ the result is obtained from initial conditions. Suppose now that

$$Q_{k-1}(x)\,G_k^\top(x) - V_{k-1}(x)\,R_k^\top(x) = C_k^{-1}\,,\ k=0,1,\dots,n-1\,.$$

Then, from the recurrence relation for $G_n^\top$ and $R_n^\top$

$$\begin{aligned} Q_{n-1}(x)\,G_n^\top(x) - V_{n-1}(x)\,R_n^\top(x) &= \left(V_{n-1}R_{n-1}^\top - Q_{n-1}\,G_{n-2}^\top\right)A_{n-2}\,C_n^{-1} \\ &\quad + \left(Q_{n-1}(x)\,G_{n-1}^\top - V_{n-1}R_{n-1}^\top\right)(x-B_{n-1})\,C_n^{-1}\,. \end{aligned}$$

Thus from Lemma 1,

$$Q_{n-1}(x)\,G_n^\top(x) - V_{n-1}(x)\,R_n^\top(x) = \left(V_{n-1}(x)\,R_{n-1}^\top(x) - Q_{n-1}(x)\,G_{n-2}^\top(x)\right)A_{n-2}\,C_n^{-1}.$$

If now we use the recurrence formulas for V_{n-1} and Q_{n-1}, then from induction hypothesis and Lemma 1, we get

$$\begin{aligned} &\Big(V_{n-1}(x)\,R_{n-1}^\top(x) - Q_{n-1}(x)\,G_{n-2}^\top(x)\Big)A_{n-2}\,C_n^{-1} \\ &\qquad = A_{n-2}^{-1}(x-B_{n-2})\left(V_{n-2}(x)\,R_{n-2}^\top(x) - Q_{n-2}(x)\,G_{n-2}^\top(x)\right)A_{n-2}\,C_n^{-1} \\ &\qquad\qquad + A_{n-2}^{-1}C_{n-2}\left(Q_{n-3}(x)\,G_{n-2}^\top(x) - V_{n-3}(x)\,R_{n-2}^\top(x)\right)A_{n-2}\,C_n^{-1} \\ &\qquad = A_{n-2}^{-1}\,C_{n-2}\left(Q_{n-3}(x)\,G_{n-2}^\top(x) - V_{n-3}(x)\,R_{n-2}^\top(x)\right)A_{n-2}\,C_n^{-1}\,, \end{aligned}$$

and the result follows from the induction hypothesis. ■

4. Casorati determinants.

Consider the matrix second-order recurrence relations

$$x\,y_n = A_n\,y_{n+1} + B_n\,y_n + C_n\,y_{n-1}\,,\quad n\geq 0\,, \tag{22}$$

$$x\,t_n = t_{n+1}\,C_{n+1} + t_n\,B_n + t_{n-1}A_{n-1}\,,\quad n\geq 0\,. \tag{23}$$

Theorem 2. *If $\{w_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are solutions of (22), then*

$$\det\left(W(w_n,v_n)\right)=\det(A_n^{-1})\det(C_n)\det\left(W(w_{n-1},v_{n-1})\right). \tag{24}$$

Proof: First of all we recall that $W(w_n,v_n)=\begin{pmatrix} w_n & v_n \\ w_{n+1} & v_{n+1}\end{pmatrix}$. Thus from Schur complement

$$\det\left(W(w_n,v_n)\right)=\det(w_n)\det\left(v_{n+1}-w_{n+1}w_n^{-1}v_n\right).$$

Since $(w_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are solutions of (22), then

$$v_{n+1}=A_n^{-1}(x\,v_n-B_n\,v_n-C_n\,v_{n-1}), w_{n+1}=A_n^{-1}(x\,w_n-B_n\,w_n-C_n\,w_{n-1}). \tag{25}$$

Thus

$$-w_{n+1}w_n^{-1}v_n=-A_n^{-1}\left(x\,v_n-B_n\,v_n-C_n\,w_{n-1}\,w_n^{-1}\,v_n\right). \tag{26}$$

If we subtract (26) from the first equation in (25)

$$v_{n+1}-w_{n+1}w_n^{-1}v_n=-A_n^{-1}C_n\left(v_{n-1}-w_{n-1}\,w_n^{-1}v_n\right). \tag{27}$$

As a consequence,

$$\begin{aligned}&\det\left(W(w_n,v_n)\right)\\ &\qquad=(-1)^p\det(w_n)\det(A_n^{-1})\det(C_n)\det(v_{n-1}-w_{n-1}w_n^{-1}v_n). \end{aligned}\tag{28}$$

On the other hand, if now we consider the matrix $W(w_{n-1},v_{n-1})$, using the fact that

$$\begin{pmatrix} w_{n-1} & v_{n-1} \\ w_n & v_n\end{pmatrix}\begin{pmatrix} -w_n^{-1}v_n & w_n^{-1} \\ I_N & 0\end{pmatrix}=\begin{pmatrix} v_{n-1}-w_{n-1}w_n^{-1}v_n & w_{n-1}w_n^{-1} \\ 0 & I_N\end{pmatrix},$$

and

$$\begin{pmatrix} w_{n-1} & v_{n-1} \\ w_n & v_n\end{pmatrix}\begin{pmatrix} I_N & 0 \\ -v_{n-1}^{-1}w_{n-1} & v_{n-1}^{-1}\end{pmatrix}=\begin{pmatrix} 0 & I_N \\ w_{n-1}-v_nv_{n-1}^{-1}w_{n-1} & v_nv_{n-1}\end{pmatrix},$$

and Proposition 1, we have that

$$\begin{aligned}\det\left(W(w_{n-1},v_{n-1})\right)&=(-1)^p\det(w_n)\det(v_{n-1}-w_{n-1}w_n^{-1}v_n)\\ &=(-1)^p\det(v_{n-1})\det(w_n-v_nv_{n-1}^{-1}w_{n-1}).\end{aligned}$$

Replacing the above in (28) we get (24). ∎

Proposition 7. *The sequences* $\{V_n\}_{n\in\mathbb{N}}$ *and* $\{V_{n-1}^{(1)}\}_{n\in\mathbb{N}}$ *with initial conditions* $(V_{-1},V_0)=(\mathbf{0},I_N)$ *and* $(V_{-2}^{(1)},V_{-1}^{(1)})=(-A_0,\mathbf{0})$ *are linearly independent solutions of* (22)*. Moreover, they constitute a basis of* $\mathbb{S}$*.*

Proof: Since $\{V_n\}_{n\in\mathbb{N}}$ and $\{V_{n-1}^{(1)}\}_{n\in\mathbb{N}}$ are solutions of (22), then using Theorem 2, we successively get

$$\begin{aligned}\det\left(W(V_n,V_{n-1}^{(1)})\right)&=\det(A_n^{-1})\det(C_n)\det\left(W(V_{n-1}(x),V_{n-2}^{(1)}(x))\right)\\&=\prod_{j=1}^{n}\det(A_j^{-1})\det(C_j)\det\left(W(V_{-1}(x),V_{-2}^{(1)}(x))\right)\\&=\det(A_0)\prod_{j=1}^{n}\det(A_j^{-1})\det(C_j).\end{aligned}$$

Now, as for every $n\in\mathbb{N}$, $\det(A_n)\neq 0$ and $\det(C_n)\neq 0$, we get that $\{V_n\}_{n\in\mathbb{N}}$ and $\{V_{n-1}^{(1)}\}_{n\in\mathbb{N}}$ are linearly independent.

Recall that if $\left(y_n(e_i)\right)_{n\in\mathbb{N}}$, $i=0,1$ are the solutions of (22) with initial conditions $(I_N,0)$ and $(0,I_N)$, respectively, then

$$V_n(x)=y_n(e_1),\quad V_{n-1}^{(1)}(x)=-y_n(e_0)A_0,$$

and so

$$\begin{pmatrix}V_{n-1}^{(1)}(x) & V_n(x)\end{pmatrix}=\begin{pmatrix}y_n(e_0) & y_n(e_1)\end{pmatrix}\begin{pmatrix}-A_0 & 0\\ 0 & I_N\end{pmatrix};$$

which implies that $\{V_n\}_{n\in\mathbb{N}}$ and $\{V_{n-1}^{(1)}\}_{n\in\mathbb{N}}$ constitute a basis for $\mathbb{S}$. ∎

From Proposition 7 it follows that every solution of (22) is a linear combination of $\{V_n\}_{n\in\mathbb{N}}$ and $\{V_{n-1}^{(1)}\}_{n\in\mathbb{N}}$. In particular

$$V_{n-k}^{(k)}(x)=V_n(x)\gamma_k+V_{n-1}^{(1)}(x)\eta_k. \tag{29}$$

Taking $n=k$ and $n=k-1$, we get the representation

$$\gamma_k=\Theta_*\begin{pmatrix}V_{k-2}^{(1)}(x) & V_{k-1}(x)\\ V_{k-1}^{(1)}(x) & V_k(x)\end{pmatrix}^{-1}\quad\text{and}\quad\eta_k=\Theta_*\begin{pmatrix}V_{k-1}(x) & V_{k-2}^{(1)}(x)\\ V_k(x) & V_{k-1}^{(1)}(x)\end{pmatrix}^{-1}.$$

In particular, if we take $k=2$, from (27) we obtain

$$x\,V_{n-1}^{(1)}(x)=V_n(x)A_0+V_{n-1}^{(1)}(x)A_0^{-1}B_0A_0+V_{n-2}^{(1)}(x)A_1^{-1}C_1A_0.$$

Proposition 8. *Let* $\{V_n\}_{n\in\mathbb{N}}$ *and* $\{V_{n-1}^{(k)}\}_{n\in\mathbb{N}}$ *satisfies* (22) *with initial conditions* $(V_{-1},V_0)=(\mathbf{0},I_N)$ *and* $(V_{-2}^{(k)},V_{-1}^{(k)})=(-C_{k-1}^{-1}A_{k-1},\mathbf{0})$*. Then,*

$$\begin{aligned} &x\,V_{n-1}^{(k)}(x)\\ &\quad= V_n^{(k-1)}(x)A_{k-1}+V_{n-1}^{(k)}(x)A_{k-1}^{-1}B_{k-1}A_{k-1}+V_{n-2}^{(k+1)}(x)A_k^{-1}C_kA_{k-1}. \end{aligned} \tag{30}$$

Proof: From (29) and taking into account (27) we get

$$\begin{aligned} V_{n-2}^{(k+1)}(x)A_k^{-1}C_k &= -V_{n+k-2}^{(1)}(x)\left(V_{k-2}^{(1)}(x)-V_{k-1}(x)(V_k(x))^{-1}V_{k-1}^{(1)}\right)^{-1}\\ &\qquad - V_{n+k-1}(x)\left(V_{k-1}(x)-V_{k-2}^{(1)}(x)(V_{k-1}^{(1)}(x))^{-1}V_k(x)\right)^{-1}. \end{aligned} \tag{31}$$

On the other hand, observe that

$$\begin{aligned} A_{k-1}^{-1}(xI_N-B_{k-1}) &= \left(V_{k-1}^{(1)}(x)+A_{k-1}^{-1}C_{k-1}V_{k-3}^{(1)}(x)\right)(V_{k-2}^{(1)}(x))^{-1},\\ A_{k-1}^{-1}(xI_N-B_{k-1}) &= \left(V_k(x)+A_{k-1}^{-1}C_{k-1}V_{k-2}(x)\right)(V_{k-1}(x))^{-1}. \end{aligned}$$

Thus, from the recurrence relation (22) and (27)

$$\begin{aligned} &\left(V_k(x)-V_{k-1}^{(1)}(x)(V_{k-2}^{(1)}(x))^{-1}V_{k-1}(x)\right)^{-1}A_{k-1}^{-1}(xI_N-B_{k-1})\\ &= \left(V_{k-2}^{(1)}(x)(V_{k-3}^{(1)})^{-1}C_{k-1}^{-1}A_{k-1}V_k(x)+\left(V_{k-2}^{(1)}(x)(V_{k-1}^{(1)}(x))^{-1}V_k(x)-V_{k-1}(x)\right)^{-1}\right.\\ &\qquad\left. -V_{k-2}^{(1)}(x)(V_{k-3}^{(1)}(x))^{-1}C_{k-1}^{-1}A_{k-1}V_{k-1}^{(1)}(x)(V_{k-2}^{(1)}(x))^{-1}V_{k-1}(x)\right)^{-1}, \end{aligned}$$

and so

$$\begin{aligned} &\left(V_k(x)-V_{k-1}^{(1)}(x)(V_{k-2}^{(1)}(x))^{-1}V_{k-1}(x)\right)^{-1}A_{k-1}^{-1}(xI_N-B_{k-1})\\ &\qquad= -\left(V_{k-1}(x)-V_{k-2}^{(1)}(x)(V_{k-1}^{(1)}(x))^{-1}V_k(x)\right)^{-1}\\ &\qquad\qquad+\left(V_{k-1}(x)-V_{k-2}^{(1)}(x)(V_{k-3}^{(1)}(x))^{-1}V_{k-2}(x)\right)^{-1}. \end{aligned} \tag{32}$$

In the same way, we obtain

$$\begin{aligned} &\left(V_{k-1}^{(1)}(x)-V_k(x)(V_{k-1}(x))^{-1}V_{k-2}^{(1)}(x)\right)^{-1}A_{k-1}^{-1}(xI_N-B_{k-1})\\ &\qquad= -\left(V_{k-2}^{(1)}(x)-V_{k-1}(x)(V_k(x))^{-1}V_{k-1}^{(1)}(x)\right)^{-1}\\ &\qquad\qquad+\left(V_{k-2}^{(1)}(x)-V_{k-1}(x)(V_{k-2}(x))^{-1}V_{k-3}^{(1)}(x)\right)^{-1}. \end{aligned} \tag{33}$$

Replacing (32) and (33) in (31)

$$V_{n-2}^{(k+1)}(x)A_k^{-1}C_k = \big(V_{n+k-1}(x)\gamma_k + V_{n+k-2}^{(1)}(x)\eta_k\big)A_{k-1}^{-1}(xI_N - B_{k-1}) \\ - \big(V_{n+k-1}(x)\gamma_{k-1} + V_{n+k-2}^{(1)}(x)\eta_{k-1}\big)$$

and so

$$V_{n-2}^{(k+1)}(x)A_k^{-1}C_k = V_{n-1}^{(k)}(x)A_{k-1}^{-1}(xI_N - B_{k-1}) - V_n^{(k-1)}(x),$$

and the result follows. ■

In a similar way, we get the following result.

Proposition 9. *The sequences of matrix polynomials, $\{G_n\}_{n\in\mathbb{N}}$ and $\{G_{n-1}^{(1)}\}_{n\in\mathbb{N}}$ are linearity independent solutions of (23). Moreover, for every $k\in\mathbb{N}$,*

$$G_{n-k}^{(k)\top}(x) = \widetilde{\gamma}_k\, G_n^\top(x) + \widetilde{\eta}_k\, G_{n-1}^{(1)\top}(x),$$

where

$$\widetilde{\gamma}_k = \Theta_*\begin{pmatrix} G_{k-2}^{(1)\top}(x) & G_{k-1}^{(1)\top}(x) \\ G_{k-1}^\top(x) & G_k^\top(x)\end{pmatrix}^{-1} \quad \text{and} \quad \widetilde{\eta}_k = \Theta_*\begin{pmatrix} G_{k-1}^\top(x) & G_k^\top(x) \\ G_{k-2}^{(1)\top}(x) & G_{k-1}^{(1)\top}(x)\end{pmatrix}^{-1};$$

moreover, the following relation holds

$$x\,G_{n-1}^{(k)\top}(x) = C_k G_n^{(k-1)\top}(x) + C_k B_{k-1} C_k^{-1} G_{n-1}^{(k)\top}(x) + C_k A_{k-1} C_{k+1}^{-1} G_{n-2}^{(k+1)\top}(x).$$

Since $\{Q_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ are also solutions of (22) with initial conditions $Q_0(y) = \left\langle I_N, \frac{I_N}{y-x}\right\rangle_u$, $Q_{-1}(y) = C_0^{-1}$, $R_0^\top(y) = \left\langle \frac{I_N}{y-x}, I_N\right\rangle_u$, and $R_{-1}^\top(y) = A_{-1}^{-1}$; then for $x\in\mathbb{C}\setminus\operatorname{supp}(u)$ it is clear that

$$Q_n(x) = V_n(x)Q_0(x) - V_{n-1}^{(1)}(x)A_0^{-1}, \tag{34}$$

$$R_n^\top(x) = R_0^\top(x)\,G_n^\top(x) - C_1^{-1}G_{n-1}^{(1)\top}(x). \tag{35}$$

From here we get

$$V_{n-1}^{(1)}(x) = \left\langle \frac{V_n(x)-V_n(y)}{x-y}, I_N\right\rangle_u A_0, \quad G_{n-1}^{(1)\top}(x) = C_1\left\langle I_N, \frac{G_n(x)-G_n(y)}{x-y}\right\rangle_u.$$

Using a similar argument as in Proposition 7,

$$\det(W(Q_n, V_{n-k}^{(k)})) = \prod_{j=k}^{n}\det(A_j^{-1})\,\det(C_j)\,\det(Q_{k-1}(x)),$$

Proposition 10. *The sequences* $\{Q_n\}_{n\in\mathbb{N}}$ *and* $\{V^{(k)}_{n-k}\}_{n\in\mathbb{N}}$ *are linearly independent solutions of* (22) *in* $x\in\mathbb{C}\setminus\operatorname{supp}(u)$. *Moreover,*

$$V^{(k)}_{n-k}(x)=V_n(x)\,\alpha_k-Q_n(x)\,\beta_k\,,$$

with

$$\alpha_k=\Theta_*\begin{pmatrix}Q_{k-1}(x) & V_{k-1}(x)\\ Q_k(x) & V_k(x)\end{pmatrix}^{-1}\quad\text{and}\quad\beta_k=\Theta_*\begin{pmatrix}V_{k-1}(x) & Q_{k-1}(x)\\ V_k(x) & Q_k(x)\end{pmatrix}^{-1}.$$

In the same way, $\{R^\top_n\}_{n\in\mathbb{N}}$ *and* $\{G^{(k)\top}_{n-k}\}_{n\in\mathbb{N}}$ *also are linearly independent solutions of* (23) *in* $x\in\mathbb{C}\setminus\operatorname{supp}(u)$; *moreover,*

$$(G^{(k)}_{n-k}(x))^\top=\widetilde{\alpha}_k\,G^\top_n(x)-\widetilde{\beta}_k\,R^\top_n(x)\,,$$

with

$$\widetilde{\alpha}_k=\Theta_*\begin{pmatrix}R^\top_{k-1}(x) & R^\top_k(x)\\ G^\top_{k-1}(x) & G^\top_k(x)\end{pmatrix}^{-1}\quad\text{and}\quad\widetilde{\beta}_k=\Theta_*\begin{pmatrix}G^\top_{k-1}(x) & G^\top_k(x)\\ R^\top_{k-1}(x) & R^\top_k(x)\end{pmatrix}^{-1}.$$

Proposition 11. *The following Christoffel-Darboux formulas hold*

$$\sum_{k=1}^{n}V^{(k)}_{n-k}(y)A^{-1}_{k-1}\,V_{k-1}(x)=\frac{V_n(x)-V_n(y)}{x-y}\,,$$
$$\sum_{k=1}^{n}G^\top_{k-1}(x)\,C^{-1}_k\,G^{(k)\top}_{n-k}(y)=\frac{G^\top_n(x)-G^\top_n(y)}{x-y}\,,$$

as well as its confluent expression

$$\sum_{k=1}^{n}V^{(k)}_{n-k}(x)A^{-1}_{k-1}V_{k-1}(x)=V'_n(x)\,,$$
$$\sum_{k=1}^{n}G^\top_{k-1}(x)\,C^{-1}_kG^{(k)\top}_{n-k}(x)=(G^\top_n(x))'\,.$$

Proof: For $k\leq n$

$$y\,V^{(k)}_{n-k}(y)A^{-1}_{k-1}=V^{(k-1)}_{n-k+1}(y)+V^{(k)}_{n+k}(y)A^{-1}_{k-1}\,B_{k-1}+V^{(k+1)}_{n-k-1}(y)A^{-1}_k\,C_k\,,$$
$$x\,A^{-1}_{k-1}V_{k-1}(x)=V_k(x)+A^{-1}_{k-1}B_{k-1}\,V_{k-1}(x)+A^{-1}_{k-1}C_{k-1}\,V_{k-2}(x)\,.$$

From here

$$(y-x)\,V^{(k)}_{n-k}(y)A^{-1}_{k-1}V_{k-1}(x)=\Big(V^{(k-1)}_{n-k+1}(y)\,V_{k-1}(x)-V^{(k)}_{n-k}(y)A_{k-1}\,C_{k-1}\,V_{k-2}(x)\Big)$$
$$-\left(V^{(k)}_{n-k}(y)\,V_{k}(x)-V^{(k+1)}_{n-k-1}(y)A_{k}\,C_{k}\,V_{k-1}(x)\right).$$

Summing the above on k from 1 to n and taking into account that for every $k\in\mathbb{N}$, $V^{(k)}_{0}(x)=I_N$ and $V^{(k)}_{-1}(x)=\mathbf{0}$ we get the result. The confluent form is obtained when y tends to x. Formulas for the sequences $\{G_n\}_{n\in\mathbb{N}}$ and $\{G^{(1)}_n\}_{n\in\mathbb{N}}$ are deduced in a similar way. ∎

As a consequence of Proposition 11, we find that for $k=1,\dots,n$,

$$V^{(k)}_{n-k}(x)A^{-1}_{k-1}=\sum_{j=1}^{n}V^{(j)}_{n-j}(x)A^{-1}_{j-1}\left\langle V_{j-1}(y),G_{k-1}(y)\right\rangle_u=\left\langle\frac{V_n(y)-V_n(x)}{y-x},G_{k-1}\right\rangle_u,$$

$$C^{-1}_{k}G^{(k)\top}_{n-k}(x)=\sum_{j=1}^{n}\left\langle V_{k-1}(y),G_{j-1}(y)\right\rangle_u C^{-1}_{j}G^{(j)\top}_{n-j}(x)$$
$$=\left\langle V_{k-1}(y),\frac{G_n(y)-G_n(x)}{y-x}\right\rangle_u.$$

When $k=1$, we recover the classical formula for $V^{(1)}_{n-1}(x)$, $G^{(1)}_{n-1}(x)$. Algebraic manipulations for the above equations yield

$$V^{(k)}_{n-k}(x)A^{-1}_{k-1}=V_n(x)R^{\top}_{k-1}(x)-Q_n(x)\,G^{\top}_{k-1}(x),$$
$$C^{-1}_{k}G^{(k)\top}_{n-k}(x)=Q_{k-1}(x)\,G^{\top}_{n}(x)-V_{k-1}(x)R^{\top}_{n}(x).$$

The above, together with Proposition 10, yield

$$R^{\top}_{k-1}A_{k-1}=\Theta_*\begin{pmatrix}Q_{k-1}(x)&V_{k-1}(x)\\Q_k(x)&V_k(x)\end{pmatrix}^{-1},\;G^{\top}_{k-1}(x)A_{k-1}=\Theta_*\begin{pmatrix}V_{k-1}(x)&Q_{k-1}(x)\\V_k(x)&Q_k(x)\end{pmatrix}^{-1},$$

$$C_k\,Q_{k-1}(x)=\Theta_*\begin{pmatrix}R^{\top}_{k-1}(x)&R^{\top}_{k}(x)\\G^{\top}_{k-1}(x)&G^{\top}_{k}(x)\end{pmatrix}^{-1},\;C_k\,V_{k-1}(x)=\Theta_*\begin{pmatrix}G^{\top}_{k-1}(x)&G^{\top}_{k}(x)\\R^{\top}_{k-1}(x)&R^{\top}_{k}(x)\end{pmatrix}^{-1}.$$

5. Outer Ratio Asymptotics

First of all, we are going to state two important theorems which can be find in [3, 4], see also [5, 7].

Theorem 3 (Markov type Theorem)**.** *Let* $\{V_n\}_{n\in\mathbb{N}}$ *and* $\{G_n\}_{n\in\mathbb{N}}$ *be the sequence of matrix biorthogonal polynomials with respect to* u *and* $\{V_n^{(1)}\}_{n\in\mathbb{N}}$ *and* $\{G_n^{(1)}\}_{n\in\mathbb{N}}$ *be the corresponding sequences of associated polynomials. Then,*

$$\lim_{n\to\infty} V_n^{-1}(x)\, V_{n-1}^{(1)}(x) = \left\langle \frac{I_N}{x-y}, I_N \right\rangle_u A_0\,, \quad x \in \mathbb{C}\setminus\Gamma^{(0)}$$

$$\lim_{n\to\infty} G_{n-1}^{(1)\top}(x)\, G_n^{-\top}(x) = C_1 \left\langle \frac{I_N}{x-y}, I_N \right\rangle_u\,, \quad x \in \mathbb{C}\setminus\Gamma^{(0)}$$

and the convergence holds uniformly on compact subsets of $\mathbb{C}\setminus\Gamma^{(0)}$.

In the sequel, given three matrices α, γ (nonsingular) and β, we define the *left-orthogonal second kind Chebyshev matrix polynomials*, $\{U_n^{\gamma,\beta,\alpha}\}_{n\in\mathbb{N}}$, by the recurrence formula

$$x\, U_n^{\gamma,\beta,\alpha}(x) = \gamma\, U_{n+1}^{\gamma,\beta,\alpha}(x) + \beta\, U_n^{\gamma,\beta,\alpha}(x) + \alpha\, U_{n-1}^{\gamma,\beta,\alpha}(x), \quad n \geq 0\,, \tag{36}$$

with initial conditions $U_0^{\gamma,\beta,\alpha}(x) = I_N$ and $U_{-1}^{\gamma,\beta,\alpha}(x) = \mathbf{0}$, as well as the *right-orthogonal second kind Chebyshev matrix polynomial*, $\{T_m^{\alpha,\beta,\gamma}(x)\}_{m\in\mathbb{N}}$, given by

$$x\, T_n^{\alpha,\beta,\gamma}(x) = T_{n+1}^{\alpha,\beta,\gamma}(x)\,\alpha + T_n^{\alpha,\beta,\gamma}(x)\,\beta + T_{n-1}^{\alpha,\beta,\gamma}(x)\,\gamma\,, \quad n \geq 0\,,$$

with initial conditions $T_0^{\alpha,\beta,\gamma}(x) = I_N$ and $T_{-1}^{\alpha,\beta,\gamma}(x) = \mathbf{0}$. We denote by $u^{\gamma,\beta,\alpha}$ the matrix of linear functionals for which the polynomials $U_n^{\gamma,\beta,\alpha}(x)$ and $T_n^{\alpha,\beta,\gamma}(x)$ are biorthogonal.

Theorem 4 (Outer Ratio Asymptotics)**.** *Let* $\{V_n\}_{n\in\mathbb{N}}$ *and* $\{G_n\}_{n\in\mathbb{N}}$ *be the sequences of matrix biorthonormal polynomials with respect to* u*. If* $A_n \to A$*,* $B_n \to B$*, and* $C_n \to C$ *with* A, C *nonsingular matrices, then*

$$\lim_{n\to\infty} V_{n-1}(x)\, V_n^{-1}(x) A_{n-1}^{-1} = \left\langle \frac{I_N}{x-y}, I_N \right\rangle_{u^{C,B,A}}\,, \quad x \in \mathbb{C}\setminus\Gamma^{(0)}\,,$$

$$\lim_{n\to\infty} C_n^{-1}\, G_n^{-\top}(x)\, G_{n-1}^{\top}(x) = \left\langle \frac{I_N}{x-y}, I_N \right\rangle_{u^{C,B,A}}\,, \quad x \in \mathbb{C}\setminus\Gamma^{(0)}\,;$$

and the convergence holds uniformly on compact subsets of $\mathbb{C}\setminus\Gamma^{(0)}$*. Moreover, if* $F_{C,B,A}(x) = \left\langle \frac{I_N}{x-y}, I_N \right\rangle_{u^{C,B,A}}$*, then* $F_{C,B,A}(x)$ *is an analytic matrix function satisfying the matrix equation*

$$C\, F_{C,B,A}(x)\, A\, F_{C,B,A}(x) + (B - xI_N)\, F_{C,B,A}(x) + I_N = \mathbf{0}\,.$$

Corollary 2. *With the conditions of Theorem 4, for $x \in \mathbb{C} \setminus \Gamma^{(0)}$, the following limits hold uniformly on compact subsets of $\mathbb{C} \setminus \Gamma^{(0)}$:*

$$\lim_{n\to\infty} V_n^{-1}(x) Q_n(x) = \mathbf{0}, \quad \lim_{n\to\infty} G_n^{-1}(x) R_n(x) = \mathbf{0}, \tag{37}$$

$$\lim_{n\to\infty} V_{n-1}^{-1}(x) C_n^{-1} G_n^{-\top}(x) = \mathbf{0}, \quad \lim_{n\to\infty} V_n^{-1} A_{n-1}^{-1} G_{n-1}^{-\top}(x) = \mathbf{0}. \tag{38}$$

Proof: The limits in (37) are obtained from (34)-(35) and Markov theorem. On the other hand, from Liouville-Ostrogradski formulas we get

$$\begin{aligned} V_{n-1}^{-1}(x) Q_{n-1}(x) - R_n^\top(x) G_n^{-\top}(x) &= V_{n-1}^{-1}(x) C_n^{-1} G_n^{-\top}, \\ R_{n-1}^\top(x) G_{n-1}^\top(x) - V_n(x) Q_n(x) &= V_n^{-1}(x) A_{n-1}^{-1} G_{n-1}^{-\top}. \end{aligned}$$

Taking the limit when $n \to \infty$ in the above identities we get (38). ■

Theorem 5. *Let assume the moment problem for u is determined and $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ be the sequence of matrix biorthogonal polynomials with respect to u, as well as $\{V_n^{(k)}\}_{n\in\mathbb{N}}$ and $\{G_n^{(k)}\}_{n\in\mathbb{N}}$ be the respective sequences of the k-th associated polynomials. Then, for all $k \in \mathbb{N}$, the following limits*

$$\begin{aligned} \lim_{n\to\infty} V_n^{-1}(x) V_{n-k}^{(k)}(x) &= R_{k-1}^\top(x) A_{k-1}, \quad x \in \mathbb{C} \setminus \Gamma^{(k)} \\ \lim_{n\to\infty} G_{n-k}^{(k)\top}(x) G_n^{-\top}(x) &= C_k Q_{k-1}(x), \quad x \in \mathbb{C} \setminus \Gamma^{(k)} \end{aligned}$$

holds uniformly on compact subsets of $\mathbb{C} \setminus \Gamma^{(k)}$.

Proof: We only prove the first formula. The second one follows in a similar way. We use induction on k. When $k = 1$ the result is a straightforward consequence of the matrix Markov theorem. Now assume that the result holds true for a k, by (30) we have that

$$V_{n-k-1}^{(k+1)}(x) = \left(x V_{n-k}^{(k)}(x) A_{k-1}^{-1} - V_{n-k+1}^{(k-1)}(x) - V_{n-k}^{(k)}(x) A_{k-1}^{-1} B_{k-1}\right) C_k^{-1} A_k,$$

so, from induction hypothesis

$$\lim_{n\to\infty} V_n^{-1}(x) V_{n-k-1}^{(k+1)}(x) = (x R_{k-1}^\top(x) - R_{k-2}^\top(x) A_{k-2} - Q_{k-1}^\top(x) B_{k-1}) C_k^{-1} A_k,$$

and so the convergence of $\{V_n^{-1}(x) V_{n-k-1}^{(k+1)}(x)\}$ holds uniformly on compact subsets of $\mathbb{C} \setminus \Gamma^{(k)}$ to $R_k^\top(x) A_k$. ■

Let $u^{(k)}$ be the matrix of linear functionals and $\{V_n^{(k)}\}_{n\in\mathbb{N}}$, $\{G_n^{(k)}\}_{n\in\mathbb{N}}$ the corresponding matrix biorthogonal polynomials with respect to u. From Markov theorem

$$\lim_{n\to\infty}(V_n^{(k)}(x))^{-1}V_{n-1}^{(k+1)}(x)=\left\langle\frac{I_N}{x-y},I_N\right\rangle_{u^{(k)}}A_k,\quad x\in\mathbb{C}\setminus\Gamma^{(k)},$$
$$\lim_{n\to\infty}G_{n-1}^{(k+1)\top}(x)(G_n^{(k)}(x))^{-\top}=C_{k+1}\left\langle\frac{I_N}{x-y},I_N\right\rangle_{u^{(k)}},\quad x\in\mathbb{C}\setminus\Gamma^{(k)},$$

and the fact that

$$(V_n^{(k)}(x))^{-1}V_{n-1}^{(k+1)}(x)=(V_n^{(k)}(x))^{-1}V_{n+k}(x)\,V_{n+k}^{-1}(x)\,V_{n-1}^{(k+1)}(x),$$
$$G_{n-1}^{(k+1)\top}(x)(G_n^{(k)}(x))^{-\top}=G_{n-1}^{(k+1)\top}(x)\,G_{n+k}^{-\top}(x)\,G_{n+k}^{\top}(x)(G_n^{(k)}(x))^{-\top},$$

we obtain that, for every $x\in\mathbb{C}\setminus\Gamma^{(k)}$,

$$\left\langle\frac{I_N}{x-y},I_N\right\rangle_{u^{(k)}}=A_{k-1}^{-1}R_{k-1}^{-\top}(x)R_k^{\top}(x),\tag{39}$$
$$\left\langle\frac{I_N}{x-y},I_p\right\rangle_{u^{(k)}}=Q_k(x)Q_{k-1}^{-1}(x)\,C_k^{-1}.$$

Theorem 6. *Let $\{V_n^{(k)}\}_{n\in\mathbb{N}}$ and $\{G_n^{(k)}\}_{n\in\mathbb{N}}$ be the sequences of k-th associated polynomials which are biorthonormal with respect to the matrix of linear functionals $u^{(k)}$. If $A_n\to A$, $B_n\to B$, and $C_n\to C$ with A,C nonsingular matrices, then*

$$\lim_{k\to\infty}\left\langle\frac{I_N}{z-x},I_p\right\rangle_{u^{(k)}}=\left\langle\frac{I_N}{z-x},I_N\right\rangle_{u^{A,B,C}},\quad z\in\mathbb{C}\setminus\bigcup_k\Gamma^{(k)},$$

and the convergence is uniform on compact subsets of $\mathbb{C}\setminus\bigcup_k\Gamma^{(k)}$.

Proof: Firs of all, we will prove that

$$\lim_{k\to\infty}\left\langle U_\ell^{A,B,C}(x),I_N\right\rangle_{u^{(k)}}=I_N\,\delta_{\ell,0},\tag{40}$$

Since $\{V_n^{(k)}\}_{n\in\mathbb{N}}$ is a basis of the bimodule of matrix polynomials, then for each $\ell\in\mathbb{N}$ there exists a set of matrices $\left(\Delta_{i,\ell,k}\right)_{i=0}^{l}$ such that

$$U_\ell^{A,B,C}(x)=\sum_{j=0}^{\ell}\Delta_{j,\ell,k}V_j^{(k)}(x).$$

From orthogonality

$$\left\langle U_\ell^{A,B,C}(x), I_N \right\rangle_{u^{(k)}} = \sum_{j=0}^{\ell} \Delta_{j,\ell,k} \left\langle V_j^{(k)}(x), I_N \right\rangle_{u^{(k)}} = \Delta_{0,\ell,k} . \tag{41}$$

Observe that (40) is proved if $\lim_{k\to\infty} \Delta_{j,\ell,k} = I_N \, \delta_{\ell,j}$. We proceed by induction on ℓ. For $\ell = 0$ the result is immediate since from (41), $\Delta_{0,0,k} = I_N$, and for $j \neq 0$, $\Delta_{j,0,k} = \mathbf{0}$. Now suppose the result is valid up to ℓ . From (36) we get

$$\begin{aligned}
\Delta_{j,\ell+1,k} &= \left\langle U_{\ell+1}^{A,B,C}(x), G_j^{(k)}(x) \right\rangle_{u^{(k)}} \\
&= \left\langle A^{-1}\big(x\, U_\ell^{A,B,C}(x) - B\, U_\ell^{A,B,C}(x) - C\, U_{\ell-1}^{A,B,C}(x)\big), G_j^{(k)}(x) \right\rangle_{u^{(k)}} \\
&= A^{-1} \left\langle x\, U_\ell^{A,B,C}(x), G_j^{(k)}(x) \right\rangle_{u^{(k)}} - A^{-1} B \Delta_{j,\ell,k} - A^{-1} C \Delta_{j,\ell-1,k} .
\end{aligned}$$

On the other hand, from the symmetry condition

$$\begin{aligned}
&A^{-1} \left\langle U_\ell^{A,B,C}(x), x\, G_j^{(k)}(x) \right\rangle_{u^{(k)}} \\
&\qquad = A^{-1} \left\langle U_\ell^{A,B,C}(x), C_{j+k+1}^{\top} G_{j+1}^{(k)}(x) + B_{j+k}^{\top} G_j^{(k)}(x) + A_{j+k-1}^{\top} G_{j-1}^{(k)}(x) \right\rangle_{u^{(k)}} \\
&\qquad = A^{-1} \big(\Delta_{j+1,\ell,k}\, C_{k+j+1} + \Delta_{j,\ell,k}\, B_{k+j} + \Delta_{j-1,\ell,k} A_{k+j-1}\big) .
\end{aligned}$$

From here we get that

$$\begin{aligned}
\Delta_{j,\ell+1,k} = A^{-1} \big(\Delta_{j+1,\ell,k}\, C_{k+j+1} + \Delta_{j,\ell,k}\, B_{k+j} + \Delta_{j-1,\ell,k} A_{k+j-1} \\
- B\, \Delta_{j,\ell,k} - C\, \Delta_{j,\ell-1,k}\big) .
\end{aligned} \tag{42}$$

Observe that for $j \leq \ell - 2$ or $j \geq \ell + 2$ the induction hypothesis and (42) show that $\lim_{k\to\infty} \Delta_{j,\ell+1,k} = \mathbf{0}$. Now, for $j = \ell - 1$, ℓ and $\ell + 1$ we get

$$\begin{aligned}
\lim_{k\to\infty} \Delta_{\ell-1,\ell+1,k} &= A^{-1} C - A^{-1} C = \mathbf{0} , \\
\lim_{k\to\infty} \Delta_{\ell,\ell+1,k} &= A^{-1} B - A^{-1} B = \mathbf{0} , \\
\lim_{k\to\infty} \Delta_{\ell+1,\ell+1,k} &= A^{-1} A = I_N .
\end{aligned}$$

We are now ready to prove that

$$\lim_{k\to\infty} \left\langle \frac{I_N}{z - x}, I_N \right\rangle_{u^{(k)}} = \left\langle \frac{I_N}{z - x}, I_N \right\rangle_{u^{A,B,C}} , \quad z \in \mathbb{C} \setminus \bigcup_k \Gamma^{(k)} .$$

If the above is not true, then there exist an $z \in \mathbb{C} \setminus \bigcup_k \Gamma^{(k)}$ and a sequence of nonnegative integers $(k_m)_{m\in\mathbb{N}}$, such that

$$\left\| \left\langle \frac{I_N}{z-x}, I_N \right\rangle_{u^{(k_m)}} - \left\langle \frac{I_N}{z-x}, I_N \right\rangle_{u^{A,B,C}} \right\|_2 > C > 0\,, \tag{43}$$

where C is a constant. Since $\{u^{(k)}\}_{k\in\mathbb{N}}$ is a sequence of matrices of linear functionals with compact support contained in $\bigcup_k \Gamma^{(k)}$ and such that $\langle I_N, I_N\rangle_{u^{(k)}} = I_N$, then from Banach-Alaoglu's theorem, there is a subsequence $(s_m)_{m\in\mathbb{N}}$ from $(k_m)_{m\in\mathbb{N}}$ such that $du^{(s_m)}$ converge to a matrix of linear functionals v with compact support contained in $\mathbf{D}_M$, for every matrix polynomial f, i.e.

$$\lim_{m\to\infty} \langle f, I_N\rangle_{u^{(s_m)}} = \langle f, I_N\rangle_v \,.$$

In particular, if we take $f = U_\ell^{A,B,C}(x)$, then $\left\langle U_\ell^{A,B,C}(x), I_N\right\rangle_v = I_N\,\delta_{\ell,0}$. Since $\{U_n^{A,B,C}\}_{n\in\mathbb{N}}$ is a basis of $C^{N\times N}[x]$ and v, $u^{A,B,C}$ have compact support, we get $v \equiv u^{A,B,C}$ but the inequality (43) is not possible. The uniform convergence follows from Stieltjes-Vitali theorem. ■

Corollary 3. *Under the hypotesis of Theorem 6 we have that,* $\{Q_n\,Q_{n-1}^{-1}\,C_n^{-1}\}_{n\in\mathbb{N}}$ *and* $\{A_{n-1}^{-1}R_{n-1}^{-\top}R_n^{\top}\}_{n\in\mathbb{N}}$ *uniformly converges to* $F_{A,B,C}$ *on compact subsets of* $\mathbb{C} \setminus \bigcup_k \Gamma^{(k)}$.

Since the recurrence relations for $\{Q_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ can be rewritten as

$$\begin{aligned} xI_N =& A_n\left(Q_{n+1}(x)\,Q_n^{-1}(x)\,C_{n+1}^{-1}\right)C_{n+1} + B_n + (Q_n(x)\,Q_{n-1}^{-1}(x)\,C_n^{-1})^{-1}\,,\\ xI_N =& A_n\left(A_n^{-1}R_n^{-\top}(x)\,Q_{n+1}^{\top}(x)\right)C_{n+1} + B_n + (A_{n-1}^{-1}R_{n-1}^{-\top}(x)\,Q_n^{\top}(x))^{-1}\,, \end{aligned}$$

then the analytic function $F_{A,B,C}$ also satisfies a matrix equation

$$A\,F_{A,B,C}(x)\,C\,F_{A,B,C}(x) + (B - xI_N)\,F_{A,B,C}(x) + I_N = \mathbf{0}\,.$$

Corollary 4. *Under the hypothesis of Theorem 6 we have that the sequences* $\{R_n^{-\top}\,V_n^{-1}\}_{n\in\mathbb{N}}$ *and* $\{G_n^{-\top}\,Q_n^{-1}\}_{n\in\mathbb{N}}$ *uniformly converge on compact subsets of* $\mathbb{C} \setminus \bigcup_k \Gamma^{(k)}$ *to* $F_{C,B,A}^{-1}(x) - A\,F_{A,B,C}(x)\,C$.

Proof: As a consequence of Christoffel-Darboux formulas when $x = y$

$$\begin{aligned} \left(G_{n+1}(x)\,G_n^{-1}(x)\right)^{\top}C_{n+1} - A_n\,Q_{n+1}(x)\,Q_n^{-1}(x) &= G_n(x)^{-\top}Q_n^{-1}(x)\\ A_n\,V_{n+1}(x)\,V_n^{-1}(x) - \left(R_{n+1}(x)\,R_n^{-1}(x)\right)^{\top}C_{n+1} &= R_n(x)^{-\top}V_n^{-1}(x)\,, \end{aligned}$$

from here the limit follows. ■

When the sesquilinear form, $\langle .,. \rangle$, is associated with a positive definite symmetric matrix of measures, μ, we have the representation

$$\langle P(x), Q(x) \rangle = \int P(x)\, d\mu\, Q^{\top}(x)\,.$$

Here we have orthonormality i.e. $V_n \equiv G_n$ and they satisfy a recurrence relation

$$x\, V_n(x) = A_n V_{n+1}(x) + B_n V_n(x) + A_{n-1}^{\top} V_{n-1}(x)\,, \quad n \geq 0\,,$$

with initial conditions $V_{-1}(x) = \mathbf{0}$ and $V_0(x) = I_N$, A_n nonsingular matrices and B_n Hermitian matrices. Thus, if $\{V_n^{(k)}\}_{n\in\mathbb{N}}$ is the sequence of k-th associated matrix polynomials which are orthonormal with respect to the matrix of measures $d\mu^{(k)}$ and $A_n \to A$, $B_n \to B$ with A a nonsingular matrix and B a Hermitian matrix, then

$$\lim_{k\to\infty} \int \frac{d\mu^{(k)}(x)}{z-x} = \int \frac{dW_{A^{\top},B^{\top}}(x)}{z-x}\,,$$

and the convergence holds uniformly on compact subsets of $\mathbb{C} \setminus \bigcup_k \Gamma^{(k)}$.

Here, $dW_{A^{\top},B^{\top}}$ denotes the matrix of measures for which the polynomials $U_n^{A^{\top},B^{\top}}(x)$ defined by the recurrence formula

$$x\, U_n^{A^{\top},B^{\top}}(x) = A\, U_{n+1}^{A^{\top},B^{\top}}(x) + B\, U_n^{A^{\top},B^{\top}}(x) + A^{\top}\, U_{n-1}^{A^{\top},B^{\top}}(x), \quad n \geq 0\,,$$

are orthonormal.

Moreover, if we assume that the matrix A is positive definite and the matrix B is Hermitian, then in [5] is showed that

$$\begin{aligned}\int \frac{dW_{A^{\top},B^{\top}}(x)}{z-x} = {} & \frac{1}{2} A^{-1}(zI_N - B)A^{-1} \\ & - \frac{1}{2} A^{-1/2} \left(\sqrt{A^{-1/2}(B - zI_N)A^{-1}(B - zI_N)A^{-1/2}} - 4I_N \right) A^{-1/2}\,,\end{aligned}$$

for $z \notin \operatorname{supp}(W_{A^{\top},B^{\top}})$.

References

[1] C. Álvarez-Fernández, G. Ariznabarreta, J. C. García-Ardila, M. Mañas and F. Marcellán, *Christoffel transformations for matrix orthogonal polynomials in the real line and the non-Abelian 2D Toda lattice hierarchy*, Int. Math. Res. Not. **2017** (2017) 5, 1285–1341.

[2] A. Branquinho, F. Marcellán and J. C. García-Ardila, *Ratio asymptotic for biorthogonal matrix polynomials with unbounded recurrence coefficients*, Preprint, 2017, Pré-publicações do DMUC **17** 27, `http://www.mat.uc.pt/preprints/ps/p1727.pdf`.
[3] A. Branquinho, F. Marcellán and A. Mendes, *Vector Interpretation of the Matrix Orthogonality on the Real Line*, Acta Appl. Math. **112** (2010), 357–383.
[4] A. Branquinho, F. Marcellán and A. Mendes, *Relative asymptotics for orthogonal matrix polynomials*, Linear Algebra Appl. **437** (2012), 1458–1481.
[5] A. J. Durán, *Ratio Asymptotics for Orthogonal Matrix Polynomials*, J. Approx. Theory **100** (1999), 304–344.
[6] A. J. Durán, *A generalization of Favard's Theorem for polynomials satisfying a recurrence relation*, J. Approx. Theory **74** (1993), 83–109.
[7] A. J. Durán, *Markov theorem for orthogonal matrix polynomials*, Canad. J. Math **48** (1996), 1180–1195.
[8] R. A. Horn and C. R. Johnson, Topics in matrix analysis, Cambridge University Press, Cambridge, 1991.
[9] P. Nevai, Orthogonal Polynomials, Memoir Amer. Math. Soc. **213**, Amer. Math. Soc., Providence, 1979.
[10] S. Lang, Algebra, Third Edition, Springer-Verlag, New York, 2005.
[11] H. Poincaré, *Sur les Equations Linéaires aux différentielles ordinaires et aux différences Finies*, Amer. J. Math. **7** (1985), 203-258.
[12] E. A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials*, Mat. Sb. **103** (105) (1977) 237-252 (in Russian); Math. USSR-Sb **32** (1977) 199-213 (English translation).
[13] L. Rowen, Ring Theory, Vol. I, Academic Press, San Diego, CA, 1988.
[14] W. Van Assche, *Orthogonal polynomials, associated polynomials and functions of the second kind*, J. Comput. Appl. Math. **37** (1991), 237-249.

A. Branquinho
CMUC and Department of Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz, 3001-501 COIMBRA, Portugal.
E-mail address: `ajplb@mat.uc.pt`

J.C. García-Ardila
Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad, 30, 28911, Leganés, Spain
E-mail address: `jugarcia@math.uc3m.es`

F. Marcellán
Instituto de Ciencias Matemáticas (ICMAT) and Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad, 30, 28911, Leganés, Spain
E-mail address: `pacomarc@ing.uc3m.es`