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EXPONENTIAL INEQUALITIES FOR MANN'S ITERATIVE SCHEME WITH FUNCTIONAL RANDOM ERRORS

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ABSTRACT: In this paper, we deal with an iteration method for approximating a fixed point of a contraction mapping using the Mann's algorithm under functional random errors. We first show its almost complete convergence to the fixed point by mean of an exponential inequality and then we specify the induced rate of convergence. We finally build a confidence set for the fixed point. Moreover, some numerical examples are considered.

KEYWORDS: Almost complete convergence; Confidence set; Fixed point-iteration; Mann's algorithm; Rate of convergence; Stochastic methods.
MATH. SUBJECT CLASSIFICATION (2000): 76M35; 60H35.

1. Introduction

The main objective of studies in fixed point theory is to find solutions for the following equation, which is commonly known as fixed point equation:

$$F(x) = x \tag{1}$$

where F is a self-map of an ambient space X and $x \in X$.

The most well-known result in fixed point theory is Banach's contraction mapping principle; it guarantees that a contraction mapping of a complete metric space to itself has a unique fixed point which may be obtained as the limit of an iteration scheme defined by repeated images under the mapping of an arbitrary starting point in the space. As such, it is a constructive fixed point theorem and hence, may be implemented for the numerical computation of the fixed point.

To solve equations given by (1), two types of methods are normally used: direct methods and iterative methods. Due to various reasons, direct methods can be impractical or fail in solving equations (1) because it leads to the inversion of a certain function, thing that is not easy to do and thus, iterative methods become a viable alternative. For this reason, the iterative

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approximation of fixed points has become one of the major and basic tools in the theory of equations.

[Mann(1953)] introduced an iterative scheme and employed it to approximate the solution of a fixed point problem defined by non-expansive mapping where Picard's iterative scheme fails to converge. Later, [Ishikawa(1974)] introduced an iterative method to obtain the convergence of a Lipschitzian pseudo-contractive operator when Mann's iterative scheme is not applicable. Many authors studied the convergence theorems and stability problems in Banach spaces and metric spaces (see; e.g. [Berinde(2007), Cegielski(2012), Chang(1998), Kang(2016), Osilike(1998), Panyanak(2007), Shahzada(2009), Xu(1992)]) using the Mann's iteration scheme or the Ishikawa's iteration scheme in deterministic frame. Some theoretical results on Mann-Ishikawa algorithm with errors can be found in various literatures (see e.g. [Agarwal(2001), Chang(2003), Huang(199), Kazmi(1997), Kim(2001), Liu(1995), Kang(2001), Kim(2002), Osilike(1997), Xu(1998)]).

In the last twenty years, many papers have been published on random fixed point theory. The study of random fixed point theory is playing an increasing role in mathematics and engineering sciences. Recently, it received considerable attention due to enormous applications in many important areas such as nonlinear analysis, probability theory and for the study of random equations arising in various engineering sciences.

[Choudhury(2003), Choudhury(1999)] has suggested and analyzed random Mann's iterative sequence in separable Hilbert spaces for finding random solutions and random fixed points for some kind of random equations and random operators. Kim and [Okeke(2015)], introduced the random Picard-Mann hybrid iterative process. They have established the strong convergence theorems and summable almost T-stability of the random Picard-Mann hybrid iterative process and the random Mann-type iterative process generated by a generalized class of random operators in separable Banach spaces.

[Chugh(2016)] studied the strong convergence and stability of a new two-step random iterative scheme with errors for accretive Lipschitzian mapping in real Banach spaces, while [Cho(2008)], has built a random Ishikawa's iterative sequence with errors for random strongly pseudo-contractive operator in separable Banach spaces and proved that under suitable conditions, this random iterative sequence with errors converges to a random fixed point of the operator.

[Saluja(2013)] proved that if a random Mann's iteration scheme defined by two random operators is convergent under some contractive inequality, the limit point is a common fixed point of each of two random operators in Banach space.

In [Arunchai(2013)], a random fixed point theorem was obtained for the sum of a weakly-strongly continuous random operator and a non-expansive random operator which contains as a special Krasnoselskii type of Edmund and O'Regan via the method of measurable selectors. We note some recent works on random fixed points in [Tufa(2016), Agrawa (2016), Beg(2014), Chugh(2014), Chugh(2016), Hussaina(2016), Okeke(2015)].

In this paper, we deal with iteration methods for approximating a fixed point of the function using the Mann's algorithm with functional random errors. We first show its complete convergence to the fixed point by mean of an exponential inequality. This inequality will allow us to specify a convergence rate and the possibility of building a confidence set for the present fixed point.

1.1. Some fixed point algorithms. Let X be a normed linear space and $F : X \to X$ a given operator. Let $x_0 \in X$ be arbitrary. The sequence $(x_n)_n \subset X$ defined by

$$x_{n+1} = F(x_n) \tag{2}$$

is called the Picard's iteration (see. [Picard(1890)]).

The sequence $(x_n)_n \subset X$ defined by

$$x_{n+1} = (1 - a_n) x_n + a_n F(x_n), n \in \mathbb{N}^* \tag{3}$$

where $(a_n)_n$ is a real sequence of positive numbers satisfying the following conditions

1. $a_0 = 1$
2. $0 \leq a_n < 1, \forall\, n \geq 1$
3. $\sum_n a_n = +\infty$

is called the Mann's iteration or Mann's iterative scheme (see. [Mann(1953)]).

The sequence $(x_n)_n \subset X$ defined by

$$\begin{aligned} x_{n+1} &= (1 - a_n) x_n + a_n F(y_n), n \geq 1 \\ y_n &= (1 - b_n) x_n + b_n F(x_n), n \geq 1 \end{aligned} \tag{4}$$

where $(a_n)_n$ and $(b_n)_n$ are real sequences of positive numbers satisfying the conditions

$$\begin{array}{ll} 1. & 0 \leq a_n, b_n < 1 \text{ for all } n \\ 2. & \lim\limits_{n \to +\infty} b_n = 0 \\ 3. & \sum\limits_n a_n b_n = +\infty \end{array}$$

and $x_0 \in X$ is arbitrary. This procedure is called the Ishikawa's iteration or Ishikawa's iterative procedure (see. [Ishikawa(1974)]).

The sequence $(x_n)_n \subset X$ defined by

$$x_{n+1} = \frac{1}{2} \left(F\left(x_n\right) + x_n \right)$$

is called the Krasnoselskii's iteration (see. [Krasnosel'skii(1955)]).

Remark 1. *For $a_n = \frac{1}{2}$, the iteration (3) reduces to the so-called Krasnoselskii's iteration while for $a_n = 1$ we obtain the Picard's iteration (2), or the method of successive approximations, as it is commonly known. Obviously, for $b_n = 0$ the Ishikawa's iteration (4) reduces to (3).*

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{B}$ a real separable Banach space. Let $(\mathbb{B}, \mathfrak{B})$ be a measurable space, where $\mathfrak{B}$ denotes the σ-algebra of all Borel subsets generated by all open subsets in $\mathbb{B}$, and $F : \mathbb{B} \to \mathbb{B}$ a contraction mapping

$$\forall\ x, y \in \mathbb{B}, \left\| F\left(x\right) - F\left(y\right) \right\| \leq c \left\| x - y \right\|, c \in (0, 1)$$

Under this condition, the Banach's fixed point theorem states that F has a unique fixed point x^*.

Let $\left(x_n\right)_n$ be a sequence obtained by a certain fixed point iteration procedure that ensures its convergence to a fixed point x^* of F. Specifically for the Mann's algorithm, when calculating $\left(x_n\right)_n$, we usually follow these steps:

(1) We choose the initial approximation $x_0 \in \mathbb{B}$.
(2) We compute $x_1 = \left(1 - a_0\right) x_0 + a_0 F\left(x_0\right)$ but, due to various errors that occur during computations (rounding errors, numerical approximations of functions, derivatives or integrals, etc.), we do not get the exact value of x_1, but a different one, say y_1, which is however close enough to x_1, i.e., $y_1 - x_1 = \xi_1$.

(3) Consequently, when computing $x_2 = (1-a_1)\,x_1 + a_1 F(x_1)$, we will actually compute x_2 as $x_2 = (1-a_1)\,y_1 + a_1 F(y_1)$ and so, instead of the theoretical value x_2, we will obtain in fact another value, say y_2, again close enough to x_2, i.e., $y_2 - x_2 = \xi_2, \cdots$, and so forth.

In this way, instead of the theoretical sequence $(x_n)_n$ defined by the given iterative method, we will practically obtain an approximate sequence $(y_n)_n$. We shall consider the given fixed point iteration method to be numerically stable if and only if, for y_n close enough (in some sense) to x_n at each stage, the approximate sequence $(y_n)_n$ still converges to the fixed point of F. That is to say,

$$x_{n+1} = (1-a_n)\,x_n + a_n F(x_n) + \xi_n$$

Unfortunately, the definitions of [Liu(1995)], which depend on the convergence of the error terms, is against the randomness of errors. Hence, we need a new definition as follows,

$$x_{n+1} = (1-a_n)\,x_n + a_n F(x_n) + b_n \xi_n$$

with $(\xi_n)_n$ a sequence of independent functional random variables denoting noise which is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values into Banach spaces $\mathbb{B}$. Moreover, assume that $(\xi_n)_n$ is zero mean and $\sup_n \mathbb{E}\,\|\xi_n\| < \infty$.

In this paper, we use the following stochastic Mann's algorithm

$$x_{n+1} = (1-a_n)\,x_n + a_n F(x_n) + b_n \xi_n \tag{5}$$

satisfying,

$$\sum_{n=1}^{\infty} a_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty$$

(The condition $\sum_{n=1}^{+\infty} a_n = +\infty$ is sometimes replaced by $\sum_{n=1}^{\infty} a_n (1-a_n) = +\infty$).

Without loss of generality, we take

$$a_n = \frac{a}{n} \text{ and } b_n = \frac{a}{n^2}$$

In this case, the stochastic Mann's algorithm (5) takes the form

$$x_{n+1} = \left(1 - \frac{a}{n}\right) x_n + \frac{a}{n}\left(F(x_n) + \frac{1}{n}\xi_n\right) \tag{6}$$

Lemma 2. *By using the formula of the algorithm (6), one obtains for*

$$\|x_1 - x^*\| \leq N,$$

the following inequality

$$\|x_{n+1} - x^*\| \leq N \prod_{i=1}^{n} \left(1 - \frac{a\,(1-c)}{i}\right) + \sum_{i=1}^{n} \frac{a}{i^2} \prod_{j=i+1}^{n} \left(1 - \frac{a\,(1-c)}{j}\right) \|\xi_i\| \quad (7)$$

Proof: By adding and subtracting x^* and using that $F(x^*) = x^*$, we obtain

$$x_{n+1} - x^* = \left(1 - \frac{a}{n}\right)(x_n - x^*) + \frac{a}{n}\left(F(x_n) - F(x^*) + \frac{1}{n}\xi_n\right)$$

Using the last formula and the contraction of F, we get

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \left(1 - \frac{a\,(1-c)}{n}\right)\|x_n - x^*\| + \frac{a}{n^2}\|\xi_n\| \\ &\leq \|x_1 - x^*\| \prod_{i=1}^{n}\left(1 - \frac{a\,(1-c)}{i}\right) + \sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1 - \frac{a\,(1-c)}{j}\right)\|\xi_i\| \\ &\leq N\prod_{i=1}^{n}\left(1 - \frac{a\,(1-c)}{i}\right) + \sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1 - \frac{a\,(1-c)}{j}\right)\|\xi_i\| \end{aligned}$$

as required. ■

Lemma 3. *For every positive constant a such that $a \in (0,1)$, we have the following inequality*

$$\prod_{j=i+1}^{n}\left(1 - \frac{a\,(1-c)}{j}\right) \leq \left(\frac{i+1}{n+1}\right)^{a(1-c)} \quad (8)$$

Proof: We have,

$$\prod_{j=i+1}^{n}\left(1 - \frac{a\,(1-c)}{j}\right) \leq \exp\left(-a\,(1-c)\sum_{j=i+1}^{n}\frac{1}{j}\right) \leq \left(\frac{i+1}{n+1}\right)^{a(1-c)}$$

which is what had to be shown. ■

3. Main results

3.1. Exponential inequalities. In this subsection, we establish an exponential inequality of Bernstein-Frechet type for the stochastic Mann's scheme.

Theorem 4. *For all $\varepsilon > 0$, if for some constants σ and $L > 0$, the inequality*

$$\mathbb{E}\left\|\xi_i\right\|^m \leq \frac{m!}{2}\sigma^2 L^{m-2} \tag{9}$$

is fulfilled, and if we denote by,

$$S_1 = \sum_{i=1}^{\infty} \frac{(i+1)^{a(1-c)}}{i^2} \text{ and } S_2 = 4a^2\sigma^2 \sum_{i=1}^{\infty} \frac{(i+1)^{2a(1-c)}}{i^4}$$

then,

$$\mathbb{P}\left(\left\|x_{n+1} - x^*\right\| > \varepsilon\right) \leq K_1 \exp\left(-K_2 n^{2a(1-c)-\rho}\varepsilon^2\right) \tag{10}$$

where,

$$0 < \rho < 2a(1-c),\ K_1 \leq \exp\left(2\left(N^2 + \left(aS_1 \max_i \mathbb{E}\left\|\xi_i\right\|\right)^2\right)\right) \text{ and } K_2 = \min\left(1, \frac{1}{16S_2}\right).$$

Proof: By using the formula (7), we get,

$$\begin{aligned}
\mathbb{P}\left(\left\|x_{n+1} - x^*\right\| > \varepsilon\right) &\leq \mathbb{P}\left(N\prod_{i=1}^{n}\left(1 - \frac{a(1-c)}{i}\right) + \sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1 - \frac{a(1-c)}{j}\right)\left\|\xi_i\right\| > \varepsilon\right) \\
&\leq \mathbb{P}\left(N\prod_{i=1}^{n}\left(1 - \frac{a(1-c)}{i}\right) + \sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1 - \frac{a(1-c)}{j}\right)\mathbb{E}\left\|\xi_i\right\| \geq \frac{\varepsilon}{2}\right) \\
&\quad + \mathbb{P}\left(\sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1 - \frac{a(1-c)}{j}\right)\left(\left\|\xi_i\right\| - \mathbb{E}\left\|\xi_i\right\|\right) > \frac{\varepsilon}{2}\right)
\end{aligned} \tag{11}$$

Let us define the random variables $(\zeta_i)_i$ as follows,

$$\zeta_i = \left\|\xi_i\right\| - \mathbb{E}\left\|\xi_i\right\|.$$

It is clear that $\mathbb{E}\zeta_i = 0$ and $\mathbb{E}\left|\zeta_i\right|^m \leq 2m!\sigma^2(2L)^{m-2}$. Firstly, we have

$$\mathbb{P}\left(N\prod_{i=1}^{n}\left(1 - \frac{a(1-c)}{i}\right) + \sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1 - \frac{a(1-c)}{j}\right)\mathbb{E}\left\|\xi_i\right\| > \frac{\varepsilon}{2}\right) \leq K_1 e^{-n^{2a(1-c)-\rho}\varepsilon^2} \tag{12}$$

where,

$$K_1 \leq \exp\left(2\left(N^2 + \left(aS_1 \max_i \mathbb{E}\left\|\xi_i\right\|\right)^2\right)\right)$$

On the other hand, under Markov inequality, we have for all $t > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{n} \frac{a}{i^2} \prod_{j=i+1}^{n} \left(1 - \frac{a\,(1-c)}{j}\right) \left(\|\xi_i\| - \mathbb{E}\,\|\xi_i\|\right) > \frac{\varepsilon}{2}\right) \\
= \; &\mathbb{P}\left(\sum_{i=1}^{n} \frac{at\,(n+1)^{a(1-c)}}{i^2} \prod_{j=i+1}^{n} \left(1 - \frac{a\,(1-c)}{j}\right) \zeta_i > \frac{\varepsilon t\,(n+1)^{a(1-c)}}{2}\right) \\
\leq \; &\exp\left(-\frac{t\varepsilon\,(n+1)^{a(1-c)}}{2}\right) \mathbb{E}\left(\exp\left(t\sum_{i=1}^{n} \frac{a\,(n+1)^{a(1-c)}}{i^2} \prod_{j=i+1}^{n} \left(1 - \frac{a\,(1-c)}{j}\right) \zeta_i\right)\right)
\end{aligned}$$

The functions $x \longmapsto \|x\|$ and $x \longmapsto e^x$ are continuous, and hence are Borel functions. Therefore, the random variables $\left(\exp\left(\sum_{i=1}^{n} \frac{at(n+1)^{a(1-c)}}{i^2} \prod_{j=i+1}^{n} \left(1 - \frac{a(1-c)}{j}\right) \zeta_i\right)\right)_i$ are also independent and we have,

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(\sum_{i=1}^{n} \frac{at\,(n+1)^{a(1-c)}}{i^2} \prod_{j=i+1}^{n} \left(1 - \frac{a\,(1-c)}{j}\right) \zeta_i\right)\right) \\
= \; &\prod_{i=1}^{n} \mathbb{E}\left(\exp\left(\frac{at\,(n+1)^{a(1-c)}}{i^2} \prod_{j=i+1}^{n} \left(1 - \frac{a\,(1-c)}{j}\right) \zeta_i\right)\right)
\end{aligned}$$

The expansion of the exponential function around zero, inequality (8) as well as Cramer's condition (9) give us,

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(\frac{at}{i^2} \prod_{j=i+1}^{n} \left(1 - \frac{a\,(1-c)}{j}\right) \zeta_i\right)\right) \leq 1 + \sum_{m=2}^{+\infty} \frac{a^m t^m \mathbb{E}\,|\zeta_i|^m}{i^{2m} m!} \left(\frac{i+1}{n+1}\right)^{a(1-c)m} \\
\leq \; &1 + \frac{2a^2t^2\sigma^2}{i^4} \left(\frac{i+1}{n+1}\right)^{2a(1-c)} \sum_{m=2}^{+\infty} \frac{a^{m-2} t^{m-2} (2L)^{m-2}}{i^{2(m-2)}} \left(\frac{i+1}{n+1}\right)^{a(1-c)(m-2)}
\end{aligned}$$

Note that the function $x \longmapsto \frac{(x+1)^{a(1-c)}}{x^2}$ is decreasing and its maximum on the interval $[1, +\infty)$ is $2^{a(1-c)}$. Thus, for suitably chosen t, take for instance,

$$t \leq \frac{(n+1)^{a(1-c)}}{2^{a(1-c)+2} aL} \tag{13}$$

and using the inequality, $1 + x \leq e^x$, we obtain,

$$\prod_{i=1}^{n} \mathbb{E}\left(\exp \frac{at}{i^2} \prod_{j=i+1}^{n} \left(1 - \frac{a\,(1-c)}{j}\zeta_i\right)\right) \leq \exp\left(\sum_{i=1}^{n} \frac{4a^2t^2\sigma^2}{i^4} \left(\frac{i+1}{n+1}\right)^{2a(1-c)}\right)$$

Consequently,

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1-\frac{a\,(1-c)}{j}\right)\zeta_i>\frac{\varepsilon}{2}\right) &\leq \exp\left(-\frac{\varepsilon t}{2}+\sum_{i=1}^{n}\frac{4a^2t^2\sigma^2}{i^4}\left(\frac{i+1}{n+1}\right)^{2a(1-c)}\right)\\ &\leq \exp\left(-\frac{\varepsilon t}{2}+\frac{t^2S_2}{n^{2a(1-c)-\rho}}\right) \qquad (14)\end{aligned}$$

The quantity on the right-hand side of (14) is minimal at

$$t^*=\frac{\varepsilon n^{2a(1-c)-\rho}}{4S_2} \qquad (15)$$

Thus, by substituting t^* in (14), we obtain

$$\mathbb{P}\left(\sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1-\frac{a\,(1-c)}{j}\right)\zeta_i>\frac{\varepsilon}{2}\right)\leq\exp\left(-\frac{\varepsilon^2n^{2a(1-c)-\rho}}{16S_2}\right). \qquad (16)$$

The conclusion of theorem (4) can be obtained from (11), (12) and (16) immediately. ■

Remark 5. *The condition (9) is known under Cramer's condition and the first example that pops into our head is the bounded random variables and also the Gaussian random variables.*

Remark 6. *Notice that both choices of t in (13) and (15) are not contradictory. Indeed,*

$$\lim_{n\to+\infty}\frac{(n+1)^{a(1-c)}}{n^{a(1-c)-\rho}}=+\infty\Longleftrightarrow\forall A\in\mathbb{R}^+,\exists\ n_0\in\mathbb{N}:n\geq n_0\Longrightarrow\frac{(n+1)^{a(1-c)}}{n^{a(1-c)-\rho}}>A$$

For $A=\frac{\varepsilon 2^{a(1-c)+2}aLS_2}{4S_2}$, we have

$$\frac{\varepsilon n^{a(1-c)-\rho}}{4S_2}<\frac{(n+1)^{a(1-c)}}{2^{a(1-c)+2}aL}$$

3.2. Almost complete convergence. As a direct consequence of theorem (4), we obtain the almost complete convergence ($a.co.$) of the Mann's stochastic scheme.

Corollary 7. *Under the assumptions of theorem (4), the algorithm (5) converges almost completely (a.co.) to the unique fixed-point x^* of F.*

Proof: Indeed, since the series of general term

$$u_n=K_1\exp\left(-K_2n^{a(1-c)-\rho}\varepsilon^2\right) \qquad (17)$$

is convergent, we have, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\|x_{n+1} - x^*\| > \varepsilon\right) < +\infty \tag{18}$$

which ensures the almost complete convergence. ■

Remark 8. *Notice that if $(x_n)_n$ converges almost completely towards x^* then it also converges almost surely to x^*. In other words, if the sequence $(x_n)_n$ converges in probability to x^* sufficiently quickly (i.e. the above sequence of tail probabilities is summable for all $\varepsilon > 0$), then the sequence $(x_n)_n$ also converges almost surely to x^*. This is a direct implication from the Borel–Cantelli's lemma.*

3.3. Confidence set. In this subsection, we build a confidence set for the fixed point of a contraction mapping determined by the stochastic Mann's algorithm.

Corollary 9. *Under the assumptions of theorem (4), for a given level α, there exists a natural integer n_α for which the fixed point x^* of F belongs to the closed ball of center $x_{n_\alpha+1}$ and radius ε with a probability greater than or equal to $1-\alpha$, i.e.,*

$$\forall\, \varepsilon > 0, \forall\, \alpha > 0, \exists\, n_\alpha \in \mathbb{N} : \mathbb{P}\left(\|x_{n_\alpha+1} - x^*\| \leq \varepsilon\right) \geq 1 - \alpha \tag{19}$$

Proof: We have,

$$\lim_{n\to+\infty} K_1 \exp\left(-K_2 n^{a(1-c)-\rho}\varepsilon^2\right) = 0. \tag{20}$$

Since there exists a natural integer n_α such that

$$\forall\, n \in \mathbb{N}, n \geq n_\alpha \Longrightarrow K_1 \exp\left(-K_2 n^{a(1-c)-\rho}\varepsilon^2\right) \leq \alpha \tag{21}$$

then, (19) arises from (10) and (21). ■

Remark 10. *Conversely, if the sample size is given, we can also determine by (10) the level of significance α required in the construction of the confidence set.*

3.4. Rate of convergence. In this subsection, we study the rate of convergence of the Mann's stochastic algorithm (6). We say that $x_n - x^* = O(r_n)$, almost completely ($a.co.$) where $(r_n)_n$ is a sequence of real positive numbers, if there exists $\epsilon_0 > 0, \epsilon_0 = O(1)$ such that

$$\sum_{n=1}^{+\infty} \mathbb{P}\left(\|x_n - x^*\| > \epsilon_0 r_n\right) < +\infty$$

Theorem 11. *For every strictly positive number* a*, satisfying* $a(1-c) < 1$*, we have*

$$x_{n+1} - x^* = O\left(\sqrt{\frac{\ln n}{n^{a(1-c)-\rho}}}\right) \qquad a.co. \tag{22}$$

Proof: Indeed, we have

$$\mathbb{P}\left(\|x_{n+1} - x^*\| > \varepsilon\right) \leq K_1 e^{-K_2 n^{a(1-c)-\rho} \varepsilon^2} \tag{23}$$

where K_1 and K_2 are positive constants. Consequently,

$$\mathbb{P}\left(\|x_{n+1} - x^*\| > \epsilon_0 \sqrt{\frac{\ln n}{n^{a(1-c)-\rho}}}\right) \leq K_1 n^{-k_2 \epsilon_0^2} \tag{24}$$

For ϵ_0 well chosen, for example $\epsilon_0 = \sqrt{\frac{1+d}{K_2}}, d > 0$, the right hand-side of the inequality (23) is a general term of a convergent series. Hence, the desired result (22) is proved. ∎

4. Numerical illustrations

In order to ask the feasibility of the presented algorithm and check the obtained results of convergence, we consider a numerical example where we take a known contraction function F and thus possessing a unique fixed point. By using the Mann's algorithm, we obtain the approximated fixed point and we compare it with the exact one by giving the absolute and relative error. Concerning the independent random errors $(\xi_n)_n$ introduced in the algorithm, we take them following a centred normal distribution.

Consider $\mathbb{B} = \mathbb{R}$ and the following function F defined by

$$\begin{aligned} F : \ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto F(x) = \tfrac{1}{1+x^2} \end{aligned}$$

It is clear that F is a contraction, moreover, we have

$$|F(x) - F(y)| \leq \frac{9}{8\sqrt{3}} |x - y| < 0.65 |x - y|$$

Consequently, the function F has a unique fixed point given by:

$$\sqrt[3]{\sqrt{\frac{31}{108}} + \frac{1}{2}} - \frac{1}{3\sqrt[3]{\sqrt{\frac{31}{108}} + \frac{1}{2}}} \simeq 0.682327803828019$$

(1) **The approximated values of the fixed point**

Here, we give the approximated values of the fixed point for different number of iterations n. To compare the fixed point to the approximated ones, we give the absolute error and relative one. For an arbitrary choice of x_1, namely $x_1 = 0.5$, the obtained numerical results are represented in the following table.

n	x_n	Absolute error	Relative error
10	0.751990575294321	0.069662771466302	0.102095753793818
100	0.689797221706968	0.007469417878949	0.010946963962254
1000	0.683175136988214	8.473331601946965 $e-004$	1.241827103397 $e-003$
10^4	0.682471996264786	1.441924367667768 $e-004$	2.113242871795981 $e-004$
10^5	0.682336379893621	8.576065601673122 $e-006$	1.256883502850006 $e-005$
10^6	0.682328662137050	8.583090307379138 $e-007$	1.257913023509519 $e-006$
10^7	0.682327889660202	8.583218269464510 $e-008$	1.257931777264628 $e-007$

Note that from $n = 1000$, the approximated fixed points are very close to the real one. These results show the efficiency of the Mann's iterative scheme, also this method is very easy to implement under the programming package Matlab.

(2) **Level of significance α**

In the following two tables, we take a level of significance α and for different values of ε, we give the order of number of iterations and hence after implementing the algorithm, we obtain the corresponding approximated fixed point and consequently a confidence interval.

(i): $\alpha = 0.01$

ε	n	Confidence interval
0.01	10^7	$[0.672327889660202, 0.692327803828019]$
0.05	10^5	$[0.632336379893621, 0.782336379893621]$
0.1	10^4	$[0.582471996264786, 0.782471996264786]$

(ii): $\alpha = 0.05$

ε	n	Confidence interval
0.01	10^6	$[0.672328662137050, 0.692328662137050]$
0.05	10^4	$[0.632471996264786, 0.732471996264786]$
0.1	10^3	$[0.583175136988214, 0.783175136988214]$

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