Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 17–42

ON A TERNARY GENERALIZATION OF JORDAN ALGEBRAS

IVAN KAYGORODOV, A. P. POZHIDAEV AND P. SARAIVA

ABSTRACT: Based on the relation between the notions of Lie triple system and Jordan algebra, we introduce n-ary Jordan algebras, an n-ary generalization of Jordan algebras obtained via the generalization of the following property $[R_x, R_y] \in Der\,(\mathcal{A})$, where $\mathcal{A}$ is an n-ary algebra. Next, we study such ternary algebras. Finally, based on the construction of a family of ternary algebras defined by means of the Cayley-Dickson algebras, we present a ternary $D_{x,y}$-derivation algebra (n-ary $D_{x,y}$-derivation algebras are non-commutative version of n-ary Jordan algebras).

KEYWORDS: Jordan algebra, non-commutative Jordan algebra, derivation, n-ary algebra, Lie triple system, generalized Lie algebra, Cayley-Dickson construction, TKK construction.
MATH. SUBJECT CLASSIFICATION (2010): 17A**.

Introduction

The notion of Jordan algebra appeared in 1934 as the underlying algebraic structure for certain operators in quantum mechanics [13]. Recall that a Jordan algebra is a commutative algebra over a field $\mathbb{F}$ ($char\,(\mathbb{F}) \neq 2$) satisfying the so-called Jordan identity

$$(xy)\,x^2 = x\left(yx^2\right). \tag{1}$$

Since then, the theory of Jordan algebras has been developed, not only in purely algebraic aspects, but also intertwined with other subjects and applications. For instance, the vast class of noncommutative Jordan algebras (it includes, *e.g.*, alternative algebras, Jordan algebras, quasiassociative algebras, quadratic flexible algebras, and anticommutative algebras) attracted a lot of attention. Schafer proved that a simple noncommutative Jordan algebra is either a simple Jordan algebra or a simple quasiassociative algebra or a simple flexible algebra of degree 2 [23]. Concerning the intervention of Jordan algebras in other areas, and just to mention a couple of these, we can find applications in differential geometry (see [5] and [27]) and in optimization methods (see [7]). For more details about a motivation and a general overview of Jordan algebras (including applications) see [18] and [10].

Received October 23, 2017.

A related issue is an attempt to generalize the Jordan algebra structure to the case of algebras with n-ary multiplication, with an emphasis to the ternary case. Mostly, these generalizations include Jordan triple systems (as in [2] and [9]), but also other ternary versions (*e.g*, [4]). In the present paper we follow a different approach.

According to [19] and [25], a *Lie triple algebra* is a commutative nonassociative algebra $\mathcal{A}$ over a field $\mathbb{F}$ ($char\,(\mathbb{F}) \neq 2$) satisfying

$$\left(a, b^2, c\right) = 2b\,(a, b, c)\,, \tag{2}$$

where $(.,.,.)$ stands for the associator

$$(a, b, c) = (ab)c - a(bc).$$

It is not difficult to check that this identity is equivalent to the operator identity

$$R_{(x,y,z)} = [R_y, [R_x, R_z]]\,, \tag{3}$$

where $[.,.]$ stands for the commutator

$$[a, b] = ab - ba,$$

and R_x is the right multiplication operator by x, *i.e.*

$$y \mapsto yR_x = yx.$$

It is easy to observe that every Jordan algebra is a Lie triple algebra, although the opposite is not necessarily true. Further, on a commutative algebra $\mathcal{A}$ the identity (3) is equivalent to

$$[R_x, R_y] \in Der\,(\mathcal{A})\,, \tag{4}$$

where $Der\,(\mathcal{A})$ stands for the Lie algebra of derivations of $\mathcal{A}$. Writing $D_{x,y}$ instead of $[R_x, R_y]$, it means that

$$D_{x,y}\,(ab) = D_{x,y}(a)b + aD_{x,y}(b). \tag{5}$$

It is also known that, in a Jordan algebra $\mathfrak{J}$,

$$Inder\,(\mathfrak{J}) = \left\{ \sum [R_{x_i}, R_{y_i}] : x_i, y_i \in \mathfrak{J} \right\},$$

where $Inder\,(\mathfrak{J})$ stands for the Lie algebra of inner derivations of $\mathfrak{J}$. Thus, in every Jordan algebra $\mathfrak{J}$ the commutator of two arbitrary right multiplication operators is a derivation (an inner derivation, to be precise) of $\mathfrak{J}$ (see [6]).

Let $\mathcal{A}$ be an n-ary algebra with a multilinear multiplication $[\![., \ldots, .]\!] : \times^n \mathbb{V} \to \mathbb{V}$, where $\mathbb{V}$ is the underlying vector space. We propose the following definition: $\mathcal{A}$ is said to be an *n-ary Jordan algebra* if

$$[\![x_{\sigma(1)}, \ldots, x_{\sigma(n)}]\!] = [\![x_1, \ldots, x_n]\!] \tag{6}$$

for every permutation $\sigma \in \mathbb{S}_n$ and for every $x_1, \ldots, x_n \in \mathbb{V}$, and if

$$\left[R_{(x_2,\ldots,x_n)}, R_{(y_2,\ldots,y_n)}\right] \in Der\,(\mathcal{A})\,, \tag{7}$$

for every $x_2, \ldots, x_n, y_2, \ldots, y_n \in \mathbb{V}$, where $[., .]$ stands again for the commutator and $R_{(x_2,\ldots,x_n)}, R_{(y_2,\ldots,y_n)}$ are the right multiplication operators, defined in the usual way

$$y \mapsto yR_{(x_2,\ldots,x_n)} = [\![y, x_2, \ldots, x_n]\!].$$

For the sake of simplicity, we often write R_x instead of $R_{(x_2,\ldots,x_n)}$ and, analogously to the binary case, $D_{x,y}$ instead of $[R_x, R_y]$. Further, every non-commutative n-ary Jordan algebra will be called a *$D_{x,y}$-derivation algebra*, *i.e.*, it is an arbitrary n-ary algebra with the operator identity (7). Under this notation (7) can be written in the following way

$$D_{x,y}\,[\![z_1, \ldots, z_n]\!] = \sum_{i=1}^{n} [\![z_1, \ldots, D_{x,y}\,(z_i)\,, \ldots, z_n]\!]. \tag{8}$$

Throughout this paper, (6) is the *total commutativity property* and (7) (or, equivalently (8)) is the *$D_{x,y}$-identity*.

The paper is organized in the following way. In the first section we consider some ternary algebras defined on the direct sum of a field and a vector space, by equipping this space with a product, which depends on three given forms. Discussing the possible cases for these forms, we obtain the first examples of ternary Jordan algebras.

The second section is devoted to a particular case of the ternary product defined in the previous section, restricted to a vector space (over a field of characteristic zero). It turns out that this provides a new example of ternary Jordan algebra, denoted by TJ_n, which is simple. We study its identities of degrees 1 and 2 concluding that these result from the total commutativity. Finally, we conclude that the proposed notion of ternary Jordan algebra doesn't coincide with the notion of Jordan triple system.

In the third section we study the derivation algebra of TJ_n concluding that it coincides with $so(n)$ and that all derivations of TJ_n are inner.

The last sections are focused on giving other examples of ternary Jordan algebras. There, dealing with matrix algebras, we obtain two non-isomorphic symmetrized matrix ternary Jordan algebras, one of which is simple. Further, defining a certain ternary product on the algebras obtained by the Cayley-Dickson doubling process, we construct a 4-dimensional $D_{x,y}$-algebra over the generalized quaternions. We also present an analog of the TKK-construction to the case of ternary algebras, obtaining new examples of ternary Jordan algebras.

Finally, we recall the concept of reduced algebras of n-ary algebras. After this, we conclude that, oppositely to some other classes of algebras, the reduced algebras of the ternary Jordan algebra TJ_n are not Jordan algebras in general. We note that other generalizations of Jordan algebras (*e.g.*, Jordan triple systems) also fail this property.

1. Ternary algebras of multilinear forms

Consider an n-dimensional vector space $\mathbb{V}$ over a field $\mathbb{F}$ equipped with two bilinear, symmetric and nondegenerate forms f and h, and also with a trilinear, symmetric, and nondegenerate form g (we also let them be zero). Given a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of $\mathbb{V}$, these forms are such that

$$f(b_i, b_j) = \delta_{ij}, \qquad h(b_i, b_j) = \delta_{ij}, \quad \text{and} \quad g(b_i, b_j, b_k) = \delta_{ijk}, \tag{9}$$

where δ_{ij} is the Kronecker delta (and δ_{ijk} by analogy).

Consider now a binary multiplication $*$ on the vector space $\mathbb{F} \oplus \mathbb{V}$ defined by

$$(\alpha + u) * (\beta + v) = \alpha\beta + f(u, v) + \alpha v + \beta u, \alpha, \beta \in \mathbb{F}, u, v \in \mathbb{V}.$$

Then we obtain a Jordan algebra of a symmetric bilinear form f, which is denoted by $J(\mathbb{V}, f)$ and is simple iff $\dim V > 1$ and f is nondegenerate.

Searching an analogue of $J(\mathbb{V}, f)$ in the case of ternary algebras, we consider the same vector space $\mathbb{F} \oplus \mathbb{V}$, where we define a (most general) trilinear product $[\![., ., .]\!]$ such that

$$\begin{aligned}
&[\![\alpha_1 + v_1, \alpha_2 + v_2, \alpha_3 + v_3]\!] = \qquad (10)\\
&= (\alpha_1\alpha_2\alpha_3 + \alpha_1 f(v_2, v_3) + \alpha_2 f(v_1, v_3) + \alpha_3 f(v_1, v_2) + g(v_1, v_2, v_3)) + \\
&+ (\alpha_2\alpha_3 + h(v_2, v_3)) v_1 + (\alpha_1\alpha_3 + h(v_1, v_3)) v_2 + (\alpha_1\alpha_2 + h(v_1, v_2)) v_3
\end{aligned}$$

for arbitrary $\alpha_i \in \mathbb{F}$ and $v_i \in \mathbb{V}$. The obtained ternary algebra will be denoted by $\mathcal{V}_{f,g,h}$.

Under this assumption, it is clear that $[\![., ., .]\!]$ is totally commutative, that is

$$[\![\alpha_{\sigma(1)} + v_{\sigma(1)}, \alpha_{\sigma(2)} + v_{\sigma(2)}, \alpha_{\sigma(3)} + v_{\sigma(3)}]\!] = [\![\alpha_1 + v_1, \alpha_2 + v_2, \alpha_3 + v_3]\!] \tag{11}$$

for all $\sigma \in \mathbb{S}_3$, $\alpha_i \in \mathbb{F}$, and $v_i \in \mathbb{V}$.

Our purpose is to check if the commutator of right multiplications defines a derivation of the ternary algebra $\mathcal{V}_{f,g,h}$, that is, considering the linear operators R_x and $D_{x,y}$ such that

$$zR_x = zR_{(x_1,x_2)} = [\![z, x_1, x_2]\!], \tag{12}$$

and

$$D_{x,y} = [R_x, R_y] = R_x R_y - R_y R_x, \tag{13}$$

we want to know if the ternary version of the $D_{x,y}$-identity

$$D_{x,y}\,[\![z_1, z_2, z_3]\!] = [\![D_{x,y}(z_1), z_2, z_3]\!] + [\![z_1, D_{x,y}(z_2), z_3]\!] + [\![z_1, z_2, D_{x,y}(z_3)]\!] \tag{14}$$

holds.

Before answering this question, observe some immediate properties of (14). It is straightforward that each linear operator $D_{(x_1,x_2),(y_1,y_2)}$ is also linear in each x_i and in each y_i. Further, we have the following symmetry properties

$$D_{(x_1,x_2),(y_1,y_2)} = D_{(x_1,x_2),(y_2,y_1)} = D_{(x_2,x_1),(y_1,y_2)} = D_{(x_2,x_1),(y_2,y_1)}.$$

Finally, it is also obvious that

$$D_{x,y} = -D_{y,x} \text{ and } D_{x,x} = 0.$$

The following result solves the above problem.

Theorem 1. *The ternary algebra $\mathcal{V}_{f,g,h}$ is a ternary Jordan algebra only in the following cases:*
I. $\mathcal{V}_{0,0,0}$; **II.** $\mathcal{V}_{0,0,h}$, *if char* $(\mathbb{F}) = 3$ *and* $\dim \mathbb{V} = 1$;
III. $\mathcal{V}_{0,g,0}$ *if char* $(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} = 1$; **IV.** $\mathcal{V}_{f,0,h}$ *if char* $(\mathbb{F}) = 2$;
V. $\mathcal{V}_{f,g,h}$ *if char* $(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} = 1$.

Proof: **I.** Prove that $\mathcal{V}_{0,0,0}$ is a ternary Jordan algebra.

Obviously, for $x = (\alpha_x + v_x, \beta_x + u_x)$, $y = (\alpha_y + v_y, \beta_y + u_y)$ and every $\alpha \in \mathbb{F}$, $z \in \mathbb{V}$, we have

$$[\![[\![\alpha + z, \alpha_x + v_x, \beta_x + u_x]\!], \alpha_y + v_y, \beta_y + u_y]\!] =$$

$$\alpha(\alpha_x\beta_x\alpha_y\beta_y + \alpha_x\beta_x\alpha_y u_y + \alpha_x\beta_x\beta_y v_y + \alpha_y\beta_y\alpha_x u_x + \alpha_y\beta_y\beta_x v_x) + \alpha_x\beta_x\alpha_y\beta_y z.$$

It is easy to see that $D_{x,y}$ is identically zero on $\mathcal{V}_{0,0,0}$ and we have a ternary Jordan algebra.

The second part of the theorem has seven cases:

$$(f \neq 0, g = 0, h = 0),\ (f = 0, g \neq 0, h = 0),\ \ldots,\ (f \neq 0, g \neq 0, h \neq 0).$$

Below we only consider those that lead to the ternary Jordan algebras in modular characteristic, since the proof for the remaining cases is similar.

II. $(f=0,\ g=0,\ h\neq 0)$. In this case, (10) is reduced to

$$[\![\alpha_1+v_1,\alpha_2+v_2,\alpha_3+v_3]\!]=\alpha_1\alpha_2\alpha_3+(\alpha_2\alpha_3+h(v_2,v_3))\,v_1$$
$$+(\alpha_1\alpha_3+h(v_1,v_3))\,v_2+(\alpha_1\alpha_2+h(v_1,v_2))\,v_3.$$

Being $\mathcal{V}_{0,0,h}=\langle 1,b\rangle_{\mathbb{F}}$, the multiplication table for the basis elements is given by

(i) $[\![1,1,1]\!]=1$, (ii) $[\![1,1,b]\!]=b$, (iii) $[\![1,b,b]\!]=0$ and (iv) $[\![b,b,b]\!]=3b$.

Thus, with respect to this basis, we have

$$R_{(1,1)}=E,\ R_{(1,b)}=e_{12},\ \text{and}\ R_{(b,b)}=3e_{22},$$

where e_{ij} are the usual matrix units. Recalling the properties of the operators $D_{x,y}$, in order to verify the $D_{x,y}$-identity it suffices to do it for

$$D=D_{(1,b),(b,b)}=R_{(1,b)}R_{(b,b)}-R_{(b,b)}R_{(1,b)}=3e_{12}.$$

Now, it is clear that $D=0$ if $char\,(\mathbb{F})=3$, and $\mathcal{V}_{0,0,h}$ is a ternary Jordan algebra. So, admit that $char\,(\mathbb{F})\neq 3$. Hereinafter, $LHS_D(z_1,z_2,z_3)$ (respectively, $RHS_D(z_1,z_2,z_3)$) denotes the left (right) hand side of (14) with $D=D_{x,y}$. It is easy to see that, concerning (iii), we have

$$LHS_D(1,b,b)=0,\ \text{while}\ RHS_D(1,b,b)=9b.$$

Therefore, $\mathcal{V}_{0,0,h}$ is not a ternary Jordan algebra.

Consider again $char\,(\mathbb{F})=3$ with $\mathcal{V}_{0,0,h}=\langle 1,b_1,\ldots,b_n\rangle_{\mathbb{F}}$, $n>1$. Then

$$\text{(i) } [\![1,1,1]\!]=1,\ \text{(ii) } [\![1,1,b_i]\!]=b_i,\ \text{(iii) } [\![1,b_i,b_i]\!]=0,\ \text{(iv) } [\![b_i,b_i,b_i]\!]=0,$$
$$\text{(v) } [\![1,b_i,b_j]\!]=0,\ (i\neq j),\ \text{(vi) } [\![b_i,b_i,b_j]\!]=b_j\ (i\neq j),$$
$$\text{and}\quad \text{(vii) } [\![b_i,b_j,b_k]\!]=0\ (i,j,k \text{ are pairwise different}).$$

Then, with respect to the considered basis, we have

$$R_{(1,1)}=E,\ R_{(1,b_i)}=e_{1,i+1},\ R_{(b_i,b_i)}=e_{j+1,j+1},\ \text{and}\ R_{(b_i,b_j)}=e_{i+1,j+1}+e_{j+1,i+1}\ (i\neq j).$$

Now, $D=D_{(1,b_1),(b_1,b_2)}=e_{13}$, so $D(1)=b_2$, and $D(b_1)=D(b_2)=0$. Concerning the product (i), it is easy to observe that

$$LHS_D(1,1,1)=b_2,\ \text{while}\ RHS_D(1,1,1)=3[\![D(1),1,1]\!]=0.$$

III. $(f=0,\ g\neq 0,\ h=0)$. Under these conditions, (10) is reduced to

$$[\![\alpha_1+v_1,\alpha_2+v_2,\alpha_3+v_3]\!]=(\alpha_1\alpha_2\alpha_3+g(v_1,v_2,v_3))+\alpha_2\alpha_3v_1+\alpha_1\alpha_3v_2+\alpha_1\alpha_2v_3.$$

Being $\mathcal{V}_{0,g,0} = \langle 1, b\rangle_{\mathbb{F}}$, the multiplication table for the basis elements is given by

(i) $[\![1,1,1]\!] = 1$, (ii) $[\![1,1,b]\!] = b$, (iii) $[\![1,b,b]\!] = 0$, and (iv) $[\![b,b,b]\!] = 1$.

Thus, with respect to this basis, we have

$$R_{(1,1)} = E,\ R_{(1,b)} = e_{12},\ \text{and } R_{(b,b)} = e_{21}.$$

Henceforth, in order to verify the $D_{x,y}$-identity it suffices to do it for

$$D = D_{(1,b),(b,b)} = R_{(1,b)}R_{(b,b)} - R_{(b,b)}R_{(1,b)} = e_{11} - e_{22}.$$

Thus, $D(1) = 1$ and $D(b) = -b$. Checking the $D_{x,y}$-identity in the four cases of the multiplication table, it is possible to observe that (10) holds iff $char\,(\mathbb{F}) = 2$.

Consider now $char\,(\mathbb{F}) = 2$ and $n = \dim \mathbb{V} = 2$. Taking $\mathcal{V}_{0,g,0} = \langle 1, b_1, b_2\rangle_{\mathbb{F}}$, we have the following multiplication table

$$[\![1,1,1]\!] = 1, \quad [\![1,1,b_1]\!] = b_1, \quad [\![1,1,b_2]\!] = b_2, \quad [\![1,b_1,b_1]\!] = [\![1,b_2,b_2]\!] = 0,$$

$$[\![b_1,b_1,b_1]\!] = [\![b_2,b_2,b_2]\!] = 1 \text{ and } [\![1,b_1,b_2]\!] = [\![b_1,b_1,b_2]\!] = [\![b_2,b_2,b_1]\!] = 0.$$

Taking $D = D_{(1,b_1),(b_1,b_1)}$, we see that $D(1) = 1$ and $D(b_2) = 0$. Therefore, $LHS_D(b_2,b_2,b_2) = 1$, while $RHS_D(b_2,b_2,b_2) = 0$. Thus, $\mathcal{V}_{0,g,0}$ is a ternary Jordan algebra only if $char\,(\mathbb{F}) = 2$ and $\dim \mathbb{V} = 1$.

IV. $(f \neq 0,\ g = 0,\ h \neq 0)$. In this case, (10) is reduced to

$$[\![\alpha_1+v_1, \alpha_2+v_2, \alpha_3+v_3]\!] = (\alpha_1 f\,(v_2,v_3) + \alpha_2 f\,(v_1,v_3) + \alpha_3 f\,(v_1,v_2) + \alpha_1\alpha_2\alpha_3)$$
$$+ (\alpha_2\alpha_3 + h\,(v_2,v_3))\,v_1 + (\alpha_1\alpha_3 + h\,(v_1,v_3))\,v_2 + (\alpha_1\alpha_2 + h\,(v_1,v_2))\,v_3.$$

Being $\mathcal{V}_{f,0,h} = \langle 1, b\rangle_{\mathbb{F}}$, the multiplication table for the basis elements is given by

(i) $[\![1,1,1]\!] = 1$, (ii) $[\![1,1,b]\!] = b$, (iii) $[\![1,b,b]\!] = 1$, and (iv) $[\![b,b,b]\!] = 3b$.

Thus, with respect to this basis, we have

$$R_{(1,1)} = E,\ R_{(1,b)} = e_{12} + e_{21},\ \text{and } R_{(b,b)} = e_{11} + 3e_{22}.$$

Now, in order to verify the $D_{x,y}$-identity it suffices to do it for

$$D = D_{(1,b),(b,b)} = R_{(1,b)}R_{(b,b)} - R_{(b,b)}R_{(1,b)} = 2(e_{12} - e_{21}).$$

Thus, $D(1) = 2b$ and $D(b) = -2$. Checking the $D_{x,y}$-identity in the four cases of the multiplication table, it is possible to observe that the identity holds iff $char\,(\mathbb{F}) = 2$.

Admit that $char\,(\mathbb{F})=2$ and $n=\dim\mathbb{V}>1$. Take $\mathcal{V}_{f,0,h}=\langle 1,b_1,\ldots,b_n\rangle_{\mathbb{F}}$, with the following multiplication table

$$\begin{aligned}&[\![1,1,1]\!]=1,\quad [\![1,1,b_i]\!]=b_i,\quad [\![1,b_i,b_i]\!]=1,\\ &[\![b_i,b_i,b_i]\!]=b_i \text{ (since } char\,(\mathbb{F})=2\text{)},\quad [\![1,b_i,b_j]\!]=0\ (i\neq j),\\ &[\![b_i,b_i,b_j]\!]=b_j\ (i\neq j),\ [\![b_i,b_j,b_k]\!]=0\ (i,j,k \text{ are pairwise different and } n\geq 3).\end{aligned}$$

Then, with respect to the considered basis we have

$$R_{(1,1)}=R_{(b_i,b_i)}=E,\ R_{(1,b_i)}=e_{1,i+1}+e_{i+1,1},\ \text{and } R_{(b_i,b_j)}=e_{i+1,j+1}+e_{j+1,i+1}\ (i\neq j).$$

In order to verify the $D_{x,y}$-identity it suffices to do it for

$$D=D_{(1,b_i),(b_i,b_j)}=e_{1,j+1}-e_{j+1,1}.$$

Then, $D(1)=b_j$, $D(b_i)=0$, $D(b_j)=-1$ ($j\neq i$), and $D(b_k)=0$ (for the pairwise different i,j,k and $n\geq 3$). Considering all possible cases of the multiplication table for elements in $\mathcal{B}$, it is not difficult to verify that the $D_{x,y}$-identity holds. Thus, in this case $\mathcal{V}_{f,0,h}$ is a ternary Jordan algebra.

V. ($f\neq 0,\ g\neq 0,\ h\neq 0$). In this case, (10) is in its most general form. Taking $\mathcal{V}_{f,g,h}=\langle 1,b\rangle_{\mathbb{F}}$, the multiplication table for the basis elements is given by

(i) $[\![1,1,1]\!]=1,$ (ii) $[\![1,1,b]\!]=b,$ (iii) $[\![1,b,b]\!]=1,$ and (iv) $[\![b,b,b]\!]=1+3b.$

Thus, with respect to this basis we have

$$R_{(1,1)}=E,\ R_{(1,b)}=e_{12}+e_{21},\ \text{and } R_{(b,b)}=e_{11}+e_{21}+3e_{22}.$$

This way, in order to verify the $D_{x,y}$-identity it suffices to do it for

$$D=D_{(1,b),(b,b)}=R_{(1,b)}R_{(b,b)}-R_{(b,b)}R_{(1,b)}=e_{11}-e_{22}+2(e_{12}-e_{21}).$$

Observe that $D(1)=1+2b$ and $D(b)=-2-b$. Concerning the above multiplication table for the basis elements, it is easy to see that the $D_{x,y}$-identity holds iff $char\,(\mathbb{F})=2$.

Assume now that $char\,(\mathbb{F})=2$ and $\dim\mathbb{V}=2$. Consider $\mathcal{V}_{f,g,h}=\langle 1,b_1,b_2\rangle_{\mathbb{F}}$. Then

$$[\![1,1,1]\!]=1,\ [\![1,1,b_i]\!]=b_i, i=1,2,\ [\![1,b_i,b_i]\!]=1, i=1,2,\ [\![b_i,b_i,b_i]\!]=1+b_i,$$
$$[\![1,b_1,b_2]\!]=0,\quad \text{and}\quad [\![b_i,b_i,b_j]\!]=b_j, i,j=1,2,\ (i\neq j).$$

Taking $D=D_{(1,b_1),(b_1,b_1)}=e_{11}-e_{22}$, we have

$$D(1)=1,\ D\,(b_1)=-b_1 \text{ and } D\,(b_2)=0.$$

Then $LHS_D(b_2,b_2,b_2)=1$, while $RHS_D(b_2,b_2,b_2)=0$. Thus, we will not obtain a ternary Jordan algebra if $char\,(\mathbb{F})=2$ and $\dim\mathbb{V}>1$.

The remaining 3 cases can be considered analogously. ■

Thus, we obtained the first examples of ternary Jordan algebras. In the case of $\mathcal{V}_{0,0,0}$, we have a vector space $\mathbb{F} \oplus \mathbb{V}$ equipped with the following ternary multiplication:

$$[\![\alpha_1 + v_1, \alpha_2 + v_2, \alpha_3 + v_3]\!] = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3 v_1 + \alpha_1\alpha_3 v_2 + \alpha_1\alpha_2 v_3,$$

where $\alpha_i \in \mathbb{F}, v_i \in \mathbb{V}$.

Recall that, given a ternary algebra $\mathcal{A}$, a *subalgebra* $\mathcal{B}$ of $\mathcal{A}$ is a vector subspace of $\mathcal{A}$ such that $[\![\mathcal{B},\mathcal{B},\mathcal{B}]\!] \subseteq \mathcal{B}$. The algebra $\mathcal{A}$ is Abelian iff $[\![\mathcal{A},\mathcal{A},\mathcal{A}]\!] = 0$. An *ideal* of $\mathcal{A}$ is a subalgebra $\mathcal{I}$ of $\mathcal{A}$ such that

$$[\![\mathcal{I},\mathcal{A},\mathcal{A}]\!] \subseteq \mathcal{I}, \quad [\![\mathcal{A},\mathcal{I},\mathcal{A}]\!] \subseteq \mathcal{I}, \quad [\![\mathcal{A},\mathcal{A},\mathcal{I}]\!] \subseteq \mathcal{I}.$$

The ideals $\{0\}$ and $\mathcal{A}$ are *trivial*. The algebra $\mathcal{A}$ is *simple* if it is not Abelian and it lacks nontrivial ideals.

Remark 2. *As we can see from the following part of the paper, the ternary algebra* $\mathcal{V}_{f,0,h}$ *has a ternary simple Jordan subalgebra.*

Lemma 3. *The ternary algebra* $\mathcal{V}_{0,0,0}$ *is not simple and every subspace of* $\mathbb{V}$ *is an ideal of* $\mathcal{V}_{0,0,0}$. *If* $\mathcal{I}$ *is a proper ideal of* $\mathcal{V}_{0,0,0}$ *then* $\mathcal{I}$ *is a subspace of* $\mathbb{V}$. *Among the modular ternary Jordan algebras obtained in the previous theorem only the following are simple*:

- $\mathcal{V}_{0,g,0}$ *with* $char\,(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} = 1$;
- $\mathcal{V}_{f,0,h}$ *with* $char\,(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} > 1$.

Proof. It is easy to see that, for every subspace $\mathbb{U}$ of $\mathbb{V}$,

$$[\![\mathbb{U}, \mathbb{F} \oplus \mathbb{V}, \mathbb{F} \oplus \mathbb{V}]\!] = [\![\mathbb{U}, \mathbb{F}, \mathbb{F}]\!] = \mathbb{U}.$$

On the other hand, let $\mathcal{I}$ be an ideal of $\mathcal{V}_{0,0,0}$. If $1+v \in \mathcal{I}$ then $[\![1+v, 1, z]\!] = z \in \mathcal{I}$ for every $z \in \mathcal{I}$, and either $\mathcal{I}$ is a subspace of $\mathbb{V}$ or $\mathcal{I} = \mathbb{F} \oplus \mathbb{V}$.

Consider the ternary Jordan algebra $\mathcal{V}_{0,0,h} = \langle 1, b \rangle_{\mathbb{F}}$ with $char\,(\mathbb{F}) = 3$. The multiplication table for the basis elements is given by

(i) $[\![1,1,1]\!] = 1$, (ii) $[\![1,1,b]\!] = b$, (iii) $[\![1,b,b]\!] = 0$, and (iv) $[\![b,b,b]\!] = 0$.

It is clear that $\mathcal{I} = \langle b \rangle_{\mathbb{F}}$ is an ideal of $\mathcal{V}_{0,0,h}$ and so this ternary Jordan algebra is not simple.

Consider the ternary Jordan algebra $\mathcal{V}_{0,g,0} = \langle 1, b \rangle_{\mathbb{F}}$ with $char\,(\mathbb{F}) = 2$. The multiplication table for the basis elements is given by

(i) $[\![1,1,1]\!] = 1$, (ii) $[\![1,1,b]\!] = b$, (iii) $[\![1,b,b]\!] = 0$, and (iv) $[\![b,b,b]\!] = 1$.

Admit that $\mathcal{I}$ is an ideal of $\mathcal{V}_{0,g,0}$ and consider $x = \alpha\, 1 + \beta b \in \mathcal{I}\backslash\{0\}$. We see from the multiplication table that if $1 \in \mathcal{I}$ or $b \in \mathcal{I}$ then $\mathcal{I} = \mathcal{V}_{0,g,0}$. So, we may assume that none of the scalars is zero. Then

$$[\![x, 1, b]\!] = \alpha b \in \mathcal{I},$$

and so $b \in \mathcal{I}$ leading to $\mathcal{I} = \mathcal{V}_{0,g,0}$. Thus, $\mathcal{V}_{0,g,0}$ is simple.

Consider now $\mathcal{V}_{f,0,h} = \langle 1, b_1, \ldots, b_n\rangle_{\mathbb{F}}$ with $char\,(\mathbb{F}) = 2$ and $n = \dim \mathbb{V}$. We divide the proof in two cases: $n = 1$ and $n > 1$. Recall that, when $n = 1$, the multiplication table with respect to the basis $\{1, b\}$ is given by

$$[\![1, 1, 1]\!] = 1, \quad [\![1, 1, b]\!] = b, \quad [\![1, b, b]\!] = 1, \quad \text{and} \quad [\![b, b, b]\!] = b.$$

It is easy to see that $\mathcal{I} = \langle 1 + b\rangle_{\mathbb{F}}$ is an ideal of $\mathcal{V}_{f,0,h}$, whence this ternary Jordan algebra is not simple. Admit now that $n > 1$. The multiplication table for the basis elements of $\mathcal{V}_{f,0,h}$ is given by

$$[\![1, 1, 1]\!] = 1, \quad [\![1, 1, b_i]\!] = b_i, \quad [\![1, b_i, b_i]\!] = 1, \quad [\![b_i, b_i, b_i]\!] = b_i,$$

$$[\![1, b_i, b_j]\!] = 0, \quad [\![b_i, b_i, b_j]\!] = b_j, \text{and} \quad [\![b_i, b_j, b_k]\!] = 0$$

for all pairwise different $i, j, k = 1, \ldots, n$. Let $\mathcal{I}$ be an ideal of $\mathcal{V}_{f,0,h}$. It is clear that if any of the basis elements lies in $\mathcal{I}$ then $\mathcal{I} = \mathcal{V}_{f,0,h}$, and this ternary Jordan algebra is simple. Consider $x = \alpha\, 1 + \sum\limits_{i=1}^{n} \beta_i\, b_i \in \mathcal{I}\backslash\{0\}$ with $\beta_j \neq 0$ for some j. Then

$$[\![x, 1, b_j]\!] = \alpha\, b_j + \beta_j \in \mathcal{I},$$

i.e., we may assume that $1 + \gamma b_j \in \mathcal{I}$ for some nonzero $\gamma \in \mathbb{F}$. Then

$$[\![1 + \gamma b_j, b_j, b_i]\!] = \gamma b_i \in \mathcal{I},$$

which proves the required simplicity.

The case $\mathcal{V}_{f,g,h}$, with $char\,(\mathbb{F}) = 2$ and $\dim \mathbb{V} = 1$, is perfectly similar to the subcase $\dim \mathbb{V} = 1$ of the previous case.

□

Lemma 4. *Let D be an arbitrary derivation of $\mathcal{V}_{0,0,0}$. Then*

(1) *if $char\,(\mathbb{F}) \neq 2$ then $Der(\mathcal{V}_{0,0,0}) \cong End(\mathbb{V})^{(-)}$;*
(2) *if $char\,(\mathbb{F}) = 2$ and $dim\mathbb{V} = 1$ then $Der(\mathcal{V}_{0,0,0}) \cong End(\mathcal{V}_{0,0,0})^{(-)}$;*
(3) *if $char\,(\mathbb{F}) = 2$ and $dim\mathbb{V} > 1$ then $D(\mathbb{V}) \subseteq \mathbb{V}, Der|_{\mathbb{V}}(\mathcal{V}_{0,0,0}) \cong End(\mathbb{V})^{(-)}$, and $D(1)$ may be an arbitrary element in $\mathcal{V}_{0,0,0}$, where $Der|_{\mathbb{V}}(\mathcal{V}_{0,0,0})$ is the algebra of derivations of $\mathcal{V}_{0,0,0}$ restricted on $\mathbb{V}$.*

Proof. Let D be a derivation of $\mathcal{V}_{0,0,0}$. If $char\,(\mathbb{F}) \neq 2$ then it is easy to see that $D(1) = 0$. Now, given an arbitrary element $v \in \mathbb{V}$, we have $D(v) = v_{\mathbb{F}} + v_D$, for some $v_{\mathbb{F}} \in \mathbb{F}, v_D \in \mathbb{V}$, and

$$D(v) = D\,[\![v, 1+v, 1+v]\!] = D(v) + 4v_{\mathbb{F}}v.$$

It follows that $D(\mathbb{V}) \subseteq \mathbb{V}$. Given $D \in End(\mathbb{V})$, extend D to a linear mapping in $End(\mathcal{V}_{0,0,0})$ such that $D(1) = 0$. Now,

$$D\,[\![\alpha_1 + v_1, \alpha_2 + v_2, \alpha_3 + v_3]\!] = D(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2 v_3 + \alpha_1\alpha_3 v_2 + \alpha_2\alpha_3 v_1) =$$

$$\alpha_1\alpha_2 D(v_3) + \alpha_1\alpha_3 D(v_2) + \alpha_2\alpha_3 D(v_1) =$$

$$[\![\alpha_1+v_1, \alpha_2+v_2, D(v_3)]\!] + [\![\alpha_1+v_1, D(v_2), \alpha_3+v_3]\!] + [\![D(v_1), \alpha_2+v_2, \alpha_3+v_3]\!] =$$

$$[\![D(\alpha_1 + v_1), \alpha_2 + v_2, \alpha_3 + v_3]\!] + [\![\alpha_1 + v_1, D(\alpha_2 + v_2), \alpha_3 + v_3]\!] +$$

$$[\![\alpha_1 v_1, \alpha_2 + v_2, D(\alpha_3 + v_3)]\!].$$

Thus, D is a derivation of $\mathcal{V}_{0,0,0}$.

Suppose that $char\,(\mathbb{F}) = 2$ and $\mathcal{V}_{0,0,0} = \langle 1, b\rangle_{\mathbb{F}}$. Then we have the following multiplication table (up to commutativity):

$$[\![1,1,1]\!] = 1, \quad [\![1,1,b]\!] = b, \quad [\![1,b,b]\!] = [\![b,b,b]\!] = 0. \tag{15}$$

Let $D \in Der\,(\mathcal{V}_{0,0,0})$, $D(1) = \alpha\ 1 + \beta\ b$ and $D(b) = \alpha'\ 1 + \beta'\ b$ for some $\alpha, \beta, \alpha', \beta' \in \mathbb{F}$. Applying D to (15) we see that D can be an arbitrary endomorphism of $\mathcal{V}_{0,0,0}$.

Consider the case $char\,(\mathbb{F}) = 2$ and $\dim \mathbb{V} > 1$. Let $\{b_1, b_2, \ldots, b_n\}$ be a basis for $\mathbb{V}$. The multiplication table in this case is the following (up to commutativity and zero products):

$$[\![1,1,1]\!] = 1, \quad [\![1,1,b_i]\!] = b_i,\ i = 1, \ldots, n.$$

Take $D \in Der\,(\mathcal{V}_{0,0,0})$. Then $D\,(b_i) = \alpha_i 1 + \beta_i b_i$ for some $\alpha_i, \beta_i \in \mathbb{F},\ i = 1, \ldots, n$. It is easy to see that only the action of D on $[\![1, b_i, b_j]\!] = 0$ gives some restrictions on α_i. Namely, from

$$D([\![1, b_i, b_j]\!]) = [\![D(1), b_i, b_j]\!] + [\![1, D(b_i), b_j]\!] + [\![1, b_i, D(b_j)]\!]$$

we infer that $\alpha_i = \alpha_j = 0$ for all $i \neq j$. Thus, $D(V) \subseteq V$, and for every b_i the image $D(b_i)$ may be an arbitrary element in $\mathbb{V}$.

In each case, the reciprocal assertion on the isomorphism is trivial.

□

2. The simple ternary Jordan algebra of a bilinear form

Restrict the algebra $\mathcal{V}_{0,0,h}$ to $\mathbb{V}$, an n-dimensional vector space over a field $\mathbb{F}$, and denote the bilinear form h by $(.,.)$, with the same properties with respect to a given basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of $\mathbb{V}$. Consider the following ternary product defined on $\mathbb{V}$:

$$[\![x, y, z]\!] = (y, z)\, x + (x, z)\, y + (x, y)\, z. \tag{16}$$

Denote the obtained ternary algebra by TJ_n. It is clear that (16) is a particular case of the general product (10).

Further, when $n = 4$ it is interesting to observe that (16) can be seen as a multiple of the symmetrization of the multiplication

$$\{x, y, z\} = \frac{1}{6}\left(-(y, z)\, x + (x, z)\, y - (x, y)\, z + [x, y, z]\right),$$

defined on the ternary Filippov algebra A_1 with anticommutative multiplication $[.,.,.]$ (see [1]). Indeed, being

$$\begin{aligned} \{x, y, z\}^{(+)} &= sym\left(\{x, y, z\}\right) \\ &= \{x, y, z\} + \{x, z, y\} + \{y, x, z\} + \{y, z, x\} + \{z, x, y\} + \{z, y, x\}, \end{aligned}$$

it is easy to see that

$$[\![x, y, z]\!] = -3\,\{x, y, z\}^{(+)}.$$

Clearly, (16) defines a totally commutative multiplication on TJ_n. Further, adopting the notations R_x and $D_{x,y}$ introduced in the previous section, now concerning the multiplication (16) in TJ_n, we have the following result.

Theorem 5. *TJ_n is a ternary Jordan algebra.*

Proof: According to the definition of ternary Jordan algebra, we must prove that

$$D_{x,y}[\![z_1, z_2, z_3]\!] = [\![D_{x,y}(z_1), z_2, z_3]\!] + [\![z_1, D_{x,y}(z_2), z_3]\!] + [\![z_1, z_2, D_{x,y}(z_3)]\!] \tag{17}$$

holds. Due to the linearity of $D_{x,y}$ (where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, with $x_i, y_i \in \mathbb{V}$) and its symmetry properties stated in the previous section, it suffices to verify (17) for $z_1, z_2, z_3 \in \mathcal{B}$ in the following cases:

1) $x_1, x_2, y_1, y_2 \in \{b_i, b_j, b_k, b_l\}$ and are all pairwise different;
2) $x_1, x_2, y_1, y_2 \in \{b_i, b_j, b_k\}$ and only two among these are equal;
3) $x_1, x_2, y_1, y_2 \in \{b_i, b_j\}$ and aren't all equal;
4) $x_1, x_2, y_1, y_2 \in \{b_i\}$.

Using the definition of $D_{x,y}$ and (16) it is immediate that in the cases 1 and 4 (17) holds trivially, since then we have

$$D_{x,y} = 0.$$

Considering the case 2, we have to check two subcases.
2.1. $x_1 = x_2 = b_i$, $y_1 = b_j$, and $y_2 = b_k$.
Under these circumstances,

$$R_{b_i,b_i} = E + 2e_{ii}, \; R_{b_j,b_k} = e_{jk} + e_{kj},$$

whence $D_{x,y} = 0$, and (17) holds trivially.
2.2. $x_1 = y_1 = b_i$, $x_2 = b_j$, and $y_2 = b_k$.
Developing $D(z) = D_{x,y}(z)$, we have

$$D(z) = (z, b_j)\, b_k - (z, b_k)\, b_j.$$

Concerning (17), it is easy to conclude that

$$\begin{aligned} LHS_D(z_1, z_2, z_3) &= [(z_1, z_2)(z_3, b_j) + (z_1, z_3)(z_2, b_j) + (z_2, z_3)(z_1, b_j)]\, b_k \\ &\quad - [(z_1, z_2)(z_3, b_k) + (z_1, z_3)(z_2, b_k) + (z_2, z_3)(z_1, b_k)]\, b_j \\ &= RHS_D(z_1, z_2, z_3). \end{aligned}$$

Thus, (17) holds.
Analyze the third case, dividing it in three subcases:

$$\begin{aligned} &3.1. \;\; x_1 = x_2 = b_i \text{ and } y_1 = y_2 = b_j; \\ &3.2. \;\; x_1 = y_1 = b_i \text{ and } x_2 = y_2 = b_j; \\ &3.3. \;\; x_1 = x_2 = y_1 = b_i \text{ and } y_2 = b_j. \end{aligned}$$

Since in the first two subcases (17) trivially holds ($D_{x,y}$ is zero), we have to check only the last subcase. We see that $D_{x,y}$ acts as in the case 2.2 (up to the scalar 2), which finishes the proof. ∎

Theorem 6. *The ternary Jordan algebra TJ_n is simple except for* $\dim \mathbb{V} = 2$ *and* $char\,(\mathbb{F}) = 2$.

Proof: Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be an orthonormal basis for $\mathbb{V}$. The assertion is trivial if $n = 1$, so admit that $n \geq 2$. The multiplication table for the basis elements is given by

$$\text{(i) } [\![b_i, b_i, b_i]\!] = 3b_i, \quad \text{(ii) } [\![b_i, b_i, b_j]\!] = b_j, \; (i \neq j),$$

$$\text{and} \quad \text{(iii) } [\![b_i, b_j, b_k]\!] = 0 \; \text{ for pairwise different } i, j, k.$$

Let $\mathbb{I} \neq \{0\}$ be an ideal of TJ_n. Clearly, it follows from (ii) in the multiplication table that if $b_i \in \mathbb{I}$ for some i then $\mathbb{I} = TJ_n$, and TJ_n is simple.

Let $z = \sum\limits_{i=1}^{p} \alpha_i b_i$ be an element in $\mathbb{I}\backslash\{0\}$ with minimal length $p \neq 1$ and $\alpha_1 \neq 0$ (otherwise, we reorder the indices). We have

$$w = [\![z, b_1, b_1]\!] = 3\alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_p b_p \in \mathbb{I}\backslash\{0\},$$

whence $w - z = 2\alpha_1 b_1 \in \mathbb{I}\backslash\{0\}$. If $char\,(\mathbb{F}) \neq 2$ then $b_1 \in \mathbb{I}$, and $\mathbb{I} = TJ_n$.

Assume that $char\,(\mathbb{F}) = 2$. Admit first that $\dim \mathbb{V} = 2$. Let $z = b_1 + b_2$. From

$$[\![z, b_1, b_1]\!] = z, \quad [\![z, b_2, b_2]\!] = z, \quad \text{and} \quad [\![z, b_1, b_2]\!] = z$$

it is clear that $\mathbb{I} = \langle z \rangle_{\mathbb{F}}$ is a non-trivial ideal of TJ_n, so TJ_n is not simple. Assume now that $\dim \mathbb{V} > 2$ and consider a non-zero ideal $\mathbb{I}$ of TJ_n. As above, take $z = \sum\limits_{r=1}^{p} \alpha_r b_r \in \mathbb{I}\backslash\{0\}$ of minimal length $p \neq 1$ and such that $\alpha_1 \neq 0$, $\alpha_i \neq 0$ for some $i \in \{2, \ldots, p\}$. Take $j \neq 1, i$. Then

$$w = [\![z, b_i, b_j]\!] = \alpha_i b_j + \alpha_j b_i \in \mathbb{I}\backslash\{0\}.$$

On the other hand, $[\![w, b_j, b_1]\!] = \alpha_i b_1 \in \mathbb{I}\backslash\{0\}$, and $\mathbb{I} = TJ_n$. ■

Recall now that an identity satisfied by a ternary algebra is said to be of *degree* (or *level*) k, with $k \in \mathbb{N}$, if k is the number of times that the multiplication appears in each term of the identity (see [1]). Next, we are going to study the multilinear identities of degrees 1 and 2, respectively, valid in the ternary Jordan algebra TJ_n of characteristic not 3.

The identities of degree 1 satisfied by TJ_n are of the following shape:

$$\sum_{\sigma \in S_3} \alpha_\sigma [\![x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]\!] = 0, \qquad \alpha_\sigma \in \mathbb{F}.$$

Due to the total commutativity of the multiplication (16), this sum is reduced to one summand and it is easy to observe that the identities of degree 1 are resumed by that property.

Again by the total commutativity of the multiplication, the degree 2 identities valid in TJ_n are of the following form:

$$\sum_{\substack{\sigma \in S_5 \\ \sigma(1) < \sigma(2) < \sigma(3) \\ \sigma(4) < \sigma(5)}} \alpha_\sigma [\![[\![x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]\!], x_{\sigma(4)}, x_{\sigma(5)}]\!] = 0, \qquad \alpha_\sigma \in \mathbb{F},$$

which can be expanded in the following way:

$$\alpha_1[\![[\![x,y,z]\!],u,v]\!]+\alpha_2[\![[\![x,y,u]\!],z,v]\!]+\alpha_3[\![[\![x,y,v]\!],z,u]\!]+\alpha_4[\![[\![x,z,u]\!],y,v]\!]+ \alpha_5[\![[\![x,z,v]\!],y,u]\!]+\alpha_6[\![[\![x,u,v]\!],y,z]\!]+\alpha_7[\![[\![y,z,u]\!],x,v]\!]+\alpha_8[\![[\![y,z,v]\!],x,u]\!]+ \alpha_9[\![[\![y,u,v]\!],x,z]\!]+\alpha_{10}[\![[\![z,u,v]\!],x,y]\!]=0. \tag{18}$$

Let us find what conditions must the α_i satisfy.

1. If $\dim\mathbb{V}=1$, being $\mathbb{V}=\langle b\rangle_{\mathbb{F}}$, since $[\![b,b,b]\!]=3b$ from (18) we get

$$\sum_{i=1}^{10}\alpha_i=0. \tag{19}$$

Now, choose an orthonormal basis $\{b_1,b_2\}$ for $\mathbb{V}$. Then

$$[\![b_i,b_i,b_i]\!]=3b_i,\ i=1,2, \quad \text{and} \quad [\![b_i,b_i,b_j]\!]=b_j,\, i\neq j.$$

In order to analyze what relations between the scalars can be derived from (18), we are going to check all non redundant possible cases with x,y,z,u,v in the considered basis.

2. Suppose that among x,y,z,u,v only four are equal (*e.g.*, to b_1). Then, we have do consider 5 subcases:

(2.1) $x=y=z=u=b_1$ and $v=b_2$; (2.2) $x=y=z=v=b_1$ and $u=b_2$;
(2.3) $x=y=u=v=b_1$ and $z=b_2$; (2.4) $x=z=u=v=b_1$ and $y=b_2$;
(2.5) $y=z=u=v=b_1$ and $x=b_2$.

Replacing in (18) for each subcase, we obtain:

$$\begin{aligned}
(2.1) &\rightarrow\ 3\alpha_1+3\alpha_2+\alpha_3+3\alpha_4+\alpha_5+\alpha_6+3\alpha_7+\alpha_8+\alpha_9+\alpha_{10}=0,\\
(2.2) &\rightarrow\ 3\alpha_1+\alpha_2+3\alpha_3+\alpha_4+3\alpha_5+\alpha_6+\alpha_7+3\alpha_8+\alpha_9+\alpha_{10}=0,\\
(2.3) &\rightarrow\ \alpha_1+3\alpha_2+3\alpha_3+\alpha_4+\alpha_5+3\alpha_6+\alpha_7+\alpha_8+3\alpha_9+\alpha_{10}=0,\\
(2.4) &\rightarrow\ \alpha_1+\alpha_2+\alpha_3+3\alpha_4+3\alpha_5+3\alpha_6+\alpha_7+\alpha_8+\alpha_9+3\alpha_{10}=0,\\
(2.5) &\rightarrow\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+3\alpha_7+3\alpha_8+3\alpha_9+3\alpha_{10}=0.
\end{aligned}$$

3. Admit now that among x,y,z,u,v only three are equal (*e.g.*, to b_1). Then, we have to consider ten subcases:

(3.1) $x=y=z=b_1$ and $u=v=b_2$; (3.2) $x=y=u=b_1$ and $z=v=b_2$;
(3.3) $x=y=v=b_1$ and $z=u=b_2$; (3.4) $x=z=u=b_1$ and $y=v=b_2$;
(3.5) $x=z=v=b_1$ and $y=u=b_2$; (3.6) $x=u=v=b_1$ and $y=z=b_2$;
(3.7) $y=z=u=b_1$ and $x=v=b_2$; (3.8) $y=z=v=b_1$ and $x=u=b_2$;
(3.9) $y=u=v=b_1$ and $x=z=b_2$; (3.10) $z=u=v=b_1$ and $x=y=b_2$.

Analogously to what we have done in the case 2., we will obtain the following equations:

$$\begin{aligned}
&(3.1) \to\ 3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + 3\alpha_6 + \alpha_7 + \alpha_8 + 3\alpha_9 + 3\alpha_{10} = 0,\\
&(3.2) \to\ \alpha_1 + 3\alpha_2 + \alpha_3 + \alpha_4 + 3\alpha_5 + \alpha_6 + \alpha_7 + 3\alpha_8 + \alpha_9 + 3\alpha_{10} = 0,\\
&(3.3) \to\ \alpha_1 + \alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5 + \alpha_6 + 3\alpha_7 + \alpha_8 + \alpha_9 + 3\alpha_{10} = 0,\\
&(3.4) \to\ \alpha_1 + \alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + 3\alpha_8 + 3\alpha_9 + \alpha_{10} = 0,\\
&(3.5) \to\ \alpha_1 + 3\alpha_2 + \alpha_3 + \alpha_4 + 3\alpha_5 + \alpha_6 + 3\alpha_7 + \alpha_8 + 3\alpha_9 + \alpha_{10} = 0,\\
&(3.6) \to\ 3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + 3\alpha_6 + 3\alpha_7 + 3\alpha_8 + \alpha_9 + \alpha_{10} = 0,\\
&(3.7) \to\ \alpha_1 + \alpha_2 + 3\alpha_3 + \alpha_4 + 3\alpha_5 + 3\alpha_6 + 3\alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} = 0,\\
&(3.8) \to\ \alpha_1 + 3\alpha_2 + \alpha_3 + 3\alpha_4 + \alpha_5 + 3\alpha_6 + \alpha_7 + 3\alpha_8 + \alpha_9 + \alpha_{10} = 0,\\
&(3.9) \to\ 3\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 + 3\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + 3\alpha_9 + \alpha_{10} = 0,\\
&(3.10) \to\ 3\alpha_1 + 3\alpha_2 + 3\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + 3\alpha_{10} = 0.
\end{aligned}$$

Now, the linear system consisting of (19) and the other 15 equations has only the trivial solution. Therefore, the only identities of degree 2 in TJ_n are those that result from lifting the identities of degree 1.

Thus, we have the following result.

Lemma 7. *All degrees* 1 *and* 2 *identities on* TJ_n *(char* $(\mathbb{F}) \neq 3$*) are a consequence of the total commutativity of* (16).

Remark 8. *Recall that a Jordan triple system* (*see* [3] *and also* [9]*, where this notion also appears under the name of "ternary Jordan algebra"*) *is a ternary algebra* A *with a ternary multiplication* $[\![.,.,.]\!]$ *satisfying a partial commutativity property*

$$[\![x,y,z]\!] = [\![z,y,x]\!]$$

and the following identity:

$$[\![[\![x,y,z]\!],u,v]\!] + [\![z,u,[\![x,y,v]\!]]\!] = [\![x,y,[\![z,u,v]\!]]\!] + [\![z,[\![y,x,u]\!],v]\!]. \qquad (20)$$

According to the previous computations, it is also clear that A *doesn't satisfy* (20)*, clarifying that we are working with a different generalization.*

3.Derivations of TJ_n

Now, we are going to describe the derivations of TJ_n. Consider a linear map $D : \mathbb{V} \to \mathbb{V}$. Using the definition of derivation of a ternary algebra and (16), one may see that $D \in Der\,(TJ_n)$ if and only if

$$\begin{gathered}\big(\,(D\,(y)\,,z) + (y, D(z))\,\big)x + \big(\,(D\,(x)\,,z) + (x, D(z))\,\big)y+ \\ \big(\,(D\,(x)\,,y) + (x, D(y))\,\big)z = 0 \end{gathered} \qquad (21)$$

for all $x, y, z \in \mathbb{V}$. It is clear that it suffices to work with (21) for all x, y, z in the orthonormal basis $\mathcal{B}$ of $\mathbb{V}$ we have chosen before.

It is easy to see that

$$(D(b_i), b_j) = -(D(b_j), b_i) \text{ with } i \neq j. \tag{22}$$

This way, we have

$$D(b_j) = \sum_{i=1}^{n} \alpha_i b_i = \sum_{i=1}^{n} (b_i, D(b_j)) b_i.$$

It means that D is a skew-symmetric operator on $\mathbb{V}$ for every derivation D with respect to our form.

Now, observe that the algebra $Inder(TJ_n)$ of inner derivations of TJ_n is just the Lie algebra generated by the right multiplication operators R_x. By definition,

$$Inder(TJ_n) = L(TJ_n) \cap Der(TJ_n),$$

where $L(TJ_n)$ is the Lie transformation algebra of TJ_n. From the proof of Theorem 5 we know the operators R_{b_i,b_j}. So, it is easy to see that $L(TJ_n) = \langle e_{ij} - e_{ji} \rangle_{\mathbb{F}}$. On the other hand, $D \in L(TJ_n)$ by (22). Thus, we have

Theorem 9. $Der(TJ_n) = Inder(TJ_n) = so(n)$.

Remark 10. *In* 1955 *Jacobson proved that if a finite-dimensional Lie algebra* L *over a field of characteristic zero has an invertible derivation then* L *is nilpotent* [12]. *The same result was proved for the Jordan algebras* [15] *but as we can see from Theorem 9 the Jacobson Theorem is not true for the ternary Jordan algebras. We can take a ternary Jordan algebra* TJ_4 *and consider the map defined by the following matrix* $\sum_{1 \leq i < j \leq 4} (e_{ij} - e_{ji})$. *As follows, there is a simple ternary Jordan algebra with an invertible derivation.*

4. Ternary symmetrized matrix algebras

Consider the following ternary algebras

$$\mathfrak{A} = (M_n(\mathbb{F}), [\![., ., .]\!]),$$

where $[\![., ., .]\!]$ is the symmetrized ternary multiplication defined by

$$[\![A_1, A_2, A_3]\!] = sym(A_1, A_2, A_3) = \sum_{\sigma \in \mathbb{S}_3} A_{\sigma(1)} A_{\sigma(2)} A_{\sigma(3)} \text{ with } A_1, A_2, A_3 \in M_n(\mathbb{F}).$$

This multiplication, also known as the ternary anticommutator, is clearly totally commutative. It is easy to see that $\mathfrak{A}$ is not a ternary Jordan algebra in general. In fact, the identity

$$D_{x,y}\,[\![A,B,C]\!] = [\![D_{x,y}(A),B,C]\!] + [\![A,D_{x,y}(B),C]\!] + [\![A,B,D_{x,y}(C)]\!]$$

does not hold in $\mathfrak{A}$. To see this, we can consider $n=3$ and evaluate both sides by the following elements of the canonical basis of $M_3(\mathbb{F})$:

$$x=(e_{23},e_{32}),\ y=(e_{22},e_{23}),\ A=e_{12},\ B=e_{23},\ \text{and } C=e_{32}.$$

Then $LHS_D(A,B,C)=0$, while $RHS_D(A,B,C)=-3e_{13}$, where $D=D_{x,y}$.

However, we have the following

Theorem 11. *Given different $i,j\in\{1,\dots,n\}$, the following 2-dimensional subalgebras of $M_n(\mathbb{F})$*

$$\mathfrak{S}_1=\langle e_{ii},e_{ij}\rangle_{\mathbb{F}} \quad \textit{and} \quad \mathfrak{S}_2=\langle e_{ij},e_{ji}\rangle_{\mathbb{F}}, \quad (i\neq j),$$

are non-isomorphic ternary Jordan algebras. Furthermore, $\mathfrak{S}_2$ is simple if $char\,\mathbb{F}\neq 2$.

Proof. The proof of the first assertion will only be done in the case of the subalgebra $\mathfrak{S}_1$, since the other case could be proved analogously.

The multiplication table for the basis elements of $\mathfrak{S}_1$ is given by

$$[\![e_{ii},e_{ii},e_{ii}]\!]=6e_{ii},\quad [\![e_{ii},e_{ii},e_{ij}]\!]=2e_{ij},\quad \text{and}\quad [\![e_{ii},e_{ij},e_{ij}]\!]=[\![e_{ij},e_{ij},e_{ij}]\!]=0.$$

Thus, considering the matrix representation of the right multiplication operators $R_{(e_{ii},e_{ii})}$, $R_{(e_{ii},e_{ij})}$, and $R_{(e_{ij},e_{ij})}$ with respect to the basis $\{f_1=e_{ii}, f_2=e_{ij}\}$, we have

$$R_{(e_{ii},e_{ii})}=6f_{11}+2f_{22},\quad R_{(e_{ii},e_{ij})}=2f_{12},\quad \text{and}\quad R_{(e_{ij},e_{ij})}=0.$$

Defining $D_{(x_1,x_2),(y_1,y_2)}$ as before, the only non-trivial case for these operators is given by

$$D=D_{(e_{ii},e_{ii}),(e_{ii},e_{ij})}=8f_{12}.$$

Now, to verify the $D_{x,y}$-identity it suffices to do it in the four cases of the multiplication table above. In the first case, we have

$$LHS_D(e_{ii},e_{ii},e_{ii})=48e_{ij}=3[\![D(e_{ii}),e_{ii},e_{ii}]\!]=RHS_D(e_{ii},e_{ii},e_{ii}).$$

For the other three cases, both sides of the identity are zero, proving that $\mathfrak{S}_1$ is a ternary Jordan algebra.

It is easy to see that $\langle e_{ij}\rangle_{\mathbb{F}}$ is an ideal of $\mathfrak{S}_1$.

Assume now that $\mathfrak{I}$ is an ideal of $\mathfrak{S}_2$ and consider $x = \alpha e_{ij} + \beta e_{ji} \in \mathfrak{I}\backslash\{0\}$. Observing that the multiplication table for the basis elements in $\mathfrak{S}_2$ is given by

$$[\![e_{ij}, e_{ij}, e_{ij}]\!] = [\![e_{ji}, e_{ji}, e_{ji}]\!] = 0,\ [\![e_{ij}, e_{ij}, e_{ji}]\!] = 2e_{ij},\ \text{and } [\![e_{ij}, e_{ji}, e_{ji}]\!] = 2e_{ji},$$

we have

$$[\![x, e_{ij}, e_{ij}]\!] = 2\beta e_{ij}.$$

If $\beta \neq 0$ then $e_{ij} \in \mathfrak{I}$. Then

$$[\![e_{ij}, e_{ji}, e_{ji}]\!] = 2e_{ji} \in \mathfrak{I}$$

which implies $e_{ji} \in \mathfrak{I}$, and $\mathfrak{I} = \mathfrak{S}_2$. If $\beta = 0$ then $x = \alpha e_{ij} \in \mathfrak{I}\backslash\{0\}$ implies $e_{ij} \in \mathfrak{I}$, and we arrive at the same conclusion.

Finally, it is clear that these two algebras are not isomorphic.

□

Concerning the identities verified in $\mathfrak{A}$, it is possible to prove the same results we have achieved about the algebra in the third section, by using similar techniques. Thus,

- the identities of degree 1 satisfied by $\mathfrak{A}$ are resumed in its total commutativity;
- all degree 2 identities satisfied by $\mathfrak{A}$ result from lifting the total commutativity of the anticommutator.

5. Ternary algebras defined on the Cayley-Dickson algebras

Recall the Cayley-Dickson doubling process ([21], [24]). Consider a unital algebra $\mathcal{A}$ over a field $\mathbb{F}$ of characteristic not 2. Assume that $\mathcal{A}$ is equipped with an involution $x \mapsto \overline{x}$ such that

$$x + \overline{x},\ x\overline{x} \in \mathbb{F}, \text{ for all } x \in \mathcal{A}.$$

Take $\mu \in \mathbb{F}\backslash\ \{0\}$ and define a new algebra $(\mathcal{A}, \mu)$ as follows:

$\mathcal{A} \oplus \mathcal{A}$,	the underlying vector space,
$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$,	the addition,
$c\,(x_1, x_2) = (cx_1, cx_2)$,	the scalar multiplication ($c \in \mathbb{F}$),
$(x_1, x_2)\,(y_1, y_2) = (x_1y_1 + \mu y_2\overline{x}_2, \overline{x}_1y_2 + y_1x_2)$,	the multiplication.

The corresponding involution in $(\mathcal{A}, \mu)$ is given by

$$\overline{(x_1, x_2)} = (\overline{x}_1, -x_2).$$

The symmetric bilinear form $\langle x, y\rangle = \frac{1}{2}(x\bar{y} + y\bar{x})$ defined on $\mathcal{A}$ is nonsingular and we can define the norm of an element $a \in \mathcal{A}$ by the rule $n(a) = \langle a, a\rangle$.

Starting with $\mathbb{F}$ such that $char\,(\mathbb{F}) \neq 2$, we obtain a sequence of 2^t-dimensional algebras denoted by $\mathcal{U}_t$, among which

$\mathcal{U}_0 = \mathbb{F}$, the scalars,
a commutative and associative algebra;
$\mathcal{U}_1 = \mathbb{C}\,(\mu) = (\mathbb{F}, \mu)$, the generalized complex numbers,
a commutative and associative algebra;
$\mathcal{U}_2 = \mathbb{H}\,(\mu, \beta) = (\mathbb{C}\,(\mu)\,, \beta)$, the generalized quaternions,
a not commutative and associative algebra;
$\mathcal{U}_3 = \mathbb{O}\,(\mu, \beta, \gamma) = (\mathbb{H}\,(\mu, \beta)\,, \gamma)$, the generalized octonions,
a not commutative, not associative and alternative algebra;

are the most notable examples. Define on each $\mathcal{U}_t$, $t = 2, 3, \ldots$, the ternary multiplication:

$$[\![x, y, z]\!] = (x\overline{y})\, z \tag{23}$$

and take

$$\mathcal{D}_t = (\mathcal{U}_t, [\![.,.,.]\!])\,.$$

Clearly, this ternary multiplication is not totally commutative, so these algebras are not ternary Jordan algebras. Before going on, recall some properties of composition algebras (thus, valid in particular in $\mathcal{U}_2$ and $\mathcal{U}_3$).

Lemma 12. *Let $\mathbb{A}$ be a composition algebra with unit 1, with an involution $^-$ and a bilinear symmetric non-degenerate form $\langle .,. \rangle$. For all $a, b, c \in \mathbb{A}$, we have*

(1) $(a\overline{a})b = a(\overline{a}b) = n(a)b = (b\overline{a})a = b(\overline{a}a)$;
(2) $(a\overline{b})c + (a\overline{c})b = 2\langle b, c\rangle a$;
(3) $a(\overline{b}c) + b(\overline{a}c) = 2\langle a, b\rangle c$.
If additionally a, b, c are different elements in an orthonormal basis then
(4) $\overline{a}b\overline{a} = -\overline{b}$;
(5) $(a\overline{b})c = -(a\overline{c})b$;
(6) $a(\overline{b}c) = -b(\overline{a}c)$.

Theorem 13. *$\mathcal{D}_2$ is a simple ternary $D_{x,y}$-derivation algebra.*

Proof: Consider arbitrary $x = (x_1, x_2)$, $y = (y_1, y_2)$, with $x_i, y_i \in \mathbb{H}(\mu, \beta)$, $i = 1, 2$. It is clear that every linear operator $D_{x,y}$ is also linear in each x_i and each y_i, so we can consider that these elements belong to $\mathcal{B} = \{1, a, b, ab\}$, the usual orthonormal basis of $\mathbb{H}(\mu, \beta)$. Recall that $\mathcal{U}_2$ is associative. State some properties of $[\![.,.,.]\!]$ and of each operator $D_{x,y}$ in $\mathcal{D}_2$.

First of all, by the previous lemma we note that for pairwise different elements $x, y, z \in \mathcal{B}$ we have

$$[\![x, y, z]\!] = -[\![y, x, z]\!] = -[\![x, z, y]\!].$$

Moreover, by Lemma 12, $R_{x,y} = -R_{y,x}$ for all $x, y \in \mathcal{B},\ x \neq y$.

Further, if $x_i, y_i \in \mathcal{B},\ x_1 \neq x_2,\ y_1 \neq y_2$, then

$$D_{(x_1,x_2),(y_1,y_2)} = -D_{(x_1,x_2),(y_2,y_1)} = -D_{(x_2,x_1),(y_1,y_2)}.$$

In order to verify the $D_{x,y}$-identity we consider the following cases:

(1) $x_1 = x_2 = y_1 = y_2$;
(2) only three elements among $\{x_1, x_2, y_1, y_2\}$ are equal;
(3) two pairs of elements among $\{x_1, x_2, y_1, y_2\}$ are equal;
(4) only two elements among $\{x_1, x_2, y_1, y_2\}$ are equal;
(5) all $\{x_1, x_2, y_1, y_2\}$ are pairwise different.

In the first case, $D_{(x_1,x_1),(x_1,x_1)} = 0$, and the $D_{x,y}$-identity trivially holds.

Admit that among $\{x_1, x_2, y_1, y_2\}$ only three are equal. By above, we may assume that $x_1 = x_2 = y_1,\ y_2 \neq x_1$. Thus,

$$D_{(x_1,x_1),(x_1,y_2)}(z) = z\overline{x_1}x_1\overline{x_1}y_2 - z\overline{x_1}y_2\overline{x_1}x_1 = 0 \text{ for all } z \in \mathcal{B},$$

and the $D_{x,y}$-identity trivially holds.

Concerning the case (3), we may assume that we have two subcases:

$$\text{(i)}\ \ x_1 = x_2,\ y_1 = y_2,\ x_1 \neq y_1;\ \ \text{(ii)}\ \ x_1 = y_1,\ x_2 = y_2.$$

The case (ii) is a trivial one. In the subcase (i) we easily have $D_{x,y} = 0$ by Lemma 12.

Analyze the case (4). By above, we may assume that we have two subcases:

$$\text{(i)}\ \ x_1 = x_2,\ x_1, y_1, y_2 \text{ are pairwise different;}$$
$$\text{(ii)}\ \ x_1 = y_1,\ x_1, x_2, y_2 \text{ are pairwise different.}$$

Concerning the subcase (i), for every $z \in \mathcal{B}$ we have

$$\begin{aligned} D_{(x_1,x_1),(y_1,y_2)}(z) &= [\![[\![z, x_1, x_1]\!], y_1, y_2]\!] - [\![[\![z, y_1, y_2]\!], x_1, x_1]\!] \\ &= z\overline{x_1}x_1\overline{y_1}y_2 - z\overline{y_1}y_2\overline{x_1}x_1 = z\overline{y_1}y_2 - z\overline{y_1}y_2 = 0 \end{aligned}$$

for all $z \in \mathcal{B}$, concluding this case.

Consider the subcase (ii). Put $D := D_{(x_1,x_2),(x_1,y_2)}$. Then for every $z \in \mathcal{B}$ we have

$$\begin{aligned} D(z) &= [\![[\![z, x_1, x_2]\!], x_1, y_2]\!] - [\![[\![z, x_1, y_2]\!], x_1, x_2]\!] = z\overline{x_1}x_2\overline{x_1}y_2 - z\overline{x_1}y_2\overline{x_1}x_2 \\ &= -z\overline{x_2}y_2 + z\overline{y_2}x_2 = -2z\overline{x_2}y_2. \end{aligned}$$

Compute both sides of the $D_{x,y}$-identity for $z_1, z_2, z_3 \in \mathcal{B}$. We have

$$D_{(x_1,x_2),(x_1,y_2)}\left([\![z_1, z_2, z_3]\!]\right) = -2z_1\overline{z_2}z_3\overline{x_2}y_2.$$

On the other hand,

$$\begin{aligned} [\![D_{(x_1,x_2),(x_1,y_2)}(z_1), z_2, z_3]\!] &= -2z_1\overline{x_2}y_2\overline{z_2}z_3, \\ [\![z_1, D_{(x_1,x_2),(x_1,y_2)}(z_2), z_3]\!] &= -2z_1\overline{y_2}x_2\overline{z_2}z_3, \\ [\![z_1, z_2, D_{(x_1,x_2),(x_1,y_2)}(z_3)]\!] &= -2z_1\overline{z_2}z_3\overline{x_2}y_2. \end{aligned}$$

Since $\overline{x_2}y_2 = -\overline{y_2}x_2$, the $D_{x,y}$-identity is proved.

The simplicity of $\mathcal{D}_2$ follows from the simplicity of $\mathcal{U}_2$ putting $y = 1$ in (23). ■

Consider $\mathbb{H}(\mu, \beta) = \langle 1 \rangle_{\mathbb{F}} \oplus \mathbb{H}(\mu, \beta)_s$, where $\mathbb{H}(\mu, \beta)_s$ is the set of all elements in $\mathbb{H}(\mu, \beta)$ that are invariant under the involution.

Theorem 14. *$D \in Der(\mathcal{D}_2)$ if and only if $D = \Phi + \Psi$ for some $\Phi, \Psi \in End\,(\mathbb{H}(\mu, \beta))$ such that $\Phi \in Der(\mathbb{H}(\mu, \beta))$, $\Psi(x) = x\Psi(1)$ for all $x \in \mathbb{H}(\mu, \beta)$, and $\Psi(1) \in \mathbb{H}(\mu, \beta)_s$.*

Proof: Admit that $D \in Der(\mathcal{D}_2)$. Then from

$$D\,[\![x, y, z]\!] = [\![D\,(x)\,, y, z]\!] + [\![x, D\,(y)\,, z]\!] + [\![x, y, D\,(z)]\!] \tag{24}$$

and (23) we obtain

$$D\,(xy) = D\,(x\overline{1}y) = D(x)y + x\overline{D(1)}y + xD(y).$$

Setting $x = y = 1$ in this equality we get

$$D(1) + \overline{D(1)} = 0,$$

which implies that $D(1) \in \mathbb{H}(\mu, \beta)_s$, and

$$D\,(xy) = D(x)y - xD(1)y + xD(y). \tag{25}$$

Consider $g \in End\,(\mathbb{H}(\mu, \beta))$ such that

$$g(x) = D(x) - xD(1). \tag{26}$$

By (25)

$$g(xy) = D(xy) - xyD(1) = D(x)y - xD(1)y + xD(y) - xyD(1) = g(x)y + xg(y)$$

for all $x, y \in \mathbb{H}(\mu, \beta)$. By (26), $D(x) = g(x) + xD(1)$.

Reciprocally, admit that $D \in End\,(\mathbb{H}(\mu, \beta))$ is such that

$$D = \Phi + \Psi$$

where $\Phi, \Psi \in End\,(\mathbb{H}(\mu, \beta))$ are such that $\Phi \in Der\,(\mathbb{H}(\mu, \beta))$, $\Psi(x) = x\Psi(1)$ for all $x \in \mathbb{H}(\mu, \beta)_s$, and $\Psi(1) \in \mathbb{H}(\mu, \beta)_s$. Note that $\Psi(x) = x\Psi(1)$ also holds if $x \in \mathbb{H}(\mu, \beta)$. Then, for all $x, y, z \in \mathbb{H}(\mu, \beta)_s$ we have

$$\Psi\,[\![x, y, z]\!] = \Psi\,(x\overline{y}z) = x\overline{y}z\;\Psi(1).$$

Furthermore,

$$\begin{aligned}
[\![\Psi\,(x)\,, y, z]\!] + [\![x, \Psi\,(y)\,, z]\!] + [\![x, y, \Psi\,(z)]\!] &= x\Psi\,(1)\,\overline{y}z + x\overline{y\Psi\,(1)}z + x\overline{y}z\Psi\,(1) \\
&= x\Psi\,(1)\,\overline{y}z - x\Psi\,(1)\,\overline{y}z + x\overline{y}z\Psi\,(1) \\
&= x\overline{y}z\;\Psi(1).
\end{aligned}$$

Note that this also holds if $x, y, z \in \langle 1\rangle_{\mathbb{F}}$, so $\Psi \in Der(\mathcal{D}_2)$.

Since $\Phi \in Der\,(\mathbb{H}(\mu, \beta))$, $\Phi(1) = 0$. It is well-known that $\Phi\,(\mathbb{H}(\mu, \beta)) \subseteq \mathbb{H}(\mu, \beta)_s$, whence for every $y = \mu\,1 + y_s$, $y_s \in \mathbb{H}(\mu, \beta)_s$ we have

$$\overline{\Phi(y)} = \overline{\Phi\,(y_s)} = -\Phi\,(y_s) = \Phi\,(\overline{y})\,.$$

Thus, if $x, y, z \in \mathbb{H}(\mu, \beta)$ then

$$\begin{aligned}
\Phi\,[\![x, y, z]\!] &= \Phi\,(x\overline{y}z) \\
&= \Phi(x)\overline{y}z + x\Phi\,(\overline{y})\,z + x\overline{y}\Phi\,(z) \\
&= \Phi(x)\overline{y}z + x\overline{\Phi\,(y)}z + x\overline{y}\Phi\,(z) \\
&= [\![\Phi\,(x)\,, y, z]\!] + [\![x, \Phi\,(y)\,, z]\!] + [\![x, y, \Phi\,(z)]\!],
\end{aligned}$$

whence $\Phi \in Der(\mathcal{D}_2)$, and the same holds for D. ■

Lemma 15. *All degree* 1 *identities of* $\mathcal{D}_2$ *are consequences of*

$$[\![y, x, x]\!] = [\![x, x, y]\!].$$

All degree 2 *identities of* $\mathcal{D}_2$ *are consequences of* (15) *and the following degree* 2 *identities*:

$$\begin{aligned}
[\![[\![x, y, z]\!], u, v]\!] &= [\![x, y, [\![z, u, v]\!]]\!], \\
[\![[\![x, y, z]\!], u, v]\!] &= [\![x, [\![u, z, y]\!], v]\!].
\end{aligned}$$

From the definition of $\mathbb{O}(\mu, \beta, \gamma)$ we have

Lemma 16. $\mathcal{D}_3$ *is not a ternary* $D_{x,y}$*-derivation algebra.*

Proof: Consider the canonical orthonormal basis $\mathcal{B} = \{1, a, b, ab, c, ac, bc, (ab)c\}$ for $\mathbb{O}(\mu, \beta, \gamma)$ with the usual multiplication in this composition algebra [24]. Take

$$x_1 = a = y_1,\ x_2 = b,\ \text{and } y_2 = c.$$

Then,

$$D(z) = D_{x,y}(z) = (((z\overline{a})\,b)\,\overline{a})\,c - (((z\overline{a})\,c)\,\overline{a})\,b \text{ for all } z \in \mathbb{O}(\mu, \beta, \gamma).$$

Take $z_1 = ab$, $z_2 = 1$, and $z_3 = c$. Then, $[\![z_1, z_2, z_3]\!] = [\![ab, 1, c]\!] = (ab)c$, and we have

$$LHS_D(z_1, z_2, z_3) = D\left((ab)c\right) = 2\alpha\beta\gamma a,$$

where $\alpha = \frac{1}{4}(4\mu + 1) \neq 0$ (see [24]). On the other hand, since $D(ab) = -2\alpha\beta ac$, $D(1) = -2\alpha bc$, and $D(c) = 2\alpha\gamma b$; therefore,

$$RHS_D(z_1, z_2, z_3) = [\![D(ab), 1, c]\!] + [\![ab, D(1), c]\!] + [\![ab, 1, D(c)]\!] = -2\alpha\beta\gamma a.$$

Thus, $\mathcal{D}_3$ is not a $D_{x,y}$-derivation algebra. ■

6. An analog of the TKK-construction for ternary algebras

We recall the Tits-Kantor-Koecher (TKK for short) construction, which connects Lie and Jordan algebras (see [14], [16] and [26]). In this section we use an analogue of this construction to define ternary multiplications and ternary Jordan algebras.

Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a 3-graded ternary algebra with the product $[x, y, z]$. By definition, we have

$$[L_i, L_j, L_k] \subseteq L_{i+j+k},$$

where the addition is considered modulo 3. Following I. Kantor, we define a ternary operation on $\mathcal{J} := L_0$ by the rule

$$[\![x, y, z]\!] = sym_{x,y,z}[[[u_{-1}, x, u_1], y, v_{-1}], z, v_1], \tag{27}$$

where $sym_{x,y,z}$ is the symmetrization operator in x, y, z, and $u_i, v_i \in L_i$, $i = -1, 1$.

Let $L = A_1$ be the simple 4-dimensional Filippov algebra over $\mathbb{C}$ with the standard basis $\{e_1, e_2, e_3, e_4\}$ and the multiplication table

$$[e_1, \ldots, \hat{e}_i, \ldots, e_4] = (-1)^i e_i.$$

Change this basis to

$$a = \frac{\mathbf{i}}{2}e_1,\ b = \frac{1}{2}e_2,\ a_{-1} = e_3 - \mathbf{i}e_4,\ a_1 = e_3 + \mathbf{i}e_4,\ \text{where } \mathbf{i}^2 = -1.$$

Then

$$\langle a_{-1}\rangle \oplus \langle a, b\rangle \oplus \langle a_1\rangle$$

is a 3-grading on A_1 with $\mathfrak{J} = L_0 = \langle a, b\rangle$. Indeed, due to the anticommutativity of the multiplication $[.,.,.]$, to reach this conclusion it suffices to observe that

$$[a, a_{-1}, a_1] = -2b \quad \text{and} \quad [b, a_{-1}, a_1] = -2a.$$

Putting $u_{-1} = v_{-1} = a_{-1},\ u_1 = v_1 = a_1$ in (27), we obtain the following multiplication table in $\mathfrak{J}$:

$$[\![a, a, a]\!] = 6b, \quad [\![a, a, b]\!] = 2a, \quad [\![a, b, b]\!] = -2b, \quad \text{and} \quad [\![b, b, b]\!] = -6a. \tag{28}$$

Then we have

$$R_{(a,a)} = 6e_{12} + 2e_{21}, \quad R_{(a,b)} = 2e_{11} - 2e_{22}, \quad \text{and} \quad R_{(b,b)} = -2e_{12} - 6e_{21}.$$

Thus,

$$D_{(a,a),(a,b)} \doteq -3e_{12}+e_{21}, \quad D_{(a,a),(b,b)} \doteq e_{11}-e_{22}, \quad \text{and} \quad D_{(a,b),(b,b)} \doteq -e_{12}+3e_{21}$$

where $\doteq$ denotes an equality up to a scalar and e_{ij} is the matrix unit in the basis $\{a, b\}$. Now, we may consider a ternary commutative algebra $\mathfrak{J}$ over an arbitrary field with the multiplication table (28). The inclusion

$$D_{x,y} \in \langle -3e_{12} + e_{21}, e_{11} - e_{22}, -e_{12} + 3e_{21}\rangle$$

is immediate. Verifying the $D_{x,y}$-identity, we conclude that it holds if and only if $char\,(\mathbb{F}) = 2$.

In the Kantor article the product was defined on the space L_{-1} by the rule $xy = [[a, x], y]$ for a fixed $a \in L_1$. We proceed analogously. Put

$$[\![x, y, z]\!] = sym_{x,y,z}[[[u_0, x, u_1], y, v_1], z, v_0],$$

where $u_i, v_i \in L_i,\ i = 0, 1,\ x, y, z \in L_{-1}$. In this case we have

$$[\![a_{-1}, a_{-1}, a_{-1}]\!] = a_{-1},$$

with

$$a_{-1} = e_3 - \mathbf{i}e_4,\ u_0 = \frac{\mathbf{i}}{4}e_1,\ v_0 = e_2,\ u_1 = v_1 = a_1 = e_3 + \mathbf{i}e_4.$$

It is easy to notice that every one-dimensional ternary algebra $\mathfrak{J}$ is a ternary Jordan algebra, and $\mathfrak{J}$ is simple if and only if $\{\mathfrak{J}, \mathfrak{J}, \mathfrak{J}\} \neq 0$.

7. On the reduced algebras of n-ary Jordan algebras

Given an arbitrary class $\mathcal{A}$ of n-ary algebras with $n > 2$ and an algebra $A \in \mathcal{A}$ with a product $[\![, \ldots,]\!]$, fix $a \in A$ and for each $i \in \{1, \ldots, n\}$ define an $(n-1)$-ary algebra $A_{i,a}$ by putting

$$[\![x_2, \ldots, x_n]\!]_{i,a} = [\![x_2, \ldots, \underbrace{a}_{i\text{-th entry}}, \ldots, x_n]\!],\ x_2, \ldots, x_n \in A.$$

Each algebra $A_{i,a}$ is called a *reduced algebra* of A. Under the total commutativity or the anticommutativity of $[\![, \ldots,]\!]$, it suffices to consider $i = 1$, which may be omitted by simply writing A_a and

$$[\![x_2, \ldots, x_n]\!]_a = [\![a, x_2, \ldots, x_n]\!],\ x_2, \ldots, x_n \in A.$$

It may happen that each reduced algebra of an n-ary algebra belongs to the same class. Indeed, it is known that

- reduced algebras of n-ary totally associative algebras are $(n-1)$-ary totally associative algebras;
- reduced algebras of n-ary totally (anti)commutative algebras are $(n-1)$-ary totally (anti)commutative algebras;
- reduced algebras of n-ary Leibniz algebras are $(n-1)$-ary Leibniz algebras;
- reduced algebras of n-ary Filippov algebras are $(n-1)$-ary Filippov algebras [8];
- reduced algebras of n-ary Malcev algebras are $(n-1)$-ary Malcev algebras [20].

So, it is natural to put the following question: whether the reduced algebras of n-ary Jordan algebras are $(n-1)$-ary Jordan algebras?

Lemma 17. *The reduced algebras of the ternary Jordan algebra TJ_n are not Jordan algebras in general.*

Proof: A counterexample may be constructed for every $n > 1$. Assume that $\{b_1, b_2\}$ belong to an orthonormal basis for TJ_n. Put $a = b_1$. Take $x = b_2$ and $y = b_1$. Then

$$x^2 = [\![a, x, x]\!] = b_1 = y,\ y^2 = 3b_1 = 3y,\ xy = b_2 = x.$$

Now, $x(yx^2) = y^2x = 3x,\ (xy)x^2 = xy = x$. Assuming that the ground field is of characteristic not 2, we see that (1) fails. ■

Remark 18. *Since the ternary multiplication of a Jordan triple system* (*see Remark* 8) *is only partially commutative, it is straightforward that its reduced algebras may not be Jordan algebras.*

Remark 19. *One of subclasses of* n*-ary Jordan algebras is the subclass of totally commutative and totally associative* n*-ary algebras. As it is easy to see from definition, the reduced algebras of every totally commutative and totally associative* n*-ary algebra are totally commutative and totally associative* $(n-1)$*-ary algebras.*

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IVAN KAYGORODOV
UNIVERSIDADE FEDERAL DO ABC, SANTO ANDRE, BRASIL

E-mail address: kaygorodov.ivan@gmail.com

A. P. POZHIDAEV
SOBOLEV INSTITUTE OF MATHEMATICS, PR. KOPTYUGA 4, NOVOSIBIRSK, RUSSIAN FEDERATION, DEPARTMENT OF MATHEMATICS AND MECHANICS, NOVOSIBIRSK STATE UNIVERSITY,, PIROGOVA 2, NOVOSIBIRSK, RUSSIAN FEDERATION

E-mail address: app@math.nsc.ru

P. SARAIVA
CMUC, CEBER AND FACULTY OF ECONOMICS, UNIVERSITY OF COIMBRA, 3004-512, COIMBRA, PORTUGAL

E-mail address: psaraiva@fe.uc.pt