Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 17–43

ON MARKOV'S THEOREM ON ZEROS OF ORTHOGONAL POLYNOMIALS REVISITED

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ABSTRACT: This paper briefly surveys Markov's theorem related to zeros of orthogonal polynomials. Monotonicity of zeros of some families of orthogonal polynomials are reviewed in detail.

KEYWORDS: Orthogonal polynomials; Zeros; Jacobi polynomials, Gegenbauer polynomials; Laguerre polynomials; Charlier polynomials; Kravchuk polynomials; Meixner polynomials; Hahn polynomial.
MATH. SUBJECT CLASSIFICATION (2000): 33C45.

1. Introduction

Markov's theorem, dating back to the late 19th century, furnishes a method for obtaining information about zeros of orthogonal polynomials from the weight function related to orthogonality. Formally, adopting modern terminology, his result is stated as follows (see [27]):

Theorem 1.1 (Markov, 1886). *Let $\{p_n(x,t)\}$ be a sequence of polynomials which are orthogonal on the interval $A = (a, b)$ with respect to the weight function $\omega(x,t)$ that depends on a parameter t, $t \in B = (c, d)$, i.e.,*

$$\int_a^b p_n(x,t)p_m(x,t)\omega(x,t)dx = 0, \quad m \neq n.$$

Suppose that $\omega(x,t)$ is positive and has a continuous first derivative with respect to t for $x \in A$, $t \in B$. Furthermore, assume that

$$\int_a^b x^k \frac{\partial \omega}{\partial t}(x,t)dx, \quad k = 0, 1, \ldots, 2n-1,$$

Received November 2, 2017.
KC is supported by the Portuguese Government through the Fundação para a Ciência e a Tecnologia (FCT) under the grant SFRH/BPD/101139/2014 and partially supported by the Centre for Mathematics of the University of Coimbra–UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020. FRR is supported by the Fundação de Amparo à Pesquisa do Estado de Minas Gerais (FAPEMIG)-Brazil under the grant PPM-00478-15.

converge uniformly for t in every compact subinterval of B. Then the zeros of $p_n(x,t)$ are increasing (decreasing) functions of t, $t \in B$, provided that

$$\frac{1}{\omega(x,t)}\frac{\partial \omega}{\partial t}(x,t)$$

is an increasing (decreasing) function of x, $x \in A$.

Markov's proof is based on the orthogonality relation (cf. [27, Equation 2]) together with the chain rule (cf. [27, Equation 5], supposing that the zeros are defined implicitly as differentiable functions of the parameter. In addition, as an application of this result, Markov established that the zeros of Jacobi polynomials, which are orthogonal in $(-1,1)$ with respect to the weight function $\omega(x,\alpha,\beta) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha,\beta > -1$, are decreasing functions of α and increasing functions of β. Later, in 1939, Szegő, in his classical book [33, Theorem 6.12.1, p. 115], provided a different proof of Markov's theorem. Szegő referred his proof of Theorem 1.1 in the following way [33, Footnote 31, p. 116]: *"This proof does not differ essentially from the original one due to A. Markov, although the present arrangement is somewhat clearer."*. Szegő's reasoning (argument, approach) is based on Gauss mechanical quadrature, which was an approach that Stieltjes suggested to handle the problem, see [32, Section 5, p. 391]. In 1971, Freud (see [12, Problem 16, p. 133]) formulated a version of Markov's theorem that is a little more general, considering sequences of polynomials orthogonal with respect to measures in the form $d\alpha(x,t) = \omega(x,t)d\nu(x)$. A proof of such result appears in Ismail [15, Theorem 3.2, p. 183] (see also in Ismail's book [17, Theorem 7.1.1, p. 204]). Ismail's argument of the proof is also based on Gauss mechanical quadrature. As consequence, Ismail provide monotonicity properties for the zeros of Hahn and Meixner polynomials (see [17, Theorem 7.1.2, p. 205]). Kroó and Peherstorfer [23, Theorem 1], in a more general context of approximation theory, extended Markov's result to zeros of polynomials which have the minimal L_p-norm. Their approach is based on the implicit function theorem.

The main concern of this work derives from Markov's classic 1886 theorem. This allows the approach to be tailored towards measures with continuous and discrete parts, thus extending the Markov's result. This point at issue was posed by Ismail in his book as an open problem [17, Problem 24.9.1, p. 660] (see also [15, Problem 1, p. 187]). The question is stated as follows:

Problem 1.1. Let μ be a positive and nontrivial Radon measure on a compact set $A \subset \mathbb{R}$. Assume that $\mathrm{d}\mu(x,t)$ has the form

$$\mathrm{d}\alpha(x,t) + \mathrm{d}\beta(x,t), \tag{1.1}$$

where $\mathrm{d}\alpha(x,t) := \omega(x,t)\mathrm{d}\nu(x)$ and $\mathrm{d}\beta(x,t) := \sum_{i=0}^{\infty} \jmath_i(t)\delta_{y_i(t)}$,* with $t \in B$, B being an open interval on $\mathbb{R}$. Determine sufficient conditions in order for the zeros of the polynomial $P_n(x,t)$ to be strictly increasing (decreasing) functions of t.

The manuscript is organized in the following way: in Section 2 the main result is stated and proved; in Section 3 some conclusions are drawn from the main result, including Markov's classic theorem, among others; finally, in Section 4, illustrative examples are given: in Subsections 4.1 and 4.2 monotonicity properties of zeros of polynomials orthogonal with respect measures with discrete parts are investigated; in Subsection 4.3 monotonicity properties of zeros of Jacobi, Gegenbauer and Laguerre orthogonal polynomials are reviewed; in Subsection 4.4, sharp monotonicity properties involving the zeros of Gegenbauer-Hermite, Jacobi-Laguerre and Laguerre-Hermite orthogonal polynomials are derived; at last, in Subsection 4.5, monotonicity properties of zeros of Charlier, Meixner, Kravchuck, and Hahn orthogonal polynomials are revisited.

2. Main result

The next result extends Markov's theorem to measure with continuous and discrete parts, giving an answer to Problem 1.1. For a result in the context of polynomials which have minimal L_p-norm see [3, Theorem 1.1].

Theorem 2.1. *Assume the notation and conditions of Problem (P). Assume further the existence and continuity for each $x \in A$ and $t \in B$ of $(\partial\omega/\partial t)(x,t)$ and, in addition, suppose that*

$$G(t) := \sum_{i=0}^{\infty} g_i(t)$$

converge at $t = t_0$ and

$$G'(t) := \sum_{i=0}^{\infty} g_i(t), \quad \frac{\partial G}{\partial x_j}(t) = \sum_{i=0}^{\infty} \frac{\partial g}{\partial x_j}(t)$$

*The Dirac measure δ_y is a positive Radon measure whose support is the set $\{y\}$.

converge uniformly for $t \in B$, where

$$g_i(t) = \jmath_i(t)(y_i(t) - x_k)^{-1} \prod_{j=1}^{n} (y_i(t) - x_j)^2$$

and $(x_1, \dots, x_n) \in \mathbb{R}^n$. Denote by $x_0(t), \dots, x_n(t)$ the zeros of $P_n(x,t)$. Fix $k \in \{1, \dots, n\}$ and set

$$d_{k,i}(t) := \begin{cases} y_i(t) - x_k(t) & \text{if } y_i(t) \neq x_k(t), \\ 1 & \text{if } y_i(t) = x_k(t). \end{cases}$$

Define the rational function

$$R_{k,i}(t) := \sum_{j=0}^{n}{}' \frac{2 - \delta_{j,k}}{y_i(t) - x_j(t)},$$

where the prime means that the sum is over all values j and t for which $y_i(t) \neq x_j(t)$. Then $x_k(t)$ is a strictly increasing function for those values of t such that

$$\frac{1}{d_{k,i}(t)} \left\{ \frac{\jmath_i'(t)}{\jmath_i(t)} + y_i'(t) R_{k,i}(t) - \frac{1}{\omega(x_k(t),t)} \frac{\partial \omega}{\partial t}(x_k(t),t) \right\} \geq 0, \tag{2.1}$$

and

$$\frac{1}{\omega(x,t)} \frac{\partial \omega}{\partial t}(x,t) \tag{2.2}$$

is an increasing function of $x \in A$, provided that at least the inequality (2.1) *be strict or the function* (2.2) *be nonconstant on A.*

Proof: The proof is based on the implicit function theorem and it is similar to the Markov's one. Since it is rather long it will be divided into several steps:
(*i*) *Differentiability of the zeros*: Let $P_n(x,t) = (x - x_1(t)) \cdots (x - x_n(t))$ be the n-th orthogonal polynomial with respect to (1.1). In other words $P_n(x,t)$ satisfies the following orthogonality relations:

$$\int_a^b q(x) P_n(x,t) \omega(x,t) d\nu(x) + \sum_{i=0}^{\infty} \jmath_i(t) q(y_i(t)) P_n(y_i(t),t) = 0 \quad (q \in \mathcal{P}_{n-1}). \tag{2.3}$$

Define the map $\mathrm{f} := (f_1, \ldots, f_n) : U \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, where one has set $\mathrm{x} := (x_1, \ldots, x_n)$, $x_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ (note that the x_j's do not depend on t), and

$$f_k(\mathrm{x}, t) := P_n(x_k, t). \tag{2.4}$$

For $j \neq k$ one has

$$\frac{\partial f_k}{\partial x_j}(\mathrm{x}, t) = 0; \tag{2.5}$$

otherwise

$$\frac{\partial f_k}{\partial x_k}(\mathrm{x}, t) = \frac{\partial P_n}{\partial x}(x_k, t). \tag{2.6}$$

Set $\mathrm{x}(t) := (x_1(t), \ldots, x_n(t))$. From (2.4), (2.5) and (2.6) one obtains $\mathrm{f}(\mathrm{x}(t_0), t_0) = 0$ and

$$\frac{\partial \mathrm{f}}{\partial \mathrm{x}}(\mathrm{x}(t_0), t_0) = \det \begin{pmatrix} \frac{\partial f_0}{\partial x_0}(\mathrm{x}(t_0), t_0) & & \\ & \ddots & \\ & & \frac{\partial f_n}{\partial x_n}(\mathrm{x}(t_0), t_0) \end{pmatrix} \neq 0.$$

According to the implicit function theorem, under these conditions the equation $\mathrm{f}(\mathrm{s}, t) = 0$ has a solution $\mathrm{s} = \mathrm{x}(t)$ in a neighborhood of $(\mathrm{x}(t_0), t_0)$ that depends differentiable on t.

(ii) Expression for the derivative of the zeros: Moreover, in view of the above result,

$$\frac{\mathrm{d}x_k}{\mathrm{d}t}(t) = -\frac{\frac{\partial f_k}{\partial t}(\mathrm{x}(t), t)}{\frac{\partial f_k}{\partial x_k}(\mathrm{x}(t), t)} = -\frac{\frac{\partial P_n}{\partial t}(x_k(t), t)}{\frac{\partial P_n}{\partial x}(x_k(t), t)}. \tag{2.7}$$

(iii) Expression for the derivative of f_k with respect to t: Take

$$q(x) = q(x, \nu) = \frac{P_n(x, \nu)}{x - x_k(\nu)} \in \mathcal{P}_{n-1},$$

and substitute it in the derivative of (2.3) with respect to t, and then let $\nu \to t$. The result is the following:

$$\int \frac{[P_n(x,t)]^2}{x-x_k(t)}\frac{\partial\omega}{\partial t}(x,t)\mathrm{d}\nu(x)+\sum_{i=0}^{\infty}\left\{\jmath_i'(t)+\jmath_i(t)y_i'(t)R_{k,i}(t)\right\}\frac{[P_n(y_i(t),t)]^2}{y_i(t)-x_k(t)}$$
$$+\int \frac{P_n(x,t)}{x-x_k(t)}\frac{\partial P_n}{\partial t}(x,t)\omega(x,t)\mathrm{d}\nu(x)$$
$$+\sum_{i=0}^{\infty}\jmath_i(t)\frac{P_n(y_i(t),t)}{y_i(t)-x_k(t)}\frac{\partial P_n}{\partial t}(y_i(t),t)=0. \quad (2.8)$$

On the other hand, if one takes

$$q(x)=q(x,t)=\left\{\frac{\partial P_n}{\partial t}(x,t)-\frac{\partial P_n}{\partial t}(x_k(t),t)\right\}\frac{1}{x-x_k(t)}\in\mathcal{P}_{n-1},$$

substitutes it in (2.3), and subtractes the result from (2.8) one derives:

$$-\frac{\partial P_n}{\partial t}(x_k(t),t)\left\{\int \frac{P_n(x,t)}{x-x_k(t)}\omega(x,t)d\nu(x)+\sum_{i=0}^{\infty}\jmath_i(t)\frac{P_n(y_i(t),t)}{y_i(t)-x_k(t)}\right\}=$$
$$\int \frac{[P_n(x,t)]^2}{x-x_k(t)}\frac{\partial\omega}{\partial t}(x,t)\mathrm{d}\nu(x)$$
$$+\sum_{i=0}^{\infty}\left\{\jmath_i'(t)+\jmath_i(t)y_i'(t)R_{k,i}(t)\right\}\frac{[P_n(y_i(t),t)]^2}{y_i(t)-x_k(t)} \quad (2.9)$$

(*iv*) *Expression for the derivative of* f_k *with respect to* x_k: Since

$$q(x)=q(x,t)=\frac{\dfrac{\partial P_n}{\partial x}(x_k(t),t)(x-x_k(t))-P_n(x,t)}{(x-x_k(t))^2}\in\mathcal{P}_{n-2},$$

by the orthogonality relation (2.3), one obtains

$$\frac{\partial P_n}{\partial x}(x_k(t),t)\left\{\int \frac{P_n(x,t)}{x-x_k(t)}\omega(x,t)d\nu(x)+\sum_{i=0}^{\infty}\jmath_i(t)\frac{P_n(y_i(t),t)}{y_i(t)-x_k(t)}\right\}=$$
$$\int \frac{[P_n(x,t)]^2}{(x-x_k(t))^2}\omega(x,t)\mathrm{d}\nu(x)+\sum_{i=0}^{\infty}\jmath_i(t)\frac{[P_n(y_i(t),t)]^2}{(y_i(t)-x_k(t))^2}. \quad (2.10)$$

Therefore, substituting (2.9) and (2.10) in (2.7) yields

$$\frac{\mathrm{d}x_k}{\mathrm{d}t}(t) = \frac{\displaystyle\int \frac{[P_n(x,t)]^2}{x-x_k(t)}\frac{\partial \omega}{\partial t}(x,t)\mathrm{d}\nu(x) + \sum_{i=0}^{\infty}\{\jmath_i'(t)+\jmath_i(t)y_i'(t)R_{k,i}(t)\}\,\frac{[P_n(y_i(t),t)]^2}{y_i(t)-x_k(t)}}{\displaystyle\int \frac{[P_n(x,t)]^2}{(x-x_k(t))^2}\omega(x,t)\mathrm{d}\nu(x) + \sum_{i=0}^{\infty}\jmath_i(t)\frac{[P_n(y_i(t),t)]^2}{(y_i(t)-x_k(t))^2}}. \tag{2.11}$$

Clearly

$$\frac{1}{\omega(x_k(t),t)}\frac{\partial \omega}{\partial t}(x_k(t),t)\int \frac{P_n(x,t)^2}{x-x_k(t)}\mathrm{d}\mu(x,t) = 0. \tag{2.12}$$

Subtracting (2.12) from the numerator of the right-hand side of (2.11) yields

$$\begin{aligned}
&\int \frac{[P_n(x,t)]^2}{x-x_k(t)}\frac{\partial \omega}{\partial t}(x,t)\mathrm{d}\nu(x) + \sum_{i=0}^{\infty}\{\jmath_i'(t)+\jmath_i(t)y_i'(t)R_{k,i}(t)\}\,\frac{[P_n(y_i(t),t)]^2}{y_i(t)-x_k(t)} \\
&= \int \frac{[P_n(x,t)]^2}{x-x_k(t)}\left(\frac{1}{\omega(x,t)}\frac{\partial \omega}{\partial t}(x,t) - \frac{1}{\omega(x_k(t),t)}\frac{\partial \omega}{\partial t}(x_k(t),t)\right)\omega(x,t)\mathrm{d}\nu(x) + \\
&\quad + \sum_{i=0}^{\infty}\left\{\jmath_i'(t)+\jmath_i(t)y_i'(t)R_{k,i}(t) - \frac{\jmath_i(t)}{\omega(x_k(t),t)}\frac{\partial \omega}{\partial t}(x_k(t),t)\right\}\frac{[P_n(y_i(t),t)]^2}{y_i(t)-x_k(t)}
\end{aligned} \tag{2.13}$$

It only remains to note that

$$\frac{1}{x-x_k(t)}\left(\frac{1}{\omega(x,t)}\frac{\partial \omega}{\partial t}(x,t) - \frac{1}{\omega(x_k(t),t)}\frac{\partial \omega}{\partial t}(x_k(t),t)\right) \geq 0.$$

Thus, the sign of $x_k'(t)$ is the same sign as the numerator of (2.11) and the desired result follows from (2.13). ■

3. Markov's theorem and its descendants

In this section, Markov's classic theorem is derived from Theorem 2.1. In addition, Markov's theorem for even weight function is revisited too, together with other results.

The next result brings us back to Markov's theorem [27] (see also [33, Theorem 6.12.1, p. 115] and [17, Theorem 7.1.1, p. 204]).

Corollary 3.1. Assume the notation and conditions of Theorem 2.1 under the constraint that $\mathrm{d}\mu(x,t) = \omega(x,t)\mathrm{d}\alpha(x)$. In this case (2.11) becomes

$$\begin{aligned}\frac{\mathrm{d}x_k}{\mathrm{d}t}(t) &= \frac{\displaystyle\int \frac{[P_n(x,t)]^2}{x-x_k(t)}\frac{\partial\omega}{\partial t}(x,t)\mathrm{d}\nu(x)}{\displaystyle\int \frac{[P_n(x,t)]^2}{(x-x_k(t))^2}\omega(x,t)\mathrm{d}\nu(x)} \\ &= \frac{\displaystyle\int \frac{[P_n(x,t)]^2}{x-x_k(t)}\left(\frac{1}{\omega(x,t)}\frac{\partial\omega}{\partial t}(x,t) - \frac{1}{\omega(x_k(t),t)}\frac{\partial\omega}{\partial t}(x_k(t),t)\right)\omega(x,t)\mathrm{d}\nu(x)}{\displaystyle\int \frac{[P_n(x,t)]^2}{(x-x_k(t))^2}\omega(x,t)\mathrm{d}\nu(x)}.\end{aligned} \tag{3.1}$$

Then $x_k(t)$ is a strictly increasing (decreasing) function of t if

$$\frac{1}{\omega(x,t)}\frac{\partial\omega}{\partial t}(x,t)$$

is an increasing (decreasing) function of $x \in A$, provided that the last function be nonconstant on A.

Markov's result concerning the zeros of polynomials orthogonal with respect to an even weight function was studied by K. Jordaan, H. Wang, and J Zhou [18, Theorem 2.1]. This case also appears in [23, Corollary 2] in a more general context. For further results about these polynomials, see [5, Chapter 1, Sections 8 and 9].

Corollary 3.2 (Markov's result for even weight function)**.** Assume the notation and conditions of Theorem 2.1 under the constraint that $\mathrm{d}\mu(x,t) = \omega(x,t)\ \mathrm{d}x$. Suppose, in addition, that $\omega(x,t)$ is an even function of x in $A = (-a,a)^\dagger$. Then, the positive zeros $x_k(t)$ are strictly increasing (decreasing) functions of t if

$$\frac{1}{\omega(x,t)}\frac{\partial\omega}{\partial t}(x,t)$$

is an increasing (decreasing) function of $x \in (0,a)$, provided that the last function be nonconstant on $(0,a)$.

†In the case of $\omega(x,t)$ is an even function in an interval of the form $(-a,a)$ it is well known that the zeros of the orthogonal polynomial are symmetric with respect to the origin, i.e., $x_k(t) = -x_{n-k+1}(t)$.

Proof: Since $\omega(x,t)$ is an even function, then $P_n(-x,t)=(-1)^nP_n(x,t)$ (for further details, see [5, Chapter 1, Section 8]). Therefore, one can write

$$P_{2m}(x,t)=S_m(x^2,t) \quad \text{and} \quad P_{2m+1}(x,t)=x\,T_m(x^2,t),$$

where S_m and T_m are polynomials of degree m. Let $y_i^{(1)}=y_i^{(1)}(t)$ and $y_i^{(2)}=y_i^{(2)}(t)$, $i=1,\dots,m$, be the zeros of the polynomials S_m and T_m, respectively. If x_i, $i=1,\dots,[n/2]$, denote the positive zeros of the polynomial P_n, then,

$$x_i=\sqrt{y_i^{(k)}}, \quad i=1,\dots,[n/2], \tag{3.2}$$

where $k=1$ if n is even and $k=2$ if n is odd. Note that

$$\begin{aligned}\int_{-a}^{a} P_{2r}(x,t)P_{2l}(x,t)\omega(x,t)dx &= \int_{-a}^{a} S_r(x^2,t)S_l(x^2,t)\omega(x,t)dx \\ &= 2\int_0^a S_r(x^2,t)S_l(x^2,t)\omega(x,t)dx = \int_0^{a^2} S_r(y,t)S_l(y,t)\frac{\omega(\sqrt{y},t)}{\sqrt{y}}dy\end{aligned}$$

and

$$\begin{aligned}\int_{-a}^{a} P_{2r+1}(x,t)P_{2l+1}(x,t)\omega(x,t)dx &= \int_{-a}^{a} xT_r(x^2,t)xT_l(x^2,t)\omega(x,t)dx \\ &= 2\int_0^a T_r(x^2,t)T_l(x^2,t)x^2\omega(x,t)dx = \int_0^{a^2} T_r(y,t)t_l(y,t)\sqrt{y}\,\omega(\sqrt{y},t)dy.\end{aligned}$$

Since $\{P_n(x,t)\}$ is a sequence of orthogonal polynomials with respect to an even weight function $\omega(x,t)$ on $(-a,a)$, it follows that $\{S_n(y,t)\}$ and $\{T_n(y,t)\}$ are sequences of orthogonal polynomials on $(0,a^2)$ with respect to the weight functions $\omega_1(y,t)=\omega(\sqrt{y},t)/\sqrt{y}$ and $\omega_2(y,t)=\sqrt{y}\,\omega(\sqrt{y},t)$, respectively. Now, it is easy to see that

$$\frac{1}{\omega_1(y,t)}\frac{\partial\omega_1(y,t)}{\partial t}=\frac{1}{\omega_2(y,t)}\frac{\partial\omega_2(y,t)}{\partial t}=\frac{1}{\omega(\sqrt{y},t)}\frac{\partial\omega(\sqrt{y},t)}{\partial t}.$$

Therefore, since the function $(\omega(x,t))^{-1}\partial\omega(x,t)/\partial t$ increases (decreases) when x increases in $(0,a)$, then the functions $(\omega_1(y,t))^{-1}\partial\omega_1(y,t)/\partial t$ and $(\omega_2(y,t))^{-1}\partial\omega_2(y,t)/\partial t$ increases (decreases) when y increases in $(0,a^2)$. So it follows from Markov's theorem that the zeros $y_i^{(k)}=y_i^{(k)}(t)$, $i=1,\dots,[n/2]$, $k=1,2$, increase (decrease) when t increases in B. Then, the result follows from (3.2). ∎

In Markov's theorem, one can consider the end points of the interval of the orthogonality as functions of the parameter, i.e., $a = a(t)$ and $b = b(t)$. From this, the following result can be derived:

Corollary 3.3. Assume the notation and conditions of Theorem 2.1 under the constraint that $\mathrm{d}\mu(x,t) = \omega(x,t)\mathrm{d}x$. Furthermore, suppose that $a = a(t)$ and $b = b(t)$ are functions of t with continuous derivatives of the first order. Then, $x_k(t)$ is a strictly increasing (decreasing) function of t if

$$\frac{1}{\omega(x,t)}\frac{\partial \omega}{\partial t}(x,t)$$

is an increasing (decreasing) function of $x \in A = (a(t), b(t))$, provided that this last function be nonconstant on A, and both $a(t)$ and $b(t)$ increase (decrease) as t increases.

Proof: By Leibniz's rule for differentiation under the integral sign one obtains that the numerator of the right hand side of (3.1) becomes

$$\int_{a(t)}^{b(t)} \frac{[P_n(x,t)]^2}{x - x_k(t)} \left(\frac{1}{\omega(x,t)}\frac{\partial \omega}{\partial t}(x,t) - \frac{1}{\omega(x_k(t),t)}\frac{\partial \omega}{\partial t}(x_k(t),t) \right) \omega(x,t)\mathrm{d}x$$
$$+ \frac{P_n^2(b(t),t)}{b(t) - x_k(t)}\omega(b(t),t)b'(t) - \frac{P_n^2(a(\tau),t)}{a(t) - x_k(t)}\omega(a(t),t)a'(t). \quad (3.3)$$

This establishes the result. ∎

In Corollary 3.3, the hypothesis that the weight function depends on the parameter t may be replaced by the hypothesis that the weight function does not depend on t, that is, $\omega = \omega(x)$. In this case if both $a(t)$ and $b(t)$ increase (decrease) with t increases (either a or b can be constant), then $x_k(t)$ is an increasing (decreasing) function of t.

Some particular cases of measures of the form

$$\mathrm{d}\mu(x,t) = \mathrm{d}\alpha(x) + \jmath(t)\delta_y \quad (3.4)$$

were frequently considered in the literature (see [7, 8, 13, 20, 21, 22, 26, 28]). See [13] for general results concerning zeros of polynomials orthogonal with respect to (3.4). A bit more general case of (3.4) is presented as follows:

Corollary 3.4. Assume the notation and conditions of Theorem 2.1 under the constraint that $\mathrm{d}\mu(x,t) = \mathrm{d}\alpha(x) + \sum_{i=0}^{\infty} \jmath_i(t)\delta_{y_i}$. Furthermore suppose that y_i, $i = 0, 1, \ldots$, are constants, and $\jmath_i'(t) = 0$ for $i \neq l$. Define the sets

$$C_l^- := \{t \in B \mid \jmath_l'(t) < 0\}, \quad C_l^+ := \{t \in B \mid \jmath_l'(t) > 0\}.$$

If $x_k(t) < y_l$ (respectively, $x_k(t) > y_l$) for each $t \in B$, then $x_k(t)$ is a strictly increasing (respectively, decreasing) function of t on C_l^+ (respectively, on C_l^-). In other words, each zero $x_k(t)$ on the left-hand side of y_l is an increasing (decreasing) function of t on C_l^+ (C_l^-), whereas each zero $x_k(t)$ on the right-hand side of y_l is a decreasing (increasing) function of t on C_l^+ (C_l^-).

Proof: In this case, (2.11) reduces to

$$\frac{\mathrm{d}x_k}{\mathrm{d}t}(t) = \frac{\jmath_l'(t)\dfrac{[P_n(y_l, t)]^2}{y_l - x_k(t)}}{\displaystyle\int \frac{[P_n(x,t)]^2}{(x - x_k(t))^2}\omega(x,t)\mathrm{d}\nu(x) + \sum_{i=0}^{\infty} \jmath_i(t)\frac{[P_n(y_i, t)]^2}{(y_i - x_k(t))^2}}.$$

This establishes the result. ■

The next result was proved firstly in [4, Theorem 2.2]. In order to derive monotonicity properties of zeros, the location of the mass point outside A is required (see Subsection 4.2 of this manuscript).

Corollary 3.5. Assume the notation and conditions of Theorem 2.1 under the constraint that $\mathrm{d}\mu(x,t) = \mathrm{d}\alpha(x) + \jmath\delta_{y(t)}$. Define the sets

$$\begin{aligned} B_- &:= \{t \in B \mid y(t) \in A^c \cap \mathbb{R} \wedge y'(t) < 0\}, \\ B_+ &:= \{t \in B \mid y(t) \in A^c \cap \mathbb{R} \wedge y'(t) > 0\}. \end{aligned}$$

Then all the zeros of $P_n(x,t)$ are strictly decreasing (respectively, increasing) functions of t on B_- (respectively, on B_+).

Proof: In this case, (2.11) reduces to

$$\frac{\mathrm{d}x_k}{\mathrm{d}t}(t) = \frac{\jmath\, y'(t)\dfrac{[P_n(y(t),t)]^2}{y(t) - x_k(t)}\displaystyle{\sum_{j=0}^{n}}{}' \frac{2 - \delta_{j,k}}{y(t) - x_j(t)}}{\displaystyle\int \frac{[P_n(x,t)]^2}{(x - x_k(t))^2}\omega(x,t)\mathrm{d}\nu(x) + \jmath\frac{[P_n(y(t),t)]^2}{(y(t) - x_k(t))^2}},$$

where the prime means that the sum is over all values j and t for which $y(t) \neq x_j(t)$. This establishes the result. ■

4. Some applications

4.1. Sharp monotonicity properties of the zeros of orthogonal polynomials derived from Corollary 3.4. Suppose that $d\mu(x,t) = dx + \jmath_1\delta_{y_1} + \jmath_2\delta_{y_2} + \jmath_3\delta_{y_3}$, where $\jmath_1 = \jmath_1(t) = t$, $\jmath_2 = \jmath_3 = 1$, $y_1 = 2$, $y_2 = 5$, and $y_3 = 7$, with $A = (-1,1)$ and $B = (0,\infty)$. Let $\{p_n\}$ be the sequence of orthogonal polynomials with respect to $d\mu$, i.e.,

$$\int_{-1}^{1} p_n(x)p_m(x)dx + tp_n(2)p_m(2) + p_n(5)p_m(5) + p_n(7)p_m(7) = 0, \quad m \neq n.$$

Then, the zeros of the polynomial p_n located on the left-hand side of $y_1 = 2$ are increasing functions of t, while the zeros of p_n on the right-hand side of $y_1 = 2$ are decreasing functions of t, in view of Corollary 3.4.

Table 1 shows the monotonicity of the zeros of p_4 from this example. Observe that two of them are increasing functions of t, while the others ones are decreasing functions of t.

4.2. Sharp monotonicity properties of the zeros of orthogonal polynomials derived from Corollary 3.5. Suppose that $d\mu(x,t) = dx + 10\delta_{y(t)}$, where $y(t) = t$, with $A = (-1,1)$ and $B = (-2,2)$. Let $\{p_n\}$ be the sequence of orthogonal polynomials with respect to $d\mu$, i.e.,

$$\int_{-1}^{1} p_n(x)p_m(x)dx + 10p_n(y(t))p_m(y(t)) = 0, \quad m \neq n.$$

Then, by Corollary 3.5, the zeros of the polynomial p_n are increasing functions of t, for $t \in (-2,-1) \cup (1,2)$. On the other hand, for $t \in (-1,1)$ one cannot guarantee the monotonicity of these zeros.

Table 2 illustrates the behavior of the zeros $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$, and $x_4 = x_4(t)$, of p_4 from this example. Note that they are not monotonic functions of t, when it varies in $(-1,1)$. In this regard, the statements of Theorem 2 and Corollary 3 in `arXiv:1501.07235 [math.CA]` appear to be incorrect.

4.3. Monotonicity of the zeros of classical continuous orthogonal polynomials derived from Corollary 3.1 (Markov's theorem). In this subsection, the classical result established by A. Markov in 1886 is reviewed. It concerns to the monotonicity of zeros of Jacobi orthogonal polynomials.

t	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$
0.0	-0.655077	0.46887	4.98364	6.99699
0.5	-0.528752	1.07504	4.83155	6.97651
1.0	-0.502388	1.33983	4.75758	6.96841
1.5	-0.491188	1.48640	4.71348	6.96408
2.0	-0.485007	1.57951	4.68410	6.96138
2.5	-0.481092	1.64394	4.66310	6.95953
3.0	-0.478390	1.69121	4.64733	6.95820
3.5	-0.476414	1.72736	4.63504	6.95718
4.0	-0.474906	1.75592	4.62520	6.95638
4.5	-0.473717	1.77906	4.61713	6.95574
5.0	-0.472756	1.79818	4.61040	6.95521
5.5	-0.471962	1.81425	4.60470	6.95477
6.0	-0.471297	1.82795	4.59981	6.95439
6.5	-0.470730	1.83977	4.59557	6.95407
7.0	-0.470242	1.85006	4.59185	6.95379
7.5	-0.469817	1.85911	4.58857	6.95354
8.0	-0.469444	1.86713	4.58565	6.95332
8.5	-0.469113	1.87428	4.58304	6.95313
9.0	-0.468819	1.88071	4.58069	6.95295
9.5	-0.468555	1.88651	4.57856	6.95279
10.0	-0.468316	1.89177	4.57662	6.95265

TABLE 1. Zeros of the polynomial p_4 as functions of t.

Moreover, results on zeros of Gegenbauer and Laguerre orthogonal polynomials are revisited too (see Szegő's book [32, Section 5], and Ismail's book [17, Chapter 7]).

Example 4.1 (Zeros of Jacobi polynomials)**.** Let $P_n^{(\alpha,\beta)}(x)$ be the n-th Jacobi polynomial which is orthogonal on $(-1,1)$ with respect to the weight function $\omega(x,\alpha,\beta) = (1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$. Then all its zeros are increasing functions of β and decreasing functions of α, for $\alpha,\beta > -1$.

Proof: Since

$$\frac{1}{\omega(x,\alpha,\beta)}\frac{\partial\omega(x,\alpha,\beta)}{\partial\alpha} = \ln(1-x)$$

t	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$
-2.0	-1.999850	-0.682492	0.142833	0.818103
-1.8	-1.799760	-0.664794	0.158989	0.821979
-1.6	-1.599570	-0.638982	0.179555	0.826752
-1.4	-1.399200	-0.598464	0.206977	0.832897
-1.2	-1.198540	-0.529080	0.246312	0.841409
-1.0	-0.998028	-0.409306	0.306336	0.854084
-0.8	-0.803116	-0.368260	0.329881	0.859152
-0.6	-0.764270	-0.572827	0.299677	0.853545
-0.4	-0.858748	-0.396400	0.335789	0.860400
-0.2	-0.854976	-0.210785	0.301057	0.856301
0.0	-0.846273	-0.098843	0.098843	0.846273
0.2	-0.856301	-0.301057	0.210785	0.854976
0.4	-0.860400	-0.335789	0.396400	0.858748
0.6	-0.853545	-0.299677	0.572827	0.764270
0.8	-0.859152	-0.329881	0.368260	0.803116
1.0	-0.854084	-0.306336	0.409306	0.998028
1.2	-0.841409	-0.246312	0.529080	1.198540
1.4	-0.832897	-0.206977	0.598464	1.399200
1.6	-0.826752	-0.179555	0.638982	1.599570
1.8	-0.821979	-0.158989	0.664794	1.799760
2.0	-0.818103	-0.142833	0.682492	1.999850

TABLE 2. Zeros of the polynomial p_4 as functions of t.

is a decreasing function of x and, otherwise,

$$\frac{1}{\omega(x,\alpha,\beta)}\frac{\partial\omega(x,\alpha,\beta)}{\partial\beta} = \ln(1+x)$$

is an increasing function of x, for $x \in (-1,1)$, by Observation 3.1 (Markov's theorem), the statements holds. ∎

Figure 1 illustrates the monotonicity of the zeros of $P_n^{(\alpha,\beta)}(x)$ with respect to β while Figure 2 shows the monotonicity of the zeros of $P_n^{(\alpha,\beta)}(x)$ with respect to α.

Example 4.2 (Zeros of Laguerre polynomials)**.** Let $L_n^{(\alpha)}(x)$ be the n-th Laguerre polynomial which is orthogonal on $(0,\infty)$ with respect to the weight

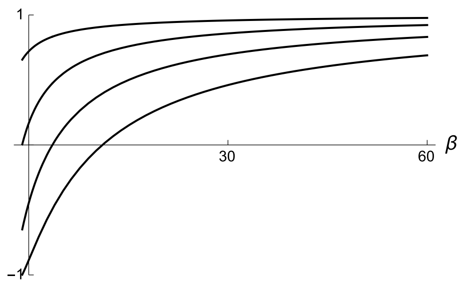

FIGURE 1. Zeros of Jacobi polynomials as functions of the parameter β. Graph of the zeros of $P_4^{(1,\beta)}(x)$.

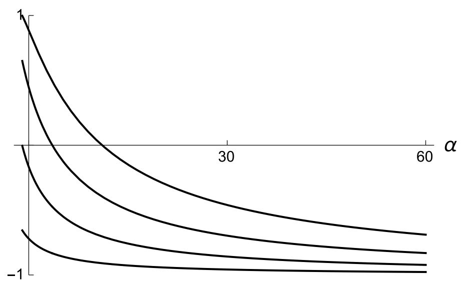

FIGURE 2. Zeros of Jacobi polynomials as functions of the parameter α. Graph of the zeros of $P_4^{(\alpha,1)}(x)$.

function $\omega(x,\alpha) = x^\alpha e^{-x}$, $\alpha > -1$. Then, all its zeros are increasing functions of α, for $\alpha > -1$.

Proof: In this case,

$$\frac{1}{\omega(x,\alpha)}\frac{\partial\omega(x,\alpha)}{\partial\alpha} = \ln x$$

is an increasing function of x, for $x \in (0,\infty)$. So, by Observation 3.1 (Markov's theorem), the statement holds. ■

Figure 3 serves to illustrate the monotonicity of the zeros of Laguerre polynomials with respect to the parameter α.

Example 4.3 (Zeros of Gegenbauer polynomials)**.** Let $P_n^{(\lambda)}(x)$ be the n-th Gegenbauer (or ultraspherical) polynomial which is orthogonal on $(-1,1)$ with respect to the weight function $\omega(x,\lambda) = (1-x^2)^{\lambda-1/2}$, $\lambda > -1/2$. Then, all its positive zeros are decreasing functions of λ, for $\lambda > -1/2$ ‡.

‡Because of the symmetry of the zeros of $P_n^{(\lambda)}(x)$, its negative zeros are increasing functions of λ, for $\lambda > -1/2$.

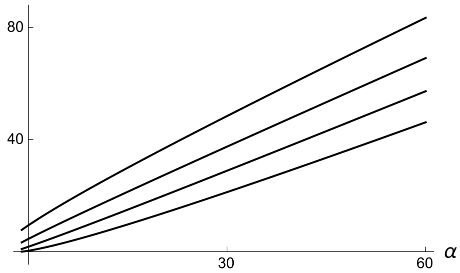

FIGURE 3. Zeros of Laguerre polynomials as functions of α. Graph of the zeros of $L_4^{(\alpha)}(x)$.

Proof: Since $\omega(x,\lambda)$ is an even function and

$$\frac{1}{\omega(x,\lambda)}\frac{\partial\omega(x,\lambda)}{\partial\lambda} = \ln(1-x^2)$$

is a decreasing function of x, for $x \in (0,1)$, then, by Observation 3.2 (Markov's theorem for even function), the statement holds. ∎

Figure 4 shows the behavior of the zeros of $P_4^{(\lambda)}(x)$ as functions of λ.

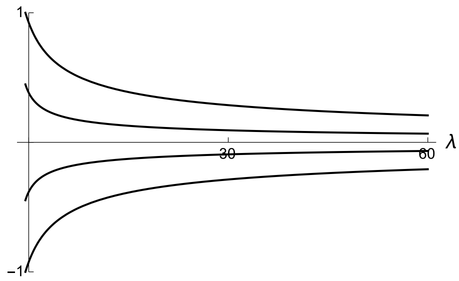

FIGURE 4. Zeros of Gegenbauer polynomials as functions of λ. Graph of the zeros of $P_4^{(\lambda)}(x)$.

4.4. Sharp monotonicity properties of the zeros of classical continuous orthogonal polynomials derived from Corollary 3.3. The motivation of the next result (Observation 4.1) goes back to a work of Laforgia [24], raised in 1981. He proved that the quantities $\lambda x_{n,k}(\lambda)$ are increasing functions of λ, for $\lambda \in (0,1)$, where $x_{n,k}(\lambda)$ are de positive zeros of Gegenbauer polynomial $P_n^{(\lambda)}(x)$. Later on, Laforgia [25] conjectured that this result remains valid for $\lambda \in (0,\infty)$. In 1988 M.E.H. Ismail and J. Letessier [16] conjectured that $(\lambda+c)^{\frac{1}{2}} x_{n,k}(\lambda)$, $k = 1,\dots,[n/2]$, increase with $\lambda > 0$ for $c = 0$.

Ismail in [15, Conjecture 3, p. 188], following a suggestion of Askey, conjectured this result for $c = 1$, leading up to the ILA conjecture of the title. E.K. Ifantis and P.H. Siafarikas [14] showed ILA for $k = 1$ and $\lambda > -1/2$, as well as D.K. Dimitrov [6]. Ahmed, Muldoon and Spigler [1] proved this monotonicity result for $c = (2n^2 + 1)/(4n + 2)$ and $-1/2 < \lambda \leq 3/2$. Elbert and Siafarikas [11] extended the result of Ahmed et al. showing thus ILA for all $\lambda > -1/2$.

Next, using one of the Markov's descendants' results, one can proof the following statement related to ILA conjecture.

Observation 4.1 (Gegenbauer - Hermite)**.** Let $x_{n,1}(\lambda) > \cdots > x_{n,n}(\lambda)$ be the zeros of the Gegenbauer polynomial $P_n^{(\lambda)}(x)$ and let $h_{n,1} > \cdots > h_{n,n}$ be the zeros of the Hermite polynomial $H_n(x)$. Then, for all $n \in \mathbb{N}$ and $c \leq -1/2$, the quantities

$$(\lambda + c)^{\frac{1}{2}} x_{n,k}(\lambda), \quad k = 1, \ldots, [n/2],$$

are increasing functions of λ and converge to $h_{n,k}$ when λ goes to infinity. So, for $c = -1/2$, one obtains

$$x_{n,k}(\lambda) \leq (\lambda - 1/2)^{-\frac{1}{2}} h_{n,k}, \quad k = 1, \ldots, [n/2].$$

Proof: One can assert the asymptotic formula [33, Section 5.6] (see also [19, formula (2.8.3)])

$$\lim_{\lambda \to \infty} \lambda^{-n/2} P_n^{(\lambda)}(\lambda^{-\frac{1}{2}} x) = \frac{H_n(x)}{n!},$$

where $H_n(x)$ denote the n-th Hermite orthogonal polynomial. Let $h_{n,k}$, $k = 1, \ldots, n$, be the zeros of $H_n(x)$ arranged in decreasing order. Thus, that gives

$$\lim_{\lambda \to \infty} \lambda^{\frac{1}{2}} x_{n,k}(\lambda) = h_{n,k}.$$

Therefore, for $f = f_n(\lambda) = (\lambda + c)^{\frac{1}{2}}$, where c is a constant that may depend on n but does not depend on λ, equivalently that gives

$$\lim_{\lambda \to \infty} (\lambda + c)^{\frac{1}{2}} x_{n,k}(\lambda) = h_{n,k}.$$

Hence, a natural question arises: is there a value of c such that all the quantities $(\lambda + c)^{\frac{1}{2}} x_{n,k}(\lambda)$, $k = 1, \ldots, [n/2]$, are monotonic (increasing or decreasing) functions of λ? To answer this question one has to perform the exchange of variables $x = z/f$ to obtain the rescaled Gegenbauer polynomial $P_n^{(\lambda)}(z/f)$ orthogonal on $(-f, f)$ with respect to the weight function $\omega(z, \lambda) =$

$(f^2 - z^2)^{\lambda - 1/2}$, $\lambda > -1/2$, and whose zeros are $z_{n,k}(\lambda) = f_n(\lambda) x_{n,k}(\lambda)$. Then, a straightforward calculation yields

$$\frac{\partial f}{\partial \lambda} = \frac{1}{2(\lambda + c)^{\frac{1}{2}}} > 0$$

and

$$\frac{\partial}{\partial z}\left[\frac{1}{\omega(z,\lambda)}\frac{\partial \omega(z,\lambda)}{\partial \lambda}\right] = \frac{2z[(z^2 - f^2) + (2\lambda - 1) f \partial f/\partial \lambda]}{(f^2 - z^2)^2} > 0$$

for $z \in (0, f)$ and $c \leq -1/2$. Therefore, having Observation 3.2 and Observation 3.3 in mind, for $c \leq -1/2$ the quantities $(\lambda + c)^{\frac{1}{2}} x_{n,k}(\lambda)$, $k = 1, \ldots, [n/2]$, are increasing functions of λ and converge to $h_{n,k}$ when λ goes to infinity. Therefore, for $c = -1/2$, one obtains

$$(\lambda - 1/2)^{\frac{1}{2}} x_{n,k}(\lambda) \leq h_{n,k}, \quad k = 1, \ldots, [n/2],$$

or equivalently

$$x_{n,k}(\lambda) \leq (\lambda - 1/2)^{-\frac{1}{2}} h_{n,k}, \quad k = 1, \ldots, [n/2].$$

■

The right-hand side of the above inequalities are upper bounds for the positive zeros of Gegenbauer polynomials, and they are sharp for large values of λ. See Figure 5.

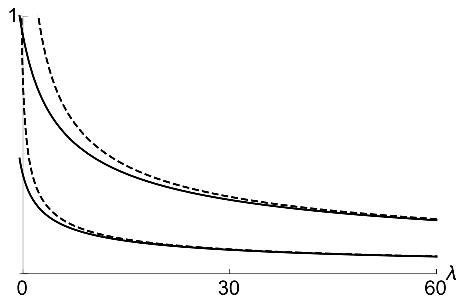

FIGURE 5. Graph of the zeros $x_{4,k}(\lambda)$ (continuous lines) and their upper bounds $(\lambda - 1/2)^{-\frac{1}{2}} h_{4,k}$ (dashed lines), for $k = 1, 2$.

The next example describes a connection between the zeros of Jacobi and Laguerre orthogonal polynomials. In [9], using Sturm's comparison theorem on solutions of Sturm-Liouville differential equation, it was shown monotonicity results for the functions $(\beta + c)(1 - x_{n,k}(\alpha, \beta))$, $k = 1, \ldots, n$, where $c = n + (\alpha + 1)/2 + (1 - \alpha^2)/(4n + 2\alpha + 2)$.

Observation 4.2 (Jacobi - Laguerre). Let $x_{n,1}(\alpha,\beta) > \cdots > x_{n,n}(\alpha,\beta)$ be the zeros of $P_n^{(\alpha,\beta)}(x)$ and let $\ell_{n,1}(\alpha) > \cdots > \ell_{n,n}(\alpha)$ be the zeros of $L_n^{(\alpha)}(x)$. Then, for every $n \in \mathbb{N}$, $1 \le k \le n$, $\alpha > -1$, and $c \le 0$, the quantities

$$(\beta + c)(1 - x_{n,k}(\alpha,\beta))/2$$

are increasing functions of β, for $\beta \in (-1,\infty)$, and converge to $\ell_{n,n-k+1}(\alpha)$ when β goes to infinity. Moreover, the inequalities

$$1 - \frac{2}{\beta}\ell_{n,n-k+1}(\alpha) \le x_{n,k}(\alpha,\beta)$$

hold.

Proof: One can find in the literature [33, Section 5.3] (see also [19, formula (2.8.1)]) the following limit relation between Jacobi and Laguerre polynomials

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1 - 2\beta^{-1}x) = L_n^{(\alpha)}(x).$$

Since the zeros are continuous functions of the coefficients of the polynomials one derives

$$\lim_{\beta\to\infty} \beta(1 - x_{n,k}(\alpha,\beta)) = 2\ell_{n,n-k+1}(\alpha), \quad k = 1,\dots,n.$$

Therefore, for $f = f_n(\alpha,\beta) = \beta + c$, where c is a constant that may depend on n and α but does not depend on β, one has equivalently that

$$\lim_{\beta\to\infty} (\beta + c)(1 - x_{n,k}(\alpha,\beta)) = 2\ell_{n,n-k+1}(\alpha), \quad k = 1,\dots,n.$$

The purpose is to find the function f in such a way that all the quantities $f(1 - x_{n,k}(\alpha,\beta))$ are monotonic (increasing or decreasing) functions of β. For this reason, performing the change of variables $x = 1 - 2z/f$ one obtains that the rescaled Jacobi polynomial $P_n^{(\alpha,\beta)}(1 - 2z/f)$, whose zeros are $z_{n,k}(\alpha,\beta) = f(1 - x_{n,k}(\alpha,\beta))/2$, $k = 1,\dots,n$, are orthogonal on $(0,f)$ with respect to the weight function $\omega(z,\alpha,\beta) = z^\alpha(f - z)^\beta$, for $\alpha,\beta > -1$.

In order to apply the Observation 3.3 for $z \in (0,f)$, one has to calculate the following derivatives:

$$\frac{\partial f}{\partial \beta} = 1 > 0$$

and

$$\frac{\partial}{\partial z}\left[\frac{1}{\omega(z,\alpha,\beta)}\frac{\partial \omega(z,\alpha,\beta)}{\partial \beta}\right] = \frac{z - f + \beta\partial f/\partial\beta}{(f - z)^2} > 0 \quad \text{if and only if} \quad c \le 0.$$

Therefore, for $c \leq 0$ the quantities $(\beta + c)(1 - x_{n,k}(\alpha, \beta))/2$ are increasing functions of β and converge to $\ell_{n,n-k+1}(\alpha)$ when β goes to infinity. Thus, for $c = 0$, one obtains

$$\beta(1 - x_{n,k}(\alpha, \beta)) \leq 2\ell_{n,n-k+1}(\alpha), \quad k = 1, \ldots, n,$$

or equivalently

$$1 - \frac{2}{\beta}\ell_{n,n-k+1}(\alpha) \leq x_{n,k}(\alpha, \beta), \quad k = 1, \ldots, n.$$

This establishes the theorem. ■

Note that the left-hand side of the above inequalities are lower bounds for the zeros of Jacobi polynomials, and they are sharp for large values of β. See Figures 6 and 7.

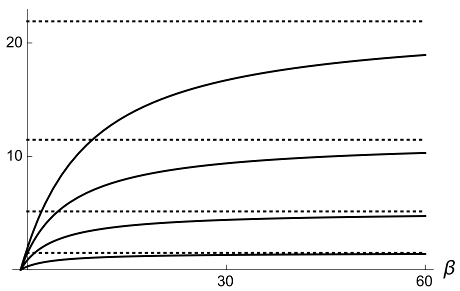

FIGURE 6. Graph of $z_{4,k}(\alpha, \beta) = \beta(1 - x_{4,k}(\alpha, \beta))/2$, $1 \leq k \leq 4$, (continuous lines). Observe that each $z_{4,k}(\alpha, \beta)$ is an increasing function of β and goes to $\ell_{4,n-k+1}(\alpha)$ as $\beta \to \infty$ (dotted lines).

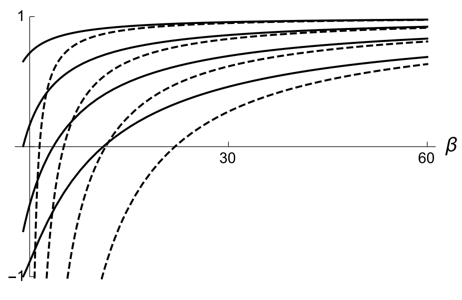

FIGURE 7. Graph of the zeros $x_{n,k}(\alpha, \beta)$ (continuous lines) and their lower bounds $1 - 2\ell_{n,n-k+1}(\alpha)/\beta$ (dashed lines), $1 \leq k \leq n$, for the case $n = 4$ and $\alpha = 1$.

The last example of this subsection shows the connection between the zeros of Laguerre and Hermite orthogonal polynomials. The first result in this topic was obtained in 1995 by Ifantis and Siafarikas [14]. They showed that $\ell_{n,1}(\alpha)/(\alpha+1)$ decreases with α, for $\alpha > -1$. In 2003, Natalini and Palumbo [29] proved that $\ell_{n,k}(\alpha)/\sqrt{\alpha+2n+1}$ are increasing functions of α, for $\alpha \in (-1,\infty)$. Moreover, they established two additional results on monotonicity of the functions of the form $\ell_{n,k}(\alpha)/\alpha^p$, with fix p, and $2 \le p \le 2n+1$. It was shown in [10] that $[\ell_{n,k}(\alpha) - (2n+\alpha-1)]/\sqrt{2(n+\alpha-1)}$ are increasing functions of α, for $\alpha \ge -1/(n-1)$. In addition, when $k = 1$, it was shown that the last function increases for every $\alpha \in (-1,\infty)$.

Observation 4.3 (Laguerre-Hermite)**.** Let $\ell_{n,1}(\alpha) > \cdots > \ell_{n,n}(\alpha)$ be the zeros of $L_n^{(\alpha)}(x)$ and let $h_{n,1} > \cdots > h_{n,n}$ be the zeros of $H_n(x)$. Then, for all $n \in \mathbb{N}$ and $1 \le k \le n$, the quantities

$$\frac{\ell_{n,k}(\alpha) - \alpha}{\sqrt{\alpha}}$$

are decreasing functions of α, for $\alpha > 0$, and moreover they converge to $\sqrt{2}h_{n,k}$ as $\alpha \to \infty$. In addition, the inequalities

$$\ell_{n,k}(\alpha) \ge \alpha + \sqrt{2\alpha}h_{n,k}$$

holds for all $\alpha > 0$.

Proof: Remember the limit relation between Laguerre and Hermite polynomials (see [19, formula (2.11.1)])

$$\lim_{\alpha\to\infty} \left(\frac{2}{\alpha}\right)^{n/2} L_n^{(\alpha)}(\alpha + (2\alpha)^{1/2}x) = \frac{(-1)^n}{n!}H_n(x).$$

Whence it follows that

$$\frac{\ell_{n,k}(\alpha) - \alpha}{\sqrt{2\alpha}} \to h_{n,k} \quad \text{as} \quad \alpha \to \infty.$$

Therefore, for any constants c and d that may depend on n but does not depend on α, one can write

$$\frac{\ell_{n,k}(\alpha) - (\alpha + c)}{\sqrt{\alpha + d}} \to \sqrt{2}h_{n,k} \quad \text{as} \quad \alpha \to \infty.$$

To obtain sharp bounds for the zeros of Laguerre polynomials one needs to determine, if possible, the best constants c and d for which the quantities $[\ell_{n,k}(\alpha) - (\alpha+c)]/\sqrt{\alpha+d}$ are monotonic (increasing or decreasing) functions of α. The best constants mean the infimum or supremum of theirs values. To go in this direction, one has to perform the change of variables $x = \sqrt{\alpha+d}z + \alpha+c$ to obtain the rescaled Laguerre polynomial $L_n^{(\alpha)}(\sqrt{\alpha+d}z+\alpha+c)$ that is orthogonal on $(-(\alpha+c)/\sqrt{\alpha+d}, +\infty)$ with respect to the weight function $\omega(z,\alpha) = (\alpha+c+\sqrt{\alpha+d}z)^{\alpha}e^{-(\alpha+c+\sqrt{\alpha+d}z)}$, $\alpha > -1$, and whose zeros are $z_{n,k}(\alpha) = [\ell_{n,k}(\alpha)-(\alpha+c)]/\sqrt{\alpha+d}$, $1 \le j \le n$. Now to apply Observation 3.3 for $z \in (-(\alpha+c)/\sqrt{\alpha+d}, +\infty)$, one has to calculate the following derivatives:

$$\frac{\partial}{\partial\alpha}\left[-\frac{\alpha+c}{\sqrt{\alpha+d}}\right] = -\frac{\alpha+2d-c}{2(\alpha+d)^{\frac{3}{2}}} \tag{4.1}$$

and

$$\frac{\partial}{\partial z}\left[\frac{1}{\omega(z,\alpha)}\frac{\partial\omega(z,\alpha)}{\partial\alpha}\right] = \frac{c(\alpha+2d-c)+2\sqrt{\alpha+d}(d-c)z-(\alpha+d)z^2}{2\sqrt{\alpha+d}(\alpha+c+\sqrt{\alpha+d}z)^2}. \tag{4.2}$$

Taking $c = d = 0$ (4.1) and (4.2) become negative for every $\alpha > 0$. Then, Observation 3.3 implies that all the quantities

$$z_{n,k}(\alpha) = \frac{\ell_{n,k}(\alpha)-\alpha}{\sqrt{\alpha}}, \quad 1 \le j \le n,$$

are decreasing functions of α, for $\alpha > 0$. It was provided that $z_{n,k}(\alpha)$ goes to $\sqrt{2}h_{n,k}$ as $\alpha \to \infty$, so $\ell_{n,k}(\alpha) \ge \alpha + \sqrt{2\alpha}h_{n,k}$ for all $\alpha > 0$ and $k = 1, \dots, n$. ■

Figure 8 exemplifies the behavior of the zeros $z_{n,k}(\alpha)$ with respect to the parameter α for $c = d = 0$. Figure 9 shows the lower bounds for the zeros $\ell_{n,k}(\alpha)$ of $L_n^{(\alpha)}(x)$.

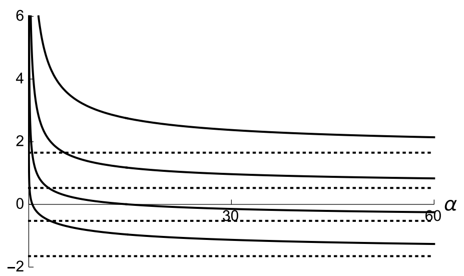

FIGURE 8. Graph of $z_{n,k}(\alpha) = \frac{\ell_{n,k}(\alpha)-\alpha}{\sqrt{2\alpha}}$, $1 \le k \le n$, for $n = 4$ (continuous lines). Observe that each $z_{4,k}(\alpha)$ is a decreasing function of α and goes to $h_{4,k}$ as $\alpha \to \infty$ (dotted lines).

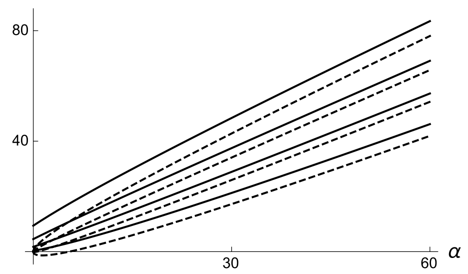

FIGURE 9. Graph of the zeros $\ell_{n,k}(\alpha)$ (continuous lines) and their lower bounds $\alpha+\sqrt{2\alpha}h_{n,k}$ (dashed lines), $1\le j\le n$, for the case $n=4$.

4.5. Monotonicity of the zeros of classical discrete orthogonal polynomials derived from Corollary 3.1 (Markov's theorem). In this subsection, the monotonicity of the zeros of the families of classical orthogonal polynomials of a discrete variable, Charlier, Meixner, Kravchuk and Hahn are revisited. Such results can be found in Ismail's book [17, Chapter 7] (see also [2]). For further information to this class of polynomials, see [30].

Example 4.4. Let $C_n^{(a)}(x)$ be the n-th Charlier orthogonal polynomial. Then all its zeros are increasing functions of a, for $a\in(0,\infty)$.

Proof: The Charlier polynomials are orthogonal with respect to $\omega(x,a)=a^x/\Gamma(x+1)$ at $x=0,1,2,\ldots$. Let consider its continuous extension on $(0,\infty)$. Then

$$\frac{1}{\omega(x,a)}\frac{\partial\omega(x,a)}{\partial a}=\frac{\partial}{\partial a}\left[\ln\frac{a^x}{\Gamma(x+1)}\right]=\frac{\partial}{\partial a}\left[x\ln(a)+\ln(\Gamma(x+1))\right]=\frac{x}{a}$$

is an increasing function of x, for $x\in(0,\infty)$. Thus, by Observation 3.1 (Markov's theorem), one concludes that the zeros of $C_n^{(a)}(x)$ are increasing function of a, for $a>0$. ■

Figure 10 presents the graph of the zeros of $C_4^{(a)}(x)$ as functions of the parameter a. Note that its zeros are increasing functions of a, for $a>0$.

Example 4.5. Let $M_n^{(\beta,c)}(x)$ be the n-th Meixner orthogonal polynomial. Then all its zeros are increasing functions of both $\beta\in(0,\infty)$ and $c\in(0,1)$.

Proof: The Meixner polynomials are orthogonal with respect to $\omega(x,\beta,c)=\Gamma(x+\beta)c^x/(\Gamma(\beta)\Gamma(x+1))$ at $x=0,1,2,\ldots$. To prove the monotonicity of the zeros of $M_n^{(\beta,c)}(x)$ with respect to the parameters β and c one has to

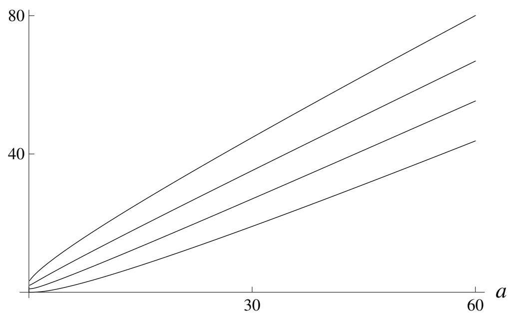

FIGURE 10. Zeros of Charlier polynomials as functions of the parameter a. Graph of the zeros of $C_4^{(a)}(x)$.

consider the analytic extension of $\omega(x,\beta,c) = \Gamma(x+\beta)c^x/(\Gamma(\beta)\Gamma(x+1))$ on $(0,\infty)$. Therefore,

$$\begin{aligned}\ln \omega(x,\beta,c) = \ln \frac{\Gamma(x+\beta)c^x}{\Gamma(\beta)\Gamma(x+1)} \\ = \ln\Gamma(x+\beta) + x\ln c - \ln\Gamma(\beta) - \ln\Gamma(x+1). \quad (4.3)\end{aligned}$$

Computing the derivative of (4.3) with respect to β one obtains[§]

$$\begin{aligned}\frac{1}{\omega(x,\beta,c)}\frac{\partial\omega(x,\beta,c)}{\partial\beta} = \frac{\partial\ln\omega(x,\beta,c)}{\partial\beta} = \frac{\Gamma'(x+\beta)}{\Gamma(x+\beta)} - \frac{\Gamma'(\beta)}{\Gamma(\beta)} \\ = \frac{x}{\beta(x+\beta)} + \sum_{n=1}^{\infty}\frac{x}{(\beta+n)(x+\beta+n)}\end{aligned}$$

which is an increasing function of x for $x \in (0,\infty)$, and $\beta > 0$. Thus, Observation 3.1 (Markov's theorem) implies that the zeros of $M_n^{(\beta,c)}(x)$ are increasing functions of β, for $\beta > 0$.

On the other hand, differentiating (4.3) with respect to c one obtains

$$\frac{1}{\omega(x,\beta,c)}\frac{\partial\omega(x,\beta,c)}{\partial c} = \frac{\partial\ln\omega(x,\beta,c)}{\partial c} = \frac{x}{c},$$

which is an increasing function of x for $x \in (0,\infty)$, and $c \in (0,1)$. Hence, by Observation 3.1 (Markov's theorem), one concludes that the zeros of $M_n^{(\beta,c)}(x)$ are also increasing function of c, for $c \in (0,1)$. ■

[§]One has the identity $\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty}\left[\frac{1}{z+n} - \frac{1}{n}\right]$, where γ is the Euler constant (see [34, Section 12.3], [31, Chapter 7]).

To exemplify the monotonicity of the zeros of Meixner polynomials as functions of β and c, one presents two graphs. See Figures 11 and 12.

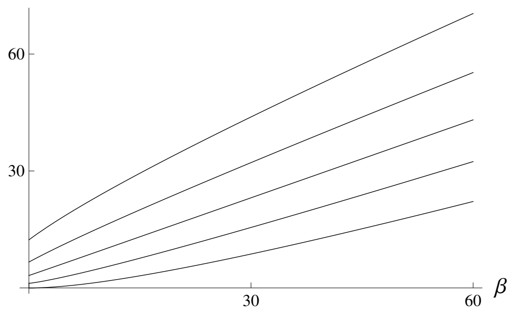

FIGURE 11. Zeros of Meixner polynomials as functions of the parameter β. Graph of the zeros of $M_5^{(\beta,0.4)}(x)$, $\beta > 0$.

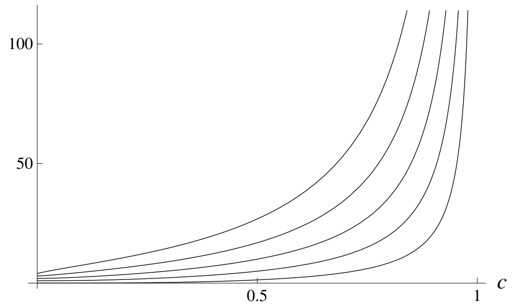

FIGURE 12. Zeros of Meixner polynomials as functions of the parameter c. Graph of the zeros of $M_5^{(6,c)}(x)$, $0 < c < 1$.

Example 4.6. Let $K_n^{(p,N)}(x)$ be the n-th Kravchuck orthogonal polynomial. Then, all its zeros are increasing functions of the parameter p, for $p \in (0,1)$.

Proof: The Kravchuck polynomials are orthogonal with respect to

$$\omega(x,p,N) = \frac{\Gamma(N+1)p^x(1-p)^{N-x}}{\Gamma(N+1-x)\Gamma(x+1)}$$

at $x = 0,1,2,\ldots,N$. Let $\omega(x,p,N)$ be the analytic extension on $(0,N)$ of the Kravchuk weight. Computing the logarithmic derivative of $\omega(x,p,N)$ with respect to p, one obtains

$$\frac{1}{\omega(x,p,N)}\frac{\partial\omega(x,p,N)}{\partial p} = \frac{\partial}{\partial p}\left[\ln\frac{\Gamma(N+1)p^x(1-p)^{N-x}}{\Gamma(N+1-x)\Gamma(x+1)}\right]$$
$$= \frac{\partial}{\partial p}\left[\ln(\Gamma(N+1)) + x\ln(p) + (N-x)\ln(1-p) - \ln(\Gamma(N+1-x))\right.$$
$$\left. - \ln(\Gamma(x+1))\right] = \frac{x}{p} - \frac{N-x}{1-p} = \frac{x-Np}{p(1-p)},$$

which is obviously an increasing function of x, for $x \in (0, N)$, and $p \in (0, 1)$. Then, by Observation 3.1 (Markov's theorem), all the zeros of $K_n^{(p,N)}(x)$ increase when p increases. See Figure 13. ■

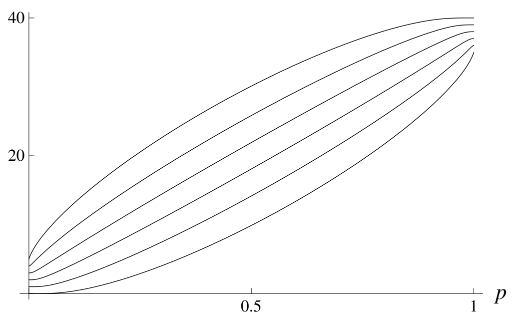

FIGURE 13. Zeros of Kravchuk polynomials as functions of the parameter p. Graph of the zeros $K_6^{(p,40)}$, $0 < p < 1$.

Example 4.7. Let $P_n^{(\alpha,\beta,N)}(x)$ be the n-th Hahn orthogonal polynomial. Then, all its zeros are increasing functions of $\alpha \in (-1, \infty)$ and decreasing functions of $\beta \in (-1, \infty)$.

Proof: The Hahn polynomials are orthogonal with respect to

$$\omega(x,\alpha,\beta,N) = \frac{\Gamma(\alpha+x+1)\Gamma(\beta+N-x+1)}{\Gamma(\alpha+1)\Gamma(x+1)\Gamma(\beta+1)\Gamma(N-x+1)}$$

at $x = 0, 1, 2, \ldots, N$. Let $\omega(x,\alpha,\beta,N)$ be the analytic extension on $(0, N)$ of the Hahn weight. Then,

$$\ln\omega(x,\alpha,\beta,N) = \ln\frac{\Gamma(\alpha+x+1)\Gamma(\beta+N-x+1)}{\Gamma(\alpha+1)\Gamma(x+1)\Gamma(\beta+1)\Gamma(N-x+1)}$$
$$= \ln\Gamma(\alpha+x+1) + \ln\Gamma(\beta+N-x+1) - \ln\Gamma(\alpha+1)$$
$$- \ln\Gamma(x+1) - \ln\Gamma(\beta+1) - \ln\Gamma(N-x+1).$$

Since

$$\frac{1}{\omega(x,\alpha,\beta,N)}\frac{\partial\omega(x,\alpha,\beta,N)}{\partial\alpha}=\frac{\partial\ln\omega(x,\alpha,\beta,N)}{\partial\alpha}=\frac{\Gamma'(\alpha+x+1)}{\Gamma(\alpha+x+1)}-\frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)}$$
$$=\frac{x}{(\alpha+1)(x+\alpha+1)}+\sum_{n=1}^{\infty}\frac{x}{(\alpha+n+1)(x+\alpha+n+1)}$$

is an increasing function of x for $x\in(0,N)$, by Observation 3.1 (Markov's theorem), one derives that the zeros of $P_n^{(\alpha,\beta,N)}(x)$ are increasing functions of α, for $\alpha\in(-1,\infty)$. On the other hand,

$$\frac{1}{\omega(x,\alpha,\beta,N)}\frac{\partial\omega(x,\alpha,\beta,N)}{\partial\beta}=\frac{\partial\ln\omega(x,\alpha,\beta,N)}{\partial\beta}$$
$$=\frac{\Gamma'(\beta+N-x+1)}{\Gamma(\beta+N-x+1)}-\frac{\Gamma'(\beta+1)}{\Gamma(\beta+1)}$$
$$=\frac{N-x}{(\beta+1)(\beta+N-x+1)}+\sum_{n=1}^{\infty}\frac{N-x}{(\beta+n+1)(\beta+N-x+1+n)}$$

is a decreasing function of x for $x\in(0,N)$. Thus, by Observation 3.1 (Markov's theorem), it implies that the zeros of $P_n^{(\alpha,\beta,N)}(x)$ are decreasing functions of β, for $\beta\in(-1,\infty)$. ■

Figures 14 and 15 illustrate these monotonicities.

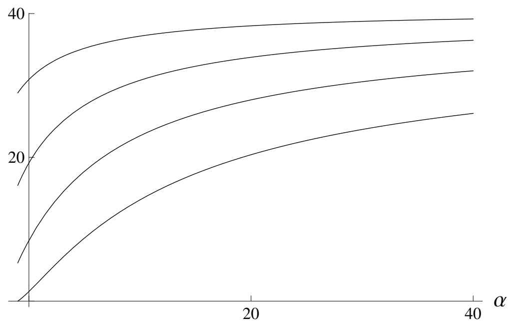

FIGURE 14. Monotonicity of zeros of Hahn polynomials. Graphic of the zeros of $P_4^{(\alpha,3,40)}(x)$ as functions of α.

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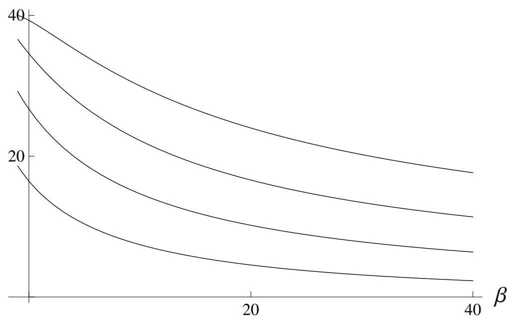

FIGURE 15. Monotonicity of zeros of Hahn polynomials. Graphic of the zeros of $P_4^{(8,\beta,40)}(x)$ as functions of β.

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