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PSEUDOALGEBRAS AND NON-CANONICAL ISOMORPHISMS

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ABSTRACT: Given a pseudomonad $\mathcal{T}$, we prove that a lax $\mathcal{T}$-morphism between pseudoalgebras is a $\mathcal{T}$-pseudomorphism if and only if there is a suitable (possibly non-canonical) invertible $\mathcal{T}$-transformation. This result encompasses several results on *non-canonical isomorphisms*, including Lack's result on normal monoidal functors between braided monoidal categories, since it is applicable in any 2-category of pseudoalgebras, such as the 2-categories of monoidal categories, cocomplete categories, pseudofunctors and so on.

KEYWORDS: pseudomonads, lax morphisms, normal monoidal functors, non-canonical isomorphisms, pseudoalgebras.
MATH. SUBJECT CLASSIFICATION (2010): 18D05, 18C15, 18C20, 18D10.

Introduction

Two-dimensional monad theory [Mar99; Lac00] gives a unifying approach to study several aspects of two-dimensional universal algebra [Kel74; BKP89; LN16a]. This fact is illustrated by the various examples of 2-categories of (lax-/pseudo)algebras in the literature. For this reason, results in 2-dimensional monad theory usually gives light to a wide range of situations, having many applications, *e.g.* [PCW00; Gar15; LN16c].

Herein, the problem of *non-canonical isomorphisms* consists of investigating whether, in a given situation, the existence of an invertible non-canonical transformation implies that a previously given canonical one is invertible as well. In order to give a glimpse of our scope, we give some examples.

The first precise example is related to the study of preservation of colimits. Given any functor $F : \mathbb{A} \to \mathbb{B}$, assuming the existence of $\mathbb{S}$-colimits, there is an induced canonical natural transformation $\mathrm{colim}_{\mathbb{B}} FD \longrightarrow F(\mathrm{colim}_{\mathbb{A}} D)$ in D, in which

$$\mathrm{colim}_{\mathbb{B}} : \mathsf{Cat}\left[\mathbb{S}, \mathbb{B}\right] \to \mathbb{B}, \qquad \mathrm{colim}_{\mathbb{A}} : \mathsf{Cat}\left[\mathbb{S}, \mathbb{A}\right] \to \mathbb{A}$$

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are the functors that give the $\mathbb{S}$-colimits. We say that F *preserves* $\mathbb{S}$*-colimits* whenever this canonical transformation is invertible. In this context, the problem of non-canonical isomorphisms, studied by Caccamo and Winksel [CW05], is to investigate under which conditions the existence of a natural isomorphism $\mathrm{colim}_{\mathbb{B}} FD \cong F(\mathrm{colim}_{\mathbb{A}} D)$ implies that F preserves $\mathbb{S}$-colimits. For instance, in the case of finite coproducts, [CW05] proves that a functor F preserves them if and only if there is a (possibly non-canonical) natural isomorphism as above and F preserves initial objects.

Three other examples of non-canonical isomorphisms are given in [Lac12]. Namely:

(1) Characterization of distributive categories: a category $\mathbb{D}$ with finite coproducts and products is distributive if, given any object x, the functor $x \times - : \mathbb{D} \to \mathbb{D}$ preserves coproducts. In this case, [Lac12] proves that the existence of an invertible natural transformation $\delta_{(x,y,z)} : (x \times y) \sqcup (x \times z) \longrightarrow x \times (y \times z)$ implies that $\mathbb{D}$ is distributive.
(2) Characterization of semi-additive categories: a category $\mathbb{B}$ with finite products and coproducts is semi-additive if (1) it is pointed and (2) the canonical natural transformation $\psi : - \sqcup - \longrightarrow - \times -$ induced by the identities and zero morphisms is invertible. In [Lac12], it is shown that the existence of any natural isomorphism $- \sqcup - \longrightarrow - \times -$ implies that $\mathbb{A}$ is semi-additive.
(3) Braided monoidal categories: Lack proved that, in the presence of a suitably defined invertible non-canonical isomorphism, a normal monoidal functor is actually a strong monoidal functor. This result encompasses the common part of both situations above.

The aim of this note is to frame the problem of non-canonical isomorphisms in the context of 2-dimensional monad theory: we show that, given a pseudomonad $\mathcal{T}$, a lax $\mathcal{T}$-morphism $\mathtt{f}$ is a $\mathcal{T}$-pseudomorphism if and only if there is a suitable (possibly "non-canonical") invertible $\mathcal{T}$-transformation as defined in 2.1. This result encompasses the four situations above, generalizing Lack's result on braided monoidal categories and being applicable to study analogues in several other instances, including monoidal categories, pseudofunctors, cocomplete categories, categories with certain types of colimites and any other example of 2-category of pseudoalgebras and lax morphisms.

In Section 1 we fix terminology, giving basic definitions and known results on 2-dimensional monad theory. Section 2 gives the main result of this note.

Finally, Section 3 gives brief comments on particular cases, showing how all the situations above are encompassed by our main theorem and how the situation is simplified for Kock-Zöberlein pseudomonads.

1. Basics

In order to fix notation, we give basic definitions in this section. We also give some results assumed in Section 2. Our setting is the tricategory $2\text{-}\mathsf{CAT}$ of 2-categories, pseudofunctors, pseudonatural transformations and modifications. We follow the notation and definitions of [LN16b]. In particular, as briefly recalled below, the definition of pseudomonads and lax algebras can be found in Section 4 of [LN16b].

Definition 1.1. [Pseudomonad] A *pseudomonad* $\mathcal{T}$ on a 2-category $\mathfrak{B}$ consists of a sextuple $(\mathcal{T}, m, \eta, \mu, \iota, \tau)$, in which $\mathcal{T} : \mathfrak{B} \to \mathfrak{B}$ is a pseudofunctor, $m : \mathcal{T}^2 \longrightarrow \mathcal{T}, \eta : \mathrm{Id}_{\mathfrak{B}} \longrightarrow \mathcal{T}$ are pseudonatural transformations and $\tau : \mathrm{Id}_{\mathcal{T}} \Longrightarrow (m)(\mathcal{T}\eta)$, $\iota : (m)(\eta\mathcal{T}) \Longrightarrow \mathrm{Id}_{\mathcal{T}}$, $\mu : m\,(\mathcal{T}m) \Longrightarrow m\,(m\mathcal{T})$ are invertible modifications satisfying coherence equations [LN16b].

Definition 1.2. [Lax algebras] Let $\mathcal{T} = (\mathcal{T}, m, \eta, \mu, \iota, \tau)$ be a pseudomonad on $\mathfrak{B}$. We define the 2-category $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell$ as follows:

- Objects: *lax $\mathcal{T}$-algebras* are defined by $\mathtt{z} = (Z, \mathfrak{alg}_{\mathtt{z}}, \underline{\overline{\mathtt{z}}}, \underline{\overline{\mathtt{z}}}_0)$ in which $\mathfrak{alg}_{\mathtt{z}} : \mathcal{T}Z \to Z$ is a morphism of $\mathfrak{B}$ and $\underline{\overline{\mathtt{z}}} : \mathfrak{alg}_{\mathtt{z}}\mathcal{T}(\mathfrak{alg}_{\mathtt{z}}) \Rightarrow \mathfrak{alg}_{\mathtt{z}} m_Z, \underline{\overline{\mathtt{z}}}_0 : \mathrm{Id}_Z \Rightarrow \mathfrak{alg}_{\mathtt{z}}\eta_Z$ are 2-cells of $\mathfrak{B}$ satisfying coherence axioms [LN16b].
  Recall that a lax $\mathcal{T}$-algebra $(Z, \mathfrak{alg}_{\mathtt{z}}, \underline{\overline{\mathtt{z}}}, \underline{\overline{\mathtt{z}}}_0)$ is a *$\mathcal{T}$-pseudoalgebra* if $\underline{\overline{\mathtt{z}}}, \underline{\overline{\mathtt{z}}}_0$ are invertible.
- Morphisms: *lax $\mathcal{T}$-morphisms* $\mathtt{f} : \mathtt{y} \to \mathtt{z}$ between lax $\mathcal{T}$-algebras $\mathtt{y} = (Y, \mathfrak{alg}_{\mathtt{y}}, \underline{\overline{\mathtt{y}}}, \underline{\overline{\mathtt{y}}}_0)$, $\mathtt{z} = (Z, \mathfrak{alg}_{\mathtt{z}}, \underline{\overline{\mathtt{z}}}, \underline{\overline{\mathtt{z}}}_0)$ are pairs $\mathtt{f} = (f, \langle\overline{\mathsf{f}}\rangle)$ in which $f : Y \to Z$ is a morphism in $\mathfrak{B}$ and $\langle\overline{\mathsf{f}}\rangle : \mathfrak{alg}_{\mathtt{z}}\mathcal{T}(f) \Rightarrow f\mathfrak{alg}_{\mathtt{y}}$ is a 2-cell of $\mathfrak{B}$ satisfying coherence axioms [LN16b].
  Recall that a lax $\mathcal{T}$-morphism $\mathtt{f} = (f, \langle\overline{\mathsf{f}}\rangle)$ is a $\mathcal{T}$-pseudomorphism if $\langle\overline{\mathsf{f}}\rangle$ is an invertible 2-cell.
- 2-cells: a *$\mathcal{T}$-transformation* $\mathfrak{m} : \mathtt{f} \Rightarrow \mathtt{h}$ between lax $\mathcal{T}$-morphisms $\mathtt{f} = (f, \langle\overline{\mathsf{f}}\rangle)$, $\mathtt{h} = (h, \langle\overline{\mathsf{h}}\rangle)$ is a 2-cell $\mathfrak{m} : f \Rightarrow h$ in $\mathfrak{B}$ subject to one natural equation [LN16b].

The compositions are defined in the obvious way and these definitions make $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell$ a 2-category. The full sub-2-category of the $\mathcal{T}$-pseudoalgebras of

$\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell$ is denoted by $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell$. Also, we consider the locally full sub-2-category $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$ consisting of $\mathcal{T}$-pseudoalgebras and $\mathcal{T}$-pseudomorphisms. Finally, the inclusion is denoted by $\widehat{\ell} : \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell$.

On one hand, if $\mathcal{T} = (\mathcal{T}, m, \eta, \mu, \iota, \tau)$ is a pseudomonad on $\mathfrak{B}$, then $\mathcal{T}$ induces the *Eilenberg-Moore biadjunction* $(L^{\mathcal{T}} \dashv U^{\mathcal{T}}, \eta, \varepsilon^{\mathcal{T}}, s^{\mathcal{T}}, t^{\mathcal{T}})$, in which $L^{\mathcal{T}}, U^{\mathcal{T}}$ are defined by

$$\begin{aligned} U^{\mathcal{T}} : \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg} &\to \mathfrak{B} \\ (Z, \mathfrak{alg}_{\mathtt{z}}, \overline{\underline{\mathtt{z}}}, \overline{\underline{\mathtt{z}}}_0) &\mapsto Z \\ (f, \langle \overline{\mathsf{f}} \rangle) &\mapsto f \\ \mathfrak{m} &\mapsto \mathfrak{m} \end{aligned} \qquad \begin{aligned} L^{\mathcal{T}} : \mathfrak{B} &\to \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \\ Z &\mapsto \left(\mathcal{T}(Z), m_Z, \mu_Z, \iota_Z^{-1}\right) \\ f &\mapsto \left(\mathcal{T}(f), m_f^{-1}\right) \\ \mathfrak{m} &\mapsto \mathcal{T}(\mathfrak{m}) \end{aligned}$$

On the other hand, this biadjunction induces a pseudocomonad on $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$, called the *Eilenberg-Moore pseudocomonad* and denoted herein by $\underline{\mathcal{T}}$ (see Remark 5.3 and Lemma 5.4 of [LN16a]). The underlying pseudofunctor of this pseudocomonad can be extended to $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell$ by composing the forgetful 2-functor $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$ with $\widehat{\ell} L^{\mathcal{T}}$. By abuse of language, we denote this extension (and its restriction to $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell$) by $\underline{\mathcal{T}}$ as well. This extension is actually the underlying pseudofunctor of what can be called a *colax comonad* that extends the pseudomonad $\mathcal{T}$. It should be observed that $\underline{\mathcal{T}}(\mathtt{z})$ is the free $\mathcal{T}$-pseudoalgebra on the underlying object of the lax $\mathcal{T}$-algebra $\mathtt{z}$.

Given a $\mathcal{T}$-pseudoalgebra $(Z, \mathfrak{alg}_{\mathtt{z}}, \overline{\underline{\mathtt{z}}}, \overline{\underline{\mathtt{z}}}_0)$, $\varepsilon^{\mathcal{T}}_{\mathtt{z}}$ is the $\mathcal{T}$-pseudomorphism defined by:

$$\varepsilon^{\mathcal{T}}_{\mathtt{z}} := (\mathfrak{alg}_{\mathtt{z}}, \overline{\underline{\mathtt{z}}}) : \underline{\mathcal{T}}\mathtt{z} \to \mathtt{z},$$

while, given a $\mathcal{T}$-pseudomorphism $\mathtt{f} = (f, \langle \overline{\mathsf{f}} \rangle)$, $\varepsilon^{\mathcal{T}}_{\mathtt{f}} := \langle \overline{\mathsf{f}} \rangle^{-1}$.

We can extend this counit to all the lax $\mathcal{T}$-algebras as long as we allow it to be only a colax natural transformation (instead of a pseudonatural one). In our scope, however, we just need to observe that, by the coherence axioms, we can actually get a pseudonatural transformation $\varepsilon^{\mathcal{T}} : \underline{\mathcal{T}} \longrightarrow \mathrm{Id}_{\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}}$ in which $\underline{\mathcal{T}}$ is the extension to $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$.

2. Non-canonical isomorphisms

In this section, we prove our main result. We start with:

Definition 2.1. [$\mathtt{f}$-isomorphism] Let $\mathcal{T}$ be a pseudomonad on a 2-category $\mathfrak{B}$. Assume that $\mathtt{f} = (f, \langle \overline{\mathsf{f}} \rangle) : \mathtt{y} \to \mathtt{z}$ is a lax $\mathcal{T}$-morphism. If it exists, an invertible $\mathcal{T}$-transformation

$$\begin{tikzcd}
\underline{\mathcal{T}}\mathtt{y} \arrow[r, "\underline{\mathcal{T}}(\mathtt{f})"] \arrow[d, "\varepsilon^{\mathcal{T}}_{\mathtt{y}}"'] & \underline{\mathcal{T}}\mathtt{z} \arrow[d, "\varepsilon^{\mathcal{T}}_{\mathtt{z}}"] \arrow[dl, phantom, "\overset{\psi}{\Longleftarrow}"] \\
\mathtt{y} \arrow[r, "\mathtt{f}"'] & \mathtt{z}
\end{tikzcd}$$

is called an $\mathtt{f}$-*isomorphism.*

Roughly, these $\mathtt{f}$-isomorphisms play the role of the *non-canonical isomorphisms* in the examples given in the introduction, except from the fact that, if the canonical transformation is invertible, then it is an $\mathtt{f}$-isomorphism as well. That is to say, the first basic result about $\mathtt{f}$-isomorphisms is the following: for each $\mathcal{T}$-pseudomorphism $\mathtt{f} = (f, \langle \overline{\mathsf{f}} \rangle)$ between lax $\mathcal{T}$-algebras $\mathtt{y}$ and $\mathtt{z}$,

$$\langle \overline{\mathsf{f}} \rangle = (\varepsilon^{\mathcal{T}}_{\mathtt{f}})^{-1} : \varepsilon^{\mathcal{T}}_{\mathtt{z}} \cdot \underline{\mathcal{T}}(\mathtt{f}) \Longrightarrow \mathtt{f} \cdot \varepsilon^{\mathcal{T}}_{\mathtt{y}}$$

is an $\mathtt{f}$-isomorphism. Theorem 2.2 gives the reciprocal to this fact for lax $\mathcal{T}$-morphisms between $\mathcal{T}$-pseudoalgebras.

Theorem 2.2 (Non-canonical isomorphisms)**.** *Let $\mathcal{T}$ be a pseudomonad on a* 2*-category $\mathfrak{B}$. A lax $\mathcal{T}$-morphism $\mathtt{f} : \mathtt{y} \to \mathtt{z}$ between $\mathcal{T}$-pseudoalgebras is a $\mathcal{T}$-pseudomorphism if and only if there is an $\mathtt{f}$-isomorphism.*

Proof: It remains only to prove that a lax $\mathcal{T}$-morphism $\mathtt{f}$ is a $\mathcal{T}$-pseudomorphism provided that there is an $\mathtt{f}$-isomorphism. We assume that the structures of the pseudomonad $\mathcal{T}$, the lax $\mathcal{T}$-morphism $\mathtt{f}$ and the $\mathcal{T}$-pseudoalgebras $\mathtt{y}$ and $\mathtt{z}$ are given as in Definition 1.2.

Assume that $\psi : \varepsilon^{\mathcal{T}}_{\mathtt{z}} \cdot \underline{\mathcal{T}}(\mathtt{f}) \Longrightarrow \mathtt{f} \cdot \varepsilon^{\mathcal{T}}_{\mathtt{y}}$ is an invertible $\mathcal{T}$-transformation. By the definition of $\mathcal{T}$-transformation, we conclude that

$$\begin{tikzcd}
\mathcal{T}Z \arrow[rr, "\mathfrak{alg}_{\mathtt{z}}"] & & Z & \\
\mathcal{T}^2Z \arrow[u, "\mathcal{T}(\mathfrak{alg}_{\mathtt{z}})"] \arrow[r, "m_Z"] \arrow[ur, phantom, "\overset{\overline{\mathtt{y}}}{\Longrightarrow}"] & \mathcal{T}Z \arrow[ur, "\mathfrak{alg}_{\mathtt{z}}"] \arrow[rr, phantom, "\overset{\psi}{\Longrightarrow}"] & & Y \arrow[ul, "f"'] \\
\mathcal{T}^2Y \arrow[u, "\mathcal{T}^2(f)"] \arrow[rr, "m_Y"'] \arrow[ur, phantom, "\overset{m_f^{-1}}{\Longrightarrow}"] & & \mathcal{T}Y \arrow[ul, "\mathcal{T}(f)"] \arrow[ur, "\mathfrak{alg}_{\mathtt{y}}"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathcal{T}Z \arrow[rr, "\mathfrak{alg}_{\mathtt{z}}"] & & Z \\
\mathcal{T}^2Z \arrow[ur, "\mathcal{T}(\mathfrak{alg}_{\mathtt{z}})"] \arrow[rr, phantom, "\overset{\widehat{\mathcal{T}(\psi)}}{\Longrightarrow}"] & & \mathcal{T}Y \arrow[ul, "\mathcal{T}(f)"'] \arrow[r, "\mathfrak{alg}_{\mathtt{y}}"] \arrow[ur, phantom, "\overset{\langle \overline{\mathsf{f}} \rangle}{\Longrightarrow}"] \arrow[dr, phantom, "\overset{\overline{\mathtt{y}}}{\Longrightarrow}"] & Y \arrow[u, "f"'] \\
 & \mathcal{T}^2Y \arrow[ul, "\mathcal{T}^2(f)"] \arrow[ur, "\mathcal{T}(\mathfrak{alg}_{\mathtt{y}})"'] \arrow[rr, "m_Y"'] & & \mathcal{T}Y \arrow[u, "\mathfrak{alg}_{\mathtt{y}}"']
\end{tikzcd}$$

holds in $\mathfrak{B}$, in which $\widehat{\mathcal{T}(\psi)} := \mathsf{t}^{-1}_{(f)(\mathfrak{alg}_{\mathsf{y}})}\mathcal{T}(\psi)\mathsf{t}_{(\mathfrak{alg}_{\mathsf{z}})(\mathcal{T}(f))}$. Since we know that all the 2-cells above but $\langle\overline{\mathsf{f}}\rangle$ are invertible, after composing with the appropriate inverses in both sides of the equation, we conclude that the horizontal composition $\langle\overline{\mathsf{f}}\rangle * \mathrm{id}_{\mathcal{T}(\mathfrak{alg}_{\mathsf{y}})}$ is invertible as well.

Therefore, defining $\widehat{\mathcal{T}\left(\underline{\overline{\mathsf{y}}}_0\right)} := \mathsf{t}^{-1}_{\left(\mathfrak{alg}_{\mathsf{y}}\right)(\eta_Y)}\mathcal{T}\left(\underline{\overline{\mathsf{y}}}_0\right)\mathsf{t}_Z$, we conclude that the left hand of the equality

$$
\begin{tikzcd}
 & \mathcal{T}Y \arrow[d, "\mathcal{T}(\eta_Y)"] \arrow[dd, equal, bend right=80, ""{name=L}] \arrow[dd, equal, bend left=80, ""{name=R}] & \\
 & \mathcal{T}^2Y \arrow[d, "\mathcal{T}(\mathfrak{alg}_{\mathsf{y}})"] \arrow[l, phantom, "\overset{\widehat{\mathcal{T}(\underline{\overline{\mathsf{y}}}_0)}^{-1}}{\Longleftarrow}"] \arrow[r, phantom, "\overset{\widehat{\mathcal{T}(\underline{\overline{\mathsf{y}}}_0)}}{\Longleftarrow}"] & \\
 & \mathcal{T}Y \arrow[dl, "\mathfrak{alg}_{\mathsf{y}}"'] \arrow[dr, "\mathcal{T}(f)"] & \\
Y \arrow[dr, "f"'] \arrow[rr, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longleftarrow}"] & & \mathcal{T}Z \arrow[dl, "\mathfrak{alg}_{\mathsf{z}}"] \\
 & Z &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{T}Y \arrow[dl, "\mathfrak{alg}_{\mathsf{y}}"'] \arrow[dr, "\mathcal{T}(f)"] & \\
Y \arrow[dr, "f"'] \arrow[rr, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longleftarrow}"] & & \mathcal{T}Z \arrow[dl, "\mathfrak{alg}_{\mathsf{z}}"] \\
 & Z &
\end{tikzcd}
$$

is a (vertical) composition of invertible 2-cells and, hence, itself invertible. ■

Remark 2.3. By doctrinal adjunction [Kel74], we conclude that a lax $\mathcal{T}$-morphism $\mathtt{f} = (f, \langle\overline{\mathsf{f}}\rangle)$ between $\mathcal{T}$-pseudoalgebras has a right adjoint in $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell$ if and only if f has a right adjoint in the base 2-category $\mathfrak{B}$ and there is an $\mathtt{f}$-isomorphism.

3. Examples

In this section, we state some examples. Instead of giving a definitive answer to every particular case, Theorem 2.2 gives a general procedure for studying non-canonical isomorphisms. Namely, in each *non-canonical isomorphism problem*, we can firstly show how this problem can be framed in our context and, then, show that such non-canonical isomorphism actually defines a suitable $\mathtt{f}$-isomorphism. For example, as it is shown below, the result on braided monoidal categories of [Lac12] is a particular instance of Theorem 2.2.

Firstly, we establish the direct corollary of Theorem 2.2 on monoidal categories. We denote by $\mathcal{F}$ the *free 2-monad* on Cat. The underlying 2-functor of this 2-monad is given by $\mathcal{F}(\mathbb{A}) = \coprod_{n=0}^{\infty} \mathbb{A}^n$, while each component of the multiplication is induced by the identities $\mathbb{A}^t \to \mathbb{A}^t$ for each t. The 2-category $\mathsf{Ps}\text{-}\mathcal{F}\text{-}\mathsf{Alg}_\ell$ is known to be the 2-category of monoidal categories, monoidal functors and monoidal transformations [BKP89; LN16b]. Recall that $\mathcal{F}$-pseudomorphisms are called *strong monoidal functors*.

Remark 3.1. We adopt the biased definition of monoidal category. Given a monoidal category $\mathbb{M} = (\mathbb{M}_0, \otimes_{\mathbb{M}}, I_{\mathbb{M}})$, the monoidal product $\otimes_{\underline{\mathcal{F}}\mathbb{M}} : (\underline{\mathcal{F}}\mathbb{M})_0 \times (\underline{\mathcal{F}}\mathbb{M})_0 \to (\underline{\mathcal{F}}\mathbb{M})_0$ of the strict monoidal category $\underline{\mathcal{F}}\mathbb{M}$ is defined firstly by taking the isomorphism $\left(\coprod_{k=0}^{\infty} \mathbb{M}_0^k\right) \times \left(\coprod_{j=0}^{\infty} \mathbb{M}_0^j\right) \cong \coprod_{k,j\in\mathbb{N}} \mathbb{M}_0^{k+j}$ and composing with the morphism induced by the inclusions of the coproduct $\coprod_{k=0}^{\infty} \mathbb{M}_0^{k+t} \to \coprod_{n=0}^{\infty} \mathbb{M}_0^n$ for each t. In other words, an object of $(\underline{\mathcal{F}}\mathbb{M})_0 \times (\underline{\mathcal{F}}\mathbb{M})_0$ is an ordered pair of words of objects in $\mathbb{M}_0$, while the tensor product is just the word obtained by juxtaposition. The empty word is the identity of the monoidal structure of $\underline{\mathcal{F}}\mathbb{M}$.

It should be noted that the component $\varepsilon^{\mathcal{F}}_{\mathbb{M}} : \underline{\mathcal{F}}\mathbb{M} \to \mathbb{M}$ is a strong monoidal functor. We consider that its underlying functor gives the monoidal product of the objects in the word respecting the order, that is to say, it is defined inductively by

$$\begin{aligned}
\mathfrak{alg}_{\mathbb{M}}() &:= I_{\mathbb{M}};\\
\mathfrak{alg}_{\mathbb{M}}(x_1) &:= x_1;\\
\mathfrak{alg}_{\mathbb{M}}(x_1, \ldots, x_{n+1}) &:= \mathfrak{alg}_{\mathbb{M}}(x_1, \ldots, x_n) \otimes_{\mathbb{M}} \mathfrak{alg}_{\mathbb{M}}(x_{n+1})
\end{aligned}$$

in which $()$ denotes the object of $\mathbb{M}_0^0$, which is the identity object of $\underline{\mathcal{F}}\mathbb{M}$, and $(x_1, \ldots, x_n)$ denotes an object of $\mathbb{M}_0^n$. Finally, for each monoidal functor $\mathtt{f} = (f, \langle\overline{\mathtt{f}}\rangle)$, $\underline{\mathcal{F}}(\mathtt{f})$ is the strict monoidal functor defined pointwise by f.

Corollary 3.2 (Monoidal Categories)**.** *Let* $\mathtt{f} = (f, \langle\overline{\mathtt{f}}\rangle) : \mathbb{M} \to \mathbb{N}$ *be a monoidal functor. There is an invertible monoidal transformation* $\psi : \varepsilon^{\mathcal{F}}_{\mathbb{N}} \cdot \underline{\mathcal{F}}(\mathtt{f}) \Longrightarrow \mathtt{f} \cdot \varepsilon^{\mathcal{F}}_{\mathbb{M}}$ *if and only if* $\mathtt{f}$ *is a strong monoidal functor or, in other words,* $\langle\overline{\mathtt{f}}\rangle$ *is invertible.*

Given a braided monoidal category $\mathbb{M} = (\mathbb{M}_0, \otimes_{\mathbb{M}}, I_{\mathbb{M}}, \lambda_{\mathbb{M}})$, we have an induced strong monoidal functor between the product $\mathbb{M} \times \mathbb{M} \to \mathbb{M}$ whose underlying functor is $\otimes_{\mathbb{M}}$ and the structure maps are induced by the isomorphisms $w \otimes x \otimes y \otimes z \to w \otimes y \otimes x \otimes z$ and $I_{\mathbb{M}} \otimes I_{\mathbb{M}} \to I_{\mathbb{M}}$ induced respectively by the braiding $\lambda_{\mathbb{M}}$ and the action of the identity. In Theorem 3.3, we denote this strong monoidal functor just by $\otimes_{\mathbb{M}}$.

Recall that a monoidal functor $\mathtt{f} = (f, \langle \overline{\mathsf{f}} \rangle)$ is called normal if the component of $\langle \overline{\mathsf{f}} \rangle$ on the empty word, denoted below by $\langle \overline{\mathsf{f}} \rangle_{()}$, is invertible.

Corollary 3.3 ([Lac12])**.** *Let* $\mathbb{M}, \mathbb{N}$ *be braided monoidal categories, and* $\mathtt{f} = (f, \langle \overline{\mathsf{f}} \rangle) : \mathbb{M} \to \mathbb{N}$ *a normal monoidal functor. If we have a invertible monoidal transformation*

$$\begin{tikzcd} \mathbb{M} \times \mathbb{M} \arrow[r, "\mathtt{f} \times \mathtt{f}"] \arrow[d, "\otimes_{\mathbb{M}}"'] & \mathbb{N} \times \mathbb{N} \arrow[d, "\otimes_{\mathbb{N}}"] \arrow[dl, phantom, "\overset{\varphi}{\Longleftarrow}"] \\ \mathbb{M} \arrow[r, "\mathtt{f}"'] & \mathbb{N} \end{tikzcd}$$

then $\mathtt{f}$ *is a strong monoidal functor.*

Proof: In fact, from φ, we define an invertible monoidal transformation $\psi : \varepsilon^{\mathcal{F}}_{\mathbb{N}} \cdot \underline{\mathcal{F}}(\mathtt{f}) \Longrightarrow \mathtt{f} \cdot \varepsilon^{\mathcal{F}}_{\mathbb{M}}$ as in Corollary 3.2 inductively as follows:

$$\begin{aligned} \psi_{()} &:= \langle \overline{\mathsf{f}} \rangle_{()} \\ \psi_{(x_1)} &:= \mathrm{id}_{f(x_1)} \\ \psi_{(x_1,\ldots,x_n,x_{n+1})} &:= \varphi_{(f(x_1)\otimes_{\mathbb{N}}\cdots\otimes_{\mathbb{N}} f(x_n), x_{n+1})} \cdot \left(\psi_{(x_1,\ldots,x_n)} \otimes_{\mathbb{M}} \mathrm{id}_{f(x_{n+1})} \right). \end{aligned}$$

■

As noted therein, Corollary 3.3 encompasses all the common parts of the examples presented in [Lac12], mentioned in the introduction. This includes the non-canonical isomorphism problem of preservation of coproducts, but it is not applicable to the case of preservation of more general conical colimits studied by Caccamo-Winskel [CW05].

3.1. Kock-Zöberlein pseudomonads.

There are situations even more simplified by Theorem 2.2. For instance, the situation of preservations of colimits can be studied as a very particular instance of:

Corollary 3.4. *Assume that* $\mathcal{T}$ *is a pseudomonad on* $\mathfrak{B}$ *such that* $\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$ *is locally fully faithful. Assume that* $\mathtt{f} = (f, \langle\overline{\mathtt{f}}\rangle) : \mathtt{y} \to \mathtt{z}$ *is a lax* $\mathcal{T}$*-morphism between the* $\mathcal{T}$*-pseudoalgebras* $\mathtt{y} = (Y, \mathfrak{alg}_{\mathtt{y}}, \underline{\overline{\mathtt{y}}}, \underline{\overline{\mathtt{y}}}_0)$ *and* $\mathtt{z} = (Z, \mathfrak{alg}_{\mathtt{z}}, \underline{\overline{\mathtt{z}}}, \underline{\overline{\mathtt{z}}}_0)$*. There is an invertible* 2*-cell*

$$\begin{tikzcd}
\mathcal{T}Y \arrow[r, "\mathcal{T}(f)"] \arrow[d, "\mathfrak{alg}_{\mathtt{y}}"'] & \mathcal{T}Z \arrow[d, "\mathfrak{alg}_{\mathtt{z}}"] \\
Y \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\psi}{\Longleftarrow}"] & Z
\end{tikzcd}$$

in $\mathfrak{B}$ *if and only if* $\mathtt{f}$ *is a* $\mathcal{T}$*-pseudomorphism.*

Proof: By hypothesis, every such a 2-cell gives an invertible $\mathcal{T}$-transformation $\varepsilon^{\mathcal{T}}_{\mathtt{z}} \cdot \underline{\mathcal{T}}(\mathtt{f}) \Longrightarrow \mathtt{f} \cdot \varepsilon^{\mathcal{T}}_{\mathtt{y}}$. Therefore the result follows from Theorem 2.2. ■

Cocompletion pseudomonads [PCW00] and, more generally, Kock-Zöberlein pseudomonads [Koc95; Mar97] satisfy, in particular, the hypothesis of the result above. This shows how Theorem 2.2 gives simple results for the examples of Kock-Zöberlein pseudomonads, encompassing in particular the cases of preservation of weighted colimits.

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