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SKEW RSK AND COINCIDENCE OF LITTLEWOOD-RICHARDSON COMMUTORS (EXTENDED ABSTRACT)

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ABSTRACT: We realise the Benkart-Sottile-Stroomer switching involution on ballot tableau pairs of partition shape as a recursive internal row insertion procedure. The key observation of such realisation is that internal row insertion operations are Knuth commutative. In fact, the Knuth class of a word encoding the sequence of internal row insertion operations preserves the P-tableau in the Sagan-Stanley skew RSK correspondence when the matrix word is empty. It confirms that all Littlewood-Richardson (LR) commutors, known up to date, are involutions and coincide as predicted by Pak and Vallejo.

KEYWORDS: ballot tableau, LR tableau, LR commutor, tableau-switching, internal insertion, skew RSK..
MATH. SUBJECT CLASSIFICATION (2000): 05A19 05E05 15E10.

1. Introduction

We give a recursive presentation of the switching involution [BSS96] on ballot (or Littlewood-Richardson) tableau pairs of partition shape based on the Sagan-Stanley *internal row insertion procedure* on skew-tableaux, [SS90] (see also [RSSW01]). Our key tool is to observe that the internal row insertion order in a skew-tableau is encoded by a word whose Knuth class preserves the P-tableau in the Sagan-Stanley skew RSK correspondence when the matrix word is empty.

Other realisations for the Littlewood-Richardson (LR) Benkart-Sottile - Stroomer (BSS) commutor are based on the Schützenberger involution [Lee98, HK06b, PV10]; or on tableau sliding, as in the Thomas-Yong infusion involution [TY16, TY08]. The latter is realised in [TY08] via Fomin's *jeu de taquin* growths [St98]. Lenart [Le08] realises the Henriques-Kamnitzer crystal commutor [HK06a] via van Leeuwen's *jeu taquin* [Lee98] generalising the Fomin's growth diagram presentation of *jeu de taquin* for Young tableaux. Danilov and Koshevoi [DK08] prove the coincidence of the LR switching commutor with the crystal commutor in type A [HK06b].

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On skew tableaux there are two types of insertion: external and internal [SS90] both of which are based on the usual Schensted insertion operation. The former proceeds very similarly to the usual Schensted insertion. The later has two main steps, firstly one chooses an inner corner (see Section 4) of the skew tableau T and bumps its entry, and, secondly, one inserts the bumped entry externally, in the row immediately below, in the usual manner. Eventually the bumping route lands at the end of some row of T where the last bumped entry settles and thus added at the end of that row of T. The internal row insertion operation adds one box, the vacant box, to the inner shape which in turn expands the outer shape in one box, the last bumped entry, but without contributing with a new element to the multiset of entries of T. The internal insertion procedure on a skew-tableau is an iteration of the internal row insertion operation and requires *a priori* in each iterative step an inner corner. Such information is encoded in the Sagan-Stanley skew RSK correspondence [SS90] by a second skew-tableau, sharing the inner border with the first. The internal row insertion order in the first skew tableau is provided by a word listing the row indices of the entries of the second skew-tableau in the standard order (see sections 2, 4). The internal insertion procedure is not independent of the order of the inner corners chosen. However, Theorem 4.3, Section 4, tells the Knuth class of an internal row insertion order word provides a set of internal insertion order words preserving the P-tableau in the Sagan-Stanley skew RSK correspondence when the matrix word is empty [SS90, RSSW01]. The Knuth commutativity of internal insertion operations is the key fact in the main Theorem 5.2, Section 5, to show that switching, on ballot tableau pairs of partition shape, can be rephrased in the language of internal row insertion.

The LR commutor based on internal (or reversal) row insertion operations was first introduced in [Aze99] and called ρ_3 (reversal version) by Pak and Vallejo [PV10] who conjectured the LR commutors coincidence. Its involutive nature is detailed in [AKT16] without making recourse of the switching involution. The paper is organised in five sections.

2. Partitions, tableaux and companion words

As usual, $[n]$ denotes the set of positive integers $\{1, \ldots, n\}$, $n \geq 1$. A partition (or normal shape) is a non negative integer vector $\mu = (\mu_1, \ldots, \mu_n)$ in weak decreasing order. A partition, usually denoted by lowercase Greek letters, is identified with the Young diagram of shape μ, in the English convention or matrix style. The box or cell of the Young diagram in row i and column j will be denoted (i,j) with $1 \leq j \leq \mu_i$. We write $|\mu| := \sum_{i=1}^{n} \mu_i$ for the number that μ partitions. The number of positive parts in this summand is the length $\ell(\mu) \leq n$ of μ. The unique partition of length 0 is the null partition (0), identified with $\emptyset$, the unique empty Young diagram.

For Young diagrams $\mu \subseteq \lambda$, the skew shape λ/μ is the set-theoretic difference $\lambda \setminus \mu$. A semistandard Young tableau (SSYT) T of shape λ/μ is a filling of the boxes of λ/μ over a finite set $[r]$ such that the labels of each row weakly increase from left to right and in each column strictly increase from top to bottom. The skew tableau T comprises an inner border, defined by the unfilled inner shape μ, a filled skew shape λ/μ, and an outer border, defined by the outer shape λ. The labels of T are often referred to elements or entries of T. If $\mu = \emptyset$, T is of partition (normal) shape. The unique empty skew tableau μ/μ, denoted $\emptyset_\mu$, is the Young diagram of shape μ. The *(row) reading word* $w(T)$ of T is the word read left to right across rows taken in turn from bottom to top. The content or weight of T is the vector which records the number of i's in T, with respect to the alphabet. We put $|T| := |\lambda| - |\mu| = |w(T)|$, the length $|w(T)|$ of $w(T)$. The set of SSYT's of shape λ/μ is denoted by $YT(\lambda/\mu)$. A tableau in $YT(\lambda/\mu)$ is said to be *standard*, SYT, if the entries are the numbers from 1 to $|\lambda| - |\mu|$, each occurring once.

The *standardisation* of $U \in YT(\lambda/\mu)$ is the standard tableau $stdU \in YT(\lambda/\mu)$ obtained by the standard renumbering of the entries of U: renumber the entries of T in numerical order from 1 to $|U|$, and, in case of equal entries, regard those to the left as smaller than those to the right. The *companion word* of $U \in YT(\lambda/\mu)$ is defined to be the word $R(U) = R(stdU) := u_{|U|} \cdots u_2 u_1$ listing the row indices of the entries of $stdU$, from the bigger to the smaller. Equivalently, to construct $u_{|U|} \cdots u_2 u_1$, for $p = 1, \ldots, |U|$, put $u_p = i$, if the number p is in the ith row of $stdU$. For example,

$$U = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 1 & 3 \\ \hline \ & \ & 2 & 4 \\ \cline{1-4} 1 & 2 & 3 \\ \cline{1-3} \end{array}\ , V = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 4 & 6 \\ \hline \ & \ & 5 & 7 \\ \cline{1-4} 4 & 5 & 6 \\ \cline{1-3} \end{array}\ , stdU = stdV = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 2 & 6 \\ \hline \ & \ & 4 & 7 \\ \cline{1-4} 1 & 3 & 5 \\ \cline{1-3} \end{array}\ ,$$

and $R(U) = R(V) = R(stdU) = R(stdV) = 2132313$. The map sending a standard tableau in $YT(\lambda/\mu)$ to its companion word is an injection.

A SSYT tableau is said to be *ballot* if the content of each suffix of its reading word is a partition. Such a word is called ballot or Yamanouchi. The ballot tableau Y_μ of shape and content μ is also called the Yamanouchi tableau of shape μ. For instance,

$$H = \begin{array}{|c|c|c|c|}\hline & & 1 & 2 \\ \hline & 1 & 3 \\ \cline{1-3} 1 & 2 \\ \cline{1-2}\end{array}\ , \quad \text{and } T = \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & 1 & 2 \\ \cline{1-3} 2 & 3 \\ \cline{1-2}\end{array}$$

are SSYT's of shape $(4,3,2)/(2,1,0)$, with reading words 121312 and 231211 respectively. The second is Yamanouchi but not the first, and so T is ballot but H is not.

3. LR rule and commutativity symmetry

Let n be a fixed positive integer and let $x = (x_1, x_2, \ldots, x_n)$ be a sequence of indeterminates. For each partition λ of length $\leq n$, there exists a Schur function $s_\lambda(x)$ which is a homogeneous symmetric polynomial in the x_k of total degree $|\lambda|$. The product of two Schur functions is explicitly given by the Littlewood-Richardson (LR) rule which amounts to finding how many SSYT's satisfy certain conditions.

Theorem 3.1. [LR34, Sch63, Tho74, Tho78] *The LR rule.* The coefficients appearing in the expansion of a product of Schur polynomials s_μ and s_ν, $s_\mu(x)\ s_\nu(x) = \sum_\lambda\ c^\lambda_{\mu\nu}\ s_\lambda(x)$, are given by

$$c^\lambda_{\mu\nu} = \#\{\text{ballot SSYT's of shape } \lambda/\mu \text{ and content } \nu\}.$$

The coefficients $c^\lambda_{\mu\nu}$ are known as Littlewood–Richardson (LR) coefficients, and the ballot SSYT's are also known as Littlewood-Richardson tableaux.

The Schubert structure coefficients of the product in $H^*(G(d,n))$, the cohomology of the Grassmannian $G(d,n)$, are also given by the LR rule $\sigma_\mu\sigma_\nu = \sum_{\lambda\subseteq d\times(n-d)} c^\lambda_{\mu\,\nu}\sigma_\lambda$. The Schur structure coefficients $c^\lambda_{\mu\,\nu}$ are also multiplicities in tensor products of $GL_n(\mathbb{C})$-representations and in induction products of $\mathfrak{S}_n$-representations.

3.1. Switching map and LR commutor. If $\mu \subseteq \lambda \subseteq \gamma$ are partitions, and V and U are SSYTs of shape γ/λ and λ/μ respectively, we say that V extends U, and we write $U \cup V$ for the object formed by gluing the outer border of U with the inner border of V. If U is of shape μ and V of shape λ/μ then $U \cup V$ is said to be of shape λ. A subtableau of a tableau U is a tableau obtained from U by discarding boxes. The subtableaux U_1 and U_2 decompose U if U_2 extends U_1 and $U = U_1 \cup U_2$. When U and V are both ballot tableaux with U of normal shape, *i.e*, $U = Y_\mu$, we have a ballot tableau pair of normal shape. The set of all ballot semistandard tableau pairs of partition shape with length $\leq n$ is denoted by $\mathcal{LR}^{(n)}$, $n \geq 1$.

Switching [BSS96] is an operation that takes a pair of tableaux $U \cup V$ and moves them through each other giving another such pair $V' \cup U'$ of the same shape, in a way that preserves Knuth equivalence, $V \equiv V'$ and $U \equiv U'$, and the shape of their union. The switching map on $U \cup V$ is processed through local moves, called *switches*, to interchanging vertically or horizontally adjacent letters u and v from U and V respectively. An interchanging of u with v is a *switch*, written u $\underset{s}{\leftrightarrow}$ v, provided the following filling conditions are simultaneously preserved in U and V: x' and x are letters from U (from V) and x' is north-west of x, $\begin{smallmatrix}\mathbf{x}' & \\ & \mathbf{x}\end{smallmatrix}$ or $\mathbf{x}'\ \mathbf{x}$, then $x \geq x'$; and within columns $\begin{smallmatrix}\mathbf{x}'\\ \mathbf{x}\end{smallmatrix}$, $x > x'$. An example of a sequence of switches

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&\color{red}{1}\\\hline 2&2&2\\\cline{1-3} \color{red}{1}&\color{red}{2}&\color{red}{2}\\\cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|}\hline 1&1&1&\color{red}{1}\\\hline 2&\color{red}{2}&\color{red}{2}\\\cline{1-3} \color{red}{1}&2&2\\\cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|}\hline 1&\color{red}{1}&1&1\\\hline 2&\color{red}{2}&\color{red}{2}\\\cline{1-3} \color{red}{1}&2&2\\\cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|}\hline \color{red}{1}&\color{red}{1}&1&1\\\hline 1&\color{red}{2}&\color{red}{2}\\\cline{1-3} 2&2&2\\\cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|}\hline \color{red}{1}&\color{red}{1}&\color{red}{2}&1\\\hline \color{red}{2}&1&1\\\cline{1-3} 2&2&2\\\cline{1-3}\end{array}\ .$$

The Switching Procedure [BSS96] *on tableau pairs*:

(1) Start with a tableau pair $U \cup V$;

(2) Switch integers from U with integers from V until it is no longer possible to do so.

This produces a new tableau pair $S \cup H$, $U \cup V \underset{s}{\longrightarrow} S \cup H$, where $U \equiv H$ and $V \equiv S$. (If $U \cup V$ has normal shape, U is the rectification of H and S the rectification of V). If subtableaux decompose U, then V can switch with U in stages. Similarly if subtableaux decompose V. The switching procedure calculates the involution map ρ_1 called *switching map*.

Tableau sliding [TY16, TY08] is a particular presentation of *switching* where an order of the switches has been imposed. The local moves are reduced to *jeu de taquin*. Tableau sliding is called *infusion* by Thomas and Yong in [TY16]. The map ρ_1 restricted to $\mathcal{LR}^{(n)}$, $n \geq 1$, is an involution, denoted $\rho_1^{(n)}$, and an LR commutor, that is, it bijectively exhibits $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$.

The map ρ_1 on ballot tableau pairs $Y_\mu \cup T \in \mathcal{LR}^{(n+1)}$, $1 \leq \ell(\mu) \leq n$, satisfies a recursion property. In a ballot tableau pair $Y_\mu \cup T$ the entries i in the ith row can not be switched upwards. Switching T with Y_μ in stages, row by row, eventually an entry $1 \leq d \leq n$ of Y_μ reaches the $(n+1)$th row of T, by lifting a letter in $[n]$. Then when T is switched with $Y_{(\mu_d,\ldots,\mu_n)}$ we stop. A bite of necessary notation. Any skew tableau $T \in YT(\lambda/\mu)$, $\ell(\lambda) \leq n$, may be trivially factored across any row i, $T = T' * T^{(i)}$, $0 \leq i \leq n$, with T' the subtableau of T of shape $(\lambda_{i+1}, \dots, \lambda_n)/(\mu_{i+1}, \dots, \mu_n)$ consisting of the last $n-i$ rows of T, and $T^{(i)}$ of shape $(\lambda_1, \dots, \lambda_i)/(\mu_1, \dots, \mu_i)$ consisting of the first i rows.

Theorem 3.2. Let $n \geq 1$ and $Y_\mu \cup T \in \mathcal{LR}^{(n+1)}$, with $\mu = (\mu_1, \dots, \mu_n, 0)$ a non zero partition, and $T \equiv Y_\nu$. Suppose the $(n+1)$th-row of T is the word $F(n+1)^{\nu_{n+1}}$ with F a non empty word in the alphabet $[n]$. Then, there exists $1 \leq d \leq n$, with $\mu_d > 0$, such that

$$Y_\mu \cup T \underset{s}{\rightarrow} Y_{(\mu_1,\dots,\mu_{d-1})} \cup S \cup Q, \tag{3.1}$$

where $S = \widehat{F}(n+1)^{\nu_{n+1}} * S^{(n)} \equiv Y_\nu$ with $\widehat{F}$ a strict subword of F, and $Q = D * Q^{(n)} \equiv Y_{(\mu_d,\dots,\mu_n)}$, over the alphabet $[d,n]$, has $Q^{(d-1)} = \emptyset$, and $D = d^{|D|}$ satisfies $|D| = |F| - |\widehat{F}| > 0$. Also, $\rho^{(n+1)}(Y_\mu \cup T) = \rho^{(n+1)}(Y_{(\mu_1,\dots,\mu_{d-1})} \cup S) \cup Q$.

Example 3.3. (a) $n = 3$, $\mu = (6,4,0,0)$, $\nu = (6,5,5,1)$, $F = 3333$, $d = 2$, $\widehat{F} = 33$, $D = 22$,

$$\begin{aligned} Y_\mu \cup T \;&=\; \begin{array}{ccccccccc} 1&1&1&1&1&1&1&1&1\\ 2&2&2&2&1&1&2&&\\ 1&2&2&2&2&3&&&\\ 3&3&3&3&4&&&& \end{array} \;\underset{s}{\rightarrow}\; \begin{array}{ccccccccc} 1&1&1&1&1&1&1&1&1\\ 1&1&1&2&2&2&2&&\\ 2&2&2&2&3&2&&&\\ 3&3&\mathbf{3}&\mathbf{3}&4&&&& \end{array} \;\underset{s}{\rightarrow}\; \begin{array}{ccccccccc} 1&1&1&1&1&1&1&1&1\\ 1&1&1&2&2&2&2&&\\ 2&2&\mathbf{3}&\mathbf{3}&3&2&&&\\ 3&3&4&2&2&&&& \end{array} \\ &=\; (Y_{(6)} \cup S) \cup Q. \end{aligned} \tag{3.2}$$

(b) $n = 2$, $\mu = (6,4,0)$, $F = 12222$, $d = 2$. Note that

$$(Y_{(6)} \cup S)^{(3)} \cup Q^{(3)} = [\widehat{G}\mathbf{33}3 * (Y_{(6)} \cup S)^{(2)}] \cup (X * Q^{(2)}),$$

with $\widehat{G} = 22$, $X = 2$. Then

$$(Y_\mu \cup T)^{(3)} \;=\; \begin{array}{ccccccccc} 1&1&1&1&1&1&1&1&1\\ 2&2&2&2&1&1&2&&\\ 1&2&2&2&2&3&&& \end{array} \;\underset{s}{\rightarrow}\; \begin{array}{ccccccccc} 1&1&1&1&1&1&1&1&1\\ 1&1&1&2&2&2&2&&\\ 2&2&2&2&3&2&&& \end{array}$$

$$\underset{s}{\rightarrow} \begin{array}{ccccccccc} 1&1&1&1&1&1&1&1&1\\ 1&1&1&2&2&2&2&&\\ 2&2&3&2&2&2&&& \end{array} \;=\; [\widehat{G}3 * (Y_{(6)} \cup S)^{(2)}] \cup (DX * Q^{(2)}). \tag{3.3}$$

4. Skew RSK and internal row insertion correspondence

Let $T \in YT(\lambda/\mu)$ with $\ell(\lambda) = n$. An inner corner of T is a cell (i,j) with $(i,j) \in \lambda/\mu$ but $(i-1,j), (i,j-1) \notin \lambda/\mu$. The *row internal insertion operator* ϕ_i is an operation over T [SS90], defined whenever $(i, \mu_i + 1)$ is an inner corner of T, which bumps the entry in the cell (i, μ_i+1) of T and then inserts the bumped element, using the usual row insertion rules, into row

$i+1$ of T. The insertion then continues in a normal fashion, ending with an element settling at the end of some row $\leq n+1$. It is also allowed an empty cell (i, μ_i+1) to be an inner corner providing $(i-1, \mu_i+1)$, $(i, \mu_i) \notin \lambda/\mu$, $1 \leq i \leq n+1$. In this case ϕ_i just adjoins the blank cell (i, μ_i+1) to the inner shape of T. Otherwise, it changes the skew-shape by adding one blank box at the end of row i of the inner shape μ, and one filled box to the outer shape λ. The new tableau $\phi_i T$ is Knuth equivalent to T. If T is a ballot tableau, $\phi_i T$ is ballot as well. Whenever the internal row insertion operator ϕ_i is defined on T, it can be easily extended to the tableau pair $Y \cup T$ with $Y = Y_\mu$, by putting

$$\bar{\phi}_i(Y \cup T) := \begin{cases} Y_{(\mu_1,\ldots,\mu_i+1,\ldots,\mu_n)} \cup \phi_i(T), \text{ if } 1 \leq i \leq n, \\ Y_{(\mu_1,\ldots,\mu_n,1)} \cup \phi_{n+1}(T), \quad \text{if } i = n+1 \text{ and } \mu_n > 0. \end{cases}$$

If $Y \cup T \in \mathcal{LR}^{(n)}$, $\bar{\phi}_i(Y \cup T) \in \mathcal{LR}^{(n+1)}$.

The bijection below is a special case of the skew RSK correspondence in [SS90], Theorem 6.11, when the matrix word $\pi = \emptyset$. The skew-insertion procedure is then reduced to the internal row insertion. It calculates a bijection between pairs of tableaux (T, U) sharing an inner border and pairs of tableaux (P, Q) sharing an outer border. Also the outer border of T equals the inner border of Q, and the outer border of U equals the inner border of P. (See also [RSSW01] for more details.)

Theorem 4.1. [SS90, RSSW01] (Internal row insertion correspondence.) Fix partitions $\mu \subseteq \lambda, \beta$. There is a bijection,

$$\begin{aligned} YT(\lambda/\mu) \times YT(\beta/\mu) \quad &\longrightarrow \quad \bigcup_{\substack{\gamma \\ |\gamma| = |\lambda| + |\beta| - |\mu|}} YT(\gamma/\beta) \times YT(\gamma/\lambda) \\ (T, U) \quad &\longrightarrow \quad (P, Q), \end{aligned} \tag{4.1}$$

where the P-tableau $P \equiv T$ is the internal row insertion of T whose sequence of inner corners containing the entries of T to be internally inserted is dictated by the entries of U in the standard order. The Q-tableau $Q \equiv U$ is the recording tableau of the internal insertion of T.

Given $T \in YT(\lambda/\mu)$, an *internal insertion order word* of T is the companion word of any skew tableau U with inner border μ. If $U \in YT(\beta/\mu)$ and $R(U) = R(std(U)) = u_{|U|} u_{|U|-1} \cdots u_2 u_1$, let $\phi_{R(U)} = \phi_{R(std(U))} := \phi_{u_{|U|}} \phi_{u_{|U|-1}} \cdots \phi_{u_2} \phi_{u_1}$ ($\phi_\emptyset = id$.) Then $\phi_{R(U)} T = \phi_{R(std(U))} T = P$ in (4.1).

Knuth relations completely characterize the words having the same P tableau in the usual RSK. It is easily checked that if $U \in YT(\beta/\mu)$ and $R(U) = ijk$, $1 \le i \le k < j \le \ell(\beta)$, then there exists a standard tableau $V \in YT(\beta/\mu)$ with $R(V) = jik \equiv R(U)$. (Similarly if $R(U) = kji$, $1 \le i < k \le j \le \ell(\beta)$.) Next lemma and theorem asserts that internal insertion words w.r.t. a fixed inner border μ are closed under Knuth relations, and Knuth equivalence of internal insertion order words means Knuth commutativity of the corresponding internal insertion operators. Our theorem below is a contribution to question (3) in [SS90], Section 9.

Lemma 4.2. Let $n-1 \le \ell(\mu) \le n$. Then $\phi_i\phi_n\phi_k T = \phi_n\phi_i\phi_k T$, $1 \le i \le k < n$, and $\phi_k\phi_i\phi_n T = \phi_k\phi_n\phi_i T$, $1 \le i < k \le n$. In addition, if $\ell(\mu) = n$, $\phi_i\phi_{n+1}\phi_k T = \phi_{n+1}\ \phi_i\phi_k T$, $1 \le i \le k < n+1$, and $\phi_k\phi_i\phi_{n+1} T = \phi_k\ \phi_{n+1}\phi_i T$, $1 \le i < k \le n+1$.

Theorem 4.3. (Knuth commutativity of row internal insertion operators.) Let u be an internal insertion order word of T and $v \equiv u$. Then

(a) v is an internal insertion order word of T.

(b) $\phi_u T = \phi_v T$.

Proof: (a) Let $T \in YT(\lambda/\mu)$ and $U \in YT(\beta/\mu)$ standard such that $R(U) = w_2ijkw_1$ w_1, w_2 words, and $1 \le i \le k < j \le \ell(\beta)$. Let $v = w_2jikw_1 \equiv R(U)$. Decompose $U = U_1 \cup M \cup U_2$ such that $R(U_1) = w_1$, $R(M) = ijk$ and $R(U_2) = w_2$. Let M' be a skew standard tableau of the same shape as M such that $R(M') = jik$. Then the standard tableau $V := U_1 \cup M' \cup U_2 \in YT(\beta/\mu)$ has $R(V) = w_2jikw_1$. (Similar for $R(M) = kji$, $1 \le i < k \le j$.)

(b) It is enough to show that $\bar\phi_k\bar\phi_i\bar\phi_j(Y \cup T) = \bar\phi_k\bar\phi_j\bar\phi_i(Y \cup T)$, $1 \le i \le k < j \le n$. (The other case follows similarly.) Let $Y \cup T = (Y' \cup T') * (Y_{(\mu_1,\ldots,\mu_j)} \cup T^{(j)})$. Lemma 4.2 guarantees that $\bar\phi_k\bar\phi_i\bar\phi_j(Y_{(\mu_1,\ldots,\mu_j)} \cup T^{(j)}) = \bar\phi_k\bar\phi_j\bar\phi_i(Y_{(\mu_1,\ldots,\mu_j)} \cup T^{(j)}), 1 \le i \le k < j$.

Let F be the restriction of $\bar\phi_k\bar\phi_i\bar\phi_j(Y_{(\mu_1,\ldots,\mu_j)} \cup T^{(j)}) = \bar\phi_k\bar\phi_j\bar\phi_i(Y_{(\mu_1,\ldots,\mu_j)} \cup T^{(j)})$ to the rows strictly below row j. If w and w' are the words consisting of the elements of $T^{(j)}$ successively bumped out from the j-th row under the action of $\phi_k\phi_i\phi_j$ and $\phi_k\phi_j\phi_i$ respectively, on T, then F is the external insertion of w and w' and $w \equiv w' \equiv F$. The action of $\bar\phi_k\bar\phi_i\bar\phi_j$ and $\bar\phi_k\bar\phi_j\bar\phi_i$ on $Y \cup T$ inserts externally the words w and w' respectively, into T' which is the same as inserting F. Therefore, $\bar\phi_k\bar\phi_i\bar\phi_j(Y \cup T) = \bar\phi_k\bar\phi_j\bar\phi_i(Y \cup T)$. $\square$

Let $T = \begin{array}{|c|c|c|}\hline & 1 & 3 \\ \hline 2 & 3 & \\ \hline\end{array}$, $U = \begin{array}{|c|c|c|c|}\hline & 1 & 3 & 5 \\ \hline 2 & 4 & & \\ \hline\end{array}$, $U' = \begin{array}{|c|c|c|c|}\hline & 1 & 3 & 4 \\ \hline 2 & 5 & & \\ \hline\end{array}$ and $R(U) = 12121 \equiv R(U') = 21121$. Then $R(U) \equiv R(U')$ implies Knuth commutativity of

$$\phi_{12121}T = \phi_{21121}T = \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline & & 3 & \\ \hline 1 & 3 & & \\ \hline 2 & & & \\ \hline\end{array} .$$

This lemma recalls the bumping route behaviour and is instrumental in the proof of the main theorem.

Lemma 4.4. [Fu97] *(Internal row insertion bumping routes.)* Consider $1 \leq i \leq j \leq n$. Let R_j and R'_i be the pair of bumping routes of $\bar{\phi}_j$ on $Y \cup T$ and $\bar{\phi}_i$ on $\bar{\phi}_j(Y \cup T)$ respectively; and let R_i and R'_j be the pair of bumping routes of $\bar{\phi}_i$ on $Y \cup T$ and $\bar{\phi}_j$ on $\bar{\phi}_i(Y \cup T)$ respectively. Let B and B' be the corresponding pair of new boxes. Then it holds:

(a) R_j is strictly left of R'_i and B is strictly left of and weakly below B', $\begin{smallmatrix} B & B' \end{smallmatrix}$ or $\begin{smallmatrix} & B' \\ B & \end{smallmatrix}$;

(b) R'_j is weakly left of R_i and B' is weakly left of and strictly below B, $\begin{smallmatrix} B \\ B' \end{smallmatrix}$ or $\begin{smallmatrix} & B \\ B' & \end{smallmatrix}$.

In particular, R'_j goes always strictly below the bottom box B of R_i either by $\bar{\phi}_j$-bumping the element in B, which happens in the case of $\begin{smallmatrix} B \\ B' \end{smallmatrix}$, or by passing strictly to the left of B. Henceforth, if x is $\bar{\phi}_i$-bumped and y is $\bar{\phi}_j$-bumped from the same row then $y < x$. Moreover, if B was created in the $(n+1)$th row then one has $\begin{smallmatrix} B \\ B' \end{smallmatrix}$ and the last $\bar{\phi}_i$-bumped element resting in B is $\bar{\phi}_j$-bumped out to be settled in B' and is strictly bigger than the element $\bar{\phi}_j$-inserted in B.

5. An internal row insertion LR commutor

Given $Y_\mu \cup T \in \mathcal{LR}^{(n)}$, $n \geq 1$, with T a ballot tableau of shape λ/μ and weight ν, define the partition $\widehat{\nu} = (\widehat{\nu}_1, \ldots, \widehat{\nu}_{n-1}, \widehat{\nu}_n = \nu_n) \subseteq \nu$ with $\widehat{\nu}_i$ the number of i's in row i of T, for $i = 1, \ldots, n$. Let G_ν be the Gelfand-Tsetlin pattern of T of shape ν. Let V_i be the ith row of T restricted to the alphabet $[i-1]$, and note that $|V_i| = \lambda_i - \mu_i - \hat{\nu}_i$, for $i = 1, \ldots, n$. Then $R(G_\nu) = V_n n^{\widehat{\nu}_n} \cdots V_3 3^{\widehat{\nu}_3} V_2 2^{\widehat{\nu}_2} 1^{\widehat{\nu}_1}$ ($V_1 = \emptyset$), and $\phi_{R(G_\nu)}(\emptyset) = \phi_{V_n}\phi_n^{\widehat{\nu}_n} \cdots \phi_{V_3}\phi_3^{\widehat{\nu}_3}$ $\phi_{V_2}\phi_2^{\widehat{\nu}_2}\phi_1^{\widehat{\nu}_1}(\emptyset) = \emptyset_\nu$, the Young diagram of shape ν. We use G_ν for the internal insertion order in $\emptyset$, and the parts μ_i of μ to be added to row i, each time $\phi_{V_i}\phi_i^{\widehat{\nu}_i}$ is applied, to construct an LR commutor as we next explain. For $i = 1, \ldots, n$, let χ_i be the operator to be iterated μ_i times over

$\phi_{V_i}\phi_i^{\widehat{\nu}_i}(\chi_{i-1}^{\mu_{i-1}}\phi_{V_{i-1}}\phi_{i-1}^{\widehat{\nu}_i})\cdots(\chi_2^{\mu_2}\phi_{V_2}\phi_2^{\widehat{\nu}_2})\ \chi_1^{\mu_1}\phi_1^{\widehat{\nu}_1}(\emptyset)$. Each iteration adds one i at the end of the ith row. Our procedure produces

$$(\chi_n^{\mu_n}\phi_{V_n}\phi_n^{\widehat{\nu}_n})\cdots(\chi_3^{\mu_3}\phi_{V_3}\phi_3^{\widehat{\nu}_3})(\chi_2^{\mu_2}\phi_{V_2}\phi_2^{\widehat{\nu}_2})(\chi_1^{\mu_1}\phi_1^{\widehat{\nu}_1})(\emptyset)=\emptyset_\nu\cup H,$$

where H is a ballot tableau of shape λ/ν and content μ.

Example 5.1. Let $\emptyset_\mu\cup T=\begin{array}{cccccc} \ & \ & \ & \ & 1 & 1\\ \ & \ & \ & 1 & 2 & \\ \ & \ & 1 & 2 & 3 & \\ \ & 2 & 3 & 4 & & \\ 2 & 3 & 4 & & & \end{array}$ of shape $(65543)/(43210)$ and weight $\nu=(44320)$, $\widehat{\nu}=(21110)$, $G_\nu=\begin{array}{cccc} 1 & 1 & 2 & 3\\ 2 & 3 & 4 & 5\\ 3 & 4 & 5 & \\ 4 & 5 & & \end{array}$ and $R(G_\nu)=234\,234\ 123$ $12\,11$. Then $\phi_{R(G_\nu)}=\phi_{234}\phi_{234}\phi_{123}\phi_{12}\phi_{11}$ and $\phi_{234}\chi_4\phi_{234}\chi_3^2\phi_{123}\chi_2^3\phi_{12}\ \chi_1^4\phi_1^2(\emptyset)$ produces:

$$\emptyset\xrightarrow[\chi_1^4\phi_1^2]{}\begin{array}{cccccc} \ & \ & 1 & 1 & 1 & 1\end{array}\xrightarrow[\chi_2^3\phi_1\phi_2]{}\begin{array}{cccccc} \ & \ & \ & 1 & 1 & 1\\ \ & 1 & 2 & 2 & 2 & \end{array}\xrightarrow[\chi_3^2\phi_1\phi_2\phi_3]{}\begin{array}{cccccc} \ & \ & \ & \ & 1 & 1\\ \ & \ & 1 & 2 & 2 & \\ \ & 1 & 2 & 3 & 3 & \end{array}$$

$$\xrightarrow[\chi_4\phi_2\phi_3\phi_4]{}\begin{array}{cccccc} \ & \ & \ & \ & 1 & 1\\ \ & \ & \ & 2 & 2 & \\ \ & \ & 1 & 3 & 3 & \\ \ & 1 & 2 & 4 & & \end{array}\xrightarrow[\phi_2\phi_3\phi_4]{}\begin{array}{cccccc} \ & \ & \ & \ & 1 & 1\\ \ & \ & \ & \ & 2 & \\ \ & \ & \ & 2 & 3 & \\ \ & \ & 1 & 3 & & \\ 1 & 2 & 4 & & & \end{array}=\emptyset_\nu\cup H\ \text{ of weight } \mu.$$

.

We now use the extended operators $\bar{\phi}$. For each $i=1,\ldots,n$, the action of $\bar{\phi}_i^{\hat{\nu}_i}$ on $(\chi_{i-1}^{\mu_{i-1}}\bar{\phi}_{V_{i-1}}\bar{\phi}_{i-1}^{\hat{\nu}_{i-1}})\cdots(\chi_2^{\mu_2}\bar{\phi}_{V_2}\bar{\phi}_2^{\hat{\nu}_2})\chi_1^{\mu_1}\bar{\phi}_1^{\hat{\nu}_1}(\emptyset)$ contributes with $\hat{\nu}_i$ i's to the ith row of the inner shape. Put $\rho_1^{(0)}(\emptyset):=\emptyset$. This allows a recursive presentation of the switching map $\rho_1^{(n)}$ in $\mathcal{LR}^{(n)}$, $n\geq 1$:

$$\begin{aligned}\rho_1^{(n)}(Y_\mu\cup T)&=(\chi_n^{\mu_n}\bar{\phi}_{V_n}\bar{\phi}_n^{\widehat{\nu}_n})\cdots(\chi_2^{\mu_2}\bar{\phi}_{V_2}\bar{\phi}_2^{\widehat{\nu}_2})(\chi_1^{\mu_1}\bar{\phi}_1^{\widehat{\nu}_1})(\emptyset)\\&=(\chi_n^{\mu_n}\bar{\phi}_{V_n}\bar{\phi}_n^{\widehat{\nu}_n})\rho^{(n-1)}[(Y_\mu\cup T)^{(n-1)}].\end{aligned}$$

In our example above $\rho_1^{(5)}(Y_\mu\cup T)=\bar{\phi}_{234}\chi_4^{\mu_4}\bar{\phi}_{23}\bar{\phi}_4\chi_3^{\mu_3}\bar{\phi}_{12}\bar{\phi}_3\chi_2^{\mu_2}\bar{\phi}_1\bar{\phi}_2\chi_1^{\mu_1}\bar{\phi}_1^2(\emptyset)$. We now check $\rho_1^{(i)}(Y_\mu\cup T)=(\chi_i^{\mu_i}\bar{\phi}_{V_i}\bar{\phi}_i^{\widehat{\nu}_i})\ \rho^{(i-1)}[(Y_\mu\cup T)^{(i-1)}]$, for $i=1,\ldots,5$:

$$(Y\cup T)^{(1)}=\ \begin{array}{cccccc}1&1&1&1&1&1\end{array}\ \xrightarrow[s]{}\ \begin{array}{cccccc}1&1&1&1&1&1\end{array}\ =\rho_1^{(1)}[(Y\cup T)^{(1)}]$$

$$\emptyset\xrightarrow[\bar{\phi}_1^2]{}\begin{array}{cc}1&1\end{array}\xrightarrow[\chi_1^{\mu_1}]{}\begin{array}{cccccc}1&1&1&1&1&1\end{array}\ =\chi_1^{\mu_1}\bar{\phi}_1^{\widehat{\nu}_1}\rho_1^{(0)}(\emptyset)=\rho_1^{(1)}[(Y\cup T)^{(1)}];$$

$$(Y\cup T)^{(2)} = \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&2&1&2& \end{array} \xrightarrow[s]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&1&2&2&2& \end{array} = \rho_1^{(2)}[(Y\cup T)^{(2)}]$$

$$\rho_1^{(1)}[(Y\cup T)^{(1)}] = \begin{array}{cccccc} 1&1&1&1&1&1 \end{array} \xrightarrow[\bar\phi_2]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&&&&& \end{array} \xrightarrow[\bar\phi_1]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&1&&&& \end{array}$$

$$\xrightarrow[\chi_2^{\mu_2}]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&1&2&2&2& \end{array} = \chi_2^{\mu_2}\bar\phi_1\bar\phi_2\rho_1^{(1)}[(Y\cup T)^{(1)}] = \rho_1^{(2)}[(Y\cup T)^{(2)}];$$

$$(Y\cup T)^{(3)} = \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&2&1&2&\\ 3&3&1&2&3& \end{array} \xrightarrow[s]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&1&2&2&\\ 3&1&2&3&3& \end{array} = \rho_1^{(3)}[(Y\cup T)^{(3)}]$$

$$\rho_1^{(2)}[(Y\cup T)^{(2)}] \xrightarrow[\bar\phi_3]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&1&2&2&2&\\ 3&&&&& \end{array} \xrightarrow[\bar\phi_2]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&2&2&2&\\ 3&1&&&& \end{array} \xrightarrow[\bar\phi_1]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&1&2&2&\\ 3&1&2&&& \end{array}$$

$$\xrightarrow[\chi_3^{\mu_3}]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&1&2&2&\\ 3&1&2&3&3& \end{array} = \chi^{\mu_3}\bar\phi_1\bar\phi_2\bar\phi_3\rho_1^{(2)}(Y\cup T)^{(2)}$$

$$= \chi^{\mu_3}\bar\phi_1\bar\phi_2\bar\phi_3\chi_2^{\mu_2}\bar\phi_1\bar\phi_2\rho_1^{(1)}[(Y\cup T)^{(1)}]$$

$$= \rho_1^{(3)}[(Y\cup T)^{(3)}];$$

$$\rho_1^{(3)}[(Y\cup T)^{(3)}] \xrightarrow[\bar\phi_4]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&1&2&2&\\ 3&1&2&3&3&\\ 4&&&&& \end{array} \xrightarrow[\bar\phi_3]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&1&2&2&\\ 3&3&2&3&3&\\ 4&1&&&& \end{array} \xrightarrow[\bar\phi_2]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&2&2&2&\\ 3&3&1&3&3&\\ 4&1&2&&& \end{array}$$

$$\xrightarrow[\chi_4^{\mu_4}]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&2&2&2&\\ 3&3&1&3&3&\\ 4&1&2&4&& \end{array} = \chi^{\mu_4}\bar\phi_2\bar\phi_3\bar\phi_4\rho_1^{(3)}[(Y\cup T)^{(3)}] = \rho_1^{(4)}[(Y\cup T)^{(4)}];$$

$$\rho_1^{(4)}[(Y\cup T)^{(4)}] \xrightarrow[\bar\phi_4]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&2&2&2&\\ 3&3&1&3&3&\\ 4&4&2&4&&\\ 1&&&&& \end{array} \xrightarrow[\bar\phi_3]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&2&2&2&\\ 3&3&3&3&3&\\ 4&4&1&4&&\\ 1&2&&&& \end{array} \xrightarrow[\bar\phi_2]{} \begin{array}{cccccc} 1&1&1&1&1&1\\ 2&2&2&2&2&\\ 3&3&3&2&3&\\ 4&4&1&3&&\\ 1&2&4&&& \end{array}$$

$$= \bar\phi_{234}\rho_1^{(4)}[(Y\cup T)^{(4)}] = \rho_1^{(5)}[Y\cup T] = Y_\nu \cup H.$$

The switching map on ballot tableau pairs of normal shape can now be translated to the language of internal row insertion operations.

Theorem 5.2. *(Main Theorem)* Let $n \geq 1$ and $Y\cup T \in \mathcal{LR}^{(n)}$ with $Y = Y_\mu$ and T a ballot tableau of shape λ/μ and weight ν. Consider the nth row word of T where V_n is the row subword restricted to the entries in $[n-1]$, and ν_n is the number of entries equal to n. Then

$$\rho_1^{(n)}(Y \cup T) = \chi_n^{\mu_n} \bar{\phi}_{V_n} \bar{\phi}_n^{\nu_n} \rho_1^{(n-1)}[(Y \cup T)^{(n-1)}]. \tag{5.1}$$

In particular, all $|V_n|$ $\bar{\phi}_{V_n}$-bumping routes are pairwise disjoint and terminate in the nth row.

Proof: Reduce to the case $\mu_n = 0$. We handle by induction on $n \geq 1$ and $|V_n| \geq 0$. For $n = 1$ is trivial. Assume the statement true for n and prove for $Y \cup T \in \mathcal{LR}^{(n+1)}$ that $\rho_1^{(n+1)}(Y \cup T) = \bar{\phi}_{V_{n+1}} \bar{\phi}_{n+1}^{\nu_{n+1}} \rho_1^{(n)}[(Y \cup T)^{(n)}]$. Detach the $(n+1)$th row in $Y \cup T = F(n+1)^{\nu_{n+1}} * (Y \cup T)^{(n)}$, $F := V_{n+1}$ a row word in the alphabet $[n]$. Put $v := |F|$. If $v = 0$, the switching reduces to $(Y \cup T)^{(n)}$, and $\rho_1^{(n+1)}(Y \cup T) = \bar{\phi}_{n+1}^{\nu_{n+1}} \rho_1^{(n)}[(Y \cup T)^{(n)}]$. Let $v \geq 1$. By Theorem 3.2,

$$Y \cup T \underset{s}{\rightarrow} [Y' \cup S] \cup [D * Q^{(n)}] = [\widehat{F}(n+1)^{\nu_{n+1}} * (Y' \cup S)^{(n)}] \cup [D * Q^{(n)}],$$

with $Y' = Y_{(\mu_1,\ldots,\mu_{d-1})}$, $\mu_d > 0$, $S \equiv T$, $D * Q^{(n)} \equiv Y_{(\mu_d,\ldots,\mu_{n-1})}$, $\widehat{F}$ a strict subword of F and $D = d^{|D|}$, $|D| = v - |\widehat{F}| > 0$. By induction on v, $\rho_1^{(n+1)}(Y' \cup S) = \bar{\phi}_{\widehat{F}} \bar{\phi}_{n+1}^{\nu_{n+1}} \rho_1^{(n)}[(Y' \cup S)^{(n)}]$, and thereby

$$\rho_1^{(n+1)}(Y \cup T) = \bar{\phi}_{\widehat{F}} \bar{\phi}_{n+1}^{\nu_{n+1}} \rho_1^{(n)}[(Y' \cup S)^{(n)}] \cup (D * Q^{(n)}), \tag{5.2}$$

where all $\bar{\phi}_{\widehat{F}}$-*bumping routes* are pairwise disjoint and terminate in the $(n+1)$th row.

Detach the nth row in $(Y' \cup S)^{(n)} = Gn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}$, where $G = ABC$ is a row word in the alphabet $[n]$, with AB in the alphabet $[n-1]$, $|A| = |\widehat{F}|$, and $C = n^{\epsilon}$ consisting of the $\epsilon \geq 0$ lifted letters n from the row word F of T. Consider the nth and $(n+1)$th rows of $(Y' \cup S) \cup (D * Q^{(n)})$:

$$[\widehat{F}(n+1)^{\nu_{n+1}} \cup D] * [Gn^{\hat{\nu}_n} \cup X] = \begin{array}{|c|c|c|c|c|c|c|} \hline A & B & C & n^{\star} & \cdots & n^{\star} & X \\ \hline \widehat{F} & (n+1)^{\star} & \cdots & (n+1)^{\star} & D & \multicolumn{2}{c}{} \\ \cline{1-5} \end{array}$$

($\star$ indicates some appropriate multiplicity). Switch $D = D_2 D_4 \cdots\ D_k$ back to the nth row where $F = F_1 F_2 F_3 \cdots F_k$, $\widehat{F} = F_1 F_3 \cdots F_{k-1}$, $|F_2| = |D_2|, \ldots,$

$|F_k| = |D_k|$, and $G = G_1F_2G_3F_4 \cdots\ G_{k-1}F_kG_{k+1}$ (G_{k+1} empty if $\epsilon > 0$):

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
G_1 & F_2 & G_3 & F_4 & \cdots & G_{k-1} & F_k & G_{k+1} & n^\star & \cdots & n^\star & X \\
\hline
F_1 & D_2 & F_3 & D_4 & \cdots & F_{k-1} & D_k & (n+1)^\star & \cdots & (n+1)^\star \\
\cline{1-10}
\end{array} \quad \overset{s}{\leftrightarrow}$$

$$\overset{s}{\leftrightarrow} \quad \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
G_1 & D_2 & G_3 & D_4 & \cdots & G_{k-1} & D_k & G_{k+1} & n^\star & \cdots & n^\star & X \\
\hline
F_1 & F_2 & F_3 & F_4 & \cdots & F_{k-1} & F_k & (n+1)^\star & \cdots & (n+1)^\star \\
\cline{1-10}
\end{array} .$$

We get in the nth row $\widehat{G} := G_1G_3\ \ldots G_{k-1}G_{k+1}$ in the alphabet $[n-1]$, a strict subword of G with $|G| = |\widehat{G}| + |D|$, such that

$$\widehat{F}(n+1)^{\nu_{n+1}}Gn^{\widehat{\nu}_n} \equiv F(n+1)^{\nu_{n+1}}\widehat{G}n^{\widehat{\nu}_n} \tag{5.3}$$

are words *Knuth equivalent* [Fu97]. Thus

$$(Y \cup T)^{(n)} \underset{s}{\to} [\widehat{G}n^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] \cup (DX * Q^{(n-1)}). \tag{5.4}$$

By induction on n,

$$\rho_1^{(n)}[(Y' \cup S)^{(n)}] \;=\; \rho_1^{(n)}[Gn^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] = \bar{\phi}_G\bar{\phi}_n^{\widehat{\nu}_n}\rho_1^{(n-1)}[(Y' \cup S)^{(n-1)}], \tag{5.5}$$

$$\rho_1^{(n)}[\widehat{G}n^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] = \bar{\phi}_{\widehat{G}}\bar{\phi}_n^{\widehat{\nu}_n}\rho_1^{(n-1)}[(Y' \cup S)^{(n-1)}], \tag{5.6}$$

where all $\bar{\phi}_G$ and $\bar{\phi}_{\widehat{G}}$-*bumping routes* are pairwise disjoint and terminate in the nth row. Therefore, since, from (5.4), one has

$$\rho_1^{(n)}[(Y \cup T)^{(n)}] = \rho_1^{(n)}[\widehat{G}n^{\hat{\nu}_n} * (Y' \cup S)^{(n-1)}] \cup (DX * Q^{(n-1)}), \tag{5.7}$$

it follows

$$\begin{aligned}
\rho_1^{(n+1)}(Y\cup T) &= \bar\phi_{\widehat F}\bar\phi_{n+1}^{\nu_{n+1}}\rho_1^{(n)}[(Y'\cup S)^{(n)}]\cup(D*Q^{(n)}),\ (5.2)\\
&= \bar\phi_{\widehat F}\bar\phi_{n+1}^{\nu_{n+1}}\bar\phi_G\bar\phi_n^{\widehat\nu_n}\rho_1^{(n-1)}[(Y'\cup S)^{(n-1)}]\cup(D*Q^{(n)}), (5.5)\\
&= \bar\phi_F\bar\phi_{n+1}^{\nu_{n+1}}\bar\phi_{\widehat G}\bar\phi_n^{\widehat\nu_n}\rho_1^{(n-1)}[(Y'\cup S)^{(n-1)}]\cup(D*X*Q^{(n-1)}),\\
&\quad (5.3),\ \text{Theorem 4.3}\\
&= \bar\phi_F\bar\phi_{n+1}^{\nu_{n+1}}\rho_1^{(n)}[\widehat G n^{\hat\nu_n}*(Y'\cup S)^{(n-1)}]\cup(D*X*Q^{(n-1)}),\\
&\quad (5.6)\\
&= \bar\phi_F\bar\phi_{n+1}^{\nu_{n+1}}\{\rho_1^{(n)}[\widehat G n^{\hat\nu_n}*(Y'\cup S)^{(n-1)}]\cup(DX*Q^{(n-1)})\},\\
&\quad \text{Lemma 4.4, } (a)\\
&= \bar\phi_F\bar\phi_{n+1}^{\nu_{n+1}}\rho_1^{(n)}[(Y\cup T)^{(n)}], (5.7).
\end{aligned}$$

The number $|G|=|\widehat G|+|D|$ $(|F|+|\widehat G|=|G|+|\widehat F|=|\widehat F|+|\widehat G|+|D|)$ of bumping routes landing in the nth row and the number $|\widehat F|$ of bumping routes landing in the $(n+1)$th row is the same for $\bar\phi_{\widehat F(n+1)^{\nu_{n+1}}Gn^{\widehat\nu_n}}$ and $\bar\phi_{F(n+1)^{\nu_{n+1}}\widehat Gn^{\widehat\nu_n}}$ $(\bar\phi_F\bar\phi_{n+1}^{\nu_{n+1}}\bar\phi_{\widehat G}\bar\phi_n^{\widehat\nu_n}=\bar\phi_{\widehat F}\bar\phi_{n+1}^{\nu_{n+1}}\bar\phi_G\bar\phi_n^{\widehat\nu_n}$, (5.3)) when acting on $\rho_1^{(n-1)}[(Y'\cup S)^{(n-1)}]$. Put $Y'\cup R:=\widehat Gn^{\hat\nu_n}*(Y'\cup S)^{(n-1)}$. Thus, in the action of $\bar\phi_F$ over $\bar\phi_{n+1}^{\nu_{n+1}}\rho_1^{(n)}[Y'\cup R]$, *$|D|$ of the $v=|F|$ pairwise disjoint bumping routes will still land and settle in the nth row, adding $|D|$ new entries at the end of the nth row of $\bar\phi_{n+1}^{\nu_{n+1}}\rho_1^{(n)}[Y'\cup R]$, while $|\widehat F|=v-|D|$ of them will land in the $(n+1)$th row.* Recall that F is a row word, from Lemma 4.4, (a), the v bumping routes are pairwise disjoint and, more importantly, the $|D|$ bumping routes settling in the nth row are necessarily the last to be executed. This means that if we attach D at the end of the nth row $\rho_1^{(n)}[Y'\cup R]$, the rightmost $|D|$ bumping routes of $\bar\phi_F$ when landing in the nth row will meet the entire row D and bumps it out to the $(n+1)$th row. $\square$

Example 5.3. In Example 3.3, (3.2), one has $F=3333$, $\widehat F=33$, $G=22\mathbf{33}$, $\widehat G=22$, $A=22$, $B=\emptyset$, $C=\mathbf{33}$, $D=2^2$, $\nu_4=1=\widehat\nu_3$, $\widehat F4^1G3^1=33422\mathbf{33}3\equiv F4^1\widehat G3^1=33\mathbf{33}4223$:

$$
\begin{aligned}
\rho_1^{(4)}(Y\cup T) &= \rho_1^{(4)}(Y_{(6)}\cup S)\cup[2^2 * Q^{(3)}] = \begin{array}{lllllllll} 1&1&1&1&1&1&1&1&1\\ 2&2&2&2&2&1&&&\\ 3&3&3&3&3&&&&\\ 4&1&1&&&&&& \end{array} \cup [2^2 * Q^{(3)}]\\
&= \begin{array}{lllllllll} 1&1&1&1&1&1&1&1&1\\ 2&2&2&2&2&1&2&&\\ 3&3&3&3&3&2&&&\\ 4&1&1&2&2&&&& \end{array} = \bar\phi_{33}\bar\phi_4\rho_1^{(3)}[(Y_{(6)}\cup S)^{(3)}]\cup[2^2 * Q^{(3)}]\\
&= \bar\phi_{33}\bar\phi_4\bar\phi_{22\mathbf{33}}\bar\phi_3\rho_1^{(2)}[(Y_{(6)}\cup S)^{(2)}]\cup[2^2 * Q^{(3)}]\\
&= \bar\phi_{33}\bar\phi_4\bar\phi_{22\mathbf{33}}\bar\phi_3 \begin{array}{lllllllll} 1&1&1&1&1&1&1&1&1\\ 2&2&2&1&1&1&&& \end{array} \cup [2^2 * Q^{(3)}]\\
&= \bar\phi_{33\mathbf{33}}\bar\phi_4\bar\phi_{22}\bar\phi_3 \begin{array}{lllllllll} 1&1&1&1&1&1&1&1&1\\ 2&2&2&1&1&1&&& \end{array} \cup [2^2 * Q^{(3)}]\\
&= \bar\phi_{33\mathbf{33}}\bar\phi_4\rho_1^{(3)}[223 * (Y_{(6)}\cup S)^{(2)}]\cup[2^2 * Q^{(3)}]\\
&= \bar\phi_{33\mathbf{33}}\bar\phi_4 \begin{array}{lllllllll} 1&1&1&1&1&1&1&1&1\\ 2&2&2&2&2&1&&&\\ 3&1&1&&&&&& \end{array} \cup [2^2 * (2 * Q^{(2)})]\\
&= \begin{array}{lllllllll} 1&1&1&1&1&1&1&1&1\\ 2&2&2&2&2&1&&&\\ 3&\mathbf{3}&\mathbf{3}&3&3&&&&\\ 4&1&1&&&&&& \end{array} \cup [2^2 * (2 * Q^{(2)})]\\
&= \bar\phi_{33\mathbf{33}}\bar\phi_4\Big\{ \begin{array}{lllllllll} 1&1&1&1&1&1&1&1&1\\ 2&2&2&2&2&1&&&\\ 3&1&1&&&&&& \end{array} \cup [2^3 * Q^{(2)}]\Big\}\\
&= \bar\phi_{33\mathbf{33}}\bar\phi_4 \begin{array}{lllllllll} 1&1&1&1&1&1&1&1&1\\ 2&2&2&2&2&1&2&&\\ 3&1&1&2&2&2&&& \end{array}\\
&= \bar\phi_{33\mathbf{33}}\bar\phi_4\{\rho_1^{(3)}[223 * (Y_{(6)}\cup S)^{(2)}]\cup[2^3 * Q^{(2)}]\}\\
&\underset{(3.3)}{=} \bar\phi_4\bar\phi_{33\mathbf{33}}\rho_1^{(3)}[(Y\cup T)^{(3)}].
\end{aligned}
$$

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