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GEVREY WELL POSEDNESS OF GOURSAT-DARBOUX PROBLEMS AND ASYMPTOTIC SOLUTIONS

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ABSTRACT: We consider the generalized Goursat-Darboux problem for a third order linear PDE with real constant coefficients. Our purpose is to find necessary conditions for the problem to be well-posed in the Gevrey classes. Since this problem can be reduced to the Cauchy problem using permutations of independent variables, we solve it for a ODE with complex coefficients and two unknown initial data. In order to prove our results, we first construct an explicit solution of a family of problems with initial data depending on a parameter $\eta > 0$ and then we obtain an asymptotic representation of a solution as η tends to infinity.

KEYWORDS: Goursat-Darboux problems, Gevrey classes, asymptotic solutions.
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1. Introduction

The generalized Goursat-Darboux problem for a third order linear PDE with real constant coefficients in the space C^∞ was studied in [2], [3]. Given an open set $\Omega \subseteq \mathbf{R^{3+m}}$, neighborhood of the origin, the most general problem is defined on Ω by

$$\begin{cases} \partial_t\partial_x\partial_y u(t,x,y,z) = \displaystyle\sum_{\substack{l+k+j+\mid\xi\mid\leq 3 \\ l\neq 3, k\neq 3, j\neq 3}} a_{l,k,j,\xi}\partial_t^l\partial_x^k\partial_y^j\partial_z^\xi u(t,x,y,z) \\ u(0,x,y,z) = f_1(x,y,z) \\ u(t,0,y,z) = f_2(t,y,z) \\ u(t,x,0,z) = f_3(t,x,z) \end{cases} \tag{1.1}$$

where initial data satisfy the necessary compatibility conditions:

$$\begin{cases} f_1(0,y,z) = f_2(0,y,z) \\ f_1(x,0,z) = f_3(0,x,z) \\ f_2(t,0,z) = f_3(t,0,z) \\ f_1(0,0,z) = f_2(0,0,z) = f_3(0,0,z)\,. \end{cases} \tag{1.2}$$

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It was showed in [3] that if the problem (1.1)-(1.2) is locally C^∞ well-posed in the neighborhood of origin then the coefficients $a_{0,0,0,\xi}$ with $\mid \xi \mid \leq 3$ are zero.
The necessary conditions for the problem to be C^∞ well-posed are very strong. Our goal is to investigate the local solvability of this problem in the classes of Gevrey functions [5].

Definition 1.1. (Gevrey classes) Let $s > 1$ be a real number and Ω be an open subset of $\mathbf{R^n}$. The Gevrey class of index s on Ω, $\Gamma^s(\Omega)$, is the space of all the functions $f \in C^\infty(\Omega)$ such that for every compact $K \subset \Omega$ there exist constants $C > 0$ and $L > 0$ satisfying

$$\sup_{x \in K} \mid \partial^\alpha f(x) \mid \leq C L^{|\alpha|} \alpha!^s \tag{1.3}$$

for all multi-index α.

It is well known that there is a scale of Gevrey classes $\Gamma^s(\Omega)$ of index $s \geq 1$:

$$1 \leq s' < s \quad \Rightarrow \quad \Gamma^{s'}(\Omega) \subset \Gamma^s(\Omega)\,.$$

In fact these classes play an important role as spaces intermediate between the spaces of real analytic functions ($s = 1$) and $C^\infty(\Omega)$. In addition we have

$$\Gamma^1(\Omega) \subset \bigcap_{s>1} \Gamma^s(\Omega) \quad ; \quad \bigcup_{s>1} \Gamma^s(\Omega) \subset C^\infty(\Omega)\,.$$

We need to give a topology for $\Gamma^s(\Omega)$. Let L be a positive constant, we denote by $\Gamma^s_{L,K}$ the space of smooth functions $f \in \Omega$ such that for every compact $K \subset \Omega$,

$$\|f\|^s_{L,K} = \sup_\alpha [L^{-|\alpha|} \alpha!^{-s} \sup_{x \in K} \mid \partial^\alpha f(x) \mid] < \infty\,.$$

We also consider the space of functions in $\Gamma^s_{L,K}$ with compact support,

$$\Gamma^s_{L,K}(\Omega) = \{f \in C^\infty(\Omega) : \ \mathrm{supp} f \subset K\,,\ \|f\|^s_{L,K} < \infty\}\,,$$

which is a Banach space endowed with the norm $\|f\|^s_{L,K}$. From a topological point of view, the Gevrey classes

$$\Gamma^s(\Omega) = \bigcup_{L>0, K \subset \Omega} \Gamma^s_{L,K}(\Omega)$$

are projective limits of inductive limits of Banach spaces [9].

2. Formulation of the generalized Goursat-Darboux problem

Let $m = 1$, without loss of generality, and let $\Omega \subseteq \mathbf{R}^4$ be an open set, neighborhood of the origin, defined by

$$\Omega = \{(t, x, y, z) : |t| < t_0 \;\;\wedge\;\; |x| < x_0 \;\;\wedge\;\; |y| < y_0 \;\;\wedge\;\; |z| < z_0\}.$$

We consider the simplest Goursat-Darboux problem on Ω for a third order linear PDE with real constant coefficients:

$$\begin{cases} \partial_t \partial_x \partial_y u(t, x, y, z) = \sum\limits_{0 \le j \le 3} A_j \partial_z^j u(t, x, y, z) \\ u(0, x, y, z) = f_1(x, y, z) \\ u(t, 0, y, z) = f_2(t, y, z) \\ u(t, x, 0, z) = f_3(t, x, z) \end{cases} \tag{2.1}$$

where initial data satisfy compatibility conditions (1.2) on characteristic hyperplanes:

$$\Sigma_1 = \left\{(t, x, y, z) \in \mathbf{R}^4 : t = 0\right\}, \; \Sigma_2 = \left\{(t, x, y, z) \in \mathbf{R}^4 : x = 0\right\}, \tag{2.2}$$

$$\Sigma_3 = \left\{(t, x, y, z) \in \mathbf{R}^4 : y = 0\right\}. \tag{2.3}$$

The problem (2.1)-(1.2) is a generalization of the problem studied by Hasegawa [7] for a second order linear PDE. It is called the Goursat problem of three faces.

Let us now introduce the definition of the well posed problem in the Gevrey classes in the sense of Hadamard [6].

Definition 2.1. (Problem well-posed in the Gevrey classes)

Let $s > 1$ be a real number and Ω be an open subset of $\mathbf{R}^\mathbf{n}$, neighborhood of origin. We say that the problem (2.1)-(1.2) is $\Gamma^s(\Omega)$ well-posed on Ω if there exists a neighborhood $\mathcal{U} \subset \Omega$ such that

- For every $f_i \in \Gamma^s(\Omega \cap \Sigma_i)$, the problem (2.1)-(1.2) has a solution $u \in \Gamma^s(\mathcal{U})$;
- It is unique;
- It depends continuously on the data. This means that for every compact $K \subset \Omega$ and every constant $L > 0$ there exist compacts K_i and constants $L_i > 0$, $i = 1, 2, 3$, and $C > 0$ such that

$$\|u\|^s_{L,K} \le C \left(\|f_1\|^s_{L_1,K_1} + \|f_2\|^s_{L_2,K_2} + \|f_3\|^s_{L_3,K_3} \right) . \tag{2.4}$$

Our purpose is to find necessary conditions for the problem (2.1)-(1.2) to be well-posed in the Gevrey classes. We will try to find some critical index s_0 such that if the Goursat-Darboux problem is well posed in Γ^s for $s > s_0$ then the coefficients of the derivatives with respect to z are zero.
We begin by showing how the problem (2.1)-(1.2) can be reduced to a Cauchy problem following the ideas of Bronshtein [1]. It is easy to see that the differential operator

$$\partial_t\partial_x\partial_y - (A_3\partial^3_{z^3} + A_2\partial^2_{z^2} + A_1\partial_z + A_0)$$

and the three characteristic hyperplanes Σ_i remain invariant under any permutation of the independent variables t, x and y. Let μ be the minimum value between t_0, x_0 and y_0 and

$$\Omega_\mu = \{(t,x,y,z) :\mid t \mid< \mu \wedge \mid x \mid< \mu \wedge \mid y \mid< \mu \wedge \mid z \mid< z_0\,\}$$

be an open set, $\Omega_\mu \subset \Omega$. From now on we suppose that the problem (2.1)-(1.2) is Γ^s well-posed on Ω. By linearity, if $u(t,x,y,z)$ is a solution of the problem (2.1)-(1.2) on Ω then

$$v(t,x,y,z) = u(t,x,y,z) + u(x,y,t,z) + u(y,t,x,z) \tag{2.5}$$

is a solution of the corresponding problem on Ω_μ

$$\begin{cases} \partial_t\partial_x\partial_y v(t,x,y,z) = \displaystyle\sum_{j\leq 3} A_j\partial_z^j v(t,x,y,z) \\ v(0,x,y,z) = f_1(x,y,z) + f_3(x,y,z) + f_2(y,x,z) \\ v(t,0,y,z) = f_2(t,y,z) + f_1(y,t,z) + f_3(y,t,z) \\ v(t,x,0,z) = f_3(t,x,z) + f_2(x,t,z) + f_1(t,x,z)\,. \end{cases} \tag{2.6}$$

We then reduce the number of the independent variables by setting $t = x = y$. We can define a function w by $w(r,z) = v(r,r,r,z)$ on

$$\tilde{\Omega} = \{(r,z) :\mid r \mid< \mu \wedge \mid z \mid< z_0\} \subseteq \mathbf{R^2}\,.$$

Its partial derivatives with respect to r are given by

$$\partial_r w(r,z) = 3\partial_t v(r,r,r,z)\,,\ \partial^2_{r^2} w(r,z) = 9\partial_t\partial_x v(r,r,r,z)\,,$$

$$\partial^3_{r^3} w(r,z) = 27\partial_t\partial_x\partial_y v(r,r,r,z)\,.$$

For every parameter $\eta > 0$, taking

$$v(0,x,y,z) = v(t,0,y,z) = v(t,x,0,z) = e^{i\eta z}$$

we are looking for a unique solution depending continuously on the data. If v_η is solution of the problem on Ω_μ

$$\begin{cases} \partial_t\partial_x\partial_y v(t,x,y,z) = \sum_{j\le 3} A_j\partial_z^j v(t,x,y,z) \\ v(0,x,y,z) = v(t,0,y,z) = v(t,x,0,z) = e^{i\eta z} \end{cases} \tag{2.7}$$

then $w_\eta(r,z) = v_\eta(r,r,r,z)$ is solution of the Cauchy problem on $\tilde{\Omega}$

$$\begin{cases} \partial^3_{r^3} w(r,z) &= 27(A_3\partial^3_{z^3} + A_2\partial^2_{z^2} + A_1\partial_z + A_0)w(r,z) \\ w(0,z) &= e^{i\eta z}\,. \end{cases} \tag{2.8}$$

Notice that there are two arbitrary data $\partial_r w(0,z)$ and $\partial^2_{r^2} w(0,z)$.

3. Solving the Cauchy problem

Applying the method of separation of variables we determine a unique solution of the Cauchy problem (2.8) in the form $w_\eta(r,z) = m_\eta(r)e^{i\eta z}$. Hence $m_\eta(r)$ is solution of the initial value problem

$$\begin{cases} m'''(r) = 27(-A_3 i\eta^3 - A_2\eta^2 + A_1 i\eta + A_0)m(r) \\ m(0) = 1 \\ m'(0) = \alpha \\ m''(0) = \beta \end{cases} \tag{3.1}$$

where α and β are unknown. In order to solve a third order linear ODE

$$m'''(r) = 27(-A_3 i\eta^3 - A_2\eta^2 + A_1 i\eta + A_0)m(r) \tag{3.2}$$

we use its characteristic equation

$$\lambda^3 - 27p(\eta) = 0 \tag{3.3}$$

where $p(\eta) = -A_3 i\eta^3 - A_2\eta^2 + A_1 i\eta + A_0$ is a polynomial with complex coefficients.

Lemma 3.1. *Let γ and $\overline{\gamma}$ be two conjugate complex roots of unity. If $A_\eta \neq 0$ is a solution of the equation (3.2) then the solution of the problem (3.1) is given by*

$$\begin{aligned} m_\eta(r) = \tfrac{1}{3}(1 + a_\eta + b_\eta)e^{A_\eta r} + \tfrac{1}{3}\left(1 + \overline{\gamma}a_\eta + \gamma b_\eta\right)e^{\gamma A_\eta r} + \\ + \tfrac{1}{3}\left(1 + \gamma a_\eta + \overline{\gamma} b_\eta\right)e^{\overline{\gamma} A_\eta r}\,. \end{aligned} \tag{3.4}$$

where $a_\eta = \dfrac{\alpha}{A_\eta}$ and $b_\eta = \dfrac{\beta}{A_\eta^2}$.

Proof: Let $A_\eta \neq 0$ be a solution of (3.2). If γ and $\overline{\gamma}$ are two conjugate complex roots of unity then by de Moivre's formula the general solution of the (3.2) is written in the form

$$m_\eta(r) = C_1 e^{A_\eta r} + C_2 e^{\gamma A_\eta r} + C_3 e^{\overline{\gamma} A_\eta r}$$

where $C_1, C_2, C_3 \in \mathbf{C}$ are arbitrary constants, which are determined from initial data of the problem (3.1) by solving a linear system. ■

If A_η is a real root of the (3.2) we simplify (3.4) by using the Euler's formula.

Theorem 3.1 (Characteristic equation with one real root). If $A_\eta \in \mathbf{R} - \{0\}$ then

$$m_\eta(r) = \tfrac{1}{3}(1 - c_\eta)e^{A_\eta r} + \tfrac{1}{3}(2 + c_\eta)\cos(\sqrt{3}A_\eta r/2)e^{-A_\eta r/2} + \\ + \tfrac{\sqrt{3}}{3} d_\eta \sin(\sqrt{3}A_\eta r/2)e^{-A_\eta r/2} \tag{3.5}$$

where $c_\eta = -a_\eta - b_\eta$ and $d_\eta = -i(a_\eta - b_\eta)$.

If A_η is a pure imaginary root of the (3.2), (3.4) can be written in a simpler expression.

Theorem 3.2 (Characteristic equation with a pure imaginary root). If $A_\eta = -iB_\eta$ with $B_\eta \in \mathbf{R} - \{0\}$ then

$$m_\eta(r) = \tfrac{1}{3}\left[(2 + c_\eta)\cosh(\sqrt{3}B_\eta r/2) + \sqrt{3}d_\eta \sinh(\sqrt{3}B_\eta r/2)\right]e^{iB_\eta r/2} + \\ + \tfrac{1}{3}(1 - c_\eta)e^{-iB_\eta r} \tag{3.6}$$

where $c_\eta = \dfrac{\beta}{B_\eta^2} - i\dfrac{\alpha}{B_\eta}$ and $d_\eta = \dfrac{\alpha}{B_\eta} - i\dfrac{\beta}{B_\eta^2}$.

4. Asymptotic representation of solutions

In previous works ([2], [3], [7]) an explicit solution of the generalized Goursat-Darboux problem involves a hypergeometric function of several variables. However some difficulties for obtaining asymptotic representations for these functions were pointed out in the paper [4].

In our work we have a linear combination of complex exponential functions as solution of the Cauchy problem. We provide asymptotic representations, as η tends to infinity, for the absolute value of complex functions m_η on a

compact, which depends on η and s. Our approach is based on asymptotic analysis of the initial data in order to have only one exponential function as dominant term, that is, when one exponential function tends to infinity and the others tend to zero.
Here $\Re(p(\eta))$ and $\Im(p(\eta))$ denote the real part of $p(\eta)$ and the imaginary part of $p(\eta)$, respectively.

Theorem 4.1. If $\Im(p(\eta)) = 0$, $A_2 \neq 0$ and $s > 3/2$ then there exist a constant $c > 0$ and a compact K_η, neighborhood of the origin, such that

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim c e^{\sqrt[3]{|A_2|}\eta^{1/s}} \tag{4.1}$$

as η tends to infinity.

Proof: By assumption $\Im(p(\eta)) = 0$ and $A_2 \neq 0$. The equation (3.2) has one real root $A_\eta = -3\sqrt[3]{A_2\eta^2 - A_0}$, then by Corollary 3.1 the solution of the problem (3.1), $m_\eta(r)$, is given by (3.5). Let's see three cases that may occur depending on complex values $c_\eta = -\dfrac{\alpha}{A_\eta} - \dfrac{\beta}{A_\eta^2}$ and $d_\eta = -i\left(\dfrac{\alpha}{A_\eta} - \dfrac{\beta}{A_\eta^2}\right)$. We first suppose that

$$\mid d_\eta \mid = O(\mid 1 - c_\eta \mid) \wedge \mid 2 + c_\eta \mid = O(\mid 1 - c_\eta \mid)\,.$$

We choose a compact K_η,

$$K_\eta = \{(r,z) : (r,z) = \pm\frac{1}{3}(\eta^{1/s - 2/3}, 0)\}\,,$$

in which

$$\sup_{r \in K_\eta} (A_\eta r) = \sqrt[3]{\mid A_2 \mid}\eta^{1/s}\,.$$

Notice that if $s > 3/2$ then K_η is a neighborhood of the origin on $\mathbf{R^2}$. Since

$$\mid (2 + c_\eta)\cos(2^{-1}\sqrt{3}\sqrt[3]{\mid A_2 \mid}\eta^{1/s}) \mid e^{-2^{-1}\sqrt[3]{|A_2|}\eta^{1/s}} = o(\mid 1 - c_\eta \mid e^{\sqrt[3]{|A_2|}\eta^{1/s}})$$

and

$$\mid d_\eta \sin(2^{-1}\sqrt{3}\sqrt[3]{\mid A_2 \mid}\eta^{1/s}) \mid e^{-2^{-1}\sqrt[3]{|A_2|}\eta^{1/s}} = o(\mid 1 - c_\eta \mid e^{\sqrt[3]{|A_2|}\eta^{1/s}})$$

we obtain

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim \frac{1}{3} \mid 1 - c_\eta \mid e^{\sqrt[3]{|A_2|}\eta^{1/s}} \tag{4.2}$$

with $\frac{1}{3} \mid 1 - c_\eta \mid \geq c > 0$, as η tends to infinity. Then we suppose that

$$\mid d_\eta \mid = O(\mid 2 + c_\eta \mid) \quad \wedge \quad \mid 1 - c_\eta \mid = O(\mid 2 + c_\eta \mid)\,.$$

If $s > 3/2$ we choose a compact K_η, neighborhood of the origin on $\mathbf{R^2}$, in which

$$\sup_{r \in K_\eta} (-A_\eta r) = 2\sqrt[3]{\mid A_2 \mid}\eta^{1/s}\,.$$

Moreover, we choose a sequence of η values satisfying $\sup\limits_{r \in K_\eta} \tan(\sqrt{3}A_\eta r/2) = 0$. Since

$$\mid 1 - c_\eta \mid e^{-2\sqrt[3]{|A_2|}\eta^{1/s}} = o(\mid 2 + c_\eta \mid e^{\sqrt[3]{|A_2|}\eta^{1/s}})$$

and

$$\mid d_\eta \sin(\sqrt{3}\sqrt[3]{\mid A_2 \mid}\eta^{1/s}) \mid = o(\mid (2 + c_\eta)\cos(\sqrt{3}\sqrt[3]{\mid A_2 \mid}\eta^{1/s}) \mid)$$

we obtain

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim \frac{1}{3} \mid 2 + c_\eta \mid e^{\sqrt[3]{|A_2|}\eta^{1/s}} \tag{4.3}$$

with $\frac{1}{3} \mid 2 + c_\eta \mid \geq c > 0$, as η tends to infinity. Finally we suppose that

$$\mid 2 + c_\eta \mid = O(\mid d_\eta \mid) \wedge \mid 1 - c_\eta \mid = O(\mid d_\eta \mid)\,.$$

If $s > 3/2$ we choose a compact K_η, neighborhood of the origin on $\mathbf{R^2}$, in which

$$\sup_{r \in K_\eta} (-A_\eta r) = 2\sqrt[3]{\mid A_2 \mid}\eta^{1/s}\,,$$

and a sequence of η values satisfying $\sup\limits_{r \in K_\eta} \cot(\sqrt{3}A_\eta r/2) = 0$. Since

$$\mid 1 - c_\eta \mid e^{-2\sqrt[3]{|A_2|}\eta^{1/s}} = o(\mid d_\eta \mid e^{\sqrt[3]{|A_2|}\eta^{1/s}})$$

and

$$\mid (2 + c_\eta)\cos(\sqrt{3}\sqrt[3]{\mid A_2 \mid}\eta^{1/s}) \mid = o(\mid d_\eta \sin(\sqrt{3}\sqrt[3]{\mid A_2 \mid}\eta^{1/s}) \mid)$$

we obtain

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim \frac{\sqrt{3}}{3} \mid d_\eta \mid e^{\sqrt[3]{|A_2|}\eta^{1/s}} \tag{4.4}$$

with $\frac{\sqrt{3}}{3} \mid d_\eta \mid \geq c > 0$, as η tends to infinity. ■

Theorem 4.2. If $\Re(p(\eta)) = 0$, $A_3 = 0$, $A_1 \neq 0$ and $s > 3$ then there exist a constant $c > 0$ and a compact K_η, neighborhood of the origin, such that

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim c e^{\sqrt[3]{|A_1|}\eta^{1/s}} \tag{4.5}$$

as η tends to infinity.

Proof: By assumption we have $p(\eta) = A_1\eta i$ with $A_1 \neq 0$. The equation (3.2) has a pure imaginary root $A_\eta = -3i\sqrt[3]{A_1\eta}$. We consider $iA_\eta = 3\sqrt[3]{A_1\eta} = B_\eta$ with $B_\eta \in \mathbf{R}$, then by Corollary 3.2 the solution of the problem (3.1), $m_\eta(r)$, is given by (3.6), where $c_\eta = \dfrac{\beta}{B_\eta^2} - i\dfrac{\alpha}{B_\eta}$ and and $d_\eta = \dfrac{\alpha}{B_\eta} - i\dfrac{\beta}{B_\eta^2}$. We notice than $2 + c_\eta$ and d_η are not null simultaneously. If we suppose that

$$\mid 1 - c_\eta \mid = O(\mid 2 + c_\eta \mid) \wedge \mid d_\eta \mid = O(\mid 2 + c_\eta \mid)\,,$$

then for $s > 3$ we choose a compact K_η,

$$K_\eta = \{(r,z) : (r,z) = \pm\frac{2}{3\sqrt{3}}(\eta^{1/s-1/3}, 0)\}\,,$$

neighborhood of origin, in which

$$\frac{\sqrt{3}}{2}\sup_{r\in K_\eta}(B_\eta r) = \sqrt[3]{\mid A_1 \mid}\eta^{1/s}\,.$$

Since

$$\mid 1 - c_\eta \mid = o(\mid 2 + c_\eta \mid \cosh(\sqrt[3]{\mid A_1 \mid \eta^{1/s}}))$$

and

$$\mid d_\eta \mid \sinh(\sqrt[3]{\mid A_1 \mid \eta^{1/s}}) = O(\mid 2 + c_\eta \mid \cosh(\sqrt[3]{\mid A_1 \mid \eta^{1/s}}))$$

we obtain

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim \frac{1}{3} \mid 2 + c_\eta \mid e^{\sqrt[3]{|A_1|}\eta^{1/s}} \tag{4.6}$$

with $\frac{1}{3} \mid 2 + c_\eta \mid \geq c > 0$, as η tends to infinity. If we suppose that

$$\mid 1 - c_\eta \mid = O(\mid d_\eta \mid) \wedge \mid 2 + c_\eta \mid = O(\mid d_\eta \mid)\,,$$

then for $s > 3$ we can choose a compact K_η, neighborhood of the origin, in which

$$\frac{\sqrt{3}}{2}\sup_{r\in K_\eta}(B_\eta r) = \sqrt[3]{\mid A_1 \mid}\eta^{1/s}\,.$$

Since

$$| 1 - c_\eta | = o(| d_\eta | \sinh(\sqrt[3]{| A_1 | \eta^{1/s}}))$$

and

$$| 2 + c_\eta | \cosh(\sqrt[3]{| A_1 | \eta^{1/s}}) = O(| d_\eta | \sinh(\sqrt[3]{| A_1 | \eta^{1/s}}))$$

we obtain

$$\sup_{r \in K_\eta} | m_\eta(r) | \sim \frac{\sqrt{3}}{3} | d_\eta | e^{\sqrt[3]{|A_1|\eta^{1/s}}} \tag{4.7}$$

with $\frac{\sqrt{3}}{3} | d_\eta | \geq c > 0$, as η tends to infinity.

■

Lemma 4.1. *Let g_1, g_2, g_3 and h with $\Re(h(\eta)) > 0$ be complex functions of the real variable η. We consider m defined by*

$$m(\eta) = g_1(\eta)e^{h(\eta)} + g_2(\eta)e^{\gamma h(\eta)} + g_3(\eta)e^{\overline{\gamma} h(\eta)}$$

where $\gamma = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. If $|g_j(\eta)| = O(|g_1(\eta)|)$, $j \neq 1$, and $|\Im(h(\eta))| < \sqrt{3}\Re(h(\eta))$ then

$$|m(\eta)| \sim |g_1(\eta)|e^{\Re(h(\eta))}$$

as η tends to infinity.

Proof: By assumption, we have $|g_j(\eta)| = O(|g_1(\eta)|)$, $j = 2, 3$, that is, there are constants $k_j > 0$, such that $|g_j(\eta)| \leq k_j|g_1(\eta)|$ for all $\eta \in]0, +\infty[$. From simple calculations we get

$$\Re((\gamma - 1)h(\eta)) = 3\Re(h(\eta) + \sqrt{3}\Im(h(\eta)) \,, \; \Re((\overline{\gamma} - 1)h(\eta)) = 3\Re(h(\eta) - \sqrt{3}\Im(h(\eta)) \,.$$

The condition $|\Im(h(\eta))| < \sqrt{3}\Re(h(\eta))$ it is equivalent to

$$\Re((\gamma - 1)h(\eta)) < 0 \quad \wedge \quad \Re((\overline{\gamma} - 1)h(\eta)) < 0 \,.$$

Since

$$|g_2(\eta)|e^{\Re(\gamma h(\eta))} = o(|g_1(\eta)|e^{\Re(h(\eta))}) \;\wedge\; |g_3(\eta)|e^{\Re(\overline{\gamma} h(\eta))} = o(|g_1(\eta)|e^{\Re(h(\eta))})$$

it implies

$$|g_2(\eta)|e^{\Re(\gamma h(\eta))} + |g_3(\eta)|e^{\Re(\overline{\gamma} h(\eta))} = o(|g_1(\eta)|e^{\Re(h(\eta))}) \,,$$

by consequence

$$|m(\eta)| \sim |g_1(\eta)|e^{\Re(h(\eta))}$$

as η tends to infinity. ■

Theorem 4.3. If $\Re(p(\eta)) \neq 0$, $A_3 \neq 0$ and $s > 1$ then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim c e^{\sqrt[3]{|A_3|}\eta^{1/s}} \tag{4.8}$$

as η tends to infinity.

Proof: We have $\Re(p(\eta)) = -A_2\eta^2 + A_0 \neq 0$ and $\Im(p(\eta)) = -A_3\eta^3 + A_1\eta$ with $A_3 \neq 0$ by assumption. Let $A_\eta = 3\sqrt[3]{p(\eta)}$ be one of the three complex roots of the equation (3.2) whose principal argument is

$$\theta_1 \in (]-\frac{\pi}{3}, 0[\cup]0, \frac{\pi}{3}[) \ \vee \ \theta_2 = \theta_1 - \frac{2\pi}{3} \ \vee \ \theta_3 = \theta_1 + \frac{2\pi}{3} .$$

Then by Lemma 3.1 the solution of the problem (3.1), $m_\eta(r)$, is given by (3.4), where $a_\eta = \dfrac{\alpha}{A_\eta}$ and $b_\eta = \dfrac{\beta}{A_\eta^2}$. If we first suppose that

$$\mid 1 + \overline{\gamma} a_\eta + \gamma b_\eta \mid = O(\mid 1 + a_\eta + b_\eta \mid) \wedge \ \mid 1 + \gamma a_\eta + \overline{\gamma} b_\eta \mid = O(\mid 1 + a_\eta + b_\eta \mid)$$

we choose a compact K_η, neighborhood of origin for $s > 1$, in which

$$h(\eta) = \sup_{r \in K_\eta} A_\eta r = (1 + i \tan\theta_1)\sqrt[3]{\mid A_3 \mid}\eta^{1/s} ,$$

for some $\theta_1 \in (]-\frac{\pi}{3}, 0[\cup]0, \frac{\pi}{3}[)$. Since $|\Im(h(\eta))| < \sqrt{3}\Re(h(\eta))$ by Lemma 4.1 we obtain

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim \frac{1}{3} \mid 1 + a_\eta + b_\eta \mid e^{\sqrt[3]{|A_3|}\eta^{1/s}} \tag{4.9}$$

with $\mid 1 + a_\eta + b_\eta \mid \geq c > 0$, as η tends to infinity. Then if we suppose that

$$\mid 1 + a_\eta + b_\eta \mid = O(\mid 1 + \overline{\gamma} a_\eta + \gamma b_\eta \mid) \wedge \ \mid 1 + \gamma a_\eta + \overline{\gamma} b_\eta \mid = O(\mid 1 + \overline{\gamma} a_\eta + \gamma b_\eta \mid)$$

we choose now a compact K_η, neighborhood of origin for $s > 1$, in which

$$h(\eta) = \sup_{r \in K_\eta} A_\eta r = \left(1 + i\frac{\tan\theta_2 - \sqrt{3}}{1 + \sqrt{3}\tan\theta_2}\right)\sqrt[3]{\mid A_3 \mid}\eta^{1/s} ,$$

for some $\theta_2 \in (]-\pi, -\frac{\pi}{2}[\cup]-\frac{\pi}{2}, -\frac{\pi}{3}[)$. Since $|\Im(h(\eta))| < \sqrt{3}\Re(h(\eta))$ by Lemma 4.1 we obtain

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim \frac{1}{3} \mid 1 + \overline{\gamma} a_\eta + \gamma b_\eta \mid e^{\sqrt[3]{|A_3|}\eta^{1/s}} \tag{4.10}$$

with $\mid 1+\overline{\gamma}a_\eta+\gamma b_\eta \mid\geq c>0$, as η tends to infinity. Finally if we suppose that

$\mid 1+a_\eta+b_\eta \mid = O(\mid 1+\gamma a_\eta+\overline{\gamma}b_\eta \mid) \wedge \mid 1+\overline{\gamma}a_\eta+\gamma b_\eta \mid = O(\mid 1+\gamma a_\eta+\overline{\gamma}b_\eta \mid)$

we take a compact K_η, neighborhood of origin for $s>1$, in which

$$h(\eta)=\sup_{r\in K_\eta} A_\eta r=\left(1+i\frac{\tan\theta_3+\sqrt{3}}{1-\sqrt{3}\tan\theta_3}\right)\sqrt[3]{\mid A_3\mid}\eta^{1/s},$$

for some $\theta_3\in(]\frac{\pi}{3},\frac{\pi}{2}[\cup]\frac{\pi}{2},\pi[)$. Since $|\Im(h(\eta))|<\sqrt{3}\Re(h(\eta))$ by Lemma 4.1 we obtain

$$\sup_{r\in K_\eta}\mid m_\eta(r)\mid\sim\frac{1}{3}\mid 1+\gamma a_\eta+\overline{\gamma}b_\eta\mid e^{\sqrt[3]{|A_3|}\eta^{1/s}} \tag{4.11}$$

with $\mid 1+\gamma a_\eta+\overline{\gamma}b_\eta \mid\geq c>0$, as η tends to infinity. ∎

Theorem 4.2. *If the problem (2.1)-(1.2) is Γ^s well-posed on Ω then*

(i):

$$s>1 \quad\Rightarrow\quad A_3=0\,; \tag{4.12}$$

(ii):

$$s>\frac{3}{2} \quad\Rightarrow\quad A_2=0\,; \tag{4.13}$$

(iii):

$$s>3 \quad\Rightarrow\quad A_1=0\,. \tag{4.14}$$

Proof: We suppose that the problem (2.1)-(1.2) is Γ^s well-posed on Ω with $s>1$. Then for every $\eta>0$ the corresponding problem (2.7) has a unique solution v_η on Ω_μ.

On the one hand, we determine *a prior* an estimation for the Gevrey norm of v_η, an upper bound, from the initial data, $\|e^{i\eta z}\|^s_{L,K}$, for every compact $K\subset\Omega$ and every constant $L>0$. The partial derivatives of $e^{i\eta z}$ with respect to multi-index (l,k,j,α), such that $l\neq 0$ or $k\neq 0$ or $j\neq 0$, are zero. Otherwise, it is clear that

$$\partial_z^\alpha(e^{i\eta z})=(i\eta)^{|\alpha|}e^{i\eta z},$$

it follows that

$$\sup_{(t,x,y,z)\in K}\mid\partial^\alpha(e^{i\eta z})\mid=\eta^{|\alpha|}$$

so that

$$\|e^{i\eta z}\|^s_{L,K}=\sup_\alpha\left(\mid\alpha\mid^{-s|\alpha|}L^{-|\alpha|}\eta^{|\alpha|}\right).$$

Since the supremum is given by $e^{se^{-1}L^{-1/s}\eta^{1/s}}$ there exist constants $c_1 = se^{-1}L^{-1/s}$ and $C > 0$ such that

$$\|v_\eta\|^s_{L,K} \leq C\|e^{i\eta z}\|^s_{L,K} \leq Ce^{c_1\eta^{1/s}} \tag{4.15}$$

for every $\eta > 0$. This is a condition for stability of solution.
On the other hand, let's prove that if each coefficient of the equation is different from zero, $A_i \neq 0$, then there is some critical index s_0 such that if $s > s_0$ then (4.15) will be violated.
In (i) we suppose that $A_3 \neq 0$ and assume $A_2 = 0$, in (ii) we suppose that $A_2 \neq 0$ and assume $A_1 = 0$ and in (iii) we suppose that $A_1 \neq 0$ and assume $A_0 = 0$. We assume that some coefficient is null because we can do suitable dependent variable changes.
By using previous propositions we construct an asymptotic representation of a solution as η tends to infinity. For every neighborhood of the origin $\mathcal{O}$ there exist a compact K_η, $K_\eta \subset \mathcal{O}$, and constants $C > 0$ and $c_2 > 0$ such that

$$\sup_{r\in K_\eta} \mid v_\eta(r,r,r,z) \mid \sim Ce^{c_2\eta^{1/s}}.$$

Notice that $K_\eta \subset \mathcal{O}$ only if $s_0 = 1$ in (i), $s_0 = 3/2$ in (ii) and $s_0 = 3$ in (iii).
We have

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid = \sup_{r\in K_\eta} \mid w_\eta(r,z) \mid = \sup_{r\in K_\eta} \mid v_\eta(r,r,r,z) \mid$$

and as we know that

$$\|v_\eta\|^s_{L',K_\eta} > \sup_{r\in K_\eta} \mid v_\eta(r,r,r,z) \mid$$

we can choose a constant $L' > 0$ such that

$$\|v_\eta\|^s_{L',K_\eta} > \|v_\eta\|^s_{L,K}$$

as η tends to infinity. The condition (4.15) fails to hold since $\|v_\eta\|^s_{L',K_\eta}$ has exponential growth of higher order to $\eta^{1/s}$ as η tends to infinity.

■

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