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SHARP REGULARITY FOR THE DEGENERATE DOUBLY NONLINEAR PARABOLIC EQUATION

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ABSTRACT: The aim of this paper is to obtain sharp regularity estimates for locally bounded solutions of the degenerate doubly nonlinear equation

$$u_t - \operatorname{div}(mu^{m-1}|\nabla u|^{p-2}\nabla u) = f,$$

where $m > 1$, $p > 2$ and $f \in L^{q,r}$. More precisely, we show that solutions are locally of class $C^{0,\beta}$, where β depends explicitly only on the optimal Hölder exponent for solutions of the homogeneous case, the integrability of f, the constants p, m and the space dimension n.

KEYWORDS: Doubly Nonlinear Equation, sharp regularity, degenerate.
MATH. SUBJECT CLASSIFICATION (2010): 35B65, 35K55, 35K65.

1. Introduction

We study sharp regularity issues for bounded weak solutions of the inhomogeneous degenerate doubly nonlinear equation (DNLE)

$$u_t - \operatorname{div}(mu^{m-1}|\nabla u|^{p-2}\nabla u) = f \in L^{q,r}(U_T) \tag{1.1}$$

for $m > 1$ and $p > 2$. The family of equations (1.1) generalizes two well-known cases: the porous media equation (PME), case $p = 2$, and the p-Laplacian equation (PLE), case $m = 1$. For the very particular case $m = 1$ and $p = 2$ we recover the standard heat equation $u_t = \Delta u$.

The main motivation for the study of this class of nonlinear evolution equations is their physical relevance, for example, in the study of non-Newtonian fluids, see [16], plasma physics, ground water problems, image-analysis, motion of viscous fluids and in the modeling of an ideal gas flowing isoentropically in a inhomogeneous porous medium [15].

The equation (1.1) exhibits a double nonlinear dependence, on both the solution u and its gradient ∇u that makes diffusion properties degenerate at points where the solution and its gradient vanish. Existence of weak solutions has been proven

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in [22, 23]. Local boundedness of the gradient for locally bounded, strictly positive weak solutions has been investigated in [20] and Harnack type inequalites for bounded weak solutions are proved in [13, 27]. Besides, in [10, 11, 18, 28], the Hölder regularity for bounded weak solutions is established. Here, we denote $0 < \alpha_* \leq 1$ the optimal Hölder exponent for solutions of the homogeneous case.

Hereafter in this paper we shall denote $U_T \equiv U \times (0,T)$, for a open and bounded set $U \subset \mathbb{R}^n$ and $T > 0$. In (1.1), we shall consider functions $f : U_T \to \mathbb{R}$ such that $f \in L^{q,r}(U_T) := L^r(0,T,L^q(U))$ satisfying conditions

$$\frac{1}{r} + \frac{n}{pq} < 1 \quad \text{and} \quad \frac{3}{r} + \frac{n}{q} > 2. \tag{1.2}$$

The first assumption is due to the standard minimal integrability condition that guarantees the existence of bounded weak solutions. The second one defines the borderline setting for the optimal Hölder regularity regime.

The greatest difficulty in the study of this equation is its doubly degeneracy. To work around this problem we adapt the techniques found in [2],[3], [4], [5] to our situation, and show the following result.

Theorem 1.1. *Let u be a locally bounded weak solution of (1.1) in G_1, with $f \in L^{q,r}(U_T)$, satisfying* (1.2)*. Then u is locally of class $C^{0,\beta}$ in space with*

$$\beta = \frac{\alpha(p-1)}{m+p-2}, \quad \textit{for} \quad \alpha = \min\left\{\alpha_\star^-, \frac{(m+p-2)[(pq-n)r-pq]}{q(p-1)[(r-1)(m+p-2)+1]}\right\}. \tag{1.3}$$

Moreover, u is locally $C^{0,\frac{\beta}{\theta}}$ in time for θ given by

$$\theta := p - \alpha(p-1)\left(1 - \frac{1}{m+p-2}\right). \tag{1.4}$$

Theorem 1.1 generalizes the cases studied in [2, 25] where the authors determined the optimal Hölder exponents for weak solutions for the p-laplacian equation and the porous media equation. Such exponents coincide with (1.3) for the cases $p = 2$ and $m = 1$ respectively.

The number β in (1.3) is obtained as follows: in the case

$$\frac{(m+p-2)[(pq-n)r-pq]}{q(p-1)(r-1)[(m+p-2)+1]} < \alpha_*, \tag{1.5}$$

we have the exponent

$$\beta = \frac{(pq-n)r - pq}{q(r-1)[(m+p-2)+1]}.$$

In the case (1.5) is not satisfied, the exponent β is any number less than

$$\frac{\alpha_*(p-1)}{m+p-2} \leq \alpha_\star.$$

Some special borderline scenarios. By making a precise analysis on the exponent in (1.3) it is possible to observe how Hölder regularity for solutions of (1.1) behaves by approaching some integrability borderline cases.

Case $q = r = \infty$. By letting $q, r \to \infty$ we observe that

$$\frac{(m+p-2)[(pq-n)r - pq]}{q(p-1)[(r-1)(m+p-2)+1]} \longrightarrow \frac{p}{p-1} > 1.$$

Therefore after a certain integrability threshold, the optimal regularity exponent of the homogeneous case prevails in (1.3). It implies that solutions of (1.1) are locally $C^{0,\beta}$ for any

$$\beta < \frac{\alpha_*(p-1)}{m+p-2} < \alpha_\star.$$

Case $r = \infty$ and $q \searrow n/p$. Here we shall observe for the next two cases, how the Hölder regularity for solutions of (1.1) deteriorates explicitly by approaching the borderline integrability conditions in (1.2). Indeed, by assuming $f \in L^{\infty, \frac{n}{p}+\varepsilon}(U_T)$, Theorem 1.1 provides that for each $\varepsilon > 0$ universally small, solutions for the problem (1.1) are locally $C^{0,\beta(\varepsilon)}$ in space where

$$\beta(\varepsilon) = \frac{\varepsilon}{\frac{n}{p}+\varepsilon} \cdot \frac{p}{m+p-2}.$$

Case $r \searrow 1$ and $q = \infty$. By considering $f \in L^{1+\varepsilon,\infty}(U_T)$, Theorem 1.1 guarantees that for each number $\varepsilon > 0$ universally small, solutions are locally $C^{0,\delta(\varepsilon)}$ in space with exponent

$$\delta(\varepsilon) = \frac{\varepsilon(m+p-2)}{\varepsilon(m+p-2)+1} \cdot \frac{p}{m+p-2}.$$

Note that in both cases, $\beta(\varepsilon)$ and $\delta(\varepsilon)$ go to 0 as $\varepsilon \to 0$. In time, solutions are $C^{0,\gamma(\varepsilon)}$ for $\gamma(\varepsilon) = \beta(\varepsilon)/\theta(\varepsilon)$ where $\theta(\varepsilon) \to p$ as $\varepsilon \to 0$ so the exponent $\gamma(\varepsilon)$ also deteriorates as $\varepsilon \to 0$.

Remark 1.1. *According to the second condition in* (1.2)*, we observe that for the last two cases such regularity is optimal.*

Organization of the paper. The paper is organised as follows. In section 2 we will define weak solutions of equation (1.1) and we develop some preliminary estimates, which are important tools for the proof of the main results of this paper. In section 3 a geometric iteration is established, using the intrinsic scale of the DNLE. In section 4 we proved the main result of the paper. In section 5 we develop optimal regularity estimates for p-Laplacian type equations that are commented in [25], section 4.

2. Definitions and preliminary results

We start with the definition of weak solution to (1.1).

Definition 2.1. *A non-negative locally bounded function*

$$u \in C_{loc}(0,T;L^2_{loc}(U)),\ u^{\frac{(m+p-1)}{p}} \in L^p_{loc}(0,T;W^{1,p}_{loc}(U))$$

is a local, weak solution to (1.1), *if for every compact set* $K \subset U$ *and every subinterval* $[t_1,t_2] \subset (0,T]$, *we have*

$$\int_K u\varphi dx\,|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \{-u\varphi_t + mu^{m-1}|\nabla u|^{p-2}\nabla u.\nabla\varphi\}dxdt = \int_{t_1}^{t_2}\int_K f\varphi dxdt$$

for all nonnegative test functions

$$\varphi \in W^{1,2}_{loc}(0,T;L^2(K)) \cap L^p_{loc}(0,T;W^{1,p}_0(K)).$$

It is clear that all integrals in the above definition are convergent, interpreting the gradient term as

$$u^{m-1}|\nabla u|^{p-2}\nabla u := \left(\frac{p}{m+p-1}\right)^{p-1} u^{\frac{m-1}{p}}|\nabla u^{\frac{m+p-1}{p}}|^{p-2}\nabla u^{\frac{m+p-1}{p}}.$$

A alternative definition makes use of the Steklov average of a function $v \in L^1(U_T)$, defined for $0 < h < T$ by

$$v_h := \begin{cases} \frac{1}{h}\int_t^{t+h} v(\cdot,\tau)d\tau, & if \quad t \in (0,T-h], \\ 0 & if \quad t \in (T-h,T]. \end{cases} \tag{2.1}$$

Definition 2.2. *A non-negative locally bounded function*

$$u \in C_{loc}(0,T;L^2_{loc}(U)),\ u^{\frac{(m+p-1)}{p}} \in L^p_{loc}(0,T;W^{1,p}_{loc}(U))$$

is a local, weak solution to (1.1), *if for every compact set* $K \subset U$ *and every* $0 < t < T-h$, *we have*

$$\int_{K\times\{t\}} \{(u_h)_t\varphi + (mu^{m-1}|\nabla u|^{p-2}\nabla u)_h.\nabla\varphi\}dx = \int_{K\times\{t\}} f_h\varphi dx, \qquad (2.2)$$

from all nonnegative $\varphi \in W_0^{1,p}(K)$.
One of the main tools we will use is the following Cacciopoli estimate.

Proposition 2.1. *Let* u *be a weak solution to* (1.1) *and* $K\times[t_1,t_2] \subset U\times[0,T]$. *There exists a constant* C, *depending only on* $n,m,p,K\times[t_1,t_2]$, *such that*

$$\begin{aligned}\sup_{t_1<t<t_2}\int_K u^2\xi^p dx + \int_{t_1}^{t_2}\int_K u^{m-1}|\nabla u|^p\xi^p dxdt \ &\leq\ C\int_{t_1}^{t_2}\int_K u^2\xi^{p-1}\xi_t dxdt \\ &+\ \int_{t_1}^{t_2}\int_K u^{m+p-1}|\nabla\xi|^p dxdt + C\|f\|^2_{L^{q,r}},\end{aligned}$$

for all $\xi \in C_0^\infty(K\times(t_1,t_2))$ *such that* $\xi \in [0,1]$.

Proof: Taking $\varphi = u_h\xi^p$ as a test function in (2.2) and $t \in (t_1,t_2]$ arbitrary, we have

$$\begin{aligned}\int_{t_1}^{t}\int_K (u_h)_t u_h\xi^p dxd\tau \ &+\ \int_{t_1}^{t}\int_K (mu^{m-1}|\nabla u|^{p-2}\nabla u)_h.\nabla u_h\xi^p dxd\tau \\ &+\ p\int_{t_1}^{t}\int_K (mu^{m-1}|\nabla u|^{p-2}\nabla u)_h.\nabla\xi\,\xi^{p-1}dxd\tau \\ &=\ \int_{t_1}^{t}\int_K f_h u_h\xi^p dxd\tau.\end{aligned}$$

Integrating by parts and passing to the limit in $h\to 0$, we get

$$\begin{aligned}\int_{t_1}^{t}\int_K (u_h)_t u_h\xi^p dxd\tau \ &=\ \frac{1}{2}\int_{t_1}^{t}\int_K (u_h^2)_t\xi^p dxd\tau \\ &\longrightarrow\ \frac{1}{2}\int_K u^2\xi^p(x,t)dx - \frac{1}{2}\int_K u^2\xi^p(x,t_1)dx \\ &-\ \int_{t_1}^{t}\int_K u^2\xi^{p-1}\xi_t dxd\tau.\end{aligned}$$

For $h \to 0$, we have

$$\int_{t_1}^{t}\int_K (mu^{m-1}|\nabla u|^{p-2}\nabla u)_h \nabla u_h \xi^p dxd\tau \;\longrightarrow\; m\int_{t_1}^{t}\int_K u^{m-1}|\nabla u|^p \xi^p dxd\tau.$$

Using Young's inequality and $h \to 0$,

$$\begin{aligned}
& p\int_{t_1}^{t}\int_K (mu^{m-1}|\nabla u|^{p-2}\nabla u)_h u_h \nabla\xi\,\xi^{p-1} dxd\tau \\
& \longrightarrow mp\int_{t_1}^{t}\int_K u^{m-1}|\nabla u|^{p-2}\nabla u u \nabla\xi\,\xi^{p-1} dxd\tau \\
& \leq \; mp\int_{t_1}^{t}\int_K u^{m-1}|\xi\nabla u|^{p-1}|u\nabla\xi| dxd\tau \\
& \leq \; \gamma(m,p)\int_{t_1}^{t}\int_K u^{m-1}\xi^p|\nabla u|^p dxd\tau \\
& + \; \gamma(m,p)\int_{t_1}^{t}\int_K u^{m+p-1}|\nabla\xi|^p dxd\tau.
\end{aligned}$$

Finally by Hölder inequality, we have

$$\begin{aligned}
\int_K f_h u_h \xi^p dx \;&\leq\; ||u_h\xi^p||_{\frac{q}{q-1},K}||f_h||_{q,K} \\
&\leq\; C(K,q)||u_h\xi^p||_{2,K}||f_h||_{q,K} \\
&\leq\; C(K,q)\left(\int_K u_h^2\xi^p dx\right)^{\frac{1}{2}}||f_h||_{q,K},
\end{aligned}$$

where in the last inequality we use the fact that $\xi^p \geq \xi^{2p}$. Therefore, passing to the limit in $h \to 0$ and using Young's inequality,

$$\begin{aligned}
\int_{t_1}^{t}\int_K fu\xi^p dxd\tau \;&\leq\; C(K,q)|t-t_1|^{\frac{r-1}{r}}\left(\int_K u^2\xi^p dx\right)^{\frac{1}{2}}||f||_{L^{q,r}} \\
&\leq\; \frac{1}{2}\int_K u^2\xi^p dx + C(t_1,t,K,q,r)||f||^2_{L^{q,r}}.
\end{aligned}$$

Taking the supremum over $t \in (t_1,t_2]$ the result follows. ∎

We start our fine regularity analysis by fixing the intrinsic geometric setting for our problem. Given $0 < \alpha < 1$, let

$$\theta := p - \alpha\left((p-1) - \frac{(p-1)}{m+p-2}\right), \tag{2.3}$$

which clearly satisfies the bounds

$$1 + \frac{(p-1)}{m+p-2} < \theta < p.$$

For such θ, define the intrinsic θ-parabolic cylinder as

$$G_\rho := (-\rho^\theta, 0) \times B_\rho(0), \quad \rho > 0.$$

In the sequel we show that for a certain smallness regime require to the parameters of the equation (1.1) that u can be approximated by homogeneous functions.

Lemma 2.1. *Given $\delta > 0$, there exists $0 < \varepsilon \ll 1$ such that if $\|f\|_{L^{q,r}(G_1)} \leq \varepsilon$ and u a weak solution of (1.1) in G_1, with $\|u\|_{\infty,G_1} \leq 1$, then there exists a ϕ such that*

$$\phi_t - \operatorname{div}(m\phi^{m-1}|\nabla\phi|^{p-2}\nabla\phi) = 0, \quad \textit{in} \quad G_{1/2} \tag{2.4}$$

and

$$\|u - \phi\|_{\infty,G_{1/2}} \leq \delta.$$

Proof: Suppose, for the sake of contradiction, that, for some $\delta_0 > 0$, there exists a sequence

$$(u^j)_j \in C_{loc}(0,T; L^2_{loc}(B_1)), \;\; (u^j)^{\frac{m+p-1}{p}} \in L^p_{loc}(0,T; W^{1,p}_{loc}(B_1))$$

and a sequence $(f^j)_j \in L^{q,r}(G_1)$ such that

$$u^j_t - \operatorname{div}(m(u^j)^{m-1}|\nabla u^j|^{p-2}\nabla u^j) = f^j \textit{ in } G_1 \tag{2.5}$$

$$\|u^j\|_{\infty,G_1} \leq 1, \tag{2.6}$$

$$\|f^j\|_{L^{q,r}(G_1)} \leq 1/j, \tag{2.7}$$

but still, for any j and any solution ϕ of the homogeneous equation (2.4) in $G_{1/2}$,

$$\|u^j - \phi\|_{\infty,G_{1/2}} > \delta_0. \tag{2.8}$$

Consider a cutoff function $\xi \in C_0^\infty(G_1)$, such that $\xi \in [0,1], \xi \equiv 1$ in $G_{1/2}$ and $\xi \equiv 0$ near $\partial_p G_1$. Thus, since u^j is a solution of (1.1), we can apply the Caccioppoli estimate of Proposition 2.1 to get

$$
\begin{aligned}
\sup_{t_1<t<t_2}\int_K u^2\xi^p dx + \int_{-1}^0\int_{B_1}(u^j)^{m-1}|\nabla u^j|^p\xi^p dxdt &\leq C\int_{-1}^0\int_{B_1}(u^j)^2\xi^{p-1}|\xi_t|dxdt \\
&+ \int_{-1}^0\int_{B_1}(u^j)^{m+p-1}|\nabla\xi|^p dxdt \\
&+ C\|f\|_{L^{q,r}}^2 \\
&\leq \tilde{c}, \qquad (2.9)
\end{aligned}
$$

where we use (2.6) and (2.7).
Define $v^j = (u^j)^{\frac{m+p-1}{p}}$; thus

$$
|\nabla v^j|^p = \left(\frac{m+p-1}{p}\right)^p (u^j)^{m-1}|\nabla u^j|^p
$$

and we get, by (2.9),

$$
\|\nabla v^j\|_{p,G_{1/2}}^p \leq \int_{-1}^0\int_{B_1}|\nabla v^j|^p\xi^p dxdt \leq \left(\frac{m+p-1}{p}\right)^p\tilde{c}. \qquad (2.10)
$$

Then, for a subsequence,

$$
\nabla v^j \rightharpoonup \psi
$$

weakly in $L^p(G_{1/2})$. Note that the equibounded sequence $(u^j)_j$ is also equicontinuous, by [18] and, by Arzel-Ascoli theorem, along a subsequence

$$
u^j \longrightarrow \phi,
$$

uniformly in $G_{1/2}$. We can identify $\psi = \nabla v$ once we have the pointwise convergence

$$
v^j = (u^j)^{\frac{m+p-1}{p}} \longrightarrow \phi^{\frac{m+p-1}{p}} =: v.
$$

Passing to the limit in (2.5), we find that ϕ solves (2.4) which contradicts (2.8) for $j \gg 1$.

$\blacksquare$

3.Geometric iteration

In this session we developed a geometric iteration in a certain intrinsic scaling. Here we consider

$$\beta = \frac{\alpha(p-1)}{m+p-2}$$

where α is defined as in (1.3). The following result provides the first step in the iteration process to be implemented.

Lemma 3.1. *Let u a weak solution of* (1.1) *in* G_1. *There exists* $\varepsilon > 0$, *and* $0 < \lambda \ll 1/2$, *depending only on* m, n, p *and* α *such that if* $\|f\|_{L^{q,r}(G_1)} \leq \varepsilon$, $\|u\|_{\infty,G_1} \leq 1$ *and*

$$|u(0,0)| \leq \frac{1}{4}\lambda^{\beta},$$

then

$$\|u\|_{\infty,G_\lambda} \leq \lambda^{\beta}.$$

Proof: Let $0 < \delta < 1$, to be chosen later using the last lemma, we obtain $0 < \varepsilon \ll 1$ and a solution ϕ of (2.4) in $G_{1/2}$ such that

$$\|u - \phi\|_{\infty,G_{1/2}} \leq \delta.$$

From the available regularity theory (see [14, 18]), ϕ is locally $C_x^{\alpha_*} \cap C_t^{\alpha_*/2}$, for some $0 < \alpha_* < 1$. Thus we obtain

$$\sup_{(x,t)\in G_\lambda} |\phi(x,t) - \phi(0,0)| \leq C\lambda^{\frac{\alpha_*(p-1)}{m+p-2}},$$

for $\lambda \ll 1$, to be chosen soon, and $C > 1$ universal. In fact, for $(x,t) \in G_\lambda$

$$\begin{aligned}
|\phi(x,t) - \phi(0,0)| &\leq |\phi(x,t) - \phi(0,t)| + |\phi(0,t) - \phi(0,0)| \\
&\leq c_1|x-0|^{\alpha_*} + c_2|t-0|^{\frac{\alpha_*}{2}} \\
&\leq c_1\lambda^{\alpha_*} + c_2\lambda^{\frac{\theta}{2}\alpha_*} \\
&\leq c_1\lambda^{\frac{\alpha_*(p-1)}{m+p-2}} + c_2\lambda^{\frac{\alpha_*(p-1)}{m+p-2}} \\
&\leq C\lambda^{\frac{\alpha_*(p-1)}{m+p-2}}
\end{aligned}$$

since $\theta > 1 + \dfrac{(p-1)}{m+p-2} > \dfrac{2(p-1)}{m+p-2}$. We will choose $\lambda \ll 1/2$ can therefore estimate

$$\begin{aligned}\sup_{G_\lambda}|u| &\leq \sup_{G_{1/2}}|u-\phi| + \sup_{G_\lambda}|\phi - \phi(0,0)| \\ &+ |\phi(0,0)-u(0,0)| + |u(0,0)| \\ &\leq 2\delta + C\lambda^{\frac{\alpha_*(p-1)}{m+p-2}} + \frac{1}{4}\lambda^\beta.\end{aligned} \tag{3.1}$$

Now finally fix the constants, choosing λ and δ as

$$\lambda := \left(\frac{1}{4C}\right)^{\frac{m+p-2}{(\alpha_*-\alpha)(p-1)}} \quad \text{and} \quad \delta := \frac{1}{4}\lambda^\beta$$

and fixing also $\varepsilon > 0$, through Lemma 2.1. The result follows from estimate (3.1). ■

Theorem 3.1. *Let u a local weak solution of* (1.1) *in G_1. There exists $\varepsilon > 0$, and $0 < \lambda \ll 1/2$, depending only on m,n,p and α, such that if $\|f\|_{L^{q,r}(G_1)} \leq \varepsilon$, $\|u\|_{\infty,G_1} \leq 1$ and*

$$|u(0,0)| \leq \frac{1}{4}(\lambda^k)^\beta,$$

then

$$\|u\|_{\infty,G_\lambda} \leq (\lambda^k)^\beta. \tag{3.2}$$

Proof: The proof is by induction on $k \in \mathbb{N}$. If $k=1$, (3.2) holds due to Lemma 3.1. Now suppose that the conclusion holds for k and let's show it also holds for $k+1$.

Consider the following function $v : G_1 \to \mathbb{R}$ defined by

$$v(x,t) = \frac{u(\lambda^k x, \lambda^{k\theta} t)}{\lambda^{\beta k}}. \tag{3.3}$$

We have that

$$v_t(x,t) = \lambda^{k\theta - \beta k} u_t(\lambda^k x, \lambda^{k\theta} t)$$

and

$$\nabla v(x,t) = \lambda^{k-\beta k} \nabla u(\lambda^k x, \lambda^{k\theta} t).$$

Thus,

$$\operatorname{div}(m(v(x,t))^{m-1}|\nabla v(x,t)|^{p-2}\nabla v(x,t))$$

$$= \lambda^{(p-\alpha(p-1))k}\mathrm{div}(m(u(\lambda^k x, \lambda^{k\theta}t))^{m-1}|\nabla u(\lambda^k x, \lambda^{k\theta}t)|^{p-2}\nabla u(\lambda^k x, \lambda^{k\theta}t)).$$

Recalling (2.3), we conclude, since u is a local weak solution of (1.1) in G_1, that

$$\begin{aligned} v_t - \mathrm{div}(m(v(x,t))^{m-1}|v(x,t)|^{p-2}v(x,t)) &= \lambda^{(p-\alpha(p-1))k} f(\lambda^k x, \lambda^{k\theta}t) \\ &= \tilde{f}(x,t). \end{aligned}$$

Now

$$\begin{aligned} \|\tilde{f}\|^r_{L^{q,r}(G_1)} &= \int_{-1}^{0}\left(\int_{B_1}\lambda^{(p-\alpha(p-1))kq}|f(\lambda^k x, \lambda^{k\theta}t)|^q dx\right)^{\frac{r}{q}} dt \\ &= \int_{-1}^{0}\left(\int_{B_{\lambda^k}}\lambda^{(p-\alpha(p-1))kq-kn}|f(x, \lambda^{k\theta}t)|^q dx\right)^{\frac{r}{q}} dt \\ &= \lambda^{[(p-\alpha(p-1))kq-kn]\frac{r}{q}}\int_{-1}^{0}\left(\int_{B_{\lambda^k}}|f(x, \lambda^{k\theta}t)|^q dx\right)^{\frac{r}{q}} dt \\ &= \lambda^{[(p-\alpha(p-1))kq-kn]\frac{r}{q}-k\theta}\int_{-\lambda^{k\theta}}^{0}\left(\int_{B_{\lambda^k}}|f(x,t)|^q dx\right)^{\frac{r}{q}} dt. \end{aligned}$$

To apply the Lemma 3.1 we need to have

$$[(p-\alpha(p-1))kq-kn]\frac{r}{q} - k\theta \geq 0,$$

that is,

$$k\left[[(p-\alpha(p-1))q-n]\frac{r}{q} - \left(p-\alpha\left((p-1)-\frac{(p-1)}{m+p-2}\right)\right)\right] \geq 0.$$

Since $k > 0$, we have

$$\alpha \leq \frac{(m+p-2)[(pq-n)r-pq]}{q(p-1)[r(m+p-2)-(m+p-3)]}.$$

Choosing the optimal

$$\alpha = \frac{(m+p-2)[(pq-n)r-pq]}{q(p-1)[r(m+p-2)-(m+p-3)]}$$

we have

$$\|\tilde{f}\|_{L^{q,r}(G_1)} = \|f\|_{L^{q,r}((-\lambda^{\theta k},0)\times B_{\lambda^k})} \leq \|f\|_{L^{q,r}(G_1)} \leq \varepsilon$$

which entitles v to Lemma 3.1. Note that $||v||_{\infty,G_1} \leq 1$, due to the induction hypothesis, and

$$|v(0,0)| = \left|\frac{u(0,0)}{(\lambda^k)^\beta}\right| \leq \left|\frac{\frac{1}{4}(\lambda^{k+1})^\beta}{(\lambda^k)^\beta}\right| \leq \frac{1}{4}\lambda^\beta.$$

It then follows that

$$||v||_{\infty,G_\lambda} \leq \lambda^\beta,$$

which is the same as

$$||u||_{\infty,G_{\lambda^{k+1}}} \leq \lambda^{\beta(k+1)}.$$

The induction is complete. ■

We next show the smallness regime required in the previous theorem is not restrictive and generalize it to cover the case of any small radius.

Theorem 3.2. *Let u be a local weak solution of (1.1) in $G_{1/2}$ then, for every $0 < r < \lambda$, if*

$$|u(0,0)| \leq \frac{1}{4}r^\beta$$

we have

$$||u||_{\infty,G_r} \leq Cr^\beta.$$

Proof: Take

$$v(x,t) = \rho u(\rho^a x, \rho^{(m-1)+(p-2)+pa} t)$$

with ρ, a to be fixed, which is a solution of (1.1) with

$$\tilde{f}(x,t) = \rho^{(m-1)+(p-1)+pa} f(\rho^a x, \rho^{(m-1)+(p-2)+pa} t).$$

In fact, let

$$v(x,t) = \rho u(\rho^a x, \rho^b t).$$

We have

$$v_t(x,t) = \rho^{1+b} u_t(\rho^a x, \rho^b t)$$

and

$$\nabla v(x,t) = \rho^{1+a} \nabla u(\rho^a x, \rho^b t).$$

So we obtain

$$\mathrm{div}(m(v(x,t))^{m-1} |\nabla v(x,t)|^{p-2} \nabla v(x,t))$$

$$= \rho^{(m-1)+(p-1)+pa} \mathrm{div}(m[u(\rho^a x, \rho^b t)]^{m-1} |\nabla u(\rho^a x, \rho^b t)|^{p-2} \nabla u(\rho^a x, \rho^b t)).$$

Now we choose b such that

$$1 + b = (m-1) + (p-1) + pa.$$

Therefore, we have

$$v_t - \mathrm{div}(m(v(x,t))^{m-1} |\nabla v(x,t)|^{p-2} \nabla v(x,t))$$

$$= \rho^{(m-1)+(p-1)+pa} f(\rho^a x, \rho^{(m-1)+(p-2)+pa} t) := \tilde{f}(x,t).$$

We have still

$$||v||_{\infty, G_1} \leq \rho ||u||_{\infty, G_1}$$

and

$$\|\tilde{f}\|^r_{L^{q,r}(G_1)} = \int_{-1}^{0} \left(\int_{B_1} \rho^{((m-1)+(p-1)+pa)q} |f(\rho^a x, \rho^{(m-1)+(p-2)+pa} t)|^q dx \right)^{\frac{r}{q}} dt$$

$$= \rho^{[((m-1)+(p-1)+pa)q - an]\frac{r}{q} - [(m-1)+(p-2)+pa]} \int_{-\rho^{(m-1)+(p-2)+pa}}^{0} \left(\int_{B_{\rho^a}} |f(x,t)|^q dx \right)^{\frac{r}{q}} dt$$

$$\leq \rho^{[(m-1)+(p-1)+pa]r - a(n\frac{r}{q}+p) - [(m-1)+(p-2)]} \|f\|^r_{L^{q,r}(G_1)}.$$

Now choosing $a > 0$ such that

$$[(m-1) + (p-1) + pa]r - a(n\frac{r}{q} + p) - [(m-1) + (p-2)] > 0,$$

which is always possible, and $0 < \rho << 1$, we enter the smallness regime required by Theorem 3.1 , i.e.,

$$\|v\|_{\infty, G_1} \leq 1$$

and

$$\|\tilde{f}\|_{L^{q,r}(G_1)} \leq \varepsilon.$$

Given $0 < r < \lambda$, there exists $k \in \mathbb{N}$ such that

$$\lambda^{k+1} < r \leq \lambda^k.$$

Since

$$|u(0,0)| \leq \frac{1}{4} r^\beta \leq \frac{1}{4} (\lambda^k)^\beta,$$

it follows from Theorem 3.1 that

$$||u||_{\infty, G_{\lambda^k}} \leq (\lambda^k)^\beta.$$

Then,

$$||u||_{\infty,G_r} \le ||u||_{\infty,G_{\lambda^k}} \le (\lambda^k)^\beta \le \left(\frac{r}{\lambda}\right)^\beta = Cr^\beta$$

where $C = \lambda^{-\beta}$. ∎

4. Proof of the Theorem 1.1

In this section we will prove the main result of our work, studying the Hölder continuity at the origin, proving there is a uniform constant B such that

$$||u - u(0,0)||_{\infty,G_r} \le Br^\beta. \tag{4.1}$$

Proof: Since u is continuous we can define

$$\kappa := (4|u(0,0)|)^{-\beta} \ge 0.$$

Now take any radius $0 < r < \lambda$. We will analyze the possible cases.

(1) If $\kappa \le r < \lambda$ then, by Theorem 3.1,

$$\sup_{G_r} |u(x,t) - u(0,0)| \le Cr^\lambda + |u(0,0)| \le \left(C + \frac{1}{4}\right) r^\beta. \tag{4.2}$$

(2) If $0 < r < \kappa$ we consider the function

$$w(x,t) := \frac{u(\kappa x, \kappa^\theta t)}{\kappa^\beta}.$$

Note that $|w(0,0)| = \frac{1}{4}$ and w solves in G_1

$$w_t - \operatorname{div}(mw^{m-1}|\nabla w|^{p-2}\nabla w) = \kappa^{(p-\alpha(p-1))} f(\kappa x, \kappa^\theta t).$$

Since $|u(0,0)| = \frac{1}{4}\kappa^\beta$, using Theorem 3.2, we have

$$||w||_{\infty,G_1} \le \kappa^{-\gamma} ||u||_{\infty,G_\kappa} \le C.$$

With this uniform estimate in hand, and using local $C^{0,\alpha}$ regularity estimates, we find that there exists a radius ρ_* depending only on the data, such that

$$|w(x,t)| \ge \frac{1}{8}, \quad \forall (x,t) \in G_{\rho_*}.$$

This implies that, in G_{ρ_*}, w solves a uniformly parabolic equation of the form

$$w_t - \operatorname{div}(a(x,t)|\nabla w|^{p-2}\nabla w) = f \in L^{q,r}$$

with continuous coefficients satisfying the bounds $0 < c \le a(x,t) < d$. By Theorem 5.1, we have

$$w \in C^{0,\gamma}(G_{\rho_*}), \quad \text{with } \gamma = \frac{(pq-n)r - pq}{q[(p-1)r-(p-2)]} > \beta.$$

Therefore,

$$\sup_{(x,t)\in G_r} |w(x,t) - w(0,0)| \le Cr^{\gamma} \quad \forall\ 0 < r < \frac{\rho_*}{2},$$

this is

$$\sup_{(x,t)\in G_r} \left| \frac{u(\kappa x, \kappa^{\theta} t)}{\kappa^{\beta}} - \frac{u(0,0)}{\kappa^{\beta}} \right| \le Cr^{\gamma} \quad \forall\ 0 < r < \frac{\rho_*}{2}.$$

Since $\beta < \gamma$, we conclude

$$\sup_{(x,t)\in G_{\kappa r}} |u(x,t) - u(0,0)| \le C(\kappa r)^{\beta} \quad \forall\ 0 < \kappa r < \kappa \frac{\rho_*}{2},$$

and, relabelling, we obtain

$$\sup_{(x,t)\in G_r} |u(x,t) - u(0,0)| \le Cr^{\beta} \quad \forall\ 0 < r < \kappa \frac{\rho_*}{2}. \tag{4.3}$$

(3) If $\kappa \frac{\rho_*}{2} \le r < \kappa$, we have

$$\begin{aligned} \sup_{(x,t)\in G_r} |u(x,t) - u(0,0)| &\le \sup_{(x,t)\in G_\kappa} |u(x,t) - u(0,0)| \\ &\le C\kappa^{\beta} \le \left(\frac{2r}{\rho_*}\right)^{\beta} = C' r^{\beta}. \end{aligned} \tag{4.4}$$

Taking $B = \max\{C + \frac{1}{4}, C'\}$, the result follows for every $0 < r < \lambda$.

■

5. Optimal regularity of solutions for p-Laplace type equations

In this section we establish optimal regularity estimates for solutions of equations

$$u_t - \operatorname{div}(\gamma(x,t)|\nabla u|^{p-2}\nabla u) = f \text{ in } U_T, \tag{5.1}$$

with continuous coefficients, i.e

$$|\gamma(z) - \gamma(z_0)| \le L\omega(d_p(z,z_0)^2) \tag{5.2}$$

where $z=(x,t), z_0=(x_0,t_0)\in\Omega_T$. The $\omega(.)$ denotes a modulus of continuity; that is, $\omega(.)$ is concave and non-decreasing such that $\lim_{s\downarrow 0}\omega(s)=0$. The parabolic metric is defined as usual by

$$d_p(z,z_0) := \max\{|x-x_0|, \sqrt{|t-t_0|}\} \approx \sqrt{|x-x_0|+|t-t_0|},$$

and also satisfying the bounds

$$0<\nu\le\gamma(x,t)\le L \tag{5.3}$$

for some structure constants $0<\nu\le 1\le L$. The function $f\in L^{q,r}(\Omega_T)$ where

$$\frac{1}{r}+\frac{n}{pq}<1 \quad\text{and}\quad \frac{2}{r}+\frac{n}{q}>1. \tag{5.4}$$

We show that weak solutions of (5.1) are $C^{0,\alpha}$ for

$$\alpha := \frac{(pq-n)r-pq}{q[(p-1)r-(p-2)]}, \tag{5.5}$$

which, by (5.4) we have $0<\alpha<1$. Let

$$\theta := \alpha+p-(p-1)\alpha = 2\alpha+(1-\alpha)p. \tag{5.6}$$

Note that $2<\theta<p$, since $0<\alpha<1$. For such θ, we define the intrinsic θ-parabolic cylinder

$$G_\tau := (-\tau^\theta,0)\times B_\tau(0), \quad \tau>0.$$

Lemma 5.1. *Let u be a local weak solution of (5.1) in G_1 where γ satisfies the conditions (5.2) and (5.3). Then, for every $\delta>0$, there exist $0<\varepsilon\ll 1$, such that if*

$$||f||_{L^{q,r}(G_1)}\le\varepsilon,\ \ ||u||_{p,avg,G_1}\le 1 \ \textit{ and }\ |\gamma(x,t)-\gamma(0,0)|\le\varepsilon \tag{5.7}$$

then there exist a function ϕ in $G_{1/2}$ solution of

$$\phi_t-\operatorname{div}(\gamma^0(0,0)|\nabla\phi|^{p-2}\nabla\phi)=0 \ \textit{ in }\ G_{1/2}, \tag{5.8}$$

such that

$$||u-\phi||_{p,avg,G_{1/2}}\le\delta. \tag{5.9}$$

Proof: Suppose, for the sake of contradiction, that the thesis of the lemma fails. That is, exist sequences $\{u^j\}$, $\{\gamma^j\}$ and $\{f^j\}$ for all $j\in\mathbb{N}$ satisfying

$$u^j_t-\operatorname{div}(\gamma^j(x,t)|\nabla u^j|^{p-2}\nabla u^j)=f^j \ \text{ in }\ G_1 \tag{5.10}$$

where

$$
\begin{aligned}
||f^j||_{L^{q,r}(G_1)} = o(j), \ \ ||u^j||_{p,avg,G_1} \le 1 \ \text{ and} \\
|\gamma^j(x,t) - \gamma^j(0,0)| = o(j),
\end{aligned}
\tag{5.11}
$$

but, for some $\delta_0 > 0$, there holds

$$
||u - \phi||_{p,avg,G_{1/2}} \ge \delta_0 \tag{5.12}
$$

for any solution ϕ of (5.8) in $G_{1/2}$. Fix a cutoff function $\xi \in C_0^\infty(G_1)$, such that $\xi \in [0,1]$, $\xi \equiv 1$ in $G_{1/2}$ and $\xi \equiv 0$ near $\partial_p G_1$. From the Cacciopoli estimate (see in [9])

$$
\begin{aligned}
\sup_{-1<t<0} \int_{B_1} u^2 \xi^p dx + C \int_{-1}^0 \int_{B_1} |\nabla u^j|^p \xi^p dxdt \ &\le \ C \int_{-1}^0 \int_{B_1} (u^j)^2 \xi^{p-1} |\xi_t| dxdt \\
&+ \ \int_{-1}^0 \int_{B_1} |u^j|^p (\xi^p + |\nabla \xi|^p) dxdt \\
&+ \ C\|f\|^2_{L^{q,r}(G_1)} \\
&\le \ \tilde{c}.
\end{aligned}
\tag{5.13}
$$

So, we have

$$
||\nabla u^j||^p_{p,G_{1/2}} \le \int_{-1}^0 \int_{B_1} |\nabla u^j|^p \xi^p dxdt \le \tilde{c}.
$$

On the other hand, a control of times derivative (see [1], section 7), gives

$$
||u^j_t||_{L^{s,1}(G_{1/2})} \le c,
$$

with $s = \min\left\{q, \frac{p}{p-1}\right\} < p$. By the classical compactness result (cf. [21], Corollary 4), with

$$
W^{1,p} \hookrightarrow L^p \subset L^s,
$$

to conclude that

$$
u^j \longrightarrow \psi \text{ in } L^p(G_{1/2})
$$

$$
\nabla u^j(x,t) \longrightarrow \nabla \psi(x,t) \ \text{ for a.e. } \ (x,t) \in G_{1/2}.
$$

Note also that the sequence $\{\gamma^j(0,0)\}$ is bounded and equicontinuous, therefore by Ascoli-Arzel, $\gamma^j(0,0) \to \gamma^0(0,0)$ uniformly in $G_{1/2}$. Hence, by (5.11) we obtain

$$\begin{aligned} |\gamma^j(x,t)|\xi|^{p-2}\xi - \gamma^0(0,0)|\xi|^{p-2}\xi| &\le |\gamma^j(x,t)-\gamma^j(0,0)||\xi|^{p-1} \\ &+ |\gamma^j(0,0)-\gamma^0(0,0)||\xi|^{p-1} = o(j) \end{aligned}$$

for any $(x,t) \in G_{1/2}$ and $\xi \in B_M$. That is, for we have verified

$$\gamma^j(x,t)|\xi|^{p-2}\xi \to \gamma^0(0,0)|\xi|^{p-2}\xi$$

uniformly in $G_{1/2} \times \mathbb{R}^n$. Using standard arguments we have that u^0 solves the constant coefficients equation

$$u_t^0 - \operatorname{div}(\gamma^0(0,0)|\nabla u^0|^{p-2}\nabla u^0) = 0 \ \text{ in } \ G_{1/2},$$

which contradicts (5.12) for $\phi = u^0$. ■

Lemma 5.2. *Let u a local weak solution of (5.1) in G_1 where γ satisfies the conditions (5.2) and (5.3) and $0 < \alpha < 1$ be fixed. There exist $\varepsilon > 0$ and $0 < \lambda \ll 1/2$ depeding only on p, n and α, such that if*

$$||f||_{L^{q,r}(G_1)} \le \varepsilon, \quad ||u||_{p,avg,G_1} \le 1 \quad \textit{and} \quad \sup_{G_1}|\gamma(x,t)-\gamma(0,0)| \le \varepsilon$$

then there exist a universally bounded constant c_0 such that

$$||u-c_0||_{p,avg,G_\lambda} \le \lambda^\alpha. \tag{5.14}$$

Proof: Take $0 < \delta < 1$, to be chosen later, and apply Lemma 5.1 to obtain $0 < \varepsilon \ll 1$ and a solution ϕ of some constant coefficient equation in $G_{1/2}$, such that

$$||u-\phi||_{p,avg,G_{1/2}} \le \delta.$$

By the local regularity estimates to solution of constant coefficients equations (see [6]), we get

$$\sup_{(x,t)\in G_\lambda} |\phi(x,t)-\phi(0,0)| \le C\lambda$$

for $C > 1$ universal. In fact, for $(x,t) \in G_\lambda$,

$$\begin{aligned} |\phi(x,t)-\phi(0,0)| &\le |\phi(x,t)-\phi(0,t)| + |\phi(0,t)-\phi(0,0)| \\ &\le C'|x-0| + C''|t-0|^{\frac{1}{2}} \\ &\le C'\lambda + C''\lambda^{\frac{\theta}{2}} \le C\lambda \end{aligned}$$

since $\theta < 2$. Therefore,

$$\begin{aligned} ||u(x,t)-u(0,0)||_{p,avg,G_\lambda} &\leq ||u(x,t)-\phi(x,t)||_{p,avg,G_{1/2}} \\ &+ ||\phi(x,t)-\phi(0,0)||_{p,avg,G_\lambda} \\ &\leq \delta + C\lambda \end{aligned} \tag{5.15}$$

where choose $\lambda \ll 1/2$. We put $c_0 := \phi(0,0)$, observing that

$$||\phi||_{p,avg,G_{1/2}} \leq ||u-\phi||_{p,avg,G_{1/2}} + ||u||_{p,avg,G_{1/2}} \leq 1+\delta \leq 2, \tag{5.16}$$

and ϕ is solution of constant coefficient equation, c_0 is universally bounded. We choose $\lambda \ll 1/2$ so small that

$$C\lambda \leq \frac{1}{2}\lambda^\alpha$$

and then we define

$$\delta = \frac{1}{2}\lambda^\alpha$$

thus fixing, via Lemma 5.1, also $\varepsilon > 0$. The lemma follows from estimate (5.15). ■

Lemma 5.3. *Under the conditions of the previous lemma, there exists a convergent sequence of real numbers $\{c_k\}_{k\geq 1}$ such that*

$$||u-c_k||_{p,avg,G_{\lambda^k}} \leq (\lambda^k)^\alpha, \tag{5.17}$$

with

$$|c_k - c_{k+1}| \leq C(\lambda^\alpha)^k \tag{5.18}$$

for some universal constant $C > 0$.

Proof: The proof is by induction on $k \in \mathbb{N}$. For $k = 1$, (5.18) holds due to Lemma 5.1, with $c_1 = c_0$. Suppose the conclusion holds for k and let's show it also holds for $k+1$. For this consider the function $v : G_1 \to \mathbb{R}$ such that

$$v(x,t) = \frac{u(\lambda^k x, \lambda^{\theta k} t) - c_k}{\lambda^{\alpha k}}. \tag{5.19}$$

We have

$$v_t(x,t) = \lambda^{k\theta - \alpha k} u_t(\lambda^k x, \lambda^{\theta k} t)$$

and

$$\nabla v(x,t) = \lambda^{k-\alpha k} \nabla u(\lambda^k x, \lambda^{\theta k} t).$$

Therefore

$$\begin{gathered} \operatorname{div}(\gamma(\lambda^k x, \lambda^{\theta k} t)|\nabla v(x,t)|^{p-2}\nabla v(x,t)) \\ = \lambda^{pk-(p-1)\alpha k}\operatorname{div}(\gamma(\lambda^k x, \lambda^{\theta k} t)|\nabla u(\lambda^k x, \lambda^{\theta k} t)|^{p-2}\nabla u(\lambda^k x, \lambda^{\theta k} t)) \end{gathered}$$

to concluded, recalling (5.6), that

$$
\begin{aligned}
v_t(x,t) - \operatorname{div}(\gamma(\lambda^k x, \lambda^{\theta k} t)|\nabla v(x,t)|^{p-2}\nabla v(x,t)) \\
= \lambda^{pk-(p-1)\alpha k} f(\lambda^k x, \lambda^{\theta k} t) := \tilde{f}(x,t).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\|\tilde{f}\|^r_{L^{q,r}(G_1)} &= \int_{-1}^{0}\left(\int_{B_1}\lambda^{[pk-(p-1)\alpha k]q}|f(\lambda^k x, \lambda^{k\theta} t)|^q dx\right)^{\frac{r}{q}} dt \\
&= \rho^{[(pk-(p-1)\alpha k)q-kn]\frac{r}{q}-k\theta}\int_{-\lambda^{k\theta}}^{0}\left(\int_{B_{\lambda^k}}|f(x,t)|^q dx\right)^{\frac{r}{q}} dt.
\end{aligned}
$$

Due to the crucial and sharp choice (2.3) of α, we have, recalling again (5.5),

$$
[(pk-(p-1)\alpha k)q-kn]\frac{r}{q} - k\theta = 0.
$$

So, we have

$$
||\tilde{f}||_{L^{q,r}(G_1)} \le ||f||_{L^{q,r}(G_{\lambda^k})} \le ||f||_{L^{q,r}(G_1)} \le \varepsilon,
$$

and $||v||_{p,avg,G_1} \le 1$, due to the induction hypothesis. By the Lemma 5.2 there exist a universally bounded constant $\tilde{c_0}$, such that

$$
||v - \tilde{c_0}||_{p,avg,G_\lambda} \le \lambda^\alpha,
$$

wich is the same as

$$
\frac{1}{|G_\lambda|}\int_{G_\lambda}\left|\frac{u(\lambda^k x, \lambda^{k\theta t}) - c_k}{\lambda^{\alpha k}} - \tilde{c_0}\right|^p dxdt \le \lambda^{\alpha p}
$$

where it follows that

$$
\frac{1}{|G_\lambda|\lambda^{\alpha k p}}\int_{G_\lambda}|u(\lambda^k x, \lambda^{k\theta t}) - c_{k+1}|^p dxdt \le \lambda^{\alpha p}
$$

for $c_{k+1} := c_k + \tilde{c_0}\lambda^{\alpha k}$. Making the following variable change $y = \lambda^k x$ and $s = \lambda^{\theta k} t$ we have

$$
\frac{1}{|G_\lambda|\lambda^{kn}\lambda^{k\theta}}\int_{G_{\lambda^{k+1}}}|u(y,s) - c_{k+1}|^p dxdt \le \lambda^{\alpha k p}\lambda^{\alpha p}.
$$

Therefore

$$
||u - c_{k+1}||_{p,avg,G_{\lambda^{k+1}}} \le \lambda^{\alpha(k+1)},
$$

where $c_{k+1} := c_k + \tilde{c_0}\lambda^{\alpha k}$ and the induction is complete. To finish we observe that

$$
|c_{k+1} - c_k| \le c(\lambda^\alpha)^k
$$

where c is a universal constant. ■

Note that

$$
\begin{aligned}
|c_k - c_{k+j}| &\leq |c_k - c_{k+1}| + |c_{k+1} - c_{k+2}| + \dots + |c_{k+j-1} - c_{k+j}| \\
&= \sum_{i=0}^{j-1} |c_{k+i} - c_{k+i+1}| \\
&\leq \sum_{i=0}^{j-1} C(\lambda^{k+i})^{\alpha} \\
&\leq C\lambda^{\alpha k} \sum_{i=0}^{\infty} (\lambda^{\alpha})^{i} \\
&= \frac{C}{1-\lambda^{\alpha}} \lambda^{\alpha k}.
\end{aligned}
$$

Due to (5.18), the sequence $\{c_k\}_{k\geq 1}$ is convergent, let's say $c_k \to \tilde{c}$. Therefore

$$
|\tilde{c} - c_k| \leq \frac{C}{1-\lambda^{\alpha}} \lambda^{\alpha k}. \tag{5.20}
$$

Theorem 5.1. *A locally bounded weak solution of (5.1), where γ satisfies the conditions (5.2) and (5.3) and $f \in L^{q,r}$, satisfying (5.4) is Hölder continuous in the space variables, with exponent*

$$
\alpha = \frac{(pq-n)r - pq}{q[(p-1)r - (p-2)]}
$$

and locally Hölder continuous in time with exponent $\frac{\alpha}{\theta}$.

Proof: Let

$$
v(x,t) = \rho u(\rho^{a} x, \rho^{(p-2)+ap} t)
$$

with ρ, a to be fixed which solves

$$
\begin{aligned}
v_t(x,t) - \operatorname{div}(\gamma(\rho^{a} x, \rho^{(p-2)+ap} t)|\nabla v(x,t)|^{p-2} \nabla v(x,t)) \\
= \rho^{(p-1)+ap} f(\rho^{a} x, \rho^{(p-2)+ap} t) := \tilde{f}(x,t).
\end{aligned}
$$

We have,

$$
||v||^{p}_{p,avg,G_1} \leq \rho^{2-a(n+p)} ||u||^{p}_{p,avg,G_1}
$$

and

$$
||\tilde{f}||^{r}_{L^{q,r}(G_1)} \leq \rho^{[(p-1)+ap]r - a(n+p) - (p-2)} ||f||^{r}_{L^{q,r}(G_1)}.
$$

We choose $a > 0$ such that

$$2 - a(n+p) > 0 \quad \text{and} \quad [(p-1)+ap]r - a(n+p) - (p-2) > 0,$$

which is always possible (observe that the second condition holds for $a = 0$ and use its continuity with respect to a), and then $0 < \rho \ll 1$ we get

$$||v||_{p,avg,G_1} \leq 1 \quad \text{and} \quad ||\tilde{f}||_{L^{q,r}(G_1)} \leq \varepsilon.$$

Now, given $0 < r < \lambda$, there exists $k \in \mathbb{N}$ such that

$$\lambda^{k+1} \leq r < \lambda^k,$$

it follows from Theorem 5.3 that

$$||u - c_k||_{p,avg,G_{\lambda^k}} \leq (\lambda^k)^\alpha. \tag{5.21}$$

Then by (5.20) we get

$$\begin{aligned}
\fint_{G_r} |u(x,t) - \tilde{c}|^p dxdt &\leq \frac{|G_{\lambda_k}|}{|G_r|}\left(\fint_{G_{\lambda^k}} |u(x,t) - c_k|^p dxdt + \fint_{G_{\lambda^k}} |\tilde{c} - c_k|^p dxdt\right) \\
&\leq \frac{1}{\lambda^{n+\theta}}\left(1 + \frac{C}{1-\lambda^\alpha}\right)(\lambda^k)^\alpha \\
&\leq \left[\left(1 + \frac{C}{1-\lambda^\alpha}\right)\frac{1}{\lambda^\alpha \lambda^{n+\theta}}\right] r^\alpha \\
&= \tilde{C} r^\alpha
\end{aligned}$$

where $\tilde{C} = \left(1 + \frac{C}{1-\lambda^\alpha}\right)\frac{1}{\lambda^\alpha\lambda^{n+\theta}}$. By standard covering arguments (see [24], Lemma 3.2) and the characterisation of Hölder continuity of Campanato-Da Prato, we have local $C^{0;\alpha,\alpha/\theta}(G_{1/2})$- continuity and thus the proof is complete. ■

The proofs adapt to more general degenerate parabolic equations

$$u_t - \operatorname{div} A(x,t,\nabla u) = f \in L^{r,q}$$

satisfying

$$\begin{cases} |A(x,t,\xi)| \leq C_1|\xi|^{p-1} + \varphi_1(x,t) \\ A(x,t,\xi)\cdot\xi \geq C_0|\xi|^p - \varphi_0(x,t) \\ |A(z,\xi) - A(z_0,\xi)| \leq C_1\omega(d_p(z,z_0)^2)|\xi|^{p-1} \end{cases} \tag{5.22}$$

for $\xi \in \mathbb{R}^n$, $z = (x,t), z_0 = (x_0,t_0) \in \Omega_T$ and C_0, C_1 are given positive constants, φ_0, φ_1 are given non-negative functions, in an appropriate function space (more details see [9]) for $p \geq 2$.

The proof is the same, just note that in Theorem 3.1 the rescaled function v defined in (3.3) now solves the equation

$$v_t - \mathrm{div} A_k(x,t,\nabla v) = \lambda^{pk-(p-1)\alpha k} f(\lambda^k x, \lambda^{k\theta} t),$$

where

$$A_k(x,t,\xi) := \left(\lambda^{-\alpha k}\right)^{1-p} A(\lambda^k x, \lambda^{k\theta} t, \lambda^{-\alpha k}\xi)$$

belongs to the same structural class of A.

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