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# NONORIENTABLE HYPERMAPS OF A GIVEN TYPE AND GENUS

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ABSTRACT: We prove that given positive integers $m, n$ with $2m^{-1} + n^{-1} < 1$ and an integer $g \geq 1$, there are infinitely many nonisomorphic compact nonorientable hypermaps of type $(m, m, n)$ and genus $g$. The technique we apply here is based on the constructions used to demonstrate the same result for orientable hypermaps, making the suitable adjustaments.

KEYWORDS: topological hypermaps, Walsh map, nonorientable surfaces.

## 1. Introduction

Topologically, a *map* is a cellular embedding of a connected graph into a closed connected surface. If that underlying surface is orientable, we say that the map is orientable. Otherwise, the map is called nonorientable. A *hypermap* is a generalization of this concept. Instead of a graph, we embed a hypergraph on a surface, allowing each (hyper)edge to be adjacent to more than two (hyper)vertices. Hypermaps are usually represented by cellular embeddings of connected trivalent graphs (James representation [1]) or by cellular embeddings of bipartite maps (following the Walsh correspondence between bipartite maps and hypermaps [2]). A Walsh map $\mathcal{W} = W(\mathcal{H})$ of a hypermap $\mathcal{H}$ is a bipartite map on the same surface as $\mathcal{H}$, with each hypervertex or hyperedge of $\mathcal{H}$ represented as a black or white vertex, each incidence between them represented as an edge between the corresponding vertices, so that each vertex has the same valency as the hypervertex or hyperedge it represents, and each hyperface of $\mathcal{H}$ is represented as a face of twice the valency (since it is bordered by alternating black and white vertices) [2]. We say that a hypermap has type $(l, m, n)$ if $l$, $m$ and $n$ are the least common multiple of the valencies of the hypervertices, hyperedges and hyperfaces, respectively.

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Algebraically, a hypermap is completely determined by a subgroup (a hypermap subgroup) of the free product $\Delta = C_2 * C_2 * C_2$.

The extended triangle group

$$\Delta[l, m, n] = \langle R_0, R_1, R_2 \mid R_i^2 = (R_1R_2)^l = (R_2R_0)^m = (R_0R_1)^n = 1\rangle$$

generated by reflections $R_0, R_1$ and $R_2$ in the sides of a triangle $T$ with angles $\pi/l, \pi/m$ and $\pi/n$ in a simply connected Riemann surface $\mathcal{U}$, where $\mathcal{U}$ is the Riemann sphere, the complex plane or the hyperbolic plane as $l^{-1} + m^{-1} + n^{-1} > 1, = 1$ or $< 1$. The orientation-preserving subgroup of index 2 in $\Delta[l, m, n]$ is the triangle group

$$\Delta = \Delta(l, m, n) = \langle X, Y, X \mid X^l = Y^m = Z^n = XYZ = 1\rangle,$$

generated by rotations $X = R_1R_2, Y = R_2R_0$ and $Z = R_0R_1$ through angles $2\pi/l, 2\pi/m$ and $2\pi/n$ around the vertices of $T$. These two groups are, respectively, the full automorphism group and the orientation-preserving automorphism group of the universal hypermap $\tilde{\mathcal{H}}$ of type $\tau = (l, m, n)$ drawn on $\mathcal{U}$. Any hypermap $\mathcal{H}$ of this type is isomorphic to the quotient of $\tilde{\mathcal{H}}$ by some subgroup $H \leq \Delta[l, m, n]$, which is unique up to conjugacy. Conversely, any conjugacy class of subgroups $H$ determines a hypermap $\mathcal{H}/H$ of type $\tau' = (l', m', n')$ where $l', m'$ and $n'$ (dividing $l, m$ and $n$) are the orders of the permutations of the cosets of $H$ induced by $X, Y$ and $Z$. Two hypermaps are isomorphic if and only if the corresponding subgroups are conjugate in $\Delta[l, m, n]$ (or in $\Delta(l, m, n)$ if we require an orientation-preserving isomorphism). Compact hypermaps $\mathcal{H}$ correspond to subgroups $H$ of finite index in $\Delta[l, m, n]$, and those on orientable surfaces without boundary correspond to subgroups $H \leq \Delta(l, m, n)$. A permutation of the triple $(l, m, n)$ corresponds to a renaming of the generators of $\Delta[l, m, n]$ and of $\Delta(l, m, n)$, or equivalently to one of Machì's operations on hypermaps, permuting hypervertices, hyperedges and hyperfaces [3].

## 2. Conjectures

**Conjecture 2.1.** *Given positive integers $l, m, n$ with $l^{-1}+m^{-1}+n^{-1} < 1$, and an integer $g \geq 0$, there are infinitely many nonisomorphic compact orientable hypermaps of type $(l, m, n)$ and of genus $g$.*

This conjecture first appeared in [4] and is a slightly stronger (and topological) version of another conjecture that arose in discussions between the authors and Jürgen Wolfart [5, 4]:

**Conjecture 2.2.** *Given positive integers $l, m, n$ with $l^{-1} + m^{-1} + n^{-1} < 1$, and an integer $g \geq 0$, the triangle group*

$$\Delta = \Delta(l, m, n) = \langle X, Y, Z \mid X^l = Y^m = Z^n = ZYZ = 1\rangle$$

*contains infinitely many subgroups of finite index and of genus $g$.*

Both of these conjectures are independent of the ordering of $l, m$ and $n$. If at least two of those parameters are equal, the conjecture was proved true [4]:

**Theorem 2.1.** *Conjectures 2.1 and 2.2 are true in all cases where at least two of $l, m$ and $n$ are equal.*

If we rewrite Conjecture 2.1 for nonorientable hypermaps, the connection between the topological approach and Conjecture 2.2 is lost. It still remains, however, an interesting conjecture about hypermaps. In this paper, we will prove that Conjecture 2.1 is also true for nonorientable hypermaps (in all cases where at least two of the parameters are equal), if we discard genus 0. In the orientable case, in order to prove the conjecture for a specific triple $(m, m, n)$, suitable copies of three different building blocks are joined together. Since those constructions play a very important role in the nonorientable case, we will now summarize the method followed in [4] (which is based on the method used in [6]).

## 3. Orientable surfaces

**3.1. General method.** Because we can permute the paremeters and we are just looking at triples in which two of the parameters are equal, it is enough to deal with hypermaps of type $\tau = (m, m, n)$ corresponding to Walsh maps of type $\{2n, m\}$ on the same surface. Hence, proving the conjecture for hypermaps of type $(m, m, n)$ is equivalent to proving it for bipartite maps of type $\{2n, m\}$ since $W(\mathcal{H}) \cong W(\mathcal{H}')$ if and only if $\mathcal{H} \cong \mathcal{H}'$.

In order to obtain a bipartite map $W$ of type $\mu = \{2n, m\}$ corresponding to a hypermap of type $(m, m, n)$, some *building blocks* are constructed. Those building blocks are three different bipartite maps on three surfaces with boundary: a *2-trisc* $\mathcal{T}$ (a torus minus two discs, with $\mathcal{X}(\mathcal{T}) = -2$), a *closed annulus* $\mathcal{A}$ (with $\mathcal{X}(\mathcal{A}) = 0$) and a *disc* $\mathcal{D}$. Bipartite maps of type $\mu = \{2n, m\}$ on each of these surfaces will be denoted, respectively, by $\mathcal{T}_\mu$, $\mathcal{A}_\mu$ and $\mathcal{D}_\mu$ or, sometimes, $\mathcal{T}_i$, $\mathcal{A}_i$ and $\mathcal{D}_i$ (with $i = n$ or $m$) if we want to emphasise the valency of the faces or vertices involved in the map (assuming the other parameter is fixed and known).

When a bipartite map exists on $\mathcal{A}$, $\mathcal{D}$ or $\mathcal{T}$, it is required that each boundary component of the surface must be a cycle in the map. If $C_0$ and $C_1$ are two cycles of the same length from two different components, an *allowed joining* between the two maps is a homeomorphism $C_0 \to C_1$ which sends vertices to vertices of the same colour so that $C_0$ and $C_1$ become a single cycle in the resulting bipartite map and adjacency is preserved. If vertices of valencies $v_0$ and $v_1$ in $C_0$ and $C_1$ are identified with each other, they give rise to a vertex of valency $v = v_0 + v_1 - 2$, so we also require that $v$ divides $m$; in fact, we will generally arrange that $v = m$.

To prove Theorem 2.1 for a specific triple, we have to join suitable many copies of the building blocks (in order to achieve the required genus) and carefully identify boundary components (so that our final hypermap has the right type). As in [4], this will be done by constructing the Walsh Map $\mathcal{W}(\mathcal{H})$ of type $\mu = \{2n, m\}$ that corresponds to the hypermap $\mathcal{H}$ of type $\tau = (m, m, n)$.

If two surfaces $X_0$ and $X_1$ are joined by identifying their boundary components $C_0$ and $C_1$, then the resulting surface has Euler characteristic $\chi(X_0 \cup X_1) = \chi(X_0) + \chi(X_1)$. Since $\chi(A) = 0$, $\chi(T) = -2$ and $\chi(D) = 1$, if $g \geq 2$ then $g - 1$ copies of $\mathcal{T}$ and an arbitrary number $h \geq 0$ of copies of $\mathcal{A}$ can be joined pairwise in some cyclic order to give an orientable bipartite map $\mathcal{W}$ of characteristic $2 - 2g$ and hence of genus $g$; by fixing $g$ and letting $h$ vary we obtain the required infinite set of nonisomorphic hypermaps $\mathcal{H}$. This method of proof is based on that used in [6]. By ignoring the vertex-colours, we can regard each $\mathcal{W} = \mathcal{W}_\mu$ as an orientable map of type $\mu$, so the same constructions prove Conjecture 2.1 for maps of this type.

In both cases (orientable and nonorientable), some methods and structures are mentioned several times:

**Multiplication of an edge**: the multiplication of an edge $e$ of the map, by an integer $k$, consists of replacing $e$ with $k$ edges between the same pair of vertices, enclosing $k - 1$ new faces of valency 2. Multiplication by 1 leaves the graph as before. If $e$ is a boundary edge then one of these new edges will also be a boundary edge (but not the other ones). The valencies of the vertices of the boundary components are relevant to describe the building blocks and to confirm that a map of a specific type is obtained when they are glued together. We say that a boundary component (denoted by $\partial^i A$, $\partial^i T$ or $\partial D$ for $i = 0, 1$) has *type* $k^{(t)}$ if it has $t$ vertices of valency $k$. If not all the vertices have the same valency, those different valencies are expliciltly given to the reader.

**Stalks**: paths of odd length $(n-1)$ with consecutive vertices $v_0, v_1, ..., v_{n-1}$ alternately black and white and with alternate edges $v_i v_{i+1}$ ($i$ odd) multiplied by $m-1$ so that $v_0$ and $v_{n-1}$ have valency 1 while the others have valency $m$. By attaching a stalk $S$ to a vertex $v$ within a face $F$ we mean identifying $v_0$ or $v_{n-1}$ with $v$, as $v$ is black or white, and embedding the rest of the stalk in $F$ without crossings. This raises the valency of the face $F$ by $2(n-1)$ and that of $v$ by 1. It also introduces $(m-2)(n-2)/2$ new faces of valency 2, together with $n-2$ vertices of valency $m$ and one of valency 1. Because these new faces have valency 2 they correspond to hyperfaces of valency 1 so they do not affect the type of the final hypermap. On the other hand, the vertex where the stalk is attached increases its valency by 1.

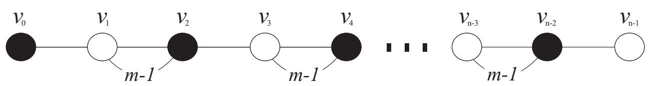



FIGURE 1. A stalk

**3.2. Proof.** For orientable surfaces, the proof is divided in ten cases for different families of hypermaps, covering all possibilities for the type [4].

- $(m, m, n)$ with $m \geq 4$, even $n \geq 4$;
- $(m, m, 2)$ with $m \geq 6$;
- $(5, 5, 2)$;
- $(3, 3, 4)$;
- $(3, 3, n)$ with even $n \geq 6$;
- $(m, m, n+1)$ with $m \geq 5$, odd $n+1 \geq 5$;
- $(m, m, 3)$ with $m \geq 5$;
- $(4, 4, 3)$;
- $(4, 4, n)$ with, odd $n \geq 5$;
- $(3, 3, n)$ with odd $n \geq 5$.

For each one of these cases, suitable building blocks have been constructed and glued.

## 4. Nonorientable Surfaces

### 4.1. Main theorem.

**Theorem 4.1.** *Given positive integers $m$, $n$ with*

$$2m^{-1} + n^{-1} < 1$$

*and an integer $g \geq 1$, there are infinitely many nonisomorphic compact nonorientable hypermaps of type $(m, m, n)$ and genus $g$.*

As we have mentioned before, this theorem is independent of the ordering of the parameters but here we will only deal with hypermaps such that the least common multiple of the valencies of the hypervertices (black vertices in the Walsh map) and the least common multiple of valencies of the hyperedges (white vertices in the Walsh map) are equal.

**4.2. The proof.** In all those cases where a boundary component of the 2-trisc or the annulus has a symmetry which reverses the cyclic order of valencies and colors of its vertices, we can generalize the constructions used on orientable surfaces and make them work on nonorientable surfaces. For each orientable hypermap of type $\tau$ and genus $g \geq 1$, we can reverse the orientation of a boundary component in one of the allowed joins, giving a nonorientable hypermap of type $\tau$ and of the same Euler characteristic $2 - 2g$, that is of nonorientable genus $p = 2g$. This means that we can use this method for $p$ even. However it does not work if the joins are in *linear* order, since reversing a boundary component would give rise to an orientable surface. We need to take $(g - 1)$ 2-triscs, an arbitrary number of annuli and no discs, joined in *cyclic* order, to construct an orientable surface of genus $g \geq 1$ (see Figure 2); then reversing one of the joins gives an nonorientable surface of the same Euler characteristic $2 - 2g \leq 0$.

However, if we want to do it also for $p$ odd, we need to construct a suitable crosscap, for instance an annulus with antipodal points of one boundary identified. The gluing process is identical to the one used in the orientable case but we need to replace one of the discs with a crosscap.

For each nonorientable case, we will use the same annulus that was constructed for the corresponding orientable case [4] and, after small changes (if

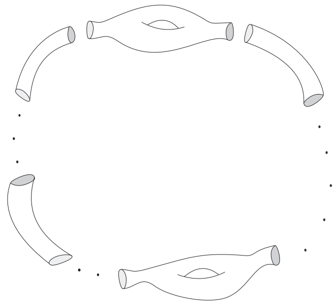


FIGURE 2. Pieces joined in cyclic order.

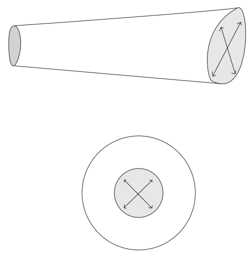


FIGURE 3. How to construct a crosscap from an annulus.

required), we will identify antipodal points of one of the boundary components, making sure we are also identifying opposite pairs of vertices in such a way that the type of the map is preserved after this procedure. In some cases, the introduction of new vertices in one of the boundary components is needed and, as a consequence, if that modification affects both boundaries, that also implies some changes on the other building blocks. When these changes are straightforward, we do not give details.

The proof for nonorientable hypermaps will be divided in several cases, according to the type of the boundary components of the annuli built for orientable hypermaps [4].

**4.2.1.** *Annnuli with boundary components of type $m_0^{(4)}$ and $m_1^{(4)}$.* This corresponds to the following types for the hypermap [4]:

- $(m, m, n)$ with $m \geq 4$, even $n \geq 4$ ;
- $(m, m, 2)$ with $m \geq 6$;
- $(5, 5, 2)$;.
- $(3, 3, 4)$.

We can assume, without loss of generality, that $m_0 \leq m_1$. Hence, to build a crosscap, we only need to multiply by $k = m_1 - m_0 + 1$ one of the edges between two adjacent vertices of the boundary component of type $m_0^{(4)}$. These two vertices will have then valency $m_1$ and after identification of opposites pairs we will get two vertices of type $m = m_1 + m_0 - 2$. The other boundary component of the annulus, and of type $m_1^{(4)}$, remains as before and without any changes can be glued to any of the other blocks.

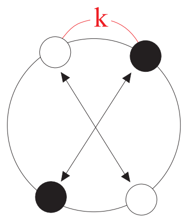



FIGURE 4. Adjustment of one the boundary components of the annulus and identification of opposite vertices.

This method also works for annuli with more than 4 vertices, if the number of black (and white) vertices is even. Instead of multiplying by $k = m_1 - m_0 + 1$ just one of the edges between consecutive vertices, we multiply by $k$ a suitable number of edges between consecutive disjoint pairs of vertices, in order to change the valency of half of the boundary vertices (leaving the other half unchanged). Hence, any annulus with vertices of the same valency in one boundary component can be easily adapted to become a crosscap by identifying opposite points on that boundary, if the number of white vertices and the number of black vertices are even (see Figure 4). This procedure will be called the *standard method*.

**4.2.2.** *Annnuli with boundary components of type* $(m_0 - 1, m_0, m_0, m_0)$ *and* $(m_1 + 1, m_1, m_1, m_1)$. This corresponds to the case:

- $(m, m, n + 1)$ with $m \geq 5$, odd $n + 1 \geq 5$;

The previous method does not work here because not all the vertices on the boundary components have the same valency. Hence, we need to change the annulus used in the orientable case by taking a less standard approach.

To construct a new map $\mathcal{A}$ on the annulus, we use a tessellation of $R = [0, 4] \times [0, n - 3]$ with vertices of the form $(i, j)$ with $j \in \{0, n - 3\}$, $i \in \{0, 1, 2, 3, 4\}$ and of the form $(i, j)$, with $j \in \{1, 2, ..., n - 4\}$, $i \in \{2, 3\}$. If $i + j$ is odd, the vertex is white, otherwise it is black. The consecutive vertices for $j = n - 3$ and $j = 0$ are linked by an horizontal edge, except $(0, 0)$ and

$(0,1)$ that are linked by two edges. For $i \in \{2,3\}$, the vertices $(i,j)$ are attached to vertices $(i,j+1)$, for $j \in \{0,...,n-5\}$, by a vertical edge. And for $j \in \{1,...,n-5\}$ there is a horizontal egde between vertices $(2,j)$ and $(3,j)$. Between $(2,n-4)$ and $(3,n-4)$ the horizontal edge is multiplied by two. We then insert a white vertex in the face of valency $2n+2$ joined by an edge to the black vertex at $(1,n-3)$. At each white vertex of the form $(2,1),(2,3),...,(2,n-5)$ we attach a stalk of length $n-3$, with all interior vertices of valency $m$, within the incident square face $i < x < i+1$, $j < y < j+1$; at each black vertex of the form $(3,1),(3,3),...,(3,n-5)$ we also attach a stalk of length $n-3$, with all interior vertices of valency $m$, within the incident square face $i-1 < x < i$, $j-1 < y < j$. It follows that each of the initial square faces has now valency $2(n+1)$. This annulus is represented in Figure 5, with stars instead of stalks of length $n-3$. Before identifying the vertical sides of the rectangle, we conveniently multiply the horizontal edges to have all interior vertices of valency $m$ and boundary components of type $(m_1+1, m_1, m_1, m_1)$, as in the orientable case (so that we can glue the other building blocks to the annulus), and of type $m_0^{(4)}$ (so that we can apply the standard method). This new map is used just once (to build a crosscap). The other annuli of our constructions remain as before (as on the orientable case).

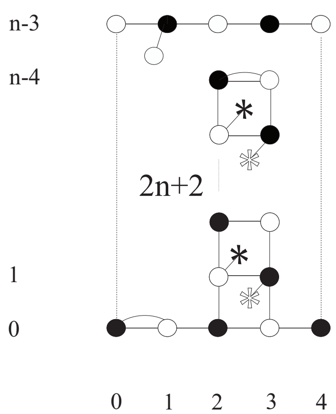



FIGURE 5. Annulus map for the odd nonorientable case (before multiplication of horizontal edges), with vertical sides of the rectangle identified.

**4.2.3.** *Annnuli with boundary components of type* $3^{(6)}$ *and* $(m-1)^{(6)}$. This corresponds to the cases:

- $(m, m, 3)$ with $m \geq 5$;
- $(4, 4, 3)$.

*Case* $(m, m, 3)$ *with* $m \geq 5$: Because, in the orientable case [4], we have six vertices in each boundary component of the building blocks, we cannot use the same method as before since we can only identify vertices of the same colour and here we have an odd number of black vertices (three black vertices) and an odd number of white vertices (three white vertices). We will have to build new blocks with eight vertices in the boundary components.

To build $\mathcal{A}$ we take the rectangle $[0, 1] \times [0, 8] \subset \mathbb{R}^2$, tesselated by eight squares. The vertices, as in the previous example, are the integer points $(i, j)$, coloured black or white as $i+j$ is even or odd, joined by edges along the sides and from $(0, j)$ to $(1, j)$ for $j = 1, ..., 8$. We then identify the horizontal sides $y = 0$ and $y = 8$ and we multiply four of the vertical edges: $x = 1, 1 \leq y \leq 2$, $3 \leq y \leq 4$, $5 \leq y \leq 6$ and $7 \leq y \leq 8$ by $m - 4$. This means that in one of the sides we are multiplying alternate edges leaving the other ones unaltered. To increase the valency of the faces and make them of valency 6 (corresponding to hyperfaces of valency 3) we also need to add a stalk of length 1 (in fact, just an edge and a vertex) at each of the six vertices with $x = 1$, those on the right side of the rectangle, in the 4-gonal face below and to the left of the vertex. Thus we get eight faces of valency 6 and the rest of valency 2, as can be easily checked with the help of Figure 6. All the six internal vertices are of valency 1 and because of that they do not interfere with the type of our final hypermap. Hence, all we had to do with the annulus (for the orientable case) was to add two more squares, with stalks, at the bottom (or at the top).

Since the number of vertices at the boundary components of the annulus has changed on both sides, we also need to change the other building blocks so that we can glue them and achieve the required hypermap type. For instance, to construct a new map on the 2-trisc, we take another rectangle $[0, 2] \times [0, 8] \subset \mathbb{R}^2$, with opposite sides identified. The vertices are again at the integer points $(i, j)$, coloured black or white as $i + j$ is even or odd but, this time, we add four more vertices: two black vertices at $(1/3, 2)$ and $(1/3, 4)$, and two white vertices at $(2/3, 2)$ and $(2/3, 4)$. There are vertical edges between $(i, j)$ and $(i, j + 1)$ for $i = 0, 1$ and $j = 0, ..., 7$ with those between $(i, 1)$ and $(i, 2)$ multiplied by $m - 3$. The horizontal edges are the ones between $(i, j)$ and $(i + 1, j)$ for $i = 0$ and $j = 0, 3, 6$, and for $i = 1$ and

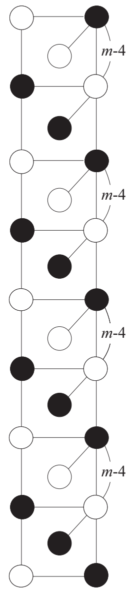


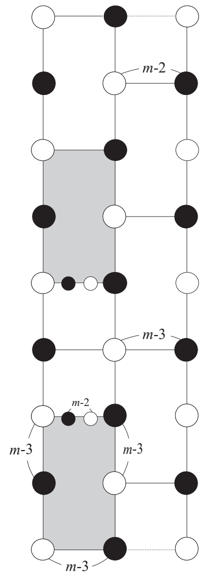

FIGURE 6. Annulus for the nonorientable case.

$j$ odd. We also have edges between $(0,2)$ and $(1/3,2)$, $(1/3,2)$ and $(2/3,2)$, $(2/3,2)$ and $(1,2)$. Some of these, the ones between $(1,7)$ and $(2,7)$, and the ones between $(1/3,2)$ and $(2/3,2)$, are multiplied by $m-2$. At the same time, we have edges between $(0,0)$ and $(1,0)$, $(0,1)$ and $(0,2)$, $(1,1)$ and $(1,2)$, $(1,3)$ and $(2,3)$, multiplied by $m-3$. We then remove two 6-gonal faces, the ones given by $0 < x < 1$ for $0 < y < 2$ and $4 < y < 6$, keeping seven faces of valency 6 (see Figure 7). With these changes, the annulus and the 2-trisc will have both boundary components of types $3^{(8)}$ and $m-1^{(8)}$.

*Case* $(4,4,3)$: We use the same 2-trisc as before (for nonorientable hypermaps of type $(m,m,3)$ with $m \geq 5$) but with a different annulus since the other one does not work for low $m = 4$ (this new annulus, represented in Figure 8, is the same as in the orientable case but with two more steps than the original *ladder*): we take a rectangle $[0,2] \times [0,8] \subset \mathbb{R}^2$, with vertices at the integer points $(i,j)$, coloured black or white as $i+j$ is even or odd. The edges are along the sides, and also from $(i,j)$ to $(i+1,j)$, for $i = 0,1$ and $j = 0,...,8$, so that the rectangle is tessellated by six faces, all of valency 6.

**4.2.4.** *Annnuli with both boundary components of type* $(2,2,3,3,2,2,3,3)$. This corresponds to the following case:

- $(3,3,n)$ for even $n \geq 6$;



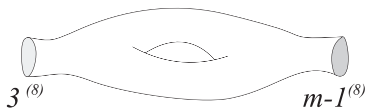




FIGURE 7. 2-trisc for hypermaps of type $(m, m, 3)$ in the nonorientable case.

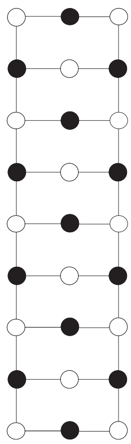


FIGURE 8. Annulus for hypermaps of type $(4, 4, 3)$ in the nonorientable case.

Although we have already built an annulus with an even number of vertices on a boundary component, it is not possible to adapt it for the nonorientable case (and construct a crosscap) using the standard method because we cannot identify opposite vertices of the same colour and, at the same time, join all vertices of order 2 with vertices of order 3. The reason is that we need to have four vertices in each half of the boundary circle (to identify opposite sides of the boundary), and because the valency sequence must be respected, vertices of valency 2 would be identified with vertices of valency 2, while vertices of valency 3 would be identified with others of valency 3. This problem can be solved by introducing four more vertices in each boundary component of the annulus. Hence, instead of two copies of the rectangle described in the orientable case, we will take three copies of it, one for $0 \leq x \leq 4$, another for $4 \leq x \leq 8$ and finally one for $8 \leq x \leq 12$ (see Figure 9). This new annulus has boundary components for $y = 0$ and $y = 2n - 6$ with type $(2, 2, 3, 3, 2, 2, 3, 3, 2, 2, 3, 3)$. It follows that we can identify opposite sides of one of the boundary components, if the vertices are regularly distributed along the border, to get six vertices of valency 4 in a crosscap.

We also need to build a new 2-trisc. For each even $n$, let $\mathcal{R}_n$ be a bipartite map on the rectangle $[0, 4] \times [0, 2n - 6] \subset \mathbb{R}^2$. This bipartite map (see Figure 10) has vertices at the points: $(0, j), (1, j), (4, j)$ for $j \in \{n - 4, ..., 2n - 6\} \cup \{0\}$, and at $(2, j), (3, j)$ for $j \in \{0, ..., n - 1\} \cup \{2n - 6\}$. The vertices $(i, j)$ are black or white if $i + j$ is even or odd, respectively. Because we want some of them to be adjacent we introduce some horizontal and vertical edges in the rectangle.

Horizontal:

$$(i, j) \times (i + 1, j) \text{ for } j \in \{0, 2n - 6\} \text{ and } i \in \{0, ..., 3\}$$

$$(0, j) \times (1, j) \text{ for } j \in \{n + 1, ..., 2n - 7\} \cup \{n - 4\}$$

$$(2, j) \times (3, j) \text{ for } j \in \{1, ..., n - 6\} \cup \{n - 1\}$$

$$(i, j) \times (i + 1, j) \text{ for } j \in \{n - 3, n - 2\} \text{ and } i \in \{1, 3\}$$

Vertical:

$$(i, j) \times (i, j + 1) \text{ for } j \in \{n - 4, ..., 2n - 7\} \text{ and } i \in \{0, 1, 4\}$$

$$(i, j) \times (i, j + 1) \text{ for } j \in \{0, ..., n - 1\} \text{ and } i \in \{2, 3\}$$

These edges enclose $2n - 7$ faces: $2n - 11$ square faces, two faces of valency $2n$ and two faces of valency 12.

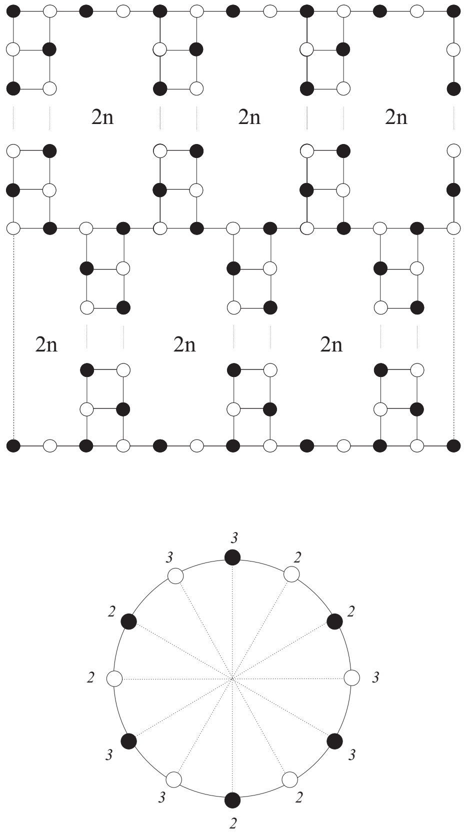



Figure 9. Annulus map and the identification of opposite vertices in one of its boundary components.

To obtain a bipartite map on the torus, we identify opposite sides in the usual way: $(4, y) = (0, y)$ for $0 \leq y \leq 2n - 6$ and $(x, 2n - 6) = (x, 0)$ for $0 \leq x \leq 4$. All the vertices have valency 3, at this stage. To build a 2-trisc $\mathcal{T}$ we need to remove two discs. We can do this by removing two of those (non-adjacent) faces, in this case, the two faces of valency 8. The map on the 2-trisc, $\mathcal{T}_n$, has now $2n - 7$ square faces and two $2n$-gonal faces. The two boundary components of $\mathcal{T}_n$ both have type $(2, 2, 3, 3, 2, 2, 3, 3, 2, 2, 3, 3)$.

Here, we have also to make important changes in the disc $\mathcal{D}_n$: for each even $n \geq 6$, we construct a tessellation $D_n$ of a closed disc $D$, with boundary type $(2, 2, 3, 3, 2, 2, 3, 3, 2, 2, 3, 3)$. We achieve this by starting with a dodecahedron, regarded as a bipartite map on $D$ with one face and with 12 vertices and 12 edges on $\partial D$. Then, we give consecutive numbers to consecutive vertices, starting with 1 in a black vertex and following a clockwise direction. Edges are added between vertices 1 and 4, and between vertices 5 and 12. This creates two faces of order 4 and one face of order 8. Inside of this

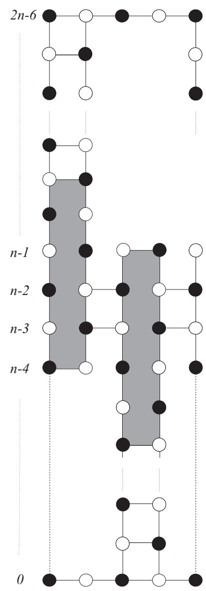



FIGURE 10. 2-trisc map with boundary components of type $(2,2,3,3,2,2,3,3,2,2,3,3)$.

face of order 8 we place a new white vertex adjacent to black vertex number 9 and we built a stalk of length $n-5$, attached to vertex 8, and with all interior vertices of order 3 or 1. Hence that face has changed its order to $8+2+2(n-5)=2n$ (see Figure 11 for $n=8$).

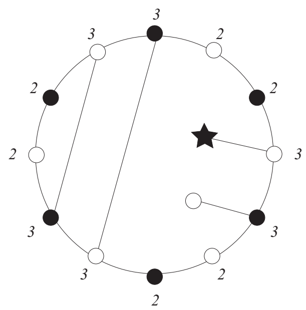



FIGURE 11. Disc map with boundary component of type $(2,2,3,3,2,2,3,3,2,2,3,3)$ for $n=8$.

**4.2.5.** *Annuli with both boundary components of types* $6^{(3)}$ *and* $(2m-4)^{(3)}$*.* . This corresponds to the following case:

- $(3,3,m)$ for odd $m\geq 5$.

In order to build the required orientable hypermap of type $(3, 3, m)$, for odd $m \geq 5$, the following method was addopted [4]: we have constructed a 2-face colourable map of type $\{3, 2m\}$, and then we have taken the dual of that map. By doing that, the vertices have become faces and the faces have become vertices. Because the original map was 2-face colourable, we could conclude that the dual was bipartite. For the nonorientable case we will follow the same idea.

Since the number of vertices in each boundary component of the annulus (in the orientable case [4]) is odd, we need to add new vertices, so that we can build a crosscap. On the boundary with valency sequence $(2m-4)^{(3)}$, we add three new vertices, each one of them between different pairs of consecutive vertices, subdividing the edge that joins them, and we add an extra edge to connect the vertices in each of those pairs, creating new faces of order 3. We will show how we can do that for the case $m = 5$. The other cases are just easy adaptations of this one. In the new annulus we will have boundaries of types $(2m, 2, 2m, 2, 2m, 2, 2m, 2)$ and $(4, 6, 6)$, while still preserving a 2-face colourable map (see Figure 12, with vertical sides identified). Identifying opposite vertices of the boundary of type $(2m, 2, 2m, 2, 2m, 2, 2m, 2)$, we will get a crosscap with interior vertices of valency $2m + 2 - 2 = 2m$. Hence the faces of the final bipartite Walsh map will have valency $2m$, corresponding to hyperfaces of valency $m$. However, adding new vertices in one of the boundaries of the old annulus also requires small adjustments in the map that affect the type of the other boundary. To be able to glue annuli and 2-triscs, and still have the required type in the final hypermap, we need to build new building blocks. In Figures 13, 14 and 15 we represent them for $m = 5$. The annuli in Figures 13 and 14 have boundaries of types $(8, 6, 6)$ and $(4, 6, 6)$, and of types $(4, 6, 6)$ and $(8, 6, 6)$, respectively. The 2-trisc in Figure 15 (where the two light grey faces are removed and opposite edges are identified) has boundaries of types $(8, 6, 6)$ and $(8, 6, 6)$. All the interior vertices have order 10. With those building blocks we can get the required constructions because conveniently gluing a boundary of type $(8, 6, 6)$ with a boundary of type $(4, 6, 6)$ will give rise to vertices of valency $8 + 4 - 2 = 6 + 6 - 2 = 10$.

**4.2.6.** *Annnuli with boundary components of type* $(4, m-1, 2, 3)$. This corresponds to the case:

- $(4, 4, n)$ with odd $n \geq 5$.

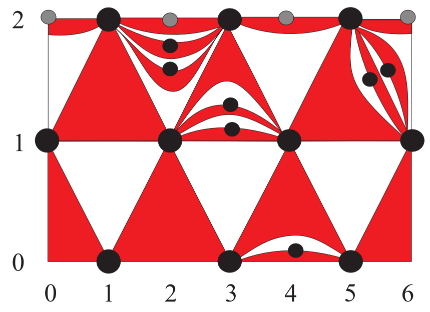



FIGURE 12. Map to build a crosscap for $m = 5$.

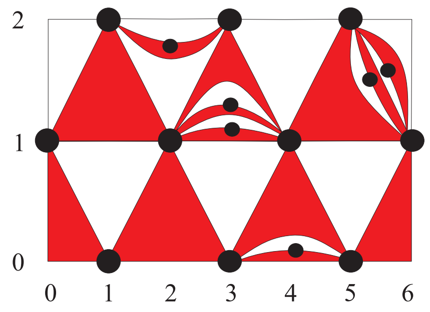



FIGURE 13. Annulus 1 for $m = 5$.

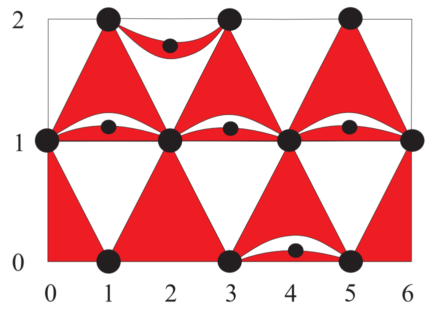



FIGURE 14. Annulus 2 for $m = 5$.

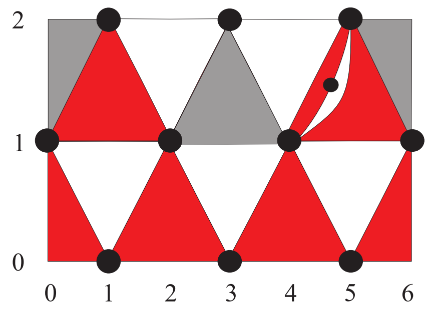



FIGURE 15. Map to build a 2-trisc for $m = 5$.

No adaptations are needed. We can identify opposite sides (identifying opposite vertices of the same colour) of one of the boundary components of the annulus without making any changes, since we get two vertices, one black of order m and one white of order 4. Because the hyperfaces have all valency 4, the map will have type $(n, 4, n)$ and, by a Machi operation [3], we can get a hypermap of type $(4, 4, m)$ from that one.

This completes the proof of Theorem 4.1. □

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