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OPTIMAL REGULARITY FOR RUPTURE SOLUTIONS OF THE INFINITY LAPLACE EQUATION WITH SINGULAR ABSORPTIONS

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Abstract: We consider the nonvariational singular equation, governed by the infinity Laplacian,

$$-\Delta_\infty u = u^{-\gamma}\chi_{\{u>0\}}, \quad \gamma > 1$$

and obtain optimal C^{0,α_γ} local regularity estimates for nonnegative viscosity solutions, where

$$\alpha_\gamma = \frac{4}{3+\gamma}.$$

Through a singular penalized approach, we further obtain the existence of minimal solutions, show they are nondegenerate and derive important geometric properties for the free boundary $\mathcal{R}(u) = \partial\{u > 0\}$, the so-called rupture set.

Keywords: Infinity Laplacian, singular absorption, viscosity solutions, sharp regularity, free boundary.
AMS Subject Classification (2010): Primary 35B65. Secondary 35J60, 35J75, 35D40.

1. Introduction

In this paper, we investigate fine analytic and geometric properties for a nonvariational elliptic equation with singular absorption terms, governed by the infinity Laplace operator

$$\Delta_\infty u := \sum_{ij} D_{ij}uD_iuD_ju.$$

In the last decades, this type of highly degenerate operator has received a great deal of attention. Infinity harmonic functions, *i.e.*, solutions of the homogeneous equation $\Delta_\infty u = 0$, are related to the best Lipschitz extension of a given boundary datum [4, 5, 6] but also with models that describe random tug-of-war games [16], among other applications. Existence and uniqueness results for viscosity solutions of the homogeneous Dirichlet problem are well

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established but the regularity of infinity harmonic functions remains one of the most challenging issues in the modern theory of nonlinear pdes. It is known that solutions are locally Lipschitz, and C^1 and $C^{1,\alpha}$ regularity holds in the plane [19, 10]. Differentiability everywhere has been proven in any dimension [11] but even the C^1 regularity is open. The two dimensional infinity harmonic function

$$x^{4/3} - y^{4/3}, \quad (x, y) \in \mathbb{R}^2$$

provided by Aronson in the nineteen sixties suggests the optimal regularity may be that infinity harmonic functions have Hölder continuous first order derivatives with exponent 1/3.

For the inhomogeneous infinity Laplace equation

$$\Delta_\infty u = f(x) \in L^\infty,$$

existence and uniqueness of viscosity solutions of the Dirichlet problem have been established in [15] under the sign condition $\inf f > 0$ (or $\sup f < 0$). For a bounded source term f, solutions are still Lipschitz continuous and everywhere differentiable [14] but no further regularity is hitherto known. For the infinity-obstacle problem, it is shown in [18] that solutions grow at the sharp rate 4/3 near the contact set. Recently, it was shown in [1] that nonnegative viscosity solutions of the nonsingular dead-core equation

$$\Delta_\infty u \sim u_+^\theta, \quad \text{for} \quad 0 \le \theta < 3$$

are surprisingly smooth along the boundary of the noncoincidence set $\partial\{u > 0\}$. The critical case $\theta \to 3$ was also considered and a strong maximum principle was shown to hold. For an optimization problem with free boundary involving the infinity Laplacian, see [20].

The main purpose of this article is to investigate qualitative properties of nonnegative viscosity solutions of the nonvariational singular elliptic equation

$$-\Delta_\infty u = u^{-\gamma}\chi_{\{u>0\}} \quad \text{in} \quad \Omega, \tag{1.1}$$

for a bounded domain $\Omega \subset \mathbb{R}^n$ and a given parameter of singularity $\gamma > 1$. This type of singular equations appears in the context of a simplified stationary model for the thickness $u \ge 0$ of a certain thin film. The *a priori* unknown set

$$\mathcal{R}(u) := \partial\{u > 0\}$$

is usually called the set of ruptures.

The singular variational setting, corresponding to the Laplacian, has been studied in [8, 12] and in [9], where the local $C^{0,\frac{2}{1+\gamma}}$ optimal regularity has been established. In this case, the diffusion process for u is modelled by rating the average value of u around a certain point. On the contrary, equation (1.1) models such diffusion process by evaluating the average in the direction of maximum/minimum growth of the solution itself.

Our first main result, obtained in section 2, asserts that viscosity solutions of (1.1) are locally C^{0,α_γ} for the precise optimal Hölder exponent

$$\alpha_\gamma := \frac{4}{3+\gamma}. \tag{1.2}$$

The optimality of the exponent α_γ is confirmed by the radial example $C_\gamma\,|x|^{\alpha_\gamma}$, where C_γ is a positive constant depending only on γ. In the proof, we have to deal with the fact that the second order operator Δ_∞ diffuses only in the direction of the gradient, which changes at each point. Simultaneously, the source term blows up along the rupture set, which further conspires to turning the study of universal geometric and analytic properties near $\mathcal{R}(u)$ a quite delicate issue. To the best of our knowledge, this is the first attempt at both studying the infinity Laplace equation with an unbounded right-hand side and of using Ishii-Lions theory to obtain optimal regularity results for this particularly degenerate nonlinear pde, an innovative approach bound to have a wide applicability. We have to emphasise here the right flavour of our optimal exponent (1.2) in the context of the infinity Laplacian regularity theory.

In section 3, we start with a brief setup for the penalised singular approach, showing, using Perron's method, the existence of approximating minimal solutions for the related Dirichlet problem. We then pass to the limit, obtaining a solution of the singular equation (1.1) with a fixed boundary datum. Furthermore, for such solutions, we we obtain the porosity of the rupture set $\mathcal{R}(u)$ and provide a precise upper/lower control away from it, *i.e.*, we show that, for $x \in \{u > 0\}$,

$$C^{-1}\operatorname{dist}(x,\mathcal{R}(u))^{\frac{4}{3+\gamma}} \le u(x) \le C\operatorname{dist}(x,\mathcal{R}(u))^{\frac{4}{3+\gamma}}.$$

Notation. Throughout the paper, Ω will be a bounded domain in $\mathbb{R}^n$, $B_r(x) \subset \mathbb{R}^n$ denotes the open n-dimensional ball with radius $r > 0$ centred at $x \in \mathbb{R}^n$, and $B_r := B_r(0)$. For a compact set $K \subset \Omega$, we define $\operatorname{dist}(K, \partial\Omega)$ to be the distance, in the usual sense, between K and the boundary of Ω, denoted by $\partial\Omega$. For a set $\mathcal{O} \subset \mathbb{R}^n$, $\mathcal{L}^n(\mathcal{O})$ denotes the n-dimensional Lebesgue measure. For a function $v : \Omega \to \mathbb{R}$ and a real number ι, we set $\{v > \iota\} := \{x \in \Omega : v(x) > \iota\}$.

2. Sharp regularity estimates

The first main result we derive is the local optimal regularity for nonnegative viscosity solutions of (1.1), for each parameter $\gamma > 1$. The proof makes use of pointwise estimates for interior maxima of viscosity solutions, an argument also known as Ishii-Lions method (see [13]). We start by defining an appropriate notion of viscosity solution for equation (1.1) but first we recall the notion of jets from [7].

Let $u : \Omega \to \mathbb{R}$ and $\hat{x} \in \Omega$. The second-order superjet of u at $\hat{x}$, $J^{2,+}_{\Omega}u(\hat{x})$, is the set of all ordered pairs $(p, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ such that

$$u(x) \leq u(\hat{x}) + \langle p, x - \hat{x}\rangle + \frac{1}{2}\langle X(x - \hat{x}), x - \hat{x}\rangle + o\left(|x - \hat{x}|^2\right),$$

as $\Omega \ni x \to \hat{x}$. The subjet is defined by $J^{2,-}_{\Omega}u(\hat{x}) = -J^{2,+}_{\Omega}(-u)(\hat{x})$.

Definition 2.1. An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity subsolution of* (1.1) if, for each $x \in \Omega$ and all $(M, \xi) \in J^{2,+}_{\Omega}v(x)$, we have

$$\begin{cases} -\langle M\xi, \xi\rangle - u(x)^{-\gamma} \leq 0 & \text{if} \quad u(x) > 0 \\ -\langle M\xi, \xi\rangle \leq 0 & \text{if} \quad u(x) \leq 0. \end{cases}$$

A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity supersolution of* (1.1) if, for each $y \in \Omega$ and all $(M, \xi) \in J^{2,-}_{\Omega}v(y)$, we have

$$\begin{cases} -\langle M\xi, \xi\rangle - u(y)^{-\gamma} \geq 0 & \text{if} \quad u(y) > 0 \\ -\langle M\xi, \xi\rangle \geq 0 & \text{if} \quad u(y) \leq 0. \end{cases}$$

We say $u : \Omega \to \mathbb{R}$ is a *viscosity solution* of (1.1) if it is both a viscosity solution and a viscosity supersolution.

We can now state the main theorem of this section.

Theorem 2.2 (Optimal regularity)**.** *A nonnegative viscosity solutions u of* (1.1) *is locally of class $C^{0,\alpha_\gamma}(\Omega)$, for*

$$\alpha_\gamma := \frac{4}{3 + \gamma}.$$

Moreover, there exists a universal constant $C > 0$, depending only on n, γ and $\operatorname{dist}(x_0, \partial\Omega)$, *such that*

$$\sup_{B_r(x_0)} \frac{|u(x) - u(y)|}{r^{\alpha_\gamma}} \leq C,$$

for $0 < r \ll \operatorname{dist}(x_0, \partial\Omega)$.

Pointwise estimates for interior maxima. We start preparing the proof of the theorem by deriving pointwise estimates involving the intrinsic structure of the infinity Laplacian operator at interior maximum points of a certain continuous function.

Lemma 2.3 (Ishii-Lions type estimate)**.** *Let $v \in C(B_1)$, $0 \leq \omega \in C^2(\mathbb{R}^+)$ and put*

$$w(x,y) := v(x) - v(y) \quad \textit{and} \quad \varphi(x,y) := L\omega(|x-y|) + \kappa\left(|x|^2 + |y|^2\right),$$

with L, κ positive constants. If the function $w - \varphi$ attains a maximum at $(x_0, y_0) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}}$, then, for each $\varepsilon > 0$, there exist $M_x, M_y \in \mathcal{S}(n)$, such that

$$(D_x\varphi(x_0,y_0), M_x) \in \overline{J}^{2,+}_{B_{1/2}} v(x_0) \quad \textit{and} \quad (-D_y\varphi(x_0,y_0), M_y) \in \overline{J}^{2,-}_{B_{1/2}} v(y_0), \tag{2.1}$$

and the estimate

$$\begin{aligned} &\langle M_x D_x\varphi(x_0,y_0), D_x\varphi(x_0,y_0)\rangle - \langle M_y D_y\varphi(x_0,y_0), D_y\varphi(x_0,y_0)\rangle \\ &\qquad \leq 4\,L\omega''(\rho)\left(L\omega'(\rho) + \kappa\rho\right)^2 + 16\kappa\left(L^2\omega'(\rho)^2 + \kappa^2\right) \end{aligned} \tag{2.2}$$

holds, where $\rho = |x_0 - y_0|$.

Proof: Under the hypothesis of the lemma, let us consider $(x_0, y_0) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}}$ a local maximum of $w - \varphi$. By [13, Theorem 3.2], for each $\varepsilon > 0$, there exist matrices $M_x, M_y \in \mathcal{S}(n)$ such that (2.1) holds and

$$\begin{pmatrix} M_x & 0 \\ 0 & -M_y \end{pmatrix} \leq A + \epsilon A^2$$

for

$$A := \begin{pmatrix} M_\omega & -M_\omega \\ -M_\omega & M_\omega \end{pmatrix} + 2\kappa\, I_{2n\times 2n},$$

where

$$M_\omega := L\,\omega''(|x_0-y_0|)\tfrac{(x_0-y_0)\otimes(x_0-y_0)}{|x_0-y_0|^2} + L\tfrac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\left(I - \tfrac{(x_0-y_0)\otimes(x_0-y_0)}{|x_0-y_0|^2}\right). \tag{2.3}$$

In particular, we have

$$\begin{aligned}\langle M_x D_x\varphi(x_0,y_0), D_x\varphi(x_0,y_0)\rangle - \langle M_y D_y\varphi(x_0,y_0), D_y\varphi(x_0,y_0)\rangle \\ \le \langle M_\omega(D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0)), D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0)\rangle \\ + 2\kappa\left(|D_x\varphi(x_0,y_0)|^2 + |D_y\varphi(x_0,y_0)|^2\right) + \epsilon\lambda \end{aligned} \tag{2.4}$$

where

$$\lambda := \langle A^2\left(D_x\varphi(x_0,y_0), D_y\varphi(x_0,y_0)\right), (D_x\varphi(x_0,y_0), D_y\varphi(x_0,y_0))\rangle.$$

Now, for $\nu := \frac{x_0-y_0}{|x_0-y_0|}$, we have

$$D_x\varphi(x_0,y_0) = L\,\omega'(\rho)\nu + 2\kappa x_0 \quad \text{and} \quad -D_y\varphi(x_0,y_0) = L\,\omega'(\rho)\nu - 2\kappa y_0$$

and thus $D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0) = \iota(x_0-y_0)$, with

$$\iota = 2(L\omega'(\rho)\rho^{-1} + \kappa).$$

It then follows from (2.3) that

$$\begin{aligned}\langle M_\omega(D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0)), D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0)\rangle \\ = \iota^2\langle M_\omega(x_0-y_0), (x_0-y_0)\rangle = \iota^2 L\omega''(\rho)\rho^2 \\ = 4L\omega''(\rho)\left(L\omega'(\rho) + \kappa\rho\right)^2. \end{aligned} \tag{2.5}$$

Moreover, observe that

$$|D_x\varphi(x_0,y_0)|^2 + |D_y\varphi(x_0,y_0)|^2 = 2L^2\omega'(\rho)^2 + 4L\kappa\omega'(\rho)\rho + 4\kappa^2(|x_0|^2 + |y_0|^2).$$

Using Cauchy's inequality, we estimate

$$4L\kappa\omega'(\rho)\rho \le \frac{(2L\omega'(\rho))^2}{2} + \frac{(2\kappa\rho)^2}{2} = 2L^2\omega'(\rho)^2 + 2\kappa^2\rho^2$$

to get

$$|D_x\varphi(x_0,y_0)|^2 + |D_y\varphi(x_0,y_0)|^2 \le 4L^2\omega'(\rho)^2 + 2\kappa^2\rho^2 + 4\kappa^2(|x_0|^2 + |y_0|^2).$$

Since $\max\{|x_0|, |y_0|, \rho\} \le 1/2$, we obtain

$$|D_x\varphi(x_0,y_0)|^2 + |D_y\varphi(x_0,y_0)|^2 \le 4(L^2\omega'(\rho)^2 + \kappa^2). \tag{2.6}$$

Finally, if $\lambda > 0$, choose

$$\epsilon = \frac{8\kappa\left(L^2\omega'(\rho)^2 + \kappa^2\right)}{\lambda},$$

otherwise choose ϵ freely. Using (2.5) and (2.6) in (2.4), together with this choice of ϵ, we obtain (2.2) and the proof is complete.

■

Building Barriers. Here, we shall derive a certain ordinary differential estimate which allows us to import geometric properties of the solutions of the ODE

$$\omega''(t)(\omega'(t))^2\omega(t)^\gamma \approx 1$$

to solutions of the singular equation (1.1). First, we consider the family of functions $\{\omega_\gamma\}_{\gamma>1}$, given by

$$\omega_\gamma(t) = t^{\alpha_\gamma}, \quad 0 < t \ll 1, \tag{2.7}$$

where $\alpha_\gamma = \frac{4}{3+\gamma} < 1$. Then, for each $L > 0$, we define the differential operator

$$\mathcal{L}_L[\theta] := aL^3\theta''\theta'^2 + bL^2\theta'^2 + d,$$

for positive parameters a, b and d, all to be chosen universally in the course of the proof of Theorem 2.2.

Proposition 2.4. *Given a constant $K > 0$, there exists $L_K \gg 1$, depending only on K, a, b, d and γ, such that*

$$L^\gamma\omega_\gamma(t)^\gamma\mathcal{L}_L[\omega_\gamma](t) < -K, \tag{2.8}$$

for all $L \geq L_K$.

Proof: Initially, by a simple computation, we obtain

$$\begin{aligned}\omega_\gamma(t)^\gamma\mathcal{L}_L[\omega_\gamma](t) \;&=\; a(\alpha_\gamma-1)\alpha_\gamma^3L^3t^{2(\alpha_\gamma-1)+(\alpha_\gamma-2)+\gamma\alpha_\gamma}\\ &\qquad + b\,\alpha_\gamma^2L^2t^{2(\alpha_\gamma-1)+\gamma\alpha_\gamma} + d\,t^{\gamma\alpha_\gamma}.\end{aligned} \tag{2.9}$$

Hence, by taking into account that

$$2(\alpha_\gamma-1) + (\alpha_\gamma-2) + \gamma\alpha_\gamma = 0$$

and

$$2(\alpha_\gamma-1) + \alpha_\gamma\gamma > 0,$$

we get

$$L^\gamma\omega_\gamma(t)^\gamma\mathcal{L}_L[\omega_\gamma](t) \leq -\tilde{a}L^{3+\gamma} + \tilde{b}L^{2+\gamma} + d\,L^\gamma$$

for positive constants $\tilde{a}, \tilde{b}, d$. Therefore, for a fixed $K > 0$, we may select $L_K \gg 1$ such that estimate (2.8) holds for every $L \geq L_K$. ■

Proof of Theorem 2.2. For the sake of clarity, we restrict to the simplest case $\Omega = B_1$. It suffices to show that there exist positive universal parameters L, κ such that

$$\sup_{B_{1/2}\times B_{1/2}} \left\{ u(x) - u(y) - L\omega_\gamma(|x-y|) - \kappa(|x|^2 + |y|^2) \right\} \le 0 \tag{2.10}$$

for ω_γ defined in (2.7). Indeed, take $y = 0$ in (2.10) to get, for all $x \in B_{1/2}$,

$$u(x) - u(0) \le L\omega_\gamma(|x|) + \kappa|x|^2 \le L|x|^{\alpha_\gamma} + \kappa|x|^2 \le C|x|^{\alpha_\gamma},$$

since $\alpha_\gamma < 2$. On the other hand, taking $x = 0$ in (2.10) we also get, for all $y \in B_{1/2}$,

$$u(0) - u(y) \le C|y|^{\alpha_\gamma}$$

and so

$$\sup_{B_{1/2}} \frac{|u(x) - u(0)|}{|x - 0|^{\alpha_\gamma}} \le C.$$

Let us then suppose, for the sake of contradiction, that there exists a point $(x_0, y_0) \in \overline{B_{1/2}} \times \overline{B_{1/2}}$ such that

$$u(x_0) - u(y_0) - L\omega_\gamma(\eta) - \kappa(|x_0|^2 + |y_0|^2) > 0, \tag{2.11}$$

for $\eta := |x_0 - y_0|$. We can assume it is a point of maximum. It immediately follows from (2.11) that $x_0 \neq y_0$ and

$$\kappa(|x_0|^2 + |y_0|^2) \le 2\|u\|_{L^\infty(B_1)}. \tag{2.12}$$

In order to guarantee that x_0, y_0 are interior points in $B_{1/2}$, take κ sufficiently large in (2.12), such that

$$\kappa \ge 8\|u\|_{L^\infty(B_1)}.$$

Now, since ω_γ is twice continuously differentiable in a neighborhood of $\eta = |x_0 - y_0| > 0$, Lemma 2.3 guarantees the existence of $(\xi_x, M_x) \in \overline{J}^{2,+}_{B_{1/2}} u(x_0)$ and $(\xi_y, M_y) \in \overline{J}^{2,-}_{B_{1/2}} u(y_0)$ satisfying

$$\begin{array}{rcl} \langle M_x\xi_x, \xi_x\rangle - \langle M_y\xi_y, \xi_y\rangle & \le & aL^3\omega_\gamma''(\eta)\omega_\gamma'(\eta)^2 + b\,L^2\omega_\gamma'(\eta)^2 + d \\ & = & \mathcal{L}_L[\omega_\gamma](\eta), \end{array} \tag{2.13}$$

for universal positive parameters a, b and d. The estimate follows from the fact that $\omega_\gamma''(\eta) < 0$, see (2.7). Also, by (2.13) and Proposition 2.4, given $K = 1$, there exists $L_* \gg 1$, such that

$$L^\gamma \omega_\gamma(\eta)^\gamma \left(\langle M_x\xi_x, \xi_x\rangle - \langle M_y\xi_y, \xi_y\rangle \right) < -1 \tag{2.14}$$

for all $L \gg L_*$.

On the other hand, using (2.11), we conclude that

$$u(x_0) \geq u(x_0) - u(y_0) > L\,\omega_\gamma(\eta) > 0. \tag{2.15}$$

We now split the analysis into two cases. If $u(y_0) > 0$, according to (2.1), (2.14) and Definition 2.1, we obtain

$$L^\gamma \omega_\gamma(\eta)^\gamma \left(-u(x_0)^{-\gamma} + u(y_0)^{-\gamma}\right) < -1.$$

Therefore, $u(y_0)^{-\gamma} < u(x_0)^{-\gamma}$, which contradicts (2.15).

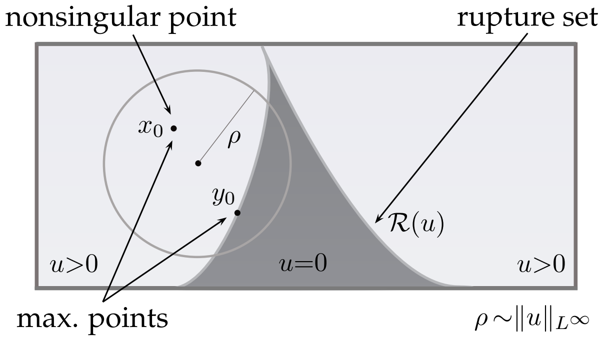

FIGURE 1. Influence of maximum points in the behavior of the solution u close to the rupture set $\mathcal{R}(u)$.

If $u(y_0) = 0$, by Definition 2.1, we have

$$-\,L^\gamma \omega_\gamma(\eta)^\gamma u(x_0)^{-\gamma} < -1. \tag{2.16}$$

Since, by (2.15), $u(x_0) > L\,\omega_\gamma(\eta)$, we again get a contradiction. The proof is complete. $\square$

3. A penalized singular approach

In this section, we consider, for each parameter $\varepsilon > 0$, the perturbed free boundary problem

$$\begin{cases} -\Delta_\infty u &= \beta_\varepsilon^\gamma(u) \quad \text{in} \quad \Omega \\ \qquad u &= f \qquad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{P_ε}$$

for a fixed nonnegative continuous boundary datum f in $\partial\Omega$. Here, $\beta_\varepsilon^\gamma : [0,\infty) \to [0,\infty)$ is the continuous function

$$\beta_\varepsilon^\gamma(t) := \begin{cases} t^{-\gamma} B_\varepsilon^\gamma(t) & \text{if} \quad t > 0 \\ \\ \quad 0 & \text{if} \quad t = 0, \end{cases} \tag{3.1}$$

where

$$B_\varepsilon^\gamma(t) := \int_0^{t\varepsilon^{-\alpha_\gamma}-\kappa_0} \zeta(s)\,ds,$$

for a normalized function $0 \le \zeta \in C_0^\infty([0,1])$, satisfying $\int_0^1 \zeta(s)\,ds = 1$, a positive constant $\kappa_0 \ll 1$ and α_γ given in (1.2). Such choices are justified in order to preserve the intrinsic scaling features of the equation.

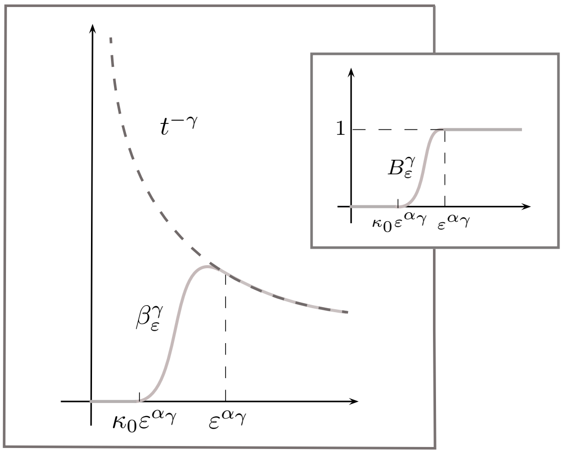

FIGURE 2. The penalization scheme given by $\beta_\varepsilon^\gamma(t) = t^{-\gamma}B_\varepsilon^\gamma(t)$.

According to [2, 17], the existence of minimal Perron solutions holds for a more general class of degenerate pdes with prescribed boundary datum.

Theorem 3.1 ([2, Thm. 2.1]). *Let $g : [0,\infty) \to \mathbb{R}$ be a bounded and uniformly Lipschitz function. Assume the operator $\mathcal{H} : \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ satisfies the monotonicity assumption*

$$\mathcal{H}(\xi, N) \le \mathcal{H}(\xi, M), \quad \text{for any } \xi \in \mathbb{R}^n \text{ and } M \le N.$$

Assume also a priori $C^{0,\alpha}$ estimates for viscosity solutions of $\mathcal{H}(\nabla v, D^2 v) = h(x) \in L^\infty(\Omega)$ and that the equation

$$\mathcal{H}(\nabla v, D^2 v) = g(v) \quad \text{in} \quad \Omega \tag{3.2}$$

admits a continuous viscosity subsolution $u_\star$ and a continuous viscosity supersolution $u^\star$, with $u_\star = u^\star = f \in W^{2,\infty}(\partial\Omega)$. Then the function

$$u(x) := \inf_{v \in \mathcal{P}} v(x) \tag{3.3}$$

is a continuous viscosity solution of (3.2) and satisfies $u = f$ in $\partial\Omega$, where

$$\mathcal{P} := \{v \in C(\overline{\Omega}) \mid v \text{ is a supersolution of (3.2) and } u_\star \le v \le u^\star\}.$$

As a consequence of Theorem 3.1, we guarantee, for each fixed $\varepsilon > 0$, the existence of a Perron solution for (P_ε) by selecting a subsolution and a supersolution of (P_ε), denoted by $u_\star$ and $u^\star$ respectively, satisfying

$$-\Delta_\infty u^\star = \sup_{\mathbb{R}^+} \beta_\varepsilon^\gamma \quad \text{and} \quad -\Delta_\infty u_\star = 0 \quad \text{in} \quad \Omega, \tag{3.4}$$

with boundary data $u^\star = u_\star = f \in \partial\Omega$. Hereafter, we denote with u_ε the nonnegative Perron solution of problem (P_ε) given by (3.3), which satisfies

$$0 \le u_\star \le u_\varepsilon \le u^\star \le \|f\|_{L^\infty(\partial\Omega)}.$$

Uniform in ε estimates for the solutions u_ε will be discussed in subsection 3.1, allowing us to obtain the existence of solutions to the genuine free boundary problem (1.1), as well as some geometric properties, to be addressed in subsection 3.2.

3.1. Optimal growth estimates. We start with a, uniform in ε, sharp interior upper bound for viscosity solutions of (P_ε). This leads to the equicontinuity of the family $\{u_\varepsilon\}_{\varepsilon>0}$, allowing the passage to the limit in (P_ε) to obtain a solution of the singular free boundary problem (1.1), to be addressed in Section 3.2.

Theorem 3.2. *Given $\Omega' \Subset \Omega$, there exists a positive constant C, depending on $\operatorname{dist}(\Omega', \partial\Omega)$, $\|f\|_\infty$, γ and dimension, but independent of ε, such that, for each solution u_ε of problem (P_ε), there holds*

$$\sup_{B_r(x_0)} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|}{r^{\alpha_\gamma}} \le C,$$

for each $x_0 \in \Omega'$ and $0 < r \ll \operatorname{dist}(\Omega', \partial\Omega)$.

Proof: Without loss of generality, we consider the simplest case $x_0 = 0$, $\Omega = B_1$ and $\Omega' = B_{1/2}$. According to the argument used in the proof of Theorem 2.2, we assume, for the sake of contradiction, the existence of $(x_\varepsilon, y_\varepsilon) \in B_{1/2} \times B_{1/2}$ such that

$$u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon) - L\omega_\gamma(\eta_\varepsilon) - \kappa(|x_\varepsilon|^2 + |y_\varepsilon|^2) > 0, \tag{3.5}$$

where $\eta_\varepsilon := |x_\varepsilon - y_\varepsilon|$, $\kappa \gg \|f\|_\infty \ge \|u_\varepsilon\|_\infty$ and $L > 0$ is a constant to be chosen.

By Lemma 2.3 and Proposition 2.4, there exists $L_* \gg 1$, independent of ε, such that

$$-(L\omega_\gamma(\eta_\varepsilon))^\gamma \beta_\varepsilon^\gamma(u_\varepsilon(x_\varepsilon)) \le (L\omega_\gamma(\eta_\varepsilon))^\gamma(-\beta_\varepsilon^\gamma(u_\varepsilon(x_\varepsilon)) + \beta_\varepsilon^\gamma(u_\varepsilon(y_\varepsilon))) \le -1 \tag{3.6}$$

for all $L \gg L_*$. On the other hand, from (3.5), we have

$$u_\varepsilon(x_\varepsilon) \geq u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon) > L\omega_\gamma(\eta_\varepsilon) > 0 \tag{3.7}$$

and so, by (3.6),

$$-(L\omega_\gamma(\eta_\varepsilon))^\gamma u_\varepsilon(x_\varepsilon)^{-\gamma} = -\|B_\varepsilon^\gamma\|_\infty (L\omega_\gamma(\eta_\varepsilon))^\gamma u_\varepsilon(x_\varepsilon)^{-\gamma} \leq -1, \tag{3.8}$$

and, again using (3.7),

$$-1 < -(L\omega_\gamma(\eta_\varepsilon))^\gamma u_\varepsilon(x_\varepsilon)^{-\gamma} \leq -1,$$

a contradiction; the proof is complete. ■

The next result establishes a lower bound for minimal solutions of the penalized problem (P_ε). Such nondegeneracy feature requires the existence of a specific supersolution whose geometric properties will be confronted with the minimality of the solution u_ε.

For a small positive parameter δ, consider the real continuous function $\Phi := \Phi_\delta : [0, \infty) \to [0, \infty)$, defined by

$$\Phi_\delta(t) = \begin{cases} \kappa_0 & \text{if } \; 0 \leq t \leq \delta \\ \\ \kappa_0 + A_\gamma (t-\delta)^{\alpha_\gamma} & \text{if } \quad t > \delta, \end{cases} \tag{3.9}$$

where

$$A_\gamma := \left(\frac{(3+\gamma)^4}{4^3(\gamma-1)} \right)^{\frac{1}{3+\gamma}}. \tag{3.10}$$

From now on, for each $x \in B_1$, we consider $\Phi_\delta(x) = \Phi_\delta(|x|)$. In particular, for the sake of clarity in the arguments, we point out that

$$-\Delta_\infty \Phi_\delta(x) = -\Phi_\delta''(|x|)(\Phi_\delta'(|x|))^2.$$

We claim that Φ_δ satisfies, in the viscosity sense,

$$-\Delta_\infty \Phi_\delta \geq \beta(\Phi_\delta) \quad \text{in } B_1, \tag{3.11}$$

where $\beta = \beta_\varepsilon$ for $\varepsilon = 1$. Indeed, in the region $0 \leq |x| \leq \delta$, we easily have

$$-\Delta_\infty \Phi_\delta(x) = 0 = \beta(\Phi(x))$$

and (3.11) holds. For $|x| > \delta$, we obtain

$$-\Delta_\infty \Phi_\delta(x) = (1-\alpha_\gamma)(A_\gamma \alpha_\gamma)^3 (|x| - \delta)^{-\gamma\alpha_\gamma}$$

and

$$\beta(\Phi_\delta(x)) \leq \left(\kappa_0 + A_\gamma (|x|-\delta)^{\alpha_\gamma}\right)^{-\gamma} \leq A_\gamma^{-\gamma} (|x|-\delta)^{-\gamma\alpha_\gamma}.$$

Therefore, since

$$(1-\alpha_\gamma)(A_\gamma \alpha_\gamma)^3 = A_\gamma^{-\gamma}$$

due to the choice (3.10), we conclude that Φ_δ satisfies (3.11).

Theorem 3.3 (Strong nondegeneracy)**.** *For each point $x \in \{u_\varepsilon \geq \varepsilon^{\alpha_\gamma}\}$, there holds*

$$\sup_{B_r(x)} u_\varepsilon \geq \frac{A_\gamma}{2^{\alpha_\gamma}} r^{\alpha_\gamma},$$

for all $0 < r \ll dist(x_0, \partial\Omega)$.

Proof: Without loss of generality, let us consider $x = 0 \in \{u_\varepsilon \geq \varepsilon^{\alpha_\gamma}\}$. For each $\varepsilon > 0$, we define the rescaled function

$$\Phi_\delta^\varepsilon(x) := \varepsilon^{\alpha_\gamma} \Phi_\delta(\varepsilon^{-1}x),$$

By (3.11), we observe that Φ_δ^ε is a viscosity supersolution of

$$-\Delta_\infty v = \beta_\varepsilon(v) \quad \text{in} \quad B_1$$

such that

$$\Phi_\delta^\varepsilon \geq A_\gamma (\delta\varepsilon)^{\alpha_\gamma} \quad \text{on } \partial B_{2\delta\varepsilon} \quad \text{and} \quad \Phi_\delta^\varepsilon(0) = \kappa_0 \varepsilon^{\alpha_\gamma}. \tag{3.12}$$

To prove the theorem, it suffices to show that, for each $0 < r \ll 1$ fixed, we have $u_\varepsilon(\xi_r) \geq \Phi_{r/2\varepsilon}^\varepsilon(\xi_r)$, for some $\xi_r \in \partial B_r$. Indeed,

$$\sup_{B_r} u_\varepsilon \geq \sup_{\partial B_r} u_\varepsilon \geq u_\varepsilon(\xi_r) \geq \Phi_{r/2\varepsilon}^\varepsilon(\xi_r) \geq A_\gamma \left(\frac{r}{2}\right)^{\alpha_\gamma} = \frac{A_\gamma}{2^{\alpha_\gamma}} r^{\alpha_\gamma}.$$

Suppose, for the sake of contradiction, that

$$u_\varepsilon < \Phi_{r/2\varepsilon}^\varepsilon \quad \text{in} \quad \partial B_r$$

and define

$$\varpi_\varepsilon := \begin{cases} \min\{u_\varepsilon, \Phi_{r/2\varepsilon}^\varepsilon\} & \text{in } \overline{B_r} \\ u_\varepsilon & \text{in } \Omega \setminus \overline{B_r}. \end{cases}$$

Note that ϖ_ε is a supersolution to (P_ε) such that $\varpi_\varepsilon = u_\varepsilon$ in $\partial\Omega$ and it is continuous since $\varpi_\varepsilon = u_\varepsilon$ on ∂B_r due to the contradiction hypothesis. However, by (3.12), we obtain

$$u_\varepsilon(0) \geq \varepsilon^{\alpha_\gamma} > \kappa_0 \varepsilon^{\alpha_\gamma} = \Phi_{r/2\varepsilon}^\varepsilon(0) = \varpi_\varepsilon(0),$$

which violates the minimality of u_ε. ■

Remark 3.4. We stress that the nondegeneracy constant $2^{-\alpha_\gamma} A_\gamma$ only depends on γ and is thus independent of ε.

Corollary 3.5. *Given $\Omega' \Subset \Omega$, there exists a constant C, depending on* $\operatorname{dist}(\Omega', \partial\Omega)$ *and universal parameters, but independent of ε, such that for $x \in \{u_\varepsilon \geq \varepsilon^{\alpha_\gamma}\} \cap \Omega'$ and $0 < r \ll \operatorname{dist}(\Omega', \partial\Omega)$, there holds*

$$C^{-1} r^{\alpha_\gamma} \leq \sup_{B_r(x)} u_\varepsilon \leq C r^{\alpha_\gamma} + u_\varepsilon(x).$$

For $x \in \Omega$, let

$$d_\varepsilon(x) := \operatorname{dist}\left(x, \partial\{u_\varepsilon > \varepsilon^{\alpha_\gamma}\}\right).$$

We now show that for points close to the perturbed free boundary $\partial\{u_\varepsilon > \varepsilon^{\alpha_\gamma}\}$ the minimal solution u_ε grows at the optimal rate $d_\varepsilon^{\alpha_\gamma}$.

Theorem 3.6. *There exist positive small constants C, d, depending on universal parameters, but independent of ε, such that*

$$C^{-1}\, d_\varepsilon(x_0)^{\alpha_\gamma} \leq u_\varepsilon(x_0) \leq C d_\varepsilon(x_0)^{\alpha_\gamma} + \varepsilon^{\alpha_\varepsilon},$$

for each $x_0 \in \partial\{u_\varepsilon > \varepsilon^{\alpha_\gamma}\}$, with $d_\varepsilon(x_0) \leq d$.

Proof: The upper estimate follows directly from Theorem 3.2.

We first prove the lower estimate for the case $\varepsilon = 1$. Put $v := u_1$ and suppose, for the sake of contradiction, that there exists a sequence $x_n \in \partial\{v > 1\}$, with $d_n := d(x_n, \partial\{v > 1\}) \longrightarrow 0$ as $n \to \infty$, and

$$v(x_n) \leq \frac{1}{n} d_n^{\alpha_\gamma}. \tag{3.13}$$

Set $v_n(y) := \dfrac{v(x_n + d_n y)}{d_n^{\alpha_\gamma}}$ and note that v_n is a minimal solution of

$$-\Delta_\infty v_n = v_n^{-\gamma} B_{1/d_n}(v_n), \tag{3.14}$$

where $B_{1/d_n}(s) = 0$, for $0 \leq s \leq \kappa_0\, d_n^{-\alpha_\gamma}$. By Theorem 3.2, for some universal constant $C > 0$, there holds

$$v_n(y) \leq C|y|^{\alpha_\gamma} + v_n(0).$$

From (3.13), by taking $n \gg 1$, we have $v_n(0) \leq \frac{\kappa_0}{2}$ and so, for

$$|y| \leq \rho := \left(\frac{\kappa_0}{2C}\right)^{1/\alpha_\gamma},$$

we have $v_n(y) \leq \kappa_0 \leq \kappa_0\, d_n^{-\alpha_\gamma}$. Therefore, by (3.14), v_n satisfies

$$-\Delta_\infty v_n = 0 \quad \text{in } B_\rho.$$

By Harnack's inequality, for some universal $\overline{C} > 0$, we have

$$\sup_{B_\rho/2} v_n \leq \overline{C}\, v_n(0) \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

which contradicts the nondegeneracy estimate of Theorem 3.3.

For the general case, we set

$$v_\varepsilon(y) := \frac{u_\varepsilon(x_0 + \varepsilon y)}{\varepsilon^{\alpha_\gamma}},$$

which is a minimal solution of $-\Delta_\infty v_\varepsilon = v_\varepsilon^{-\gamma} B_1(v_\varepsilon)$. From the previous case, for some universal constant $c > 0$, we obtain

$$v_\varepsilon(0) \geq c\,\mathrm{dist}(0, \partial\{v_\varepsilon > 1\})^{\alpha_\gamma} = c\,(\varepsilon^{-1} d_\varepsilon(x_0))^{\alpha_\gamma}$$

and so $u_\varepsilon(x_0) \geq c\, d_\varepsilon(x_0)^{\alpha_\gamma}$. ■

3.2. The limiting free boundary problem. Here, we address the genuine free boundary problem (1.1), by letting $\varepsilon \to 0$ in the penalized problem (P_ε). Such analysis provides an existence result as well as geometric and analytic properties for the limit problem.

Due to Theorem 3.2, the family $\{u_\varepsilon\}_{\varepsilon>0}$ is C^{α_γ}-equicontinuous. Therefore, up to a subsequence,

$$\lim_{\varepsilon \to 0} u_\varepsilon =: u_0 \in C_{loc}^{\frac{4}{3+\gamma}}(\Omega),$$

uniformly in ε. This implies that u_0 solves the singular free boundary problem

$$\begin{cases} -\Delta_\infty u_0 &= u_0^{-\gamma} \chi_{\{u_0 > 0\}} \quad \text{in} \quad \Omega \\ \\ u &= f \qquad\qquad\quad \text{on} \quad \partial\Omega. \end{cases} \tag{P_0}$$

Indeed, for each $x_0 \in \{u_0 > 0\}$ fixed, we set $u_0(x_0) =: \iota$. By continuity, there exists a small $\rho > 0$ such that

$$u_0 \geq \iota/4 \quad \text{in } B_\rho(x_0)$$

and so, since u_ε converges to u_0 uniformly on compact sets, for $\varepsilon \ll 1$, we have

$$u_\varepsilon \geq \iota/8 \geq (1 + \kappa_0)\varepsilon^{\alpha_\gamma} \quad \text{in } B_{\rho/2}(x_0).$$

Therefore, u_ε solves explicitly the equation $-\Delta_\infty u_\varepsilon = u_\varepsilon^{-\gamma}$ in $B_{\rho/2}(x_0)$. By the stability of viscosity solutions under uniform limits, we finally conclude that u_0 solves (P_0).

Due to Corollary 3.5 and the fact that u_ε converges locally uniformly to u_0 in C^{α_γ}, we are able to obtain strong upper and lower bounds for limiting solutions.

Theorem 3.7. *There exists a a universal constant C such that, for each*

$$x \in \overline{\{u_0 > 0\}} \cap B_{1/2},$$

and $0 < r \ll 1$, there holds

$$C^{-1} r^{\alpha_\gamma} \leq \sup_{B_r(x)} u_0 \leq C r^{\alpha_\gamma} + u_0(x).$$

As a classical consequence of Theorem 3.7 (see [3] for more details) we obtain that the free boundary $\mathcal{R}(u_0)$ is locally a porous subset in B_1.

Corollary 3.8. *There exists a universal small number $\delta > 0$ such that $\mathcal{R}(u_0)$ is a δ-porous set. More precisely, for all $x \in \mathcal{R}(u_0)$ and $0 < r \ll 1$, there exists $y \in B_r(x)$, such that*

$$B_{\delta r}(y) \subset B_r(x) \setminus \mathcal{R}(u_0).$$

Moreover, there exists a universal small constant c such that

$$\frac{\mathcal{L}^n(B_r(x) \cap \{u_0 > 0\})}{\mathcal{L}^n(B_r(x))} \geq c > 0.$$

Finally, Theorem 3.6 provides one of the most important consequences of the uniform convergence: the optimal growth rate for the limiting solution u_0, for points close to the singular set $\mathcal{R}(u_0) := \partial\{u_0 > 0\}$. Here, we set $d_0(x) := \operatorname{dist}(x, \mathcal{R}(u_0))$.

Theorem 3.9. *There exist positive universally small constants C, d such that*

$$C^{-1}\, d_0(x_0)^{\alpha_\gamma} \leq u_0(x_0) \leq C d_0(x_0)^{\alpha_\gamma},$$

for each $x_0 \in \{u_0 > 0\}$ with $d_0(x_0) \leq d$.

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