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NASH EQUILIBRIA IN N-PERSON GAMES WITH SUPER-QUADRATIC HAMILTONIANS

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Abstract: We consider the Hamilton-Jacobi-Bellman system

$$\partial_t \mathbf{u} - \Delta \mathbf{u} = \mathbf{H}(\mathbf{u}, \nabla \mathbf{u}) + \mathbf{f}$$

for $\mathbf{u} \in \mathbb{R}^N$, where the Hamiltonian $\mathbf{H}(\mathbf{u}, \nabla \mathbf{u})$ satisfies a super-quadratic growth condition with respect to $|\nabla \mathbf{u}|$. Such a nonlinear parabolic system corresponds to a stochastic differential game with N players. We obtain the existence of bounded weak solutions and prove regularity results in Sobolev spaces for the Dirichlet problem.

Keywords: Hamilton-Jacobi-Bellman system, super-quadratic growth, stochastic games, existence and regularity.
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1. Introduction

The purpose of this paper is the study of the Hamilton-Jacobi-Bellman system

$$\partial_t \mathbf{u} - \Delta \mathbf{u} = \mathbf{H}(\mathbf{u}, \nabla \mathbf{u}) + \mathbf{f} \quad \text{in} \quad \Omega \times (0,T], \tag{1.1}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega \times (0,T], \tag{1.2}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u_0} \quad \text{in} \quad \Omega, \tag{1.3}$$

where $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^N$ ($N \geq 1$) is a vector valued function, $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) is a bounded domain with a $C^{1,1}$-boundary, $T < \infty$, and $\partial_t = \frac{\partial}{\partial t}$. We assume the Hamiltonian has the form

$$\mathbf{H}(\mathbf{u}, \nabla \mathbf{u}) = (\mathbf{g}(\nabla \mathbf{u}) - \mathbf{u}) \, |\nabla \mathbf{u}|^q \,,$$

where $\mathbf{g} \in \mathbb{R}^N$ is a bounded function and $q > 2$ may be arbitrarily large, and thus satisfies a super-quadratic growth condition with respect to $|\nabla \mathbf{u}|$.

This type of nonlinear parabolic system, for Hamiltonians with sub-quadratic, quadratic or super-quadratic growth with respect to $|\nabla \mathbf{u}|$, occur in

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the theory of stochastic differential games, portfolio theory, economic finance, and geometry. An example of a simplified N-person game with a super-quadratic Hamiltonian is given in Section 2.

Our goal is to prove the existence of bounded weak solutions $\mathbf{u}$ of the system (1.1)–(1.3) in the super-quadratic case ($q > 2$), having the additional regularity

$$\mathbf{u} \in L^2\left(0,T;W^{2,2}(\Omega;\mathbb{R}^N)\right) \cap L^\infty\left(0,T;W^{1,\infty}(\Omega;\mathbb{R}^N)\right)$$

and

$$\partial_t \mathbf{u} \in L^2\left(0,T;L^2(\Omega;\mathbb{R}^N)\right).$$

The existence theory in the sub-quadratic growth case ($q < 2$) is simple (cf. [21]) but the case of quadratic growth ($q = 2$) is already challenging. In game theory, the existence of weak solutions for N-person games, with $N = 1$, is treated in [12, 14], and N-person games, for $N > 1$ arbitrarily large, are studied, e.g., in [7, 8, 11, 16].

The uniqueness of weak solutions in the quadratic case ($q = 2$) is only known for bounded weak solutions. In fact, there may be several weak solutions, but at most one that is bounded; a counter-example is given in [4]. Uniqueness results can be found in [4] for elliptic equations, and in [19] for parabolic systems with N players.

Essential for our analysis is the fact that $\mathbf{H}$ has a negative definite Jacobian $\mathbf{H_u}$ in the case that $|\nabla \mathbf{u}| \neq 0$. More precisely, it is well known that weak solutions may show finite time blow up behaviour if $\mathbf{H_u}$ is positive definite; see, e.g., [13] for the quadratic case. In the case that $\mathbf{H_u} = \mathbf{0}$ and $q > 2$ a loss of boundary conditions may occur; cf. [3]. Moreover, in [2] it is shown that classical solutions may blow up in finite time. Another non-existence result can be found in [1], where the initial data are measures. Existence results for viscous solutions of the Dirichlet problem in the case that $\mathbf{H_u} = \mathbf{0}$ and $q > 2$ are given in [6, 22], the Neumann boundary value problem is treated in [5], and the existence of bounded solutions under periodic boundary value conditions is shown in [10]. Our method of proof uses test functions of power-law type, similar to those in [19, 20], and a difference quotient technique. We obtain an energy estimate by taking a discrete p-Laplacian as test function, where p has to be sufficiently large.

The paper is organised as follows. In Section 2, we discuss stochastic games as the main motivation for our system and provide an example of a simplified N-person game with a super-quadratic Hamiltonian. Section 3 contains the assumptions on the data, the main result and a motivation to the use of power-law test functions. In Section 4, a regularised Hamilton-Jacobi-Bellman system is studied and the basic energy estimate is obtained, leading to the proof of the main result for smooth domains. Section 5 gives a generalisation of our result to a class of Lipschitzian domains, such as convex polyhedrons. Finally, the case of inhomogeneous Dirichlet boundary value conditions is discussed in the last section.

2. Nash equilibria in differential games with discount control

Consider stochastic games with N players, where each player can influence the drift m of the dynamical system

$$dx(t) = m(x, \mathbf{v})\, dt + \sigma\, dw_t, \quad x(0) = y \tag{2.1}$$

for $x(t) \in \mathbb{R}^d$, $t \in [0, T]$ and $T > 0$. Here, $m \in \mathbb{R}^d$ is a given function, the diffusion term σ is a constant $N \times N$ matrix, dw_t a Wiener process, and $y \in \mathbb{R}^d$ is the initial state of x. The intention of the k-th player ($1 \leq k \leq N$) is to maximize the cost functional

$$\begin{aligned} J^k\big(\mathbf{v}\big) = \mathbb{E}^k\Big[\int_0^T l_k\big(x(t), \mathbf{v}(t)\big)\, \exp\Big(- \int_0^t c_k\big(x(r), \mathbf{v}(r)\big)\, dr\Big)\, dt + \\ + \phi_k\big(x(T)\big)\, \exp\Big(- \int_0^T c_k\big(x(t), \mathbf{v}(t)\big)\, dt\Big)\Big], \end{aligned} \tag{2.2}$$

where $\mathbf{v} = (v^1, \ldots, v^N)^\top$ and $\mathbb{E}^k$ is the expectation of the k-th player. Moreover, l_k, c_k, and ϕ_k are prescribed functions. The stochastic process $x(t)$ ($0 \leq t \leq T$) describes the state of the underlying dynamical system.

For stochastic control problems the method of dynamic programming leads to an analytical problem, called the Hamilton-Jacobi-Bellman system, whose solution allows to derive an optimal stochastic control. The concept of a Nash point is the following: find functions $\hat{v}^1, \ldots, \hat{v}^N$ such that

$$J^k\big(\hat{v}^1, \ldots, \hat{v}^k, \ldots, \hat{v}^N\big) \geq J^k\big(\hat{v}^1, \ldots, \bar{v}, \ldots, \hat{v}^N\big),$$

for all $\bar{v}$ that are admissible for the k-th player. For one player, that is, $N = 1$, the problem reduces to the classical stochastic control problem. The factor

$$\exp\Big(- \int_0^T c_k\big(x(r), \mathbf{v}(r)\big)\, dr\Big)$$

is the discount factor influenced by the k-th player. To the k-th player there is an associated Lagrange functional

$$L^k(x, \lambda_k, \xi_k, \mathbf{v}) = l_k(x, \mathbf{v}) + \xi_k \cdot m(x, \mathbf{v}) - \lambda_k \, c_k(x, \mathbf{v}) \,, \tag{2.3}$$

where $l_k, \lambda_k, c_k \in \mathbb{R}$, $\xi_k \in \mathbb{R}^d$, and $\lambda_k c_k$ is the discount control.

Fixing x, λ_k and ξ_k, we look for a Nash point $\hat{\mathbf{v}}(x, \lambda, \xi)$ for the functionals L^k. Here, λ stands for $(\lambda_1, \ldots, \lambda_N)$, ξ for $(\xi_1, \ldots, \xi_N)$, and $\hat{\mathbf{v}}$ for $(\hat{v}^1, \ldots, \hat{v}^N)$. We define the Hamiltonian functions

$$H^k(\lambda, \xi) := L^k\big(x, \lambda_k, \xi_k, \hat{\mathbf{v}}(x, \lambda, \xi)\big)$$

and consider the following nonlinear system, for $1 \le k \le N$,

$$\partial_t u^k - A u^k = H^k(x, t, \mathbf{u}, \nabla \mathbf{u}) \quad \text{in } \Omega \times (0, T] \,, \tag{2.4}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^d$, where

$$A = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

and the coefficients a_{ij} are the components of the matrix $\frac{1}{2}\sigma\sigma^T$.

We will limit our presentation to stochastic processes which are killed at the exit of the domain, which leads to an homogeneous Dirichlet boundary value problem. Let there be a sufficiently smooth solution $\mathbf{u}$, say

$$\mathbf{u} \in L^\infty(0, T; W^{2,r}(\Omega; \mathbb{R}^N))$$

and $r > d$. Now we obtain an optimal feedback for the k-th player, in the sense that $\hat{v}^k(t) = \hat{v}^k(x(t))$ is a solution to (2.1). Therefore the problem of finding a smooth solution to the system (2.4) is the tool to obtain Nash equilibrium points for the stochastic differential game. For more details we refer to [9, 15].

We now give an example of a simplified N-person game with a super-quadratic Hamiltonian. Let the drift m in equation (2.1) be a linear function with respect to $\mathbf{v}$, that is,

$$m(x, \mathbf{v}) = \sum_{\nu} a_\nu(x) \, v^\nu + a_0(x),$$

where a_ν $(0 \le \nu \le N)$ are $\mathbb{R}^d$-functions. Let us introduce the Lagrange functionals $(1 \le k \le N)$

$$L^k(x, \lambda_k, \xi_k, \mathbf{v}) = l_k(x, \mathbf{v}) + \xi_k \cdot m(x, \mathbf{v}) - \lambda_k \, c(x, \mathbf{v}). \tag{2.5}$$

We define

$$l_k(x, \mathbf{v}) = \frac{1}{\alpha}|v^k|^\alpha + b(x),$$

where b is a scalar function and $\alpha \in (1, 2)$. Moreover, the discount is given by

$$c(x, \mathbf{v}) = 1 + \frac{1}{\alpha}\sum_\nu |v^\nu|^\alpha.$$

In order to find a Nash point $\hat{\mathbf{v}}$, we solve the equations

$$\frac{\partial}{\partial v^k} L^k = 0 \qquad (1 \le k \le N).$$

This implies that

$$|v^k|^{\alpha-2}v^k - \lambda_k |v^k|^{\alpha-2}v^k + \xi_k \cdot a_k = 0$$

and we find the solutions

$$|\hat{v}^k|^{\alpha-2}\hat{v}^k = (\lambda_k - 1)^{-1}\xi_k \cdot a_k. \tag{2.6}$$

Due to (2.5), we obtain the Hamiltonian

$$\begin{aligned} H^k(\mathbf{u}, \nabla\mathbf{u}) &= L^k(x, u^k, \nabla u^k, \hat{\mathbf{v}}) \\ &= \left[\frac{1}{\alpha}|\hat{v}^k|^\alpha + b(x) + \nabla u^k \cdot \left(\sum_\nu a_\nu(x)\,\hat{v}^\nu + a_0(x)\right)\right] \\ &\quad -u^k\left[1 + \frac{1}{\alpha}\sum_\nu |\hat{v}^\nu|^\alpha\right]. \end{aligned}$$

In view of (2.6) we get

$$|\hat{v}^k|^\alpha = (|u^k - 1|^{-1}|\nabla u^k \cdot a_k|)^{\frac{\alpha}{\alpha-1}},$$

that is, $|\hat{\mathbf{v}}|^\alpha \sim |\nabla\mathbf{u}|^{\frac{\alpha}{\alpha-1}}$ and $|\nabla\mathbf{u}|\,|\hat{\mathbf{v}}| \sim |\nabla\mathbf{u}|^{\frac{\alpha}{\alpha-1}}$. Thus, the Hamiltonian has the form

$$\mathbf{H}(\mathbf{u}, \nabla\mathbf{u}) = \mathbf{H_0}(\mathbf{u}, \nabla\mathbf{u}) - \mathbf{u}\,\mathbf{H_1}(\mathbf{u}, \nabla\mathbf{u}),$$

where $\mathbf{H_0}(\mathbf{u}, \nabla\mathbf{u}) \sim |\nabla\mathbf{u}|^{\frac{\alpha}{\alpha-1}}$ and $\mathbf{H_1}(\mathbf{u}, \nabla\mathbf{u}) \sim |\nabla\mathbf{u}|^{\frac{\alpha}{\alpha-1}}$, if $|\nabla\mathbf{u}|$ is large. Moreover, we have $\mathbf{H_1}(\mathbf{u}, \nabla\mathbf{u}) \ge 1$. Altogether, we obtain an Hamilton-Jacobi-Bellman system of the form

$$\partial_t\mathbf{u} - \Delta\mathbf{u} + \mathbf{u}\,\mathbf{H_1}(\mathbf{u}, \nabla\mathbf{u}) = \mathbf{H_0}(\mathbf{u}, \nabla\mathbf{u}),$$

where the Hamiltonians $\mathbf{H_0}(\mathbf{u}, \nabla\mathbf{u})$ and $\mathbf{H_1}(\mathbf{u}, \nabla\mathbf{u})$ satisfy a super-quadratic growth condition with respect to $|\nabla\mathbf{u}|$, since $\frac{\alpha}{\alpha-1} > 2$.

3. The main result

We assume the following set of assumptions on the data.

(H1) $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with a $C^{1,1}$–boundary;

(H2) $T < \infty$, $q > 2$, and $N \geq 1$;

(H3) $\mathbf{f}, \mathbf{g} \in L^\infty(0,T; L^\infty(\Omega;\mathbb{R}^N))$ and $\mathbf{u_0} \in W^{1,\infty}(\Omega;\mathbb{R}^N)$.

We next make precise the notion of solution we are dealing with.

Definition 1. *We say* $\mathbf{u}(x,t)$ *is a bounded weak solution of the system (1.1)–(1.3) if* $\mathbf{u} \in L^2\left(0,T; W_0^{1,q}(\Omega;\mathbb{R}^N)\right) \cap L^\infty\left(0,T; L^\infty(\Omega;\mathbb{R}^N)\right)$, $\partial_t \mathbf{u} \in L^2\left(0,T; H^{-1}(\Omega;\mathbb{R}^N)\right)$, $\mathbf{u}$ *fulfills the initial condition (1.3) in the sense of* $L^2(\Omega;\mathbb{R}^N)$, *and*

$$\int_0^T \int_\Omega \langle \partial_t \mathbf{u}, \boldsymbol{\varphi} \rangle + \int_0^T \int_\Omega \nabla \mathbf{u} \cdot \nabla \boldsymbol{\varphi} = \int_0^T \int_\Omega \mathbf{H}(\mathbf{u}, \nabla \mathbf{u}) \cdot \boldsymbol{\varphi} + \int_0^T \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi},$$

for all $\boldsymbol{\varphi} \in L^2(0,T; H_0^1(\Omega;\mathbb{R}^N)) \cap L^\infty(0,T; L^\infty(\Omega;\mathbb{R}^N))$. *Here,* $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $H^{-1}(\Omega;\mathbb{R}^N)$ *and* $H_0^1(\Omega;\mathbb{R}^N)$.

The existence of a bounded weak solution $\mathbf{u}$ such that $\nabla \mathbf{u}$ is bounded and $\nabla^2 \mathbf{u}$ and $\partial_t \mathbf{u}$ are L^2-functions is the main result of this paper.

Theorem 1. *There exists a bounded weak solution* $\mathbf{u}$ *of the system (1.1)–(1.3) satisfying*

$$\mathbf{u} \in L^2\left(0,T; W^{2,2}(\Omega;\mathbb{R}^N)\right) \cap L^\infty\left(0,T; W^{1,\infty}(\Omega;\mathbb{R}^N)\right)$$

and

$$\partial_t \mathbf{u} \in L^2\left(0,T; L^2(\Omega;\mathbb{R}^N)\right).$$

Energy estimates for Hamilton-Jacobi-Bellman systems can not be obtained via standard approaches. Therefore, we make use of test functions of power-law type. In order to explain our approach we sketch the basic idea introduced in [19, 20] for the case of Hamiltonians with quadratic growth. In the following example we treat the case of just one player, that is $N = 1$, and give the proof of the basic energy estimate.

Let us consider the equation

$$\partial_t u - \Delta u + u|\nabla u|^2 = |\nabla u|^2 + f.$$

In order to obtain an energy estimate, cf. (3.1), we multiply the equation by

$$\varphi = (1 + |u|^{p-1})u,$$

where $p > 2$ is sufficiently large. Let us note that

$$\begin{aligned} \int_0^T \int_\Omega \partial_t u \, |u|^{p-1} u &= \frac{1}{p+1} \int_0^T \int_\Omega \partial_t |u|^{p+1} \\ &= \frac{1}{p+1} \, \|u(\cdot, T)\|_{L^{p+1}(\Omega)}^{p+1} - \frac{1}{p+1} \, \|u_0\|_{L^{p+1}(\Omega)}^{p+1} \end{aligned}$$

and

$$\int_0^T \int_\Omega \nabla u \cdot \nabla((1 + |u|^{p-1})u) = p \int_0^T \int_\Omega |u|^{p-1} |\nabla u|^2 + \int_0^T \int_\Omega |\nabla u|^2.$$

Moreover, on the left-hand side of the equation we have the integral

$$\int_0^T \int_\Omega u |\nabla u|^2 \varphi = \int_0^T \int_\Omega |u|^{p+1} |\nabla u|^2.$$

Now let us estimate the integrals on the right-hand side of the equation. Applying Young's inequality, we find

$$\int_0^T \int_\Omega |u|^{\frac{p+1}{2}} |u|^{\frac{p-1}{2}} |\nabla u|^2 \leq \delta \int_0^T \int_\Omega |u|^{p+1} |\nabla u|^2 + c_\delta \int_0^T \int_\Omega |u|^{p-1} |\nabla u|^2,$$

for some sufficiently small $\delta > 0$. Thus, we can absorb the first integral on the right-hand side into the left-hand side. Furthermore, we may absorb the second integral as well, if p is suitably large.

Next, we introduce the set

$$\Omega_\delta(t) := \left\{ x \in \Omega : |u(x,t)| \geq \frac{1}{\delta} \right\}.$$

Let us assume that f is bounded. It follows that

$$\int_0^T \int_{\Omega_\delta(t)} |f| \, |u|^p \leq c \, \delta \int_0^T \int_{\Omega_\delta(t)} |u|^{p+1} \leq c \, \delta \, \|u\|_{L^{p+1}(0,T;L^{p+1}(\Omega))}^{p+1} \, .$$

Choosing δ sufficiently small, we may absorb this integral into the left-hand side. Further, the integral

$$\int_0^T \int_{\Omega \setminus \Omega_\delta(t)} |f| \, |u|^p \leq c \, \delta^{-p}$$

is bounded. Collecting results, we arrive at

$$\|u\|^{p+1}_{L^\infty(0,T;L^{p+1}(\Omega))} + \|\nabla u\|^2_{L^2(0,T;L^2(\Omega))} \le c_p\,.$$

Extracting the p-th rooth and sending $p \to \infty$, we obtain the energy estimate

$$\|u\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\nabla u\|_{L^2(0,T;L^2(\Omega))} \le c\,. \tag{3.1}$$

To treat the super-quadratic case, the basic idea is to use a power-law test function like $-\Delta_p u = -\mathrm{div}\left((1+|\nabla u|^{p-1})\nabla u\right)$; cf. (4.6). Noting that

$$\int_0^T\int_\Omega \partial_t\nabla u\cdot|\nabla u|^{p-1}\nabla u = \frac{1}{p+1}\int_0^T\int_\Omega \partial_t|\nabla u|^{p+1},$$

extracting the p-th root and sending $p\to\infty$, we are able to obtain the energy estimate

$$\|\nabla u\|_{L^\infty(0,T;L^\infty(\Omega))} + \left\|\nabla^2 u\right\|_{L^2(0,T;L^2(\Omega))} \le c,$$

cf. Theorem 1.

Let us remark that our method of proof can be applied to a more general class of Lipschitzian domains such as convex polyhedrons. This will be discussed in Section 5.

4. The auxiliary problem

In this section, we study the regularised system

$$\partial_t\mathbf{u}_\varepsilon - \Delta\mathbf{u}_\varepsilon = \mathbf{H}_\varepsilon(\mathbf{u}_\varepsilon,\nabla\mathbf{u}_\varepsilon) + \mathbf{f} \quad \text{in} \quad \Omega\times(0,T], \tag{4.1}$$

$$\mathbf{u}_\varepsilon = \mathbf{0} \quad \text{on} \quad \partial\Omega\times(0,T], \tag{4.2}$$

$$\mathbf{u}_\varepsilon(\cdot,0) = \mathbf{u_0} \quad \text{in} \quad \Omega, \tag{4.3}$$

where

$$\mathbf{H}_\varepsilon(\mathbf{u}_\varepsilon,\nabla\mathbf{u}_\varepsilon) = (\mathbf{g}(\nabla\mathbf{u}_\varepsilon) - \mathbf{u}_\varepsilon)\,\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)$$

and $\mu_\varepsilon(s) = (1+\varepsilon s)^{-1}s$, for $s\in\mathbb{R}$. Note that $\mu_\varepsilon(|s|^q)$ is bounded.

Due to the standard theory of parabolic systems, cf. [21], the system(4.1)–(4.3) has a weak solution fulfilling

$$\mathbf{u}_\varepsilon \in L^2\left(0,T;W^{2,2}(\Omega;\mathbb{R}^N)\right)\cap W^{1,2}\left(0,T;L^2(\Omega;\mathbb{R}^N)\right).$$

Moreover, it holds that

$$\nabla\mathbf{u}_\varepsilon \in L^\infty\left(0,T;L^\infty(\Omega;\mathbb{R}^{dN})\right). \tag{4.4}$$

In fact, arguing as in the previous section, it follows that $\mathbf{u}_\varepsilon$ is bounded. Thus, the right-hand side of equation (4.1) is bounded. It follows that

$$\Delta \mathbf{u}_\varepsilon \in L^\infty\left(0,T;L^s(\Omega;\mathbb{R}^N)\right),$$

for $s > d$. Applying Sobolev's imbedding theorem, we obtain (4.4).

Our goal is to prove the basic energy estimate for the regularised solution $\mathbf{u}_\varepsilon$ and we apply a difference quotient technique. Let us introduce some notations. Let $R > 0$, $h \in \mathbb{R}^d$ be a vector, $|h| \in (0, \frac{R}{2})$, $T_{\pm h}v(x) = v(x \pm h)$,

$$D_h v(x) = \frac{T_h v(x) - v(x)}{|h|}, \quad \text{and} \quad D_{-h}v(x) = \frac{v(x) - T_{-h}v(x)}{|h|}.$$

That is, $D_h v$ is the forward difference quotient of v in the direction h and $D_{-h}v$ is the backward difference quotient.

Further, let $B_R(P) = \{x \in \mathbb{R}^d : |P - x| < R\}$. To shorten our writing, we use the abbreviations $B_R = B_R(P)$ and $\Omega_R = \Omega \cap B_R(P)$. The function $\eta \in W^{2,\infty}(\mathbb{R}^d)$ is a radial-symmetric cut–off function satisfying $\eta \equiv 1$ in B_R, $\operatorname{supp}\eta = \overline{B_{2R}}$, and $0 \le \eta \le 1$ in $\mathbb{R}^d$. Moreover, let $\eta_{\pm h}(x) = \eta(x \pm \frac{1}{2}h)$.

Our proof consists of several steps. We will give an interior energy estimate (Lemma 1) and will then show the local regularity up to a flat boundary portion (Lemma 2). Global regularity in a smooth domain is proved in Proposition 1. The case of a domain with a non-smooth boundary with corner points is discussed in Section 5.

Lemma 1. *Let $B_{3R} \subset \Omega$. Then there are constants C_p, C_R, C_0 and $p > 2$ such that*

$$\begin{aligned}\sup_{0<t\le T}\int_{\Omega_R}|\nabla \mathbf{u}_\varepsilon|^p + \int_0^T\int_{\Omega_R}|\nabla^2\mathbf{u}_\varepsilon|^2 + C_R \\ \le C_p\Big(\int_{\Omega_{3R}}|\nabla \mathbf{u_0}|^p + \int_0^T\int_{\Omega_{3R}}|\mathbf{f}|^2 + C_0\Big) + \kappa\, C_{3R},\end{aligned} \tag{4.5}$$

where $C_R = \int_0^T\int_{\Omega_R}\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\left(|\nabla \mathbf{u}_\varepsilon|^2 + |\nabla \mathbf{u}_\varepsilon|^p\right)$ and $\kappa > 0$ is sufficiently small.

Proof: Multiplying the system (4.1) by a test function $\boldsymbol{\varphi} \in L^2(0,T;W^{1,2}(\Omega))$, we get

$$\begin{aligned}
&J_1 + J_2 + J_3 \\
&:= \int_0^T\int_{\Omega_{3R}} \partial_t \mathbf{u}_\varepsilon \cdot \boldsymbol{\varphi} + \int_0^T\int_{\Omega_{3R}} \nabla\mathbf{u}_\varepsilon \cdot \nabla\boldsymbol{\varphi} + \int_0^T\int_{\Omega_{3R}} \mathbf{u}_\varepsilon\, \mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q) \cdot \boldsymbol{\varphi} \\
&= -\int_0^T\int_{\Omega_{3R}} \mathbf{g}\, \mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q) \cdot \boldsymbol{\varphi} + \int_0^T\int_{\Omega_{3R}} \mathbf{f}\cdot\boldsymbol{\varphi} =: J_4 + J_5.
\end{aligned}$$

We set

$$\boldsymbol{\varphi} = -D_h(\eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) - D_{-h}(\eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon), \quad (4.6)$$

for some $p > 2$. This test function can be seen as a discretisation of the non-degenerate p-Laplacian. Since $B_{3R} \subset \Omega$ and supp $\eta = \overline{B_{2R}}$ it holds that $\boldsymbol{\varphi} = 0$ on $\partial\Omega$.

To begin with, let us estimate the integral J_1. Due to Leibniz rules

$$D_h(vw) = (D_h v)(T_h w) + vD_h w \quad \text{and} \quad D_{-h}(vw) = (D_{-h}v)(T_{-h}w) + vD_{-h}w,$$

there holds the identity

$$\begin{aligned}
J_1 &= \int_0^T\int_{\Omega_{3R}} D_h\partial_t\mathbf{u}_\varepsilon \cdot \eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon \\
&\quad + \int_0^T\int_{\Omega_{3R}} D_{-h}\partial_t\mathbf{u}_\varepsilon \cdot \eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon \\
&\quad - \int_0^T\int_{\Omega_{3R}} D_h(\partial_t\mathbf{u}_\varepsilon \cdot \eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) \\
&\quad - \int_0^T\int_{\Omega_{3R}} D_{-h}(\partial_t\mathbf{u}_\varepsilon \cdot \eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon) \\
&=: J_{11} + \cdots + J_{14}.
\end{aligned}$$

In view of the fact that supp $\eta = \overline{B_{2R}}$, we have $\eta = 0$ in the sets $\{\Omega_{3R} \pm h \backslash \Omega_{3R}\}$ and $\{\Omega_{3R} \setminus \Omega_{3R} \pm h\}$. Hence, it follows that

$$\begin{aligned}
J_{13} &= -\frac{1}{|h|}\int_0^T\int_{\Omega_{3R}+h\backslash\Omega_{3R}} \partial_t\mathbf{u}_\varepsilon \cdot \eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon \\
&\quad + \frac{1}{|h|}\int_0^T\int_{\Omega_{3R}\backslash\Omega_{3R}+h} \partial_t\mathbf{u}_\varepsilon \cdot \eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon = 0
\end{aligned}$$

and

$$\begin{aligned} J_{14} &= -\frac{1}{|h|}\int_0^T\int_{\Omega_{3R}\setminus\Omega_{3R}-h} \partial_t \mathbf{u}_\varepsilon \cdot \eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon \\ &\quad +\frac{1}{|h|}\int_0^T\int_{\Omega_{3R}-h\setminus\Omega_{3R}} \partial_t \mathbf{u}_\varepsilon \cdot \eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon = 0. \end{aligned}$$

Further, noting that $\partial_t|v|^2 = 2vv_t$ and $\partial_t|v|^p = p|v|^{p-2}vv_t$, we have

$$J_{11} = \frac{1}{2}\int_0^T\int_{\Omega_{3R}} \eta_h^2\,\partial_t|D_h\mathbf{u}_\varepsilon|^2 + \frac{1}{p}\int_0^T\int_{\Omega_{3R}} \eta_h^2\,\partial_t|D_h\mathbf{u}_\varepsilon|^p.$$

Altogether, we find

$$\begin{aligned} J_1 &= \frac{1}{2}\int_{\Omega_{3R}} \big(\eta_h^2|D_h\mathbf{u}_\varepsilon(\cdot,T)|^2 + \eta_{-h}^2|D_{-h}\mathbf{u}_\varepsilon(\cdot,T)|^2\big) \\ &\quad +\frac{1}{p}\int_{\Omega_{3R}} \big(\eta_h^2|D_h\mathbf{u}_\varepsilon(\cdot,T)|^p + \eta_{-h}^2|D_{-h}\mathbf{u}_\varepsilon(\cdot,T)|^p\big) \\ &\quad -\frac{1}{2}\int_{\Omega_{3R}} \big(\eta_h^2|D_h\mathbf{u_0}|^2 + \eta_{-h}^2|D_{-h}\mathbf{u_0}|^2\big) \\ &\quad -\frac{1}{p}\int_{\Omega_{3R}} \big(\eta_h^2|D_h\mathbf{u_0}|^p + \eta_{-h}^2|D_{-h}\mathbf{u_0}|^p\big). \end{aligned}$$

To simplify our writing, we suppose that there is a constant $c > 0$ such that

$$\int_{\Omega_{3R}} \eta_{\pm h}^2|D_{\pm h}\mathbf{u}_\varepsilon(\cdot,T)|^2 + \eta_{\pm h}^2|D_{\pm h}\mathbf{u}_\varepsilon(\cdot,T)|^p$$

$$\geq c\sup_{0<t\leq T}\int_{\Omega_{3R}} \eta_{\pm h}^2|D_{\pm h}\mathbf{u}_\varepsilon|^2 + \eta_{\pm h}^2|D_{\pm h}\mathbf{u}_\varepsilon|^p.$$

Next, we have

$$\begin{aligned} J_2 &= \int_0^T\int_{\Omega_{3R}} D_h\nabla\mathbf{u}_\varepsilon\cdot\nabla(\eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon) \\ &\quad+\int_0^T\int_{\Omega_{3R}} D_{-h}\nabla\mathbf{u}_\varepsilon\cdot\nabla(\eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) \\ &\quad-\int_0^T\int_{\Omega_{3R}} D_h(\nabla\mathbf{u}_\varepsilon\cdot\nabla(\eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon)) \\ &\quad-\int_0^T\int_{\Omega_{3R}} D_{-h}(\nabla\mathbf{u}_\varepsilon\cdot\nabla(\eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon)) \\ &=: J_{21}+\cdots+J_{24}. \end{aligned}$$

Let us note that $J_{23}=J_{24}=0$. Moreover, it holds that

$$\begin{aligned} J_{21} &= \int_0^T\int_{\Omega_{3R}} D_h\nabla\mathbf{u}_\varepsilon\cdot\nabla\eta_h^2\,(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon \\ &\quad+\int_0^T\int_{\Omega_{3R}}\eta_h^2\,|D_h\nabla\mathbf{u}_\varepsilon|^2+(p-1)\int_0^T\int_{\Omega_{3R}}\eta_h^2\,|D_h\nabla\mathbf{u}_\varepsilon|^2|D_h\mathbf{u}_\varepsilon|^{p-2}. \end{aligned}$$

Noting that $\nabla\eta_h^2=2\eta_h\nabla\eta_h$ and using Young's inequality, we get

$$\begin{aligned} \int_0^T\int_{\Omega_{3R}} |D_h\nabla\mathbf{u}_\varepsilon\cdot\nabla\eta_h^2\,D_h\mathbf{u}_\varepsilon| &\le \delta\int_0^T\int_{\Omega_{3R}}\eta_h^2\,|\nabla D_h\mathbf{u}_\varepsilon|^2 \\ &\quad+c_\delta\int_0^T\int_{\Omega_{3R}}|2\nabla\eta_h|^2\,|D_h\mathbf{u}_\varepsilon|^2. \end{aligned}$$

The first integral on the right-hand side may by absorbed into J_{21}. Let us estimate the second integral. For $\kappa>0$, we define the set

$$\Omega_\kappa(t)=\left\{x\in\Omega_{3R}:|\nabla\eta(x)|^2\le\kappa\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon(x,t)|^q)\right\}.$$

Notice that $|\nabla\mathbf{u}_\varepsilon(x,t)|$ is bounded in $\Omega_{3R}\setminus\Omega_\kappa(t)$. Hence, it follows that

$$\int_0^T\int_{\Omega_{3R}}|\nabla\eta_h|^2\,|D_h\mathbf{u}_\varepsilon|^2\le\kappa\int_0^T\int_{\Omega_\kappa(t)}\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)\,|D_h\mathbf{u}_\varepsilon|^2+c_\kappa,$$

for some constant c_κ. Choosing κ sufficiently small, we may absorb the integral on the right-hand side into the left-hand side. This will be discussed later. In the same manner, we estimate

$$\begin{aligned}
&\int_0^T\int_{\Omega_{3R}} |D_h\nabla\mathbf{u}_\varepsilon|\,|\nabla\eta_h^2|\,|D_h\mathbf{u}_\varepsilon|^{\frac{p-2}{2}}\,|D_h\mathbf{u}_\varepsilon|^{\frac{p}{2}} \\
&\quad\le \delta\int_0^T\int_{\Omega_{3R}} \eta_h^2\,|\nabla D_h\mathbf{u}_\varepsilon|^2\,|D_h\mathbf{u}_\varepsilon|^{p-2} + c_\delta\int_0^T\int_{\Omega_{3R}} |2\nabla\eta_h|^2\,|D_h\mathbf{u}_\varepsilon|^p
\end{aligned}$$

and

$$\int_0^T\int_{\Omega_{3R}} |\nabla\eta_h|^2\,|D_h\mathbf{u}_\varepsilon|^p \le \kappa\int_0^T\int_{\Omega_\kappa(t)} \mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)\,|D_h\mathbf{u}_\varepsilon|^p + c_\kappa.$$

Thus, for some sufficiently small numbers $\delta,\kappa>0$, we conclude that

$$\begin{aligned}
J_2 \;\ge\; & \frac{1}{2}\int_0^T\int_{\Omega_{3R}} (\eta_h^2\,|\nabla D_h\mathbf{u}_\varepsilon|^2 + \eta_{-h}^2\,|\nabla D_{-h}\mathbf{u}_\varepsilon|^2) \\
& + \frac{p-1}{2}\int_0^T\int_{\Omega_{3R}} \Big(\eta_h^2\,|\nabla D_h\mathbf{u}_\varepsilon|^2\,|D_h\mathbf{u}_\varepsilon|^{p-2} \\
& \qquad\qquad + \eta_{-h}^2\,|\nabla D_{-h}\mathbf{u}_\varepsilon|^2\,|D_h\mathbf{u}_\varepsilon|^{p-2}\Big) \\
& - \kappa\int_0^T\int_{\Omega_{3R}} \mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)\,(|D_h\mathbf{u}_\varepsilon|^2 + |D_h\mathbf{u}_\varepsilon|^p) - c_\kappa.
\end{aligned}$$

Now let us estimate the integral J_3 from below. It holds that

$$\begin{aligned}
J_3 \;=\; & \int_0^T\int_{\Omega_{3R}} D_h(\mathbf{u}_\varepsilon\,\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q))\cdot\eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon \\
& + \int_0^T\int_{\Omega_{3R}} D_{-h}(\mathbf{u}_\varepsilon\,\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q))\cdot\eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon \\
& - \int_0^T\int_{\Omega_{3R}} D_h(\mathbf{u}_\varepsilon\,\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)\cdot\eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) \\
& - \int_0^T\int_{\Omega_{3R}} D_{-h}(\mathbf{u}_\varepsilon\,\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)\cdot\eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon) \\
=:\; & J_{31}+\cdots+J_{34},
\end{aligned}$$

where $J_{33} = J_{34} = 0$. The Leibniz rule $D_h(vw) = (D_h v)w + (T_h v)D_h w$ yields

$$\begin{aligned} J_{31} &= \int_0^T\int_{\Omega_{3R}} \eta_h^2\,\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)(|D_h\mathbf{u}_\varepsilon|^2 + |D_h\mathbf{u}_\varepsilon|^p) \\ &\quad + \int_0^T\int_{\Omega_{3R}} \eta_h^2\,T_h\mathbf{u}_\varepsilon\, D_h\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\cdot(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon \\ &=: J_{35} + J_{36}. \end{aligned}$$

Noting that $|\mu_\varepsilon'| \le 1$ and applying Tailor's expansion, we deduce

$$|D_h\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)| \le q\,|D_h\nabla\mathbf{u}_\varepsilon|\,(|\nabla T_h\mathbf{u}_\varepsilon|^{q-1} + |\nabla\mathbf{u}_\varepsilon|^{q-1})$$

and thus

$$\begin{aligned} |J_{36}| &\le \delta(p-1)\int_0^T\int_{\Omega_{3R}} \eta_h^2\,|\nabla D_h\mathbf{u}_\varepsilon|^2\,(1+|D_h\mathbf{u}_\varepsilon|^{p-2}) \\ &\quad + \frac{c_\delta}{p-1}\int_0^T\int_{\Omega_{3R}} \eta_h^2\,(|D_h\mathbf{u}_\varepsilon|^2 + |D_h\mathbf{u}_\varepsilon|^p)\,|T_h\mathbf{u}_\varepsilon|^2 \times \\ &\qquad\qquad \times \left(|\nabla T_h\mathbf{u}_\varepsilon|^{2q-2} + |\nabla\mathbf{u}_\varepsilon|^{2q-2}\right) \\ &=: J_{37} + J_{38}. \end{aligned}$$

For simplicity, we suppose that $|D_h\mathbf{u}_\varepsilon| \ge 1$ in Ω_{3R} and absorb J_{37} into J_2. Using again Young's inequality, we estimate

$$(\lambda^{-1}x^q)^{\frac{1}{2}}(\lambda x^{3q-4})^{\frac{1}{2}} \le \delta\lambda^{-1}x^q + c_\delta\lambda x^{3q-4}$$

and obtain

$$\begin{aligned} J_{38} &\le \delta\int_0^T\int_{\Omega_{3R}} \eta_h^2\,\lambda^{-1}\,(|\nabla T_h\mathbf{u}_\varepsilon|^q + |\nabla\mathbf{u}_\varepsilon|^q)(|D_h\mathbf{u}_\varepsilon|^2 + |D_h\mathbf{u}_\varepsilon|^p) \\ &\quad + \frac{c_\delta}{(p-1)^2}\int_0^T\int_{\Omega_{3R}} \eta_h^2\,(|D_h\mathbf{u}_\varepsilon|^2 + |D_h\mathbf{u}_\varepsilon|^p)\,\times \\ &\qquad\qquad \times\lambda\left(|\nabla T_h\mathbf{u}_\varepsilon|^{3q-4} + |\nabla\mathbf{u}_\varepsilon|^{3q-4}\right)\,|T_h\mathbf{u}_\varepsilon|^4. \end{aligned}$$

Let $\lambda := 1 + \varepsilon|\nabla T_h\mathbf{u}_\varepsilon|^q + \varepsilon|\nabla\mathbf{u}_\varepsilon|^q$. Due to (4.4), it holds that $\|\nabla\mathbf{u}_\varepsilon\|_\infty$ and $\|\mathbf{u}_\varepsilon\|_\infty$ are bounded. Hence, we may absorb the second integral on the right-hand side into J_1, if p is sufficiently large, and the first one into J_3. In the same manner, we estimate the integral

$$\begin{aligned} J_{32} &= \int_0^T\int_{\Omega_{3R}} \eta_{-h}^2\, \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)(|D_{-h}\mathbf{u}_\varepsilon|^2 + |D_{-h}\mathbf{u}_\varepsilon|^p) \\ &\quad + \int_0^T\int_{\Omega_{3R}} \eta_{-h}^2\, T_{-h}\mathbf{u}_\varepsilon\, D_{-h}\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\cdot(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon. \end{aligned}$$

Let us now consider J_4. We have

$$\begin{aligned} J_4 &= \int_0^T\int_{\Omega_{3R}} \mathbf{g}\,\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\cdot D_h(\eta_{-h}^2\,(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) \\ &\quad + \int_0^T\int_{\Omega_{3R}} \mathbf{g}\,\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\cdot D_{-h}(\eta_h^2\,(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon) \\ &=: J_{41}+J_{42}. \end{aligned}$$

The Leibniz rule $D_h(vw) = (D_hv)w + (T_hv)D_hw$ yields

$$\begin{aligned} J_{41} &= \int_0^T\int_{\Omega_{3R}} \mathbf{g}\,\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\cdot D_h\eta_{-h}^2\,(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon \\ &\quad + \int_0^T\int_{\Omega_{3R}} \mathbf{g}\,\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\cdot \eta_h^2\, D_h((1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) \\ &=: J_{43}+J_{44}. \end{aligned}$$

Let $\kappa \in (0,1)$ be sufficiently small and

$$\Omega_\kappa^\pm(t) = \{x \in \Omega_{3R} : |D_{\pm h}\mathbf{u}_\varepsilon(x,t)| \geq \kappa^{-1}\}.$$

Notice that $|\mathbf{g}|$ and $|D_h\eta_{-h}^2|$ are bounded and it thus follows that $|J_{43}|$ is bounded in $\Omega_{3R}\setminus\Omega_\kappa^-(t)$ by a constant c_κ, if $|h|$ is sufficiently small. Further, let us note that

$$(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})|D_{-h}\mathbf{u}_\varepsilon| \leq \kappa(|D_{-h}\mathbf{u}_\varepsilon|^2 + |D_{-h}\mathbf{u}_\varepsilon|^p) \quad \text{in } \Omega_\kappa^-(t).$$

Altogether, we deduce

$$|J_{43}| \leq c\kappa \int_0^T\int_{\Omega_\kappa(t)} \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\,(|D_{-h}\mathbf{u}_\varepsilon|^2 + |D_{-h}\mathbf{u}_\varepsilon|^p) + c_\kappa.$$

Next, let us estimate $|J_{44}|$. We define the function

$$\mathbf{y}(\mathbf{s}) := (1+|\mathbf{s}|^{p-2})\mathbf{s}, \quad \mathbf{s}\in\mathbb{R}^N$$

and use Taylor's expansion to get

$$y^\nu(\mathbf{s}) - y^\nu(\bar{\mathbf{s}}) = (s^m - \bar{s}^m) \int_0^1 y^\nu_{s^m}(z\mathbf{s} + (1-z)\bar{\mathbf{s}})\, dz.$$

Here we have used the notation $y^\nu_{s^m}(\mathbf{s}) = \frac{\partial}{\partial_{s^m}} y^\nu(\mathbf{s})$, where $\mathbf{y}$ and $\mathbf{s}$ have the components y^ν $(1 \le \nu \le N)$ and s^m $(1 \le m \le d)$. We deduce

$$|J_{44}| \le c \int_0^T \int_{\Omega_{3R}} \eta_h^2\, \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\, |D_h D_{-h} \mathbf{u}_\varepsilon|\, \big(1+(p-1)(|D_h \mathbf{u}_\varepsilon|^{p-2} + |D_{-h} \mathbf{u}_\varepsilon|^{p-2})\big).$$

We estimate

$$\eta_h(\eta_h \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)|D_h D_{-h}\mathbf{u}_\varepsilon|) \le \delta \eta_h^2 |\nabla \mathbf{u}_\varepsilon|^{2q} |D_h D_{-h} \mathbf{u}_\varepsilon|^2 + c_\delta \eta_h^2$$

and notice that

$$|\nabla \mathbf{u}_\varepsilon|^{2q} \le c_\kappa \quad \text{in} \;\; \Omega_{3R} \setminus \Omega_\kappa^\pm(t),$$

if $|h|$ is small, and

$$|\nabla \mathbf{u}_\varepsilon|^{2q} \le \delta |D_h \mathbf{u}_\varepsilon|^{p-2} \quad \text{in} \;\; \Omega_\kappa^\pm(t),$$

where $\delta > 0$ is small, if p is large. Thus, we get

$$\int_0^T \int_{\Omega_{3R}} \eta_h^2\, \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\, |D_h D_{-h} \mathbf{u}_\varepsilon| \le \delta \int_0^T \int_{\Omega_{3R}} \eta_h^2\, |D_h D_{-h} \mathbf{u}_\varepsilon|^2 |D_h \mathbf{u}_\varepsilon|^{p-2} + C$$

and absorb the integral on the right-hand side into J_2. Furthermore, Hölder's inequality entails

$$\begin{aligned} (p-1) &\int_0^T \int_{\Omega_{3R}} \eta_h^2\, \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\, |D_h D_{-h} \mathbf{u}_\varepsilon|\, |D_h \mathbf{u}_\varepsilon|^{\frac{p-2}{2}}\, |D_h \mathbf{u}_\varepsilon|^{\frac{p-2}{2}} \\ \le\;\; &\frac{\delta(p-1)}{2} \int_0^T \int_{\Omega_{3R}} \eta_h^2\, \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\, |D_h D_{-h} \mathbf{u}_\varepsilon|^2\, |D_h \mathbf{u}_\varepsilon|^{p-2} \\ &+ \frac{p-1}{2\delta} \int_0^T \int_{\Omega_{3R}} \eta_h^2\, \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\, |D_h \mathbf{u}_\varepsilon|^{p-2}. \end{aligned}$$

Let us note that $\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)$ is bounded. Hence, we choose δ sufficiently small in order to absorb the first integral on the right-hand side into J_2. Next, we use the estimates

$$\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q) |D_h \mathbf{u}_\varepsilon|^{p-2} \le |\nabla \mathbf{u}_\varepsilon|^q |D_h \mathbf{u}_\varepsilon|^{p-2} \le c_\kappa \quad \text{in } \Omega_{3R} \setminus \Omega_\kappa^+(t)$$

and

$$\mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q) |D_h \mathbf{u}_\varepsilon|^{p-2} \le \kappa^2 \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q) |D_h \mathbf{u}_\varepsilon|^p \quad \text{in } \Omega_\kappa^+(t)$$

and absorb the second integral into J_3.

In the same manner we deal with J_{42}. Now, let us consider

$$\begin{aligned} J_5 &= -\int_0^T\int_{\Omega_{3R}} \mathbf{f}\cdot D_h(\eta_{-h}^2\,(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) \\ &\quad -\int_0^T\int_{\Omega_{3R}} \mathbf{f}\cdot D_{-h}(\eta_h^2\,(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon) =: J_{51}+J_{52}. \end{aligned}$$

Due to Leibniz's rule $D_h(vw)=(D_hv)w+(T_hv)D_hw$, we obtain

$$\begin{aligned} J_{51} &= -\int_0^T\int_{\Omega_{3R}} \mathbf{f}\cdot D_h\eta_{-h}^2\,(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon \\ &\quad -\int_0^T\int_{\Omega_{3R}} \mathbf{f}\cdot \eta_h^2\,D_h((1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) =: J_{53}+J_{54}. \end{aligned}$$

Since $|D_h\eta_{-h}^2|$ and $|\mathbf{f}|$ are bounded we conclude that $|J_{53}|\le c_\kappa$ in $\Omega_{3R}\setminus\Omega_\kappa^-(t)$, if $|h|$ is sufficiently small. Moreover, in $\Omega_\kappa^-(t)$ it holds that

$$(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})|D_{-h}\mathbf{u}_\varepsilon|\le\kappa(|D_{-h}\mathbf{u}_\varepsilon|^2+|D_{-h}\mathbf{u}_\varepsilon|^p),$$

and also $\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)\ge 1$, if ε is small. Thus, it follows that

$$|J_{53}|\le c\kappa\int_0^T\int_{\Omega_{3R}}\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)\,(|D_{-h}\mathbf{u}_\varepsilon|^2+|D_{-h}\mathbf{u}_\varepsilon|^p)+c_\kappa.$$

Further, using the Taylor expansion of the function $\mathbf{y}(\mathbf{s}):=(1+|\mathbf{s}|^{p-2})\mathbf{s}$, we deduce

$$|J_{54}|\le c\int_0^T\int_{\Omega_{3R}}\eta_h^2\,|\mathbf{f}|\,|D_hD_{-h}\mathbf{u}_\varepsilon|\,\big(1+(p-1)(|D_h\mathbf{u}_\varepsilon|^{p-2}+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})\big).$$

We estimate

$$\int_0^T\int_{\Omega_{3R}}\eta_h^2\,|\mathbf{f}|\,|D_hD_{-h}\mathbf{u}_\varepsilon|\le\delta\int_0^T\int_{\Omega_{3R}}\eta_h^2\,|D_hD_{-h}\mathbf{u}_\varepsilon|^2+c_\delta\int_0^T\int_{\Omega_{3R}}\eta_h^2\,|\mathbf{f}|^2$$

and absorb the first integral on the right-hand side into J_2. Moreover, Hölder's inequality yields

$$
\begin{aligned}
&(p-1)\int_0^T\int_{\Omega_{3R}} \eta_h^2\,|\mathbf{f}|\,|D_hD_{-h}\mathbf{u}_\varepsilon|\,|D_h\mathbf{u}_\varepsilon|^{\frac{p-2}{2}}\,|D_h\mathbf{u}_\varepsilon|^{\frac{p-2}{2}}\\
&\le \frac{\delta(p-1)}{2}\int_0^T\int_{\Omega_{3R}} \eta_h^2\,|\mathbf{f}|\,|D_hD_{-h}\mathbf{u}_\varepsilon|^2\,|D_h\mathbf{u}_\varepsilon|^{p-2}\\
&\quad+\frac{p-1}{2\delta}\int_0^T\int_{\Omega_{3R}} \eta_h^2\,|\mathbf{f}|\,|D_h\mathbf{u}_\varepsilon|^{p-2}.
\end{aligned}
$$

Since $|\mathbf{f}|$ is bounded, we may absorb the first integral on the right-hand side into J_2. Further, it holds that

$$
|D_h\mathbf{u}_\varepsilon| \le c_\kappa \quad \text{in } \Omega_{3R}\setminus\Omega_\kappa^+(t)
$$

and

$$
|D_h\mathbf{u}_\varepsilon|^{p-2} \le \kappa^2\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)|D_h\mathbf{u}_\varepsilon|^p \quad \text{in } \Omega_\kappa^+(t).
$$

Here we have used the fact that $\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q) \ge 1$ in $\Omega_\kappa^+(t)$ if ε and $|h|$ are sufficiently small. Now we absorb the second integral into J_3. In the same manner, we estimate J_{52}.

Let h be parallel to the k-th unit vector in $\mathbb{R}^d$. Taking $|h|\to 0$, noting that $|x|^2 \le 1+|x|^p$, and collecting results we arrive at

$$
\begin{aligned}
&\sup_{0<t\le T}\int_{\Omega_R}|\partial_k\mathbf{u}_\varepsilon|^p+\int_0^T\int_{\Omega_R}|\nabla\partial_k\mathbf{u}_\varepsilon|^2+\int_0^T\int_{\Omega_R}\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)\,(|\partial_k\mathbf{u}_\varepsilon|^2+|\partial_k\mathbf{u}_\varepsilon|^p)\\
&\le c_p\Big(\int_{\Omega_{3R}}|\partial_k\mathbf{u_0}|^p+\int_0^T\int_{\Omega_{3R}}|\mathbf{f}|^2+C\Big)\\
&\quad+\kappa\int_0^T\int_{\Omega_{3R}}\mu_\varepsilon(|\nabla\mathbf{u}_\varepsilon|^q)\,(|\partial_k\mathbf{u}_\varepsilon|^2+|\partial_k\mathbf{u}_\varepsilon|^p) \qquad (4.7)
\end{aligned}
$$

for all $k\in\{1,\dots,d\}$ and $\kappa>0$ sufficiently small. This yields the assertion. ∎

In our next lemma we investigate the local regularity of $\mathbf{u}_\varepsilon$ up to a flat boundary portion. More precisely, we suppose that $P\in\partial\Omega$ and

$$
\partial\Omega\cap B_{3R}(P)=E_{d-1}\cap B_{3R}(P),
$$

where E_{d-1} is a $(d-1)$-dimensional hyperplane.

Let e be the inner unit normal of $\partial\Omega \cap B_{3R}$, $z \in \partial\Omega \cap B_{3R}$, and $\lambda > 0$. Thus, it holds that $z + \lambda e \in \Omega_{3R}$ and $z - \lambda e \in B_{3R} \setminus \overline{\Omega}$. We define extensions of the functions $\mathbf{u}_\varepsilon$ and $\mathbf{f}$ by setting

$$\mathbf{u}_\varepsilon(z + \lambda e) := -\mathbf{u}_\varepsilon(z - \lambda e) \qquad \text{and} \qquad \mathbf{f}(z + \lambda e) := \mathbf{f}(z - \lambda e)\,. \tag{4.8}$$

Moreover, let $\eta(z + \lambda e) := \eta(z - \lambda e)$. Thus, $\mathbf{u}_\varepsilon$ is odd with respect to the boundary, and the functions $\mathbf{f}$ and η are even.

Lemma 2. *Let $P \in \partial\Omega$ and $\partial\Omega \cap B_{3R}(P) = E_{d-1} \cap B_{3R}(P)$, where E_{d-1} is a $(d-1)$-dimensional hyperplane. Then there are constants C_p, C_R, C_0 and $p > 2$ such that*

$$\begin{aligned}\sup_{0<t\le T} \int_{\Omega_R} |\nabla \mathbf{u}_\varepsilon|^p + \int_0^T\!\!\int_{\Omega_R} |\nabla^2 \mathbf{u}_\varepsilon|^2 + C_R \\ \le C_p\Big(\int_{\Omega_{3R}} |\nabla \mathbf{u_0}|^p + \int_0^T\!\!\int_{\Omega_{3R}} |\mathbf{f}|^2 + C_0\Big) + \kappa\, C_{3R},\end{aligned} \tag{4.9}$$

where $C_R = \int_0^T\!\int_{\Omega_R} \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\,(|\nabla \mathbf{u}_\varepsilon|^2 + |\nabla \mathbf{u}_\varepsilon|^p)$ and $\kappa > 0$ is sufficiently small.

Proof: We proceed as in the proof of Lemma 1. Without loss of generality, we suppose that $\Omega_{3R} = \{x \in B_{3R} : x_1 > 0\}$. Thus, the unit vector e_1 is the inner unit normal of $\partial\Omega \cap B_{3R}$.

In the case that the vector h is parallel to $\partial\Omega \cap B_{3R}$, the test function

$$\boldsymbol{\varphi} = -D_h(\eta_{-h}^2(1 + |D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) - D_{-h}(\eta_h^2(1 + |D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon)$$

is admissible, for it holds that $\boldsymbol{\varphi} = 0$ on $\partial\Omega \cap B_{3R}$. Arguing as in the proof of Lemma 1 we obtain the estimate (4.7) for all $k \in \{2, \dots, d\}$.

Next, let us discuss the case $k = 1$. We choose h as an outer normal of $\partial\Omega \cap B_{3R}$. Notice that $\boldsymbol{\varphi}$ is an admissible test function. In fact, it holds that

$$D_h(\eta_{-h}^2(1 + |D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon) = D_{-h}(\eta_h^2(1 + |D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon) = 0$$

on $\partial\Omega \cap B_{3R}$. For instance, on $\partial\Omega$ we have

$$\begin{aligned}&\eta^2(x + \tfrac{1}{2}h)\,(1 + |T_h\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon|^{p-2})(T_h\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon) \\ &\quad -\eta^2(x - \tfrac{1}{2}h)\,(1 + |\mathbf{u}_\varepsilon - T_{-h}\mathbf{u}_\varepsilon|^{p-2})(\mathbf{u}_\varepsilon - T_{-h}\mathbf{u}_\varepsilon) \\ &= \eta^2(x + \tfrac{1}{2}h)\,(1 + |T_h\mathbf{u}_\varepsilon|^{p-2})T_h\mathbf{u}_\varepsilon - \eta^2(x - \tfrac{1}{2}h)\,(1 + |T_{-h}\mathbf{u}_\varepsilon|^{p-2})T_{-h}\mathbf{u}_\varepsilon = 0,\end{aligned}$$

since $\mathbf{u}_\varepsilon$ is odd and η is even with respect to the boundary.

We now multiply the system (4.1) by $\boldsymbol{\varphi}$. Then the calculations run as in the proof of Lemma 1; we only have to show that the boundary integrals vanish.

To begin with, let us show that $J_{13} + J_{14} = 0$. We have

$$\begin{aligned} J_{13} &= -\frac{1}{|h|}\int_0^T\int_{\Omega_{3R}+h\setminus\Omega_{3R}} \partial_t\mathbf{u}_\varepsilon\cdot\eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon \\ &\quad +\frac{1}{|h|}\int_0^T\int_{\Omega_{3R}\setminus\Omega_{3R}+h} \partial_t\mathbf{u}_\varepsilon\cdot\eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon. \end{aligned}$$

Due to the fact that supp $\eta = \overline{B_{2R}}$, the second integral on the right-hand side vanishes. Further, we find

$$J_{14} = -\frac{1}{|h|}\int_0^T\int_{\Omega_{3R}\setminus\Omega_{3R}-h} \partial_t\mathbf{u}_\varepsilon\cdot\eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon,$$

since

$$\int_0^T\int_{\Omega_{3R}-h\setminus\Omega_{3R}} \partial_t\mathbf{u}_\varepsilon\cdot\eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon = 0.$$

Notice that $\Omega_{3R}\setminus\Omega_{3R}-h$ is the reflection of $\Omega_{3R}+h\setminus\Omega_{3R}$ with respect to the hyperplane $\{x\in\mathbb{R}^d : x_1 = 0\}$. The functions $\eta_{\pm h}^2$, $(1+|D_{\pm h}\mathbf{u}_\varepsilon|^{p-2})$, and $D_{\pm h}\mathbf{u}_\varepsilon$ are even with respect to the hyperplane, in the sence that, e.g., $\eta_{-h}^2(x) = \eta_h^2(x^*)$ for all $x\in\Omega_{3R}+h\setminus\Omega_{3R}$, where $x^*\in\Omega_{3R}\setminus\Omega_{3R}-h$ is the reflection of the point x with respect to the hyperplane. Moreover, the function $\partial_t\mathbf{u}_\varepsilon$ is odd. Hence, the integrand is an odd function and it follows that $J_{13}+J_{14}=0$.

Next, we consider J_{23} and J_{24}. We find

$$J_{23} = -\frac{1}{|h|}\int_0^T\int_{\Omega_{3R}+h\setminus\Omega_{3R}} \nabla\mathbf{u}_\varepsilon\cdot\nabla(\eta_{-h}^2(1+|D_{-h}\mathbf{u}_\varepsilon|^{p-2})D_{-h}\mathbf{u}_\varepsilon)$$

and

$$J_{24} = -\frac{1}{|h|}\int_0^T\int_{\Omega_{3R}\setminus\Omega_{3R}-h} \nabla\mathbf{u}_\varepsilon\cdot\nabla(\eta_h^2(1+|D_h\mathbf{u}_\varepsilon|^{p-2})D_h\mathbf{u}_\varepsilon).$$

The function $\mathbf{u}_\varepsilon$ is odd with respect to the hyperplane $\{x\in\mathbb{R}^d : x_1 = 0\}$. Thus, the derivative $\partial_1\mathbf{u}_\varepsilon$ is an even function, and $\partial_k\mathbf{u}_\varepsilon$ $(2\le k\le d)$ are odd functions. Moreover, since $\eta_{\pm h}^2(1+|D_{\pm h}\mathbf{u}_\varepsilon|^{p-2})D_{\pm h}\mathbf{u}_\varepsilon$ is even, its derivative with respect to x_k is odd for $k = 1$ and even for $k \neq 1$. Hence, the integrand is an odd function and it holds that $J_{23} + J_{24} = 0$.

Moreover, we find

$$\begin{aligned} J_{33} + J_{34} &= -\frac{1}{|h|} \int_0^T \int_{\Omega_{3R+h} \setminus \Omega_{3R}} \mathbf{u}_\varepsilon \, \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q) \cdot \eta_{-h}^2 (1 + |D_{-h} \mathbf{u}_\varepsilon|^{p-2}) D_{-h} \mathbf{u}_\varepsilon \\ & \quad - \frac{1}{|h|} \int_0^T \int_{\Omega_{3R} \setminus \Omega_{3R-h}} \mathbf{u}_\varepsilon \, \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q) \cdot \eta_h^2 (1 + |D_h \mathbf{u}_\varepsilon|^{p-2}) D_h \mathbf{u}_\varepsilon . \end{aligned}$$

Noting that $\mathbf{u}_\varepsilon$ is odd and all the other functions are even with respect to the hyperplane, we deduce $J_{33} + J_{34} = 0$.

Altogether, we conclude that the estimate (4.7) holds for all $k \in \{1, \dots, d\}$ and the assertion (4.9) follows. ■

Now we discuss the global regularity of $\mathbf{u}_\varepsilon$ in a smooth domain Ω.

Proposition 1. *There are constants C_p, C_0, and $p > 2$ such that*

$$\sup_{0<t\le T} \int_0^T \int_\Omega |\nabla \mathbf{u}_\varepsilon|^p + \int_0^T \int_\Omega |\nabla^2 \mathbf{u}_\varepsilon|^2 \le C_p \Big(\int_0^T \int_\Omega |\nabla \mathbf{u_0}|^p + \int_0^T \int_\Omega |\mathbf{f}|^2 + C_0 \Big). \tag{4.10}$$

Proof: Let us cover Ω by a finite number of balls $B_{R_i}(P_i)$, $i = 1, 2, \dots$, such that either $B_{3R_i}(P_i) \subset \Omega$ or $P_i \in \partial\Omega$. In the case that $B_{3R_i}(P_i) \subset \Omega$, the proof of Lemma 1 yields

$$\begin{aligned} & \sup_{0<t\le T} \int_0^T \int_{\Omega_{R_i}} |\nabla \mathbf{u}_\varepsilon|^p + \int_0^T \int_{\Omega_{R_i}} |\nabla^2 \mathbf{u}_\varepsilon|^2 + C_R \\ & \quad \le C_p \Big(\int_0^T \int_{\Omega_{3R_i}} |\nabla \mathbf{u_0}|^p + \int_0^T \int_{\Omega_{3R_i}} |\mathbf{f}|^2 + C_0 \Big) + \kappa \, C_{3R_i}, \end{aligned} \tag{4.11}$$

where

$$C_{R_i} = \int_0^T \int_{\Omega_{R_i}} \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q) \, (|\nabla \mathbf{u}_\varepsilon|^2 + |\nabla \mathbf{u}_\varepsilon|^p).$$

In the case that $P_i \in \partial\Omega$ and $\partial\Omega \cap B_{3R_i}(P) = E_{d-1} \cap B_{3R_i}(P)$, where E_{d-1} is a $(d-1)$-dimensional hyperplane, the estimate (4.11) follows from Lemma 2

Now let us discuss the case that $P_i \in \partial\Omega$ and $\partial\Omega \cap B_{3R_i}(P)$ is not a hyperplane. Since $\partial\Omega$ is smooth, there is a $W^{2,\infty}$–mapping ϕ_i and a ball $B_{3R_i}(\phi_i(P_i))$ such that $B_{3R_i}(\phi_i(P_i)) \cap \phi_i(\partial\Omega)$ is the intersection of $B_{3R_i}(\phi_i(P_i))$

and a $(d-1)$-dimensional hyperplane. Let $\hat{x} = \phi_i(x)$. The function $\hat{\mathbf{u}}_\varepsilon(\hat{x}) := \mathbf{u}_\varepsilon(\phi_i^{-1}(\hat{x}))$ is the weak solution of

$$\begin{aligned} \partial_t \hat{\mathbf{u}}_\varepsilon - \hat{\Delta}\hat{\mathbf{u}}_\varepsilon = \mathbf{H}_\varepsilon(\hat{\mathbf{u}}_\varepsilon, \hat{\nabla}\hat{\mathbf{u}}_\varepsilon) + \hat{\mathbf{f}} \quad & \text{in} \quad \hat{\Omega} \times (0,T], \\ \hat{\mathbf{u}}_\varepsilon = \mathbf{0} \quad & \text{on} \quad \partial\hat{\Omega} \times (0,T], \\ \hat{\mathbf{u}}_\varepsilon(\cdot, 0) = \hat{\mathbf{u}}_\mathbf{0} \quad & \text{in} \quad \hat{\Omega}, \end{aligned}$$

where $\hat{\nabla} = M\nabla$, $\hat{\Delta} = \hat{\partial}_l\hat{\partial}_l$, and M has the components $m_{jk} = \partial_j \phi_i^k$, where ϕ_i^k is the k-th component of ϕ_i. Arguing as above, applying the substitution rule for integrals, and noting that M is positive definite, we obtain an analogous of estimate (4.11).

We now cover Ω by a finite number of appropriate sets $\phi_i^{-1}(B_{3R_i}(\phi_i(P_i)))$, $i = 1, 2, \ldots$, and proceed as above. Choosing $\kappa > 0$ sufficiently small yields the assertion. ■

We conclude this section with the proof of the main result for domains with a smooth boundary.

Proof of Theorem 1: Estimate (4.10) holds for a number p sufficiently large. Let us extract the p-th root and send $p \to \infty$. It follows that there is a constant C, independent of ε, such that

$$\|\nabla \mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C.$$

Now we take $\lim_{\varepsilon\to 0}$. We extract a subsequence, again denoted by $\mathbf{u}_\varepsilon$, such that $\mathbf{u}_\varepsilon \to \mathbf{u}$ uniformly and

$$\nabla \mathbf{u}_\varepsilon \to \nabla \mathbf{u} \quad \text{weakly-* in } L^\infty(0,T; L^\infty(\Omega; \mathbb{R}^N)).$$

Due to standard arguments, we deduce that the weak limit $\mathbf{u}$ satisfies the primal system (1.1)–(1.3) in the weak sense.

Since $\nabla \mathbf{u}$ is a $L^\infty(0,T; L^\infty(\Omega; \mathbb{R}^N))$-function we may proceed as above and obtain an analogous of estimate (4.10) for $\mathbf{u}$. Thus, there is a constant C_p, depending only on p, such that

$$\left\|\nabla^2 \mathbf{u}\right\|_{L^2(0,T;L^2(\Omega))} \leq C_p,$$

for p sufficiently large. Hence, $\Delta \mathbf{u}$ is a $L^2(0,T; L^2(\Omega; \mathbb{R}^N))$-function. Noting that the Hamiltonian and $\mathbf{f}$ are bounded, it follows that

$$\partial_t \mathbf{u} \in L^2(0,T; L^2(\Omega; \mathbb{R}^N))$$

as well. □

5. Generalisation to Lipschitzian domains

Our difference quotient technique can be applied to a more general class of Lipschitzian domains, such as convex polyhedrons; cf. [17, 18]. More precisely, let us assume that Ω can be mapped in a smooth way onto a convex polyhedron. We make the following assumptions.

($\mathbf{\Omega}$1) $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) is a bounded open set;

($\mathbf{\Omega}$2) for each $P \in \partial\Omega$, there exists a mapping ϕ and a ball $B_R(\phi(P))$ such that

i) $\phi(\Omega) \cap B_R(\phi(P))$ is the intersection of $B_R(\phi(P))$ and a convex polyhedron,

ii) ϕ, $\phi^{-1} \in W^{2,\infty}_{loc}(\mathbb{R}^d)$ and the Jacobian of ϕ is positive definite;

($\mathbf{\Omega}$3) $\partial\Omega = \bigcup_{1\leq k\leq M} \overline{\Gamma_k}$, where Γ_k are open $(d-1)$–dimensional Lipschitzian domains for $k = 1, \dots, M$, and $\Gamma_i \cap \Gamma_k = \emptyset$ for $i \neq k$;

($\mathbf{\Omega}$4) $\partial\Gamma_{k_1} \cap \dots \cap \partial\Gamma_{k_j} = \emptyset$, if $j > d$ and $k_1 < \dots < k_j$ (that is, there are at most d adjacent faces Γ_k).

We now state the following corollary of Theorem 1.

Corollary 1. *Let Ω satisfy assumptions* **($\mathbf{\Omega}$1)**–**($\mathbf{\Omega}$4)**. *Then there exists a weak solution* $\mathbf{u}$ *of the system (1.1)-(1.3), satisfying*

$$\mathbf{u} \in L^2(0,T; W^{2,2}(\Omega;\mathbb{R}^N)) \cap L^\infty(0,T; W^{1,\infty}(\Omega;\mathbb{R}^N))$$

and

$$\mathbf{u}_t \in L^2(0,T; L^2(\Omega;\mathbb{R}^N)).$$

Proof: We proceed as in the proof of Theorem 1. First, let us discuss the case when Ω is a convex polyhedron. We can cover Ω by a finite number of balls $B_{R_i}(P_i)$, $i = 1, 2, \dots$, such that either $B_{3R_i}(P_i) \subset \Omega$ or $P_i \in \partial\Omega$. For each $P_i \in \partial\Omega$, there is an index set Λ_i such that $\Gamma_k \cap B_{3R_i}(P_i) \neq \emptyset$, for all $k \in \Lambda_i$ and $\partial\Omega \cap B_{3R_i}(P_i) = \bigcup_{k\in\Lambda_i} \overline{\Gamma_k} \cap B_{3R_i}(P_i)$. We suppose that $P_i \in \bigcap_{k\in\Lambda_i} \overline{\Gamma_k}$.

In the case that $B_{3R_i}(P_i) \subset \Omega$, the proof of Lemma 1 yields constants C_p, C_R, C_0 and $p > 2$ such that

$$\begin{aligned} &\sup_{0<t\leq T} \int_0^T \int_{\Omega_{R_i}} |\nabla \mathbf{u}_\varepsilon|^p + \int_0^T \int_{\Omega_{R_i}} |\nabla^2 \mathbf{u}_\varepsilon|^2 + C_R \\ &\qquad \leq C_p \Big(\int_0^T \int_{\Omega_{3R_i}} |\nabla \mathbf{u_0}|^p + \int_0^T \int_{\Omega_{3R_i}} |\mathbf{f}|^2 + C_0 \Big) + \kappa\, C_{3R_i}, \qquad (5.1) \end{aligned}$$

where

$$C_{R_i} = \int_0^T \int_{\Omega_{R_i}} \mu_\varepsilon(|\nabla \mathbf{u}_\varepsilon|^q)\,(|\nabla \mathbf{u}_\varepsilon|^2 + |\nabla \mathbf{u}_\varepsilon|^p).$$

In the case that $P_i \in \partial\Omega$ and $\partial\Omega \cap B_{3R_i}$ is contained in an $(d-1)$-dimensional hyperplane, we can find $d-1$ linearly independent vectors h parallel to $\partial\Omega \cap B_{3R_i}$ and one that is normal to $\partial\Omega \cap B_{3R_i}$. Then the proof of estimate (5.1) runs as in the proof of Lemma 2.

Now let us discuss the case when $\partial\Omega \cap B_{3R_i}(P_i)$ is not contained in a hyperplane. Let $k_0 \in \Lambda$ and $e \in \mathbb{R}^d$ be a unit vector parallel to $(\partial\Omega \cap B_{3R_i}) \setminus \Gamma_{k_0}$ satisfying $z + \lambda e \in \overline{\Omega}$, for all $z \in \partial\Omega \cap B_{3R_i}$ and $0 < \lambda < R$. Furthermore, let e^* be the reflection of e with respect to the hyperplane containing $\partial\Omega \cap B_{3R}$, and Ω_{3R}^* be the reflection of $\Omega_{3R} \setminus \Omega_{3R} + h$ with respect to the hyperplane. We now define the extensions of the functions $\mathbf{u}_\varepsilon$, $\mathbf{f}$, and η onto Ω_{3R}^* by setting

$$\mathbf{u}_\varepsilon(z + \lambda e^*) := -\mathbf{u}_\varepsilon(z + \lambda e),$$

$\mathbf{f}(z + \lambda e^*) := \mathbf{f}(z + \lambda e)$, and $\eta(z + \lambda e^*) := \eta(z + \lambda e)$.

We make use of the shift operator $T_{\pm h}^* v(x) := v(\psi(\lambda_0 \pm h))$ and define

$$D_h^* v(x) = \frac{T_h^* v(x) - v(x)}{|h|} \quad \text{and} \quad D_{-h}^* v(x) = \frac{v(x) - T_{-h}^* v(x)}{|h|},$$

where

$$\psi(\lambda) := \begin{cases} z + \lambda\, e & \text{for } \lambda \geq 0, \\ z - \lambda\, e^* & \text{for } \lambda < 0, \end{cases}$$

and $x = \psi(\lambda_0) = z + \lambda_0 e \in \Omega_{3R_i}$.

Taking the test function

$$\boldsymbol{\varphi} := -D_h^*(\eta_{-h}^2(1 + |D_{-h}^* \mathbf{u}_\varepsilon|^{p-2}) D_{-h}^* \mathbf{u}_\varepsilon) - D_{-h}^*(\eta_h^2(1 + |D_h^* \mathbf{u}_\varepsilon|^{p-2}) D_h^* \mathbf{u}_\varepsilon),$$

where $\eta_{\pm h}(x) = T_{\pm \frac{1}{2} h}^* \eta(x)$, and proceeding as in the proof of Lemma 2, estimate (5.1) follows.

Finally, let us consider the case that Ω is not a convex polyhedron. Then there is a $W^{2,\infty}$–mapping ϕ_i and a ball $B_{3R_i}(\phi_i(P_i))$ such that $B_{3R_i}(\phi_i(P_i)) \cap \phi_i(\partial\Omega)$ is the intersection of $B_{3R_i}(\phi_i(P_i))$ and a $(d-1)$-dimensional hyperplane. Now the proof of estimate (5.1) runs as in the proof of Proposition 1.

Collecting results and arguing as in the proof of Theorem 1, we obtain the assertion. ∎

6. Inhomogeneous Dirichlet boundary conditions

In this section we discuss the case of inhomogeneous Dirichlet boundary value conditions. The dynamical system

$$dy = m(y, \mathbf{v})\, dt + \sigma(y)\, dw_t, \qquad y(0) = x, \tag{6.1}$$

is modified by the N players through the controls $v^k(t)$ $(1 \le k \le N)$. The k-th player chooses its own control to maximize its cost functional

$$\begin{aligned} J^k_{x,t}(\mathbf{v}) \;&=\; \mathbb{E}^k_{x,t}\Bigg[\int_t^T l_k(y(\tau), \mathbf{v}(\tau))\, e^{-\int_t^\tau c_k(y(s),\mathbf{v}(s))\,ds}\, d\tau \\ &\qquad\qquad + \phi_k(y(T)) e^{-\int_t^T c_k(y(s),\mathbf{v}(s))\,ds}\Bigg]. \end{aligned}$$

Let us assume that T is the exit time of $y(t)$ from the domain Ω. Defining the payoff functions

$$u^k(x,t) := J^k_{x,t}(\mathbf{v}) \tag{6.2}$$

and setting $t = T$, we obtain the boundary value conditions $u^k(x,T) = \phi_k(y(T))$.

Let us discount the payoff function $u^k(x,t)$ back to the time point $t+\delta$. Thus, the payoff is given by $u^k(y(t+\delta), t+\delta)$. Let $\mathbf{v}$ be the optimal choice maximising the cost functional. This leads to the payoff functions

$$\begin{aligned} u^k(x,t) \;&=\; \mathbb{E}^k_{x,t}\Bigg[\int_t^{t+\delta} l_k(y(\tau), \mathbf{v}(\tau))\, e^{-\int_t^\tau c_k(y(s),\mathbf{v}(s))\,ds}\, d\tau \\ &\qquad + u^k(y(t+\delta), t+\delta) e^{-\int_t^{t+\delta} c_k(y(s),\mathbf{v}(s))\,ds}\Bigg]. \end{aligned}$$

Subtracting $u^k(x,t)$ and dividing by δ we arrive at

$$\begin{aligned} 0 \;&=\; \mathbb{E}^k_{x,t}\Bigg[\frac{1}{\delta}\int_t^{t+\delta} l_k(y(\tau), \mathbf{v}(\tau))\, e^{-\int_t^\tau c_k(y(s),\mathbf{v}(s))\,ds}\, d\tau \\ &\qquad + \frac{1}{\delta}\Big(u^k(y(t+\delta), t+\delta) e^{-\int_t^{t+\delta} c_k(y(s),\mathbf{v}(s))\,ds} - u^k(x,t)\Big)\Bigg]. \end{aligned}$$

The mean value theorem yields, for sufficiently smooth functions,

$$\lim_{\delta\to 0} \frac{1}{\delta}\int_t^{t+\delta} l_k(y(\tau), \mathbf{v}(\tau))\, e^{-\int_t^\tau c_k(y(s),\mathbf{v}(s))\,ds}\, d\tau = l_k(y(t), \mathbf{v}(t))\,.$$

Moreover, due to the rules of stochastic differential calculus it holds that

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\Big(u^k(y(t+\delta),t+\delta)e^{-\int_t^{t+\delta}c_k(y(s),\mathbf{v}(s))\,ds}-u^k(x,t)\Big)\\
&\quad = -u^k(x,t)\,c_k(y(t),\mathbf{v}(t))+\nabla u^k(x,t)\cdot m(y(t),\mathbf{v}(t))\\
&\qquad +Au^k(x,t)+\partial_t u^k(x,t)\,,
\end{aligned}$$

where $Au^k=\sum_{i,j}a_{ij}\partial_i\partial_j u^k$ and $a_{ij}=\frac{1}{2}\sum_k \sigma_{ik}\sigma_{kj}$. Thus, $\mathbf{u}=(u^1,\dots,u^N)$ is a solution of the Hamilton-Jacobi-Bellmann equation (2.4).

Assuming that there is an exit time from the domain, the solution $\mathbf{u}$ satisfies an inhomogeneous Dirichlet boundary value condition.

Remark 1. *Our difference quotient method can be applied to the case of inhomogeneous Dirichlet boundary value conditions as well. Let $\boldsymbol{\phi}^*$ be a sufficiently smooth function satisfying $\boldsymbol{\phi}^*(x,T)=\boldsymbol{\phi}(x,T)$ on $\partial\Omega$ in the sense of traces. Using the test function*

$$\begin{aligned}
\boldsymbol{\varphi} &= -D_h(\eta_{-h}^2(1+|D_{-h}(\mathbf{u}_\varepsilon-\boldsymbol{\phi}^*)|^{p-2})D_{-h}(\mathbf{u}_\varepsilon-\boldsymbol{\phi}^*)\\
&\quad -D_{-h}(\eta_h^2(1+|D_h(\mathbf{u}_\varepsilon-\boldsymbol{\phi}^*)|^{p-2})D_h(\mathbf{u}_\varepsilon-\boldsymbol{\phi}^*)
\end{aligned}$$

the proof runs as above.

References

[1] N. Alaa, *Solutions faibles d'équations paraboliques quasilinéaires avec données initiales mesures*, Ann. Math. Blaise Pascal 3 (1996), 1-15.

[2] J. M. Arrieta, A. Rodriguez-Bernal, Ph. Souplet, *Boundedness of global solutions for nonlinear parabolic equations involving gradient blow-up phenomena*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 3 (2004), 1-15.

[3] G. Barles, F. Da Lio, *On the generalised Dirichlet problem for viscous Hamilton–Jacobi equations*, J. Math. Pures Appl. (9) 83 (2004), 53-75.

[4] G. Barles, F. Murat, *Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions*, Arch. Rational Mech. Anal. 133 (1995), 77-101.

[5] S. Benachour, S. Dabuleanu, *Large time behaviour for a viscous Hamilton-Jacobi equation with Neumann boundary condition*, J. Differential Equations 216 (2005), 223-258.

[6] S. Benachour, S. Dăbuleanu-Hapca, Ph. Laurençot, *Decay estimates for a viscous Hamilton–Jacobi equation with homogeneous Dirichlet boundary conditions*, Asymptot. Anal. 51 (2007), 209-229.

[7] A. Bensoussan, J. Frehse, *Ergodic Bellman systems for stochastic games in arbitrary dimension*, Proc. Roy. Soc. London Ser. A 449 (1995), 65-77.

[8] A. Bensoussan, J. Frehse, *Stochastic games with risk sensitive pay offs for N-players*, Matematiche (Catania) 55 (2000), suppl. 2, 5-54 (2001).

[9] A. Bensoussan, J. Frehse, *Regularity results for nonlinear elliptic systems and applications*, Applied Mathematical Sciences, 151. Springer-Verlag, Berlin, 2002. xii+441 pp.

[10] A. Bensoussan, J. Frehse, *On diagonal elliptic and parabolic systems with super-quadratic Hamiltonians*, Commun. Pure Appl. Anal. 8 (2009), 83-94.

[11] A. Bensoussan, J. Frehse, J. Vogelgesang, *Systems of Bellman equations to stochastic differential games with non-compact coupling*, Discrete Contin. Dyn. Syst. 27 (2010), 1375-1389.

[12] L. Boccardo, S. Segura de Léon, C. Trombetti, *Bounded and unbounded solutions for a class of quasi-linear elliptic problems with a quadratic gradient term*, J. Math. Pures Appl. (9) 80 (2001), 919-940.

[13] K.-C. Chang, W.-Y. Ding, R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geom. 36 (1992), 507-515.

[14] Y.-M. Chen, *The weak solutions to the evolution problems of harmonic maps*, Math. Z. 201 (1989) , 69-74.

[15] E. Dockner, S. Jørgensen, N.V. Long, G. Sorger, *Differential games in economics and management science*, University Press, Cambridge, 2000. xii+382 pp.

[16] B. Düring, A. Jüngel, *Existence and uniqueness of solutions to a quasilinear parabolic equation with quadratic gradients in financial markets*, Nonlinear Anal. 62 (2005), 519-544.

[17] C. Ebmeyer, *Regularity in Sobolev spaces for the fast diffusion and the porous medium equation*, J. Math. Anal. Appl. 307 (2005), 134-152.

[18] C. Ebmeyer, W.B. Liu, M. Steinhauer, *Global regularity in fractional order Sobolev spaces for the p–Laplace equation on polyhedral domains*, Z. Anal. Anwendungen 24 (2005), 353-374.

[19] C. Ebmeyer, J. Vogelgesang, *Uniqueness for nonlinear parabolic systems in stochastic game theory with application to financial economics*, Differential Integral Equations 22 (2009), 601-615.

[20] C. Ebmeyer, J. Vogelgesang, *Finite element approximation of discounted stochastic differential games with non-market interactions for N-players*, submitted.

[21] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1968 xi+648 pp.

[22] T. Tabet Tchamba, *Large time behaviour of solutions of viscous Hamilton–Jacobi equations with super-quadratic Hamiltonian*, Asymptot. Anal. 66 (2010), 161-186.

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