NASH EQUILIBRIA IN N-PERSON GAMES WITH SUPER-QUADRATIC HAMILTONIANS

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Abstract: We consider the Hamilton-Jacobi-Bellman system
\[ \partial_t u - \Delta u = H(u, \nabla u) + f \]
for \( u \in \mathbb{R}^N \), where the Hamiltonian \( H(u, \nabla u) \) satisfies a super-quadratic growth condition with respect to \( |\nabla u| \). Such a nonlinear parabolic system corresponds to a stochastic differential game with \( N \) players. We obtain the existence of bounded weak solutions and prove regularity results in Sobolev spaces for the Dirichlet problem.

Keywords: Hamilton-Jacobi-Bellman system, super-quadratic growth, stochastic games, existence and regularity.

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1. Introduction

The purpose of this paper is the study of the Hamilton-Jacobi-Bellman system
\[ \partial_t u - \Delta u = H(u, \nabla u) + f \quad \text{in} \quad \Omega \times (0, T], \]
\[ u = 0 \quad \text{on} \quad \partial \Omega \times (0, T], \]
\[ u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega, \]
where \( u : \Omega \times [0, T] \to \mathbb{R}^N \) is a vector valued function, \( \Omega \subset \mathbb{R}^d \) \( (d \geq 2) \) is a bounded domain with a \( C^{1,1} \)-boundary, \( T < \infty \), and \( \partial_t = \frac{\partial}{\partial t} \). We assume the Hamiltonian has the form
\[ H(u, \nabla u) = (g(\nabla u) - u) |\nabla u|^q, \]
where \( g \in \mathbb{R}^N \) is a bounded function and \( q > 2 \) may be arbitrarily large, and thus satisfies a super-quadratic growth condition with respect to \( |\nabla u| \).

This type of nonlinear parabolic system, for Hamiltonians with sub-quadratic, quadratic or super-quadratic growth with respect to \( |\nabla u| \), occur in

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the theory of stochastic differential games, portfolio theory, economic finance, and geometry. An example of a simplified $N$-person game with a super-quadratic Hamiltonian is given in Section 2.

Our goal is to prove the existence of bounded weak solutions $u$ of the system (1.1)–(1.3) in the super-quadratic case ($q > 2$), having the additional regularity

$$u \in L^2 (0, T; W^{2,2}(\Omega; \mathbb{R}^N)) \cap L^\infty (0, T; W^{1,\infty}(\Omega; \mathbb{R}^N))$$

and

$$\partial_t u \in L^2 (0, T; L^2(\Omega; \mathbb{R}^N)) .$$

The existence theory in the sub-quadratic growth case ($q < 2$) is simple (cf. [21]) but the case of quadratic growth ($q = 2$) is already challenging. In game theory, the existence of weak solutions for $N$-person games, with $N = 1$, is treated in [12, 14], and $N$-person games, for $N > 1$ arbitrarily large, are studied, e.g., in [7, 8, 11, 16].

The uniqueness of weak solutions in the quadratic case ($q = 2$) is only known for bounded weak solutions. In fact, there may be several weak solutions, but at most one that is bounded; a counter-example is given in [4]. Uniqueness results can be found in [4] for elliptic equations, and in [19] for parabolic systems with $N$ players.

Essential for our analysis is the fact that $H$ has a negative definite Jacobian $H_u$ in the case that $|\nabla u| \neq 0$. More precisely, it is well known that weak solutions may show finite time blow up behaviour if $H_u$ is positive definite; see, e.g., [13] for the quadratic case. In the case that $H_u = 0$ and $q > 2$ a loss of boundary conditions may occur; cf. [3]. Moreover, in [2] it is shown that classical solutions may blow up in finite time. Another non-existence result can be found in [1], where the initial data are measures. Existence results for viscous solutions of the Dirichlet problem in the case that $H_u = 0$ and $q > 2$ are given in [6, 22], the Neumann boundary value problem is treated in [5], and the existence of bounded solutions under periodic boundary value conditions is shown in [10]. Our method of proof uses test functions of power-law type, similar to those in [19, 20], and a difference quotient technique. We obtain an energy estimate by taking a discrete $p$-Laplacian as test function, where $p$ has to be sufficiently large.

The paper is organised as follows. In Section 2, we discuss stochastic games as the main motivation for our system and provide an example of a simplified $N$-person game with a super-quadratic Hamiltonian. Section 3
contains the assumptions on the data, the main result and a motivation to the use of power-law test functions. In Section 4, a regularised Hamilton-Jacobi-Bellman system is studied and the basic energy estimate is obtained, leading to the proof of the main result for smooth domains. Section 5 gives a generalisation of our result to a class of Lipschitzian domains, such as convex polyhedrons. Finally, the case of inhomogeneous Dirichlet boundary value conditions is discussed in the last section.

2. Nash equilibria in differential games with discount control

Consider stochastic games with \( N \) players, where each player can influence the drift \( m \) of the dynamical system

\[
dx(t) = m(x, v) \, dt + \sigma \, dw_t, \quad x(0) = y
\]

for \( x(t) \in \mathbb{R}^d \), \( t \in [0, T] \) and \( T > 0 \). Here, \( m \in \mathbb{R}^d \) is a given function, the diffusion term \( \sigma \) is a constant \( N \times N \) matrix, \( dw_t \) a Wiener process, and \( y \in \mathbb{R}^d \) is the initial state of \( x \). The intention of the \( k \)-th player \( (1 \leq k \leq N) \) is to maximize the cost functional

\[
J^k(v) = \mathbb{E}^k \left[ \int_0^T l_k(x(t), v(t)) \exp \left( - \int_0^t c_k(x(r), v(r)) \, dr \right) \, dt + \phi_k(x(T)) \exp \left( - \int_0^T c_k(x(t), v(t)) \, dt \right) \right],
\]

where \( v = (v^1, \ldots, v^N)^\top \) and \( \mathbb{E}^k \) is the expectation of the \( k \)-th player. Moreover, \( l_k, c_k, \) and \( \phi_k \) are prescribed functions. The stochastic process \( x(t) \) \( (0 \leq t \leq T) \) describes the state of the underlying dynamical system.

For stochastic control problems the method of dynamic programming leads to an analytical problem, called the Hamilton-Jacobi-Bellman system, whose solution allows to derive an optimal stochastic control. The concept of a Nash point is the following: find functions \( \hat{v}^1, \ldots, \hat{v}^N \) such that

\[
J^k(\hat{v}^1, \ldots, \hat{v}^k, \ldots, \hat{v}^N) \geq J^k(\hat{v}^1, \ldots, \bar{v}, \ldots, \hat{v}^N),
\]

for all \( \bar{v} \) that are admissible for the \( k \)-th player. For one player, that is, \( N = 1 \), the problem reduces to the classical stochastic control problem. The factor

\[
\exp \left( - \int_0^T c_k(x(r), v(r)) \, dr \right)
\]
is the discount factor influenced by the $k$-th player. To the $k$-th player there is an associated Lagrange functional

$$L^k(x, \lambda_k, \xi_k, \mathbf{v}) = l_k(x, \mathbf{v}) + \xi_k \cdot m(x, \mathbf{v}) - \lambda_k c_k(x, \mathbf{v}),$$

(2.3)

where $l_k, \lambda_k, c_k \in \mathbb{R}$, $\xi_k \in \mathbb{R}^d$, and $\lambda_k c_k$ is the discount control.

Fixing $x, \lambda_k$ and $\xi_k$, we look for a Nash point $\hat{\mathbf{v}}(x, \lambda, \xi)$ for the functionals $L^k$. Here, $\lambda$ stands for $(\lambda_1, \ldots, \lambda_N)$, $\xi$ for $(\xi_1, \ldots, \xi_N)$, and $\hat{\mathbf{v}}$ for $(\hat{v}^1, \ldots, \hat{v}^N)$. We define the Hamiltonian functions

$$H^k(\lambda, \xi) := L^k(x, \lambda_k, \xi_k, \hat{\mathbf{v}}(x, \lambda, \xi))$$

and consider the following nonlinear system, for $1 \leq k \leq N$,

$$\partial_t u^k - Au^k = H^k(x, t, \mathbf{u}, \nabla \mathbf{u}) \quad \text{in } \Omega \times (0, T],$$

(2.4)

in a bounded smooth domain $\Omega \subset \mathbb{R}^d$, where

$$A = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

and the coefficients $a_{ij}$ are the components of the matrix $\frac{1}{2} \sigma \sigma^T$.

We will limit our presentation to stochastic processes which are killed at the exit of the domain, which leads to an homogeneous Dirichlet boundary value problem. Let there be a sufficiently smooth solution $\mathbf{u}$, say

$$\mathbf{u} \in L^\infty(0, T; W^{2,r}(\Omega; \mathbb{R}^N))$$

and $r > d$. Now we obtain an optimal feedback for the $k$-th player, in the sense that $\hat{\mathbf{v}}^k(t) = \hat{\mathbf{v}}^k(x(t))$ is a solution to (2.1). Therefore the problem of finding a smooth solution to the system (2.4) is the tool to obtain Nash equilibrium points for the stochastic differential game. For more details we refer to [9, 15].

We now give an example of a simplified $N$-person game with a super-quadratic Hamiltonian. Let the drift $m$ in equation (2.1) be a linear function with respect to $\mathbf{v}$, that is,

$$m(x, \mathbf{v}) = \sum_{\nu} a_{\nu}(x) v^\nu + a_0(x),$$

where $a_{\nu}$ $(0 \leq \nu \leq N)$ are $\mathbb{R}^d$-functions. Let us introduce the Lagrange functionals $(1 \leq k \leq N)$

$$L^k(x, \lambda_k, \xi_k, \mathbf{v}) = l_k(x, \mathbf{v}) + \xi_k \cdot m(x, \mathbf{v}) - \lambda_k c(x, \mathbf{v}).$$

(2.5)
We define
\[ l_k(x, v) = \frac{1}{\alpha} |v^k|^\alpha + b(x), \]
where \( b \) is a scalar function and \( \alpha \in (1, 2) \). Moreover, the discount is given by
\[ c(x, v) = 1 + \frac{1}{\alpha} \sum_{\nu} |v^\nu|^\alpha. \]
In order to find a Nash point \( \hat{v} \), we solve the equations
\[ \frac{\partial}{\partial v^k} L^k = 0 \quad (1 \leq k \leq N). \]
This implies that
\[ |v^k|^{\alpha-2} \hat{v}^k - \lambda_k |v^k|^{\alpha-2} \hat{v}^k + \xi_k \cdot a_k = 0 \]
and we find the solutions
\[ |\hat{v}^k|^{\alpha-2} \hat{v}^k = (\lambda_k - 1)^{-1} \xi_k \cdot a_k. \tag{2.6} \]
Due to (2.5), we obtain the Hamiltonian
\[ H^k(u, \nabla u) = L^k(x, u^k, \nabla u^k, \hat{v}) \]
\[ = \left[ \frac{1}{\alpha} |\hat{v}^k|^\alpha + b(x) + \nabla u^k \cdot \left( \sum_{\nu} a_{\nu}(x) \hat{v}^\nu + a_0(x) \right) \right] \]
\[ - u^k \left[ 1 + \frac{1}{\alpha} \sum_{\nu} |\hat{v}^\nu|^\alpha \right]. \]
In view of (2.6) we get
\[ |\hat{v}^k|^\alpha = (|u^k - 1|^{-1} |\nabla u^k \cdot a_k|)^{\frac{\alpha}{\alpha-2}}, \]
that is, \( |\hat{v}|^{\alpha} \sim |\nabla u|^{\frac{\alpha}{\alpha-2}} \) and \( |\nabla u| |\hat{v}| \sim |\nabla u|^{\frac{\alpha}{\alpha-2}} \). Thus, the Hamiltonian has the form
\[ H(u, \nabla u) = H_0(u, \nabla u) - u H_1(u, \nabla u), \]
where \( H_0(u, \nabla u) \sim |\nabla u|^{\frac{\alpha}{\alpha-2}} \) and \( H_1(u, \nabla u) \sim |\nabla u|^{\frac{\alpha}{\alpha-2}} \), if \( |\nabla u| \) is large. Moreover, we have \( H_1(u, \nabla u) \geq 1 \). Altogether, we obtain an Hamilton-Jacobi-Bellman system of the form
\[ \partial_t u - \Delta u + u H_1(u, \nabla u) = H_0(u, \nabla u), \]
where the Hamiltonians \( H_0(u, \nabla u) \) and \( H_1(u, \nabla u) \) satisfy a super-quadratic growth condition with respect to \( |\nabla u| \), since \( \frac{\alpha}{\alpha-1} > 2 \).
3. The main result

We assume the following set of assumptions on the data.

(H1) $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with a $C^{1,1}$-boundary;
(H2) $T < \infty$, $q > 2$, and $N \geq 1$;
(H3) $f, g \in L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^N))$ and $u_0 \in W^{1,\infty}(\Omega; \mathbb{R}^N)$.

We next make precise the notion of solution we are dealing with.

Definition 1. We say $u(x, t)$ is a bounded weak solution of the system (1.1)–(1.3) if $u \in L^2(0, T; W^{1,q}_0(\Omega; \mathbb{R}^N)) \cap L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^N))$, $\partial_t u \in L^2(0, T; H^{-1}(\Omega; \mathbb{R}^N))$, $u$ fulfills the initial condition (1.3) in the sense of $L^2(\Omega; \mathbb{R}^N)$, and
\[
\int_0^T \int_{\Omega} \langle \partial_t u, \varphi \rangle + \int_0^T \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_0^T \int_{\Omega} H(u, \nabla u) \cdot \varphi + \int_0^T \int_{\Omega} f \cdot \varphi,
\]
for all $\varphi \in L^2(0, T; H^1_0(\Omega; \mathbb{R}^N)) \cap L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^N))$. Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega; \mathbb{R}^N)$ and $H^1_0(\Omega; \mathbb{R}^N)$.

The existence of a bounded weak solution $u$ such that $\nabla u$ is bounded and $\nabla^2 u$ and $\partial_t u$ are $L^2$-functions is the main result of this paper.

Theorem 1. There exists a bounded weak solution $u$ of the system (1.1)–(1.3) satisfying
\[
u \in L^2(0, T; W^{2,2}(\Omega; \mathbb{R}^N)) \cap L^\infty(0, T; W^{1,\infty}(\Omega; \mathbb{R}^N))
\]
and
\[
\partial_t u \in L^2(0, T; L^2(\Omega; \mathbb{R}^N)).
\]

Energy estimates for Hamilton-Jacobi-Bellman systems can not be obtained via standard approaches. Therefore, we make use of test functions of power-law type. In order to explain our approach we sketch the basic idea introduced in [19, 20] for the case of Hamiltonians with quadratic growth. In the following example we treat the case of just one player, that is $N = 1$, and give the proof of the basic energy estimate.

Let us consider the equation
\[
\partial_t u - \Delta u + u|\nabla u|^2 = |\nabla u|^2 + f.
\]
In order to obtain an energy estimate, cf. (3.1), we multiply the equation by
\[ \varphi = (1 + |u|^{p-1})u, \]
where \( p > 2 \) is sufficiently large. Let us note that
\[
\int_0^T \int_\Omega \partial_t u |u|^{p-1} u = \frac{1}{p+1} \int_0^T \int_\Omega \partial_t |u|^{p+1} = \frac{1}{p+1} \|u(\cdot, T)\|_{L^{p+1}(\Omega)}^{p+1} - \frac{1}{p+1} \|u_0\|_{L^{p+1}(\Omega)}^{p+1}
\]
and
\[
\int_0^T \int_\Omega \nabla u \cdot \nabla ((1 + |u|^{p-1})u) = p \int_0^T \int_\Omega |u|^{p-1} |\nabla u|^2 + \int_0^T \int_\Omega |\nabla u|^2.
\]
Moreover, on the left-hand side of the equation we have the integral
\[
\int_0^T \int_\Omega u |\nabla u|^2 \varphi = \int_0^T \int_\Omega |u|^{p+1} |\nabla u|^2.
\]
Now let us estimate the integrals on the right-hand side of the equation. Applying Young’s inequality, we find
\[
\int_0^T \int_\Omega |u|^{p+1} |u|^{\frac{p-1}{2}} |\nabla u|^2 \leq \delta \int_0^T \int_\Omega |u|^{p+1} |\nabla u|^2 + c_\delta \int_0^T \int_\Omega |u|^{p-1} |\nabla u|^2,
\]
for some sufficiently small \( \delta > 0 \). Thus, we can absorb the first integral on the right-hand side into the left-hand side. Furthermore, we may absorb the second integral as well, if \( p \) is suitably large.

Next, we introduce the set
\[
\Omega_\delta(t) := \left\{ x \in \Omega : |u(x, t)| \geq \frac{1}{\delta} \right\}.
\]
Let us assume that \( f \) is bounded. It follows that
\[
\int_0^T \int_{\Omega_\delta(t)} |f| |u|^p \leq c \delta \int_0^T \int_{\Omega_\delta(t)} |u|^{p+1} \leq c \delta \|u\|_{L^{p+1}(0, T; L^{p+1}(\Omega))}^{p+1}.
\]
Choosing \( \delta \) sufficiently small, we may absorb this integral into the left-hand side. Further, the integral
\[
\int_0^T \int_{\Omega \setminus \Omega_\delta(t)} |f| |u|^p \leq c \delta^{-p}
\]
is bounded. Collecting results, we arrive at
\[ \|u\|_{L^{p+1}(0,T;L^p\Omega)}^{p+1} + \|\nabla u\|^2_{L^2(0,T;L^2\Omega)} \leq c_p. \]
Extracting the \( p \)-th root and sending \( p \to \infty \), we obtain the energy estimate
\[ \|u\|_{L^\infty(0,T;L^\infty\Omega)} + \|\nabla u\|_{L^2(0,T;L^2\Omega)} \leq c. \quad (3.1) \]
To treat the super-quadratic case, the basic idea is to use a power-law test function like
\[ -\Delta_p u = -\text{div} \left( (1 + |\nabla u|^{-1}) \nabla u \right); \text{ cf. (4.6)} \]
Noting that
\[ \int_0^T \int_\Omega \partial_t \nabla u \cdot |\nabla u|^{p-1} \nabla u = \frac{1}{p+1} \int_0^T \int_\Omega \partial_t |\nabla u|^{p+1}, \]
extracting the \( p \)-th root and sending \( p \to \infty \), we are able to obtain the energy estimate
\[ \|\nabla u\|_{L^\infty(0,T;L^\infty\Omega)} + \|\nabla^2 u\|_{L^2(0,T;L^2\Omega)} \leq c, \]
\text{ cf. Theorem 1.}
Let us remark that our method of proof can be applied to a more general class of Lipschitzian domains such as convex polyhedrons. This will be discussed in Section 5.

4. The auxiliary problem
In this section, we study the regularised system
\[ \partial_t u_\varepsilon - \Delta u_\varepsilon = H_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) + f \quad \text{in} \quad \Omega \times (0,T], \quad (4.1) \]
\[ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \times (0,T], \quad (4.2) \]
\[ u_\varepsilon(\cdot,0) = u_0 \quad \text{in} \quad \Omega, \quad (4.3) \]
where
\[ H_\varepsilon (u_\varepsilon, \nabla u_\varepsilon) = (g(\nabla u_\varepsilon) - u_\varepsilon) \mu_\varepsilon(|\nabla u_\varepsilon|^{q}) \]
and \( \mu_\varepsilon(s) = (1 + \varepsilon s)^{-1}s, \) for \( s \in \mathbb{R} \). Note that \( \mu_\varepsilon(|s|^{q}) \) is bounded.

Due to the standard theory of parabolic systems, cf. [21], the system (4.1)–(4.3) has a weak solution fulfilling
\[ u_\varepsilon \in L^2(0,T;W^{2,2}(\Omega;\mathbb{R}^N)) \cap W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^N)). \]
Moreover, it holds that
\[ \nabla u_\varepsilon \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^dN)). \quad (4.4) \]
In fact, arguing as in the previous section, it follows that $u_\varepsilon$ is bounded. Thus, the right-hand side of equation (4.1) is bounded. It follows that

$$\Delta u_\varepsilon \in L^\infty \left( 0, T; L^s(\Omega; \mathbb{R}^N) \right),$$

for $s > d$. Applying Sobolev’s imbedding theorem, we obtain (4.4).

Our goal is to prove the basic energy estimate for the regularised solution $u_\varepsilon$ and we apply a difference quotient technique. Let us introduce some notations. Let $R > 0$, $h \in \mathbb{R}^d$ be a vector, $|h| \in (0, R/2)$, $T_{\pm h}v(x) = v(x \pm h)$,

$$D_h v(x) = \frac{T_h v(x) - v(x)}{|h|}, \quad \text{and} \quad D_{-h} v(x) = \frac{v(x) - T_{-h} v(x)}{|h|}. $$

That is, $D_h v$ is the forward difference quotient of $v$ in the direction $h$ and $D_{-h} v$ is the backward difference quotient.

Further, let $B_R(P) = \{ x \in \mathbb{R}^d : |P - x| < R \}$. To shorten our writing, we use the abbreviations $B_R = B_R(P)$ and $\Omega_R = \Omega \cap B_R(P)$. The function $\eta \in W^{2,\infty}(\mathbb{R}^d)$ is a radial-symmetric cut–off function satisfying $\eta \equiv 1$ in $B_R$, $\text{supp} \eta = \overline{B}_{2R}$, and $0 \leq \eta \leq 1$ in $\mathbb{R}^d$. Moreover, let $\eta_{\pm h}(x) = \eta(x \pm \frac{1}{2}h)$.

Our proof consists of several steps. We will give an interior energy estimate (Lemma 1) and will then show the local regularity up to a flat boundary portion (Lemma 2). Global regularity in a smooth domain is proved in Proposition 1. The case of a domain with a non-smooth boundary with corner points is discussed in Section 5.

**Lemma 1.** Let $B_{3R} \subset \Omega$. Then there are constants $C_p$, $C_R$, $C_0$ and $p > 2$ such that

$$\sup_{0 < t \leq T} \int_{\Omega_R} |\nabla u_\varepsilon|^p + \int_0^T \int_{\Omega_R} |\nabla^2 u_\varepsilon|^2 + C_R \leq C_p \left( \int_{\Omega_{3R}} |\nabla u_0|^p + \int_0^T \int_{\Omega_{3R}} |f|^2 + C_0 \right) + \kappa C_{3R}, \quad (4.5)$$

where $C_R = \int_0^T \int_{\Omega_R} \mu_\varepsilon (|\nabla u_\varepsilon|^q) (|\nabla u_\varepsilon|^2 + |\nabla u_\varepsilon|^p)$ and $\kappa > 0$ is sufficiently small.
Proof: Multiplying the system (4.1) by a test function \( \varphi \in L^2(0,T; W^{1,2}(\Omega)) \), we get

\[
J_1 + J_2 + J_3 = \int_0^T \int_{\Omega_{3R}} \partial_t \mathbf{u}_\varepsilon \cdot \varphi + \int_0^T \int_{\Omega_{3R}} \nabla \mathbf{u}_\varepsilon \cdot \nabla \varphi + \int_0^T \int_{\Omega_{3R}} \mathbf{u}_\varepsilon \mu_\varepsilon (|\nabla \mathbf{u}_\varepsilon|^q) \cdot \varphi
\]

\[
= \int_0^T \int_{\Omega_{3R}} D_h(\nabla \mathbf{u}_\varepsilon) \cdot \nabla \varphi + \int_0^T \int_{\Omega_{3R}} f \cdot \varphi =: J_4 + J_5.
\]

We set

\[
\varphi = -D_h\left( \eta^2_h(1 + |D_{-h} \mathbf{u}_\varepsilon|^{p-2})D_{-h} \mathbf{u}_\varepsilon \right) - D_{-h}\left( \eta^2_h(1 + |D_{-h} \mathbf{u}_\varepsilon|^{p-2})D_{-h} \mathbf{u}_\varepsilon \right),
\]

for some \( p > 2 \). This test function can be seen as a discretisation of the non-degenerate \( p \)-Laplacian. Since \( B_{3R} \subset \Omega \) and \( \text{supp} \eta = \overline{B_{2R}} \), it holds that \( \varphi = 0 \) on \( \partial \Omega \).

To begin with, let us estimate the integral \( J_1 \). Due to Leibniz rules

\[
D_h(vw) = (D_h v)(T_h w) + vD_h w \quad \text{and} \quad D_{-h}(vw) = (D_{-h} v)(T_{-h} w) + vD_{-h} w,
\]

there holds the identity

\[
J_1 = \int_0^T \int_{\Omega_{3R}} D_h \partial_t \mathbf{u}_\varepsilon \cdot \eta^2_h(1 + |D_{-h} \mathbf{u}_\varepsilon|^{p-2})D_{-h} \mathbf{u}_\varepsilon
\]

\[
\quad + \int_0^T \int_{\Omega_{3R}} D_{-h} \partial_t \mathbf{u}_\varepsilon \cdot \eta^2_h(1 + |D_{-h} \mathbf{u}_\varepsilon|^{p-2})D_{-h} \mathbf{u}_\varepsilon
\]

\[
\quad - \int_0^T \int_{\Omega_{3R}} D_h(\partial_t \mathbf{u}_\varepsilon \cdot \eta^2_h(1 + |D_{-h} \mathbf{u}_\varepsilon|^{p-2})D_{-h} \mathbf{u}_\varepsilon
\]

\[
\quad - \int_0^T \int_{\Omega_{3R}} D_{-h}(\partial_t \mathbf{u}_\varepsilon \cdot \eta^2_h(1 + |D_{-h} \mathbf{u}_\varepsilon|^{p-2})D_{-h} \mathbf{u}_\varepsilon
\]

\[
= J_{11} + \cdots + J_{14}.
\]

In view of the fact that \( \text{supp} \eta = \overline{B_{2R}} \), we have \( \eta = 0 \) in the sets \( \{ \Omega_{3R} \pm h \setminus \Omega_{3R} \} \) and \( \{ \Omega_{3R} \setminus \Omega_{3R} \pm h \} \). Hence, it follows that

\[
J_{13} = -\frac{1}{|h|} \int_0^T \int_{\Omega_{3R}^+ \setminus \Omega_{3R}} \partial_t \mathbf{u}_\varepsilon \cdot \eta^2_h(1 + |D_{-h} \mathbf{u}_\varepsilon|^{p-2})D_{-h} \mathbf{u}_\varepsilon
\]

\[
+ \frac{1}{|h|} \int_0^T \int_{\Omega_{3R} \setminus \Omega_{3R}^+} \partial_t \mathbf{u}_\varepsilon \cdot \eta^2_h(1 + |D_{-h} \mathbf{u}_\varepsilon|^{p-2})D_{-h} \mathbf{u}_\varepsilon = 0
\]
and

\[ J_{14} = -\frac{1}{|h|} \int_{0}^{T} \int_{\Omega_{3R} \setminus \Omega_{3R} - h} \partial_{t} u_{\varepsilon} \cdot \eta_{h}^{2}(1 + |D_{h} u_{\varepsilon}|^{p-2}) D_{h} u_{\varepsilon} \]

\[ + \frac{1}{|h|} \int_{0}^{T} \int_{\Omega_{3R} - h \setminus \Omega_{3R}} \partial_{t} u_{\varepsilon} \cdot \eta_{h}^{2}(1 + |D_{h} u_{\varepsilon}|^{p-2}) D_{h} u_{\varepsilon} = 0. \]

Further, noting that \( \partial_{t} |v|^{2} = 2v v_{t} \) and \( \partial_{t} |v|^{p} = p|v|^{p-2}v v_{t} \), we have

\[ J_{11} = \frac{1}{2} \int_{0}^{T} \int_{\Omega_{3R}} \eta_{h}^{2} \partial_{t} |D_{h} u_{\varepsilon}|^{2} + \frac{1}{p} \int_{0}^{T} \int_{\Omega_{3R}} \eta_{h}^{2} \partial_{t} |D_{h} u_{\varepsilon}|^{p}. \]

Altogether, we find

\[ J_{1} = \frac{1}{2} \int_{\Omega_{3R}} \left( \eta_{h}^{2} |D_{h} u_{\varepsilon}(\cdot, T)|^{2} + \eta_{-h}^{2} |D_{-h} u_{\varepsilon}(\cdot, T)|^{2} \right) \]

\[ + \frac{1}{p} \int_{\Omega_{3R}} \left( \eta_{h}^{2} |D_{h} u_{\varepsilon}(\cdot, T)|^{p} + \eta_{-h}^{2} |D_{-h} u_{\varepsilon}(\cdot, T)|^{p} \right) \]

\[ - \frac{1}{2} \int_{\Omega_{3R}} \left( \eta_{h}^{2} |D_{h} u_{0}|^{2} + \eta_{-h}^{2} |D_{-h} u_{0}|^{2} \right) \]

\[ - \frac{1}{p} \int_{\Omega_{3R}} \left( \eta_{h}^{2} |D_{h} u_{0}|^{p} + \eta_{-h}^{2} |D_{-h} u_{0}|^{p} \right). \]

To simplify our writing, we suppose that there is a constant \( c > 0 \) such that

\[ \int_{\Omega_{3R}} \eta_{\pm h}^{2} |D_{\pm h} u_{\varepsilon}(\cdot, T)|^{2} + \eta_{\pm h}^{2} |D_{\pm h} u_{\varepsilon}(\cdot, T)|^{p} \]

\[ \geq c \sup_{0 \leq t \leq T} \int_{\Omega_{3R}} \eta_{\pm h}^{2} |D_{\pm h} u_{\varepsilon}|^{2} + \eta_{\pm h}^{2} |D_{\pm h} u_{\varepsilon}|^{p}. \]
Next, we have

\[
J_2 = \int_0^T \int_{\Omega_{3R}} D_h \nabla u_\varepsilon \cdot \nabla (\eta_h^2 (1 + |D_h u_\varepsilon|^{p-2}) D_h u_\varepsilon) \\
+ \int_0^T \int_{\Omega_{3R}} D_{-h} \nabla u_\varepsilon \cdot \nabla (\eta_{-h}^2 (1 + |D_{-h} u_\varepsilon|^{p-2}) D_{-h} u_\varepsilon) \\
- \int_0^T \int_{\Omega_{3R}} D_h (\nabla u_\varepsilon \cdot \nabla (\eta_{-h}^2 (1 + |D_{-h} u_\varepsilon|^{p-2}) D_{-h} u_\varepsilon)) \\
- \int_0^T \int_{\Omega_{3R}} D_{-h} (\nabla u_\varepsilon \cdot \nabla (\eta_h^2 (1 + |D_h u_\varepsilon|^{p-2}) D_h u_\varepsilon)) \\
=: J_{21} + \cdots + J_{24}.
\]

Let us note that \(J_{23} = J_{24} = 0\). Moreover, it holds that

\[
J_{21} = \int_0^T \int_{\Omega_{3R}} D_h \nabla u_\varepsilon \cdot \nabla \eta_h^2 (1 + |D_h u_\varepsilon|^{p-2}) D_h u_\varepsilon \\
+ \int_0^T \int_{\Omega_{3R}} \eta_h^2 |D_h \nabla u_\varepsilon|^2 + (p-1) \int_0^T \int_{\Omega_{3R}} \eta_h^2 |D_h \nabla u_\varepsilon|^2 |D_h u_\varepsilon|^{p-2}.
\]

Noting that \(\nabla \eta_h^2 = 2\eta_h \nabla \eta_h\) and using Young’s inequality, we get

\[
\int_0^T \int_{\Omega_{3R}} |D_h \nabla u_\varepsilon \cdot \nabla \eta_h^2 D_h u_\varepsilon| \leq \delta \int_0^T \int_{\Omega_{3R}} \eta_h^2 |\nabla D_h u_\varepsilon|^2 \\
+ c_\delta \int_0^T \int_{\Omega_{3R}} |2\nabla \eta_h|^2 |D_h u_\varepsilon|^2.
\]

The first integral on the right-hand side may by absorbed into \(J_{21}\). Let us estimate the second integral. For \(\kappa > 0\), we define the set

\[
\Omega_\kappa(t) = \{ x \in \Omega_{3R} : |\nabla \eta(x)|^2 \leq \kappa \mu_\varepsilon(|\nabla u_\varepsilon(x,t)|^q) \}.
\]

Notice that \(|\nabla u_\varepsilon(x,t)|\) is bounded in \(\Omega_{3R} \setminus \Omega_\kappa(t)\). Hence, it follows that

\[
\int_0^T \int_{\Omega_{3R}} |\nabla \eta_h|^2 |D_h u_\varepsilon|^2 \leq \kappa \int_0^T \int_{\Omega_\kappa(t)} \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h u_\varepsilon|^2 + c_\kappa,
\]

for some constant \(c_\kappa\). Choosing \(\kappa\) sufficiently small, we may absorb the integral on the right-hand side into the left-hand side. This will be discussed
later. In the same manner, we estimate

\[ \int_0^T \int_{\Omega_3} |D_h \nabla u_\varepsilon| |\nabla \eta_h^2| |D_h u_\varepsilon|^2 + |D_h u_\varepsilon|^p \]

\[ \leq \delta \int_0^T \int_{\Omega_3} \eta_h^2 |\nabla D_h u_\varepsilon|^2 |D_h u_\varepsilon|^{p-2} + c_\delta \int_0^T \int_{\Omega_3} |2 \nabla \eta_h|^2 |D_h u_\varepsilon|^p \]

and

\[ \int_0^T \int_{\Omega_3} |\nabla \eta_h|^2 |D_h u_\varepsilon|^p \leq \kappa \int_0^T \int_{\Omega_3} \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h u_\varepsilon|^p + c_\kappa. \]

Thus, for some sufficiently small numbers \( \delta, \kappa > 0 \), we conclude that

\[ J_2 \geq \frac{1}{2} \int_0^T \int_{\Omega_3} (\eta_h^2 |\nabla D_h u_\varepsilon|^2 + \eta_{-h}^2 |\nabla D_{-h} u_\varepsilon|^2) \]

\[ + \frac{p-1}{2} \int_0^T \int_{\Omega_3} \left( \eta_h^2 |\nabla D_h u_\varepsilon|^2 |D_h u_\varepsilon|^{p-2} \right. \]

\[ + \eta_{-h}^2 |\nabla D_{-h} u_\varepsilon|^2 |D_h u_\varepsilon|^{p-2}) \]

\[ - \kappa \int_0^T \int_{\Omega_3} \mu_\varepsilon(|\nabla u_\varepsilon|^q) \left( |D_h u_\varepsilon|^2 + |D_h u_\varepsilon|^p \right) - c_\kappa. \]

Now let us estimate the integral \( J_3 \) from below. It holds that

\[ J_3 = \int_0^T \int_{\Omega_3} D_h(u_\varepsilon \mu_\varepsilon(|\nabla u_\varepsilon|^q)) \cdot \eta_h^2(1 + |D_h u_\varepsilon|^{p-2}) D_h u_\varepsilon \]

\[ + \int_0^T \int_{\Omega_3} D_{-h}(u_\varepsilon \mu_\varepsilon(|\nabla u_\varepsilon|^q)) \cdot \eta_{-h}^2(1 + |D_{-h} u_\varepsilon|^{p-2}) D_{-h} u_\varepsilon \]

\[ - \int_0^T \int_{\Omega_3} D_h(u_\varepsilon \mu_\varepsilon(|\nabla u_\varepsilon|^q)) \cdot \eta_{-h}^2(1 + |D_{-h} u_\varepsilon|^{p-2}) D_{-h} u_\varepsilon \]

\[ - \int_0^T \int_{\Omega_3} D_{-h}(u_\varepsilon \mu_\varepsilon(|\nabla u_\varepsilon|^q)) \cdot \eta_h^2(1 + |D_h u_\varepsilon|^{p-2}) D_h u_\varepsilon \]

\[ =: J_{31} + \cdots + J_{34}, \]
where \( J_{33} = J_{34} = 0 \). The Leibniz rule \( D_h(vw) = (D_hv)w + (T_hw)D_hw \) yields

\[
J_{31} = \int_0^T \int_{\Omega_{3R}} \eta_h^2 \mu_\varepsilon(|\nabla u_\varepsilon|^q)(|D_hu_\varepsilon|^2 + |D_hu_\varepsilon|^p) \\
+ \int_0^T \int_{\Omega_{3R}} \eta_h^2 T_hu_\varepsilon D_h\mu_\varepsilon(|\nabla u_\varepsilon|^q) \cdot (1 + |D_hu_\varepsilon|^{p-2})D_hu_\varepsilon \\
=: J_{35} + J_{36}.
\]

Noting that \(|\mu'_\varepsilon| \leq 1\) and applying Tailor’s expansion, we deduce

\[
|D_h\mu_\varepsilon(|\nabla u_\varepsilon|^q)| \leq q |D_h| \nabla u_\varepsilon| (|\nabla T_hu_\varepsilon|^q - 1 + |\nabla u_\varepsilon|^{q-1})
\]

and thus

\[
|J_{36}| \leq \delta (p-1) \int_0^T \int_{\Omega_{3R}} \eta_h^2 |\nabla D_hu_\varepsilon|^2 (1 + |D_hu_\varepsilon|^{p-2}) \\
+ \frac{c_\delta}{p-1} \int_0^T \int_{\Omega_{3R}} \eta_h^2 (|D_hu_\varepsilon|^2 + |D_hu_\varepsilon|^p) |T_hu_\varepsilon|^2 \times \\
\times (|\nabla T_hu_\varepsilon|^{2q-2} + |\nabla u_\varepsilon|^{2q-2}) \\
=: J_{37} + J_{38}.
\]

For simplicity, we suppose that \(|D_hu_\varepsilon| \geq 1\) in \( \Omega_{3R} \) and absorb \( J_{37} \) into \( J_2 \). Using again Young’s inequality, we estimate

\[
(\lambda^{-1}x^q)^{\frac{1}{2}}(\lambda x^{3q-4})^{\frac{1}{2}} \leq \delta \lambda^{-1}x^q + c_\delta \lambda x^{3q-4}
\]

and obtain

\[
J_{38} \leq \delta \int_0^T \int_{\Omega_{3R}} \eta_h^2 \lambda^{-1} (|\nabla T_hu_\varepsilon|^q + |\nabla u_\varepsilon|^q)(|D_hu_\varepsilon|^2 + |D_hu_\varepsilon|^p) \\
+ \frac{c_\delta}{(p-1)^2} \int_0^T \int_{\Omega_{3R}} \eta_h^2 (|D_hu_\varepsilon|^2 + |D_hu_\varepsilon|^p) \times \\
\times \lambda (|\nabla T_hu_\varepsilon|^{3q-4} + |\nabla u_\varepsilon|^{3q-4}) |T_hu_\varepsilon|^4.
\]

Let \( \lambda := 1 + \varepsilon |\nabla T_hu_\varepsilon|^q + \varepsilon |\nabla u_\varepsilon|^q \). Due to (4.4), it holds that \( \|\nabla u_\varepsilon\|_\infty \) and \( \|u_\varepsilon\|_\infty \) are bounded. Hence, we may absorb the second integral on the right-hand side into \( J_1 \), if \( p \) is sufficiently large, and the first one into \( J_3 \). In the
same manner, we estimate the integral

\[ J_{32} = \int_0^T \int_{\Omega_{3R}} \eta_{-h} \mu_{\varepsilon} (|\nabla u_\varepsilon|^q) (|D_{-h}u_\varepsilon|^2 + |D_{-h}u_\varepsilon|^p) \]

\[ + \int_0^T \int_{\Omega_{3R}} \eta_{-h} T_{-h} u_\varepsilon D_{-h} \mu_{\varepsilon} (|\nabla u_\varepsilon|^q) \cdot (1 + |D_{-h}u_\varepsilon|^{p-2}) D_{-h}u_\varepsilon. \]

Let us now consider \( J_4 \). We have

\[ J_4 = \int_0^T \int_{\Omega_{3R}} g \mu_{\varepsilon} (|\nabla u_\varepsilon|^q) \cdot D_h (\eta_{-h}^2 (1 + |D_{-h}u_\varepsilon|^{p-2}) D_{-h}u_\varepsilon) \]

\[ + \int_0^T \int_{\Omega_{3R}} g \mu_{\varepsilon} (|\nabla u_\varepsilon|^q) \cdot D_{-h} (\eta_h^2 (1 + |D_{-h}u_\varepsilon|^{p-2}) D_{-h}u_\varepsilon) \]

\[ =: J_{41} + J_{42}. \]

The Leibniz rule \( D_h (vw) = (D_h v)w + (T_h v)D_h w \) yields

\[ J_{41} = \int_0^T \int_{\Omega_{3R}} g \mu_{\varepsilon} (|\nabla u_\varepsilon|^q) \cdot D_h (\eta_{-h}^2 (1 + |D_{-h}u_\varepsilon|^{p-2}) D_{-h}u_\varepsilon) \]

\[ + \int_0^T \int_{\Omega_{3R}} g \mu_{\varepsilon} (|\nabla u_\varepsilon|^q) \cdot \eta_{-h} D_h ((1 + |D_{-h}u_\varepsilon|^{p-2}) D_{-h}u_\varepsilon) \]

\[ =: J_{43} + J_{44}. \]

Let \( \kappa \in (0, 1) \) be sufficiently small and

\[ \Omega_{\kappa}^{\pm} (t) = \{ x \in \Omega_{3R} : |D_{\pm h} u_\varepsilon (x, t) | \geq \kappa^{-1} \}. \]

Notice that \( |g| \) and \( |D_{-h} T_{-h}^2| \) are bounded and it thus follows that \( |J_{43}| \) is bounded in \( \Omega_{3R} \setminus \Omega_{\kappa}^- (t) \) by a constant \( c_\kappa \), if \( |h| \) is sufficiently small. Further, let us note that

\[ (1 + |D_{-h}u_\varepsilon|^{p-2})|D_{-h}u_\varepsilon| \leq \kappa (|D_{-h}u_\varepsilon|^2 + |D_{-h}u_\varepsilon|^p) \text{ in } \Omega_{\kappa}^- (t). \]

Altogether, we deduce

\[ |J_{43}| \leq c\kappa \int_0^T \int_{\Omega_{\kappa}(t)} \mu_{\varepsilon} (|\nabla u_\varepsilon|^q) (|D_{-h}u_\varepsilon|^2 + |D_{-h}u_\varepsilon|^p) + c_\kappa. \]

Next, let us estimate \( |J_{44}| \). We define the function

\[ y(s) := (1 + |s|^{p-2}) s, \quad s \in \mathbb{R}^N \]
and use Taylor’s expansion to get:

\[ y^\nu(s) - y^\nu(\bar{s}) = (s^m - \bar{s}^m) \int_0^1 y^\nu_{s^m}(z s + (1 - z)\bar{s}) \, dz. \]

Here we have used the notation \( y^\nu_{s^m}(s) = \frac{\partial y^\nu}{\partial s^m}(s) \), where \( y \) and \( s \) have the components \( y^\nu \) (1 ≤ \( \nu \) ≤ \( N \)) and \( s^m \) (1 ≤ \( m \) ≤ \( d \)). We deduce

\[ |J_{14}| \leq c \int_0^T \int_{\Omega_{3R}} \eta_h \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h D_{-h} u_\varepsilon| \left( 1 + (p-1)(|D_h u_\varepsilon|^{p-2} + |D_{-h} u_\varepsilon|^{p-2}) \right). \]

We estimate

\[ \eta_h (\eta_h \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h D_{-h} u_\varepsilon|) \leq \delta \eta_h^2 |\nabla u_\varepsilon|^{2q} |D_h D_{-h} u_\varepsilon|^2 + c_\delta \eta_h^2 \]

and notice that

\[ |\nabla u_\varepsilon|^{2q} \leq c_\kappa \quad \text{in} \quad \Omega_{3R} \setminus \Omega_\kappa^\pm(t), \]

if \( |h| \) is small, and

\[ |\nabla u_\varepsilon|^{2q} \leq \delta |D_h u_\varepsilon|^{p-2} \quad \text{in} \quad \Omega_\kappa^\pm(t), \]

where \( \delta > 0 \) is small, if \( p \) is large. Thus, we get

\[ \int_0^T \int_{\Omega_{3R}} \eta_h^2 \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h D_{-h} u_\varepsilon| \leq \delta \int_0^T \int_{\Omega_{3R}} \eta_h^2 |D_h D_{-h} u_\varepsilon|^2 |D_h u_\varepsilon|^{p-2} + C \]

and absorb the integral on the right-hand side into \( J_2 \). Furthermore, Hölder’s inequality entails

\[ (p - 1) \int_0^T \int_{\Omega_{3R}} \eta_h^2 \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h D_{-h} u_\varepsilon| |D_h u_\varepsilon|^{\frac{p-2}{2}} |D_h u_\varepsilon|^{\frac{p-2}{2}} \]

\[ \leq \frac{\delta(p - 1)}{2} \int_0^T \int_{\Omega_{3R}} \eta_h^2 \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h D_{-h} u_\varepsilon|^2 |D_h u_\varepsilon|^{p-2} \]

\[ + \frac{p-1}{2\delta} \int_0^T \int_{\Omega_{3R}} \eta_h^2 \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h u_\varepsilon|^{p-2}. \]

Let us note that \( \mu_\varepsilon(|\nabla u_\varepsilon|^q) \) is bounded. Hence, we choose \( \delta \) sufficiently small in order to absorb the first integral on the right-hand side into \( J_2 \). Next, we use the estimates

\[ \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h u_\varepsilon|^{p-2} \leq |\nabla u_\varepsilon|^q |D_h u_\varepsilon|^{p-2} \leq c_\kappa \quad \text{in} \quad \Omega_{3R} \setminus \Omega_\kappa^\pm(t) \]

and

\[ \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h u_\varepsilon|^{p-2} \leq \kappa^2 \mu_\varepsilon(|\nabla u_\varepsilon|^q) |D_h u_\varepsilon|^p \quad \text{in} \quad \Omega_\kappa^\pm(t) \]
and absorb the second integral into $J_3$.

In the same manner we deal with $J_{42}$. Now, let us consider

$$ J_5 = - \int_0^T \int_{\Omega_{3R}} f \cdot D_h (\eta_{-h}^2 (1 + |D_{-h}u_\varepsilon|^{p-2}) D_{-h}u_\varepsilon) $$

$$ - \int_0^T \int_{\Omega_{3R}} f \cdot D_{-h} (\eta_h^2 (1 + |D_hu_\varepsilon|^{p-2}) D_hu_\varepsilon) =: J_{51} + J_{52}. $$

Due to Leibniz’s rule $D_h(vw) = (D_hv)w + (T_hv)D_hw$, we obtain

$$ J_{51} = - \int_0^T \int_{\Omega_{3R}} f \cdot D_h \eta_{-h}^2 (1 + |D_{-h}u_\varepsilon|^{p-2}) D_{-h}u_\varepsilon $$

$$ - \int_0^T \int_{\Omega_{3R}} f \cdot \eta_h^2 D_h ((1 + |D_{-h}u_\varepsilon|^{p-2}) D_{-h}u_\varepsilon) =: J_{53} + J_{54}. $$

Since $|D_h\eta_{-h}^2|$ and $|f|$ are bounded we conclude that $|J_{53}| \leq c_\kappa$ in $\Omega_{3R} \setminus \Omega_\kappa^-(t)$, if $|h|$ is sufficiently small. Moreover, in $\Omega_\kappa^-(t)$ it holds that

$$(1 + |D_{-h}u_\varepsilon|^{p-2})|D_{-h}u_\varepsilon| \leq \kappa(|D_{-h}u_\varepsilon|^2 + |D_{-h}u_\varepsilon|^p),$$

and also $\mu_\varepsilon(|\nabla u_\varepsilon|^q) \geq 1$, if $\varepsilon$ is small. Thus, it follows that

$$ |J_{53}| \leq c\kappa \int_0^T \int_{\Omega_{3R}} \mu_\varepsilon(|\nabla u_\varepsilon|^q) (|D_{-h}u_\varepsilon|^2 + |D_{-h}u_\varepsilon|^p) + c_\kappa. $$

Further, using the Taylor expansion of the function $y(s) := (1 + |s|^{p-2})s$, we deduce

$$ |J_{54}| \leq c \int_0^T \int_{\Omega_{3R}} \eta_h^2 |f| |D_hD_{-h}u_\varepsilon| (1 + (p-1)(|D_hu_\varepsilon|^{p-2} + |D_{-h}u_\varepsilon|^{p-2})). $$

We estimate

$$ \int_0^T \int_{\Omega_{3R}} \eta_h^2 |f| |D_hD_{-h}u_\varepsilon| \leq \delta \int_0^T \int_{\Omega_{3R}} \eta_h^2 |D_hD_{-h}u_\varepsilon|^2 + c_\delta \int_0^T \int_{\Omega_{3R}} \eta_h^2 |f|^2 $$
and absorb the first integral on the right-hand side into $J_2$. Moreover, Hölder’s inequality yields

\[
(p - 1) \int_0^T \int_{\Omega_{3R}} \eta_h^2 |f| |D_h D_{-h}u_\varepsilon| |D_h u_\varepsilon|^{p-2} |D_h u_\varepsilon|^{p-2} \leq \frac{\delta (p - 1)}{2} \int_0^T \int_{\Omega_{3R}} \eta_h^2 |f| |D_h D_{-h}u_\varepsilon|^2 |D_h u_\varepsilon|^{p-2} \leq \frac{p - 1}{2\delta} \int_0^T \int_{\Omega_{3R}} \eta_h^2 |f| |D_h u_\varepsilon|^{p-2}.
\]

Since $|f|$ is bounded, we may absorb the first integral on the right-hand side into $J_2$. Further, it holds that

\[
|D_h u_\varepsilon| \leq c_\kappa \text{ in } \Omega_{3R} \setminus \Omega_{\kappa}(t)
\]

and

\[
|D_h u_\varepsilon|^{p-2} \leq \kappa^2 \mu_\varepsilon(|\nabla u_\varepsilon|) |D_h u_\varepsilon|^p \text{ in } \Omega_{\kappa}(t).
\]

Here we have used the fact that $\mu_\varepsilon(|\nabla u_\varepsilon|) \geq 1$ in $\Omega_{\kappa}(t)$ if $\varepsilon$ and $|h|$ are sufficiently small. Now we absorb the second integral into $J_3$. In the same manner, we estimate $J_{32}$.

Let $h$ be parallel to the $k$-th unit vector in $\mathbb{R}^d$. Taking $|h| \to 0$, noting that $|x|^2 \leq 1 + |x|^p$, and collecting results we arrive at

\[
\sup_{0 < t \leq T} \int_{\Omega_{R}} |\partial_k u_\varepsilon|^p + \int_0^T \int_{\Omega_{R}} |\nabla \partial_k u_\varepsilon|^2 + \int_0^T \int_{\Omega_{R}} \mu_\varepsilon(|\nabla u_\varepsilon|^q) (|\partial_k u_\varepsilon|^2 + |\partial_k u_\varepsilon|^p) \\
\leq c_p \left( \int_{\Omega_{3R}} |\partial_k u_0|^p + \int_0^T \int_{\Omega_{3R}} |f|^2 + C \right) \\
+ \kappa \int_0^T \int_{\Omega_{3R}} \mu_\varepsilon(|\nabla u_\varepsilon|^q) (|\partial_k u_\varepsilon|^2 + |\partial_k u_\varepsilon|^p)
\]

(4.7)

for all $k \in \{1, \ldots, d\}$ and $\kappa > 0$ sufficiently small. This yields the assertion.

In our next lemma we investigate the local regularity of $u_\varepsilon$ up to a flat boundary portion. More precisely, we suppose that $P \in \partial\Omega$ and $\partial\Omega \cap B_{3R}(P) = E_{d-1} \cap B_{3R}(P)$, where $E_{d-1}$ is a $(d - 1)$-dimensional hyperplane.
Let $e$ be the inner unit normal of $\partial \Omega \cap B_{3R}$, $z \in \partial \Omega \cap B_{3R}$, and $\lambda > 0$. Thus, it holds that $z + \lambda e \in \Omega_{3R}$ and $z - \lambda e \in B_{3R} \setminus \overline{\Omega}$. We define extensions of the functions $u_\varepsilon$ and $f$ by setting

$$u_\varepsilon(z + \lambda e) := -u_\varepsilon(z - \lambda e) \quad \text{and} \quad f(z + \lambda e) := f(z - \lambda e). \quad (4.8)$$

Moreover, let $\eta(z + \lambda e) := \eta(z - \lambda e)$. Thus, $u_\varepsilon$ is odd with respect to the boundary, and the functions $f$ and $\eta$ are even.

**Lemma 2.** Let $P \in \partial \Omega$ and $\partial \Omega \cap B_{3R}(P) = E_{d-1} \cap B_{3R}(P)$, where $E_{d-1}$ is a $(d - 1)$-dimensional hyperplane. Then there are constants $C_p, C_R, C_0$ and $p > 2$ such that

$$\sup_{0 < t \leq T} \int_{\Omega_R} |\nabla u_\varepsilon|^p + \int_0^T \int_{\Omega_R} |\nabla^2 u_\varepsilon|^2 + C_R \leq C_p \left( \int_{\Omega_{3R}} |\nabla u_0|^p + \int_0^T \int_{\Omega_{3R}} |f|^2 + C_0 \right) + \kappa C_{3R}, \quad (4.9)$$

where $C_R = \int_0^T \int_{\Omega_R} \mu_\varepsilon(\nabla u_\varepsilon)^q (|\nabla u_\varepsilon|^2 + |\nabla u_\varepsilon|^p)$ and $\kappa > 0$ is sufficiently small.

**Proof:** We proceed as in the proof of Lemma 1. Without loss of generality, we suppose that $\Omega_{3R} = \{ x \in B_{3R} : x_1 > 0 \}$. Thus, the unit vector $e_1$ is the inner unit normal of $\partial \Omega \cap B_{3R}$.

In the case that the vector $h$ is parallel to $\partial \Omega \cap B_{3R}$, the test function

$$\varphi = -D_h(\eta_{-h}^2(1 + |D_{-h} u_\varepsilon|^{p-2})D_{-h} u_\varepsilon) - D_{-h}(\eta_{h}^2(1 + |D_{h} u_\varepsilon|^{p-2})D_{h} u_\varepsilon)$$

is admissible, for it holds that $\varphi = 0$ on $\partial \Omega \cap B_{3R}$. Arguing as in the proof of Lemma 1 we obtain the estimate (4.7) for all $k \in \{2, \ldots, d\}$.

Next, let us discuss the case $k = 1$. We choose $h$ as an outer normal of $\partial \Omega \cap B_{3R}$. Notice that $\varphi$ is an admissible test function. In fact, it holds that

$$D_h(\eta_{-h}^2(1 + |D_{-h} u_\varepsilon|^{p-2})D_{-h} u_\varepsilon) = D_{-h}(\eta_{h}^2(1 + |D_{h} u_\varepsilon|^{p-2})D_{h} u_\varepsilon) = 0$$

on $\partial \Omega \cap B_{3R}$. For instance, on $\partial \Omega$ we have

$$\eta^2(x + \frac{1}{2}h)(1 + |T_h u_\varepsilon - u_\varepsilon|^{p-2})(T_h u_\varepsilon - u_\varepsilon)$$

$$-\eta^2(x - \frac{1}{2}h)(1 + |u_\varepsilon - T_{-h} u_\varepsilon|^{p-2})(u_\varepsilon - T_{-h} u_\varepsilon)$$

$$= \eta^2(x + \frac{1}{2}h)(1 + |T_h u_\varepsilon|^{p-2})T_h u_\varepsilon - \eta^2(x - \frac{1}{2}h)(1 + |T_{-h} u_\varepsilon|^{p-2})T_{-h} u_\varepsilon = 0,$$
since \( u_\varepsilon \) is odd and \( \eta \) is even with respect to the boundary.

We now multiply the system (4.1) by \( \varphi \). Then the calculations run as in the proof of Lemma 1; we only have to show that the boundary integrals vanish.

To begin with, let us show that \( J_{13} + J_{14} = 0 \). We have

\[
J_{13} = -\frac{1}{|h|} \int_0^T \int_{\Omega_{3R} + h \setminus \Omega_{3R}} \partial_t u_\varepsilon \cdot \eta^2 (1 + |D_{-h} u_\varepsilon|^{p-2}) D_{-h} u_\varepsilon
+ \frac{1}{|h|} \int_0^T \int_{\Omega_{3R} \setminus (\Omega_{3R} + h)} \partial_t u_\varepsilon \cdot \eta^2 (1 + |D_{-h} u_\varepsilon|^{p-2}) D_{-h} u_\varepsilon.
\]

Due to the fact that \( \text{supp} \ \eta = \overline{B_{2R}} \), the second integral on the right-hand side vanishes. Further, we find

\[
J_{14} = -\frac{1}{|h|} \int_0^T \int_{\Omega_{3R} \setminus \Omega_{3R} - h} \partial_t u_\varepsilon \cdot \eta^2 (1 + |D_{h} u_\varepsilon|^{p-2}) D_{h} u_\varepsilon,
\]

since

\[
\int_0^T \int_{\Omega_{3R} - h \setminus \Omega_{3R}} \partial_t u_\varepsilon \cdot \eta^2 (1 + |D_{h} u_\varepsilon|^{p-2}) D_{h} u_\varepsilon = 0.
\]

Notice that \( \Omega_{3R} \setminus \Omega_{3R} - h \) is the reflection of \( \Omega_{3R} + h \setminus \Omega_{3R} \) with respect to the hyperplane \( \{ x \in \mathbb{R}^d : x_1 = 0 \} \). The functions \( \eta_{\pm h}^2, (1 + |D_{\pm h} u_\varepsilon|^{p-2}) \), and \( D_{\pm h} u_\varepsilon \) are even with respect to the hyperplane, in the sense that, e.g., \( \eta_{-h}^2(x) = \eta_{h}^2(x^*) \) for all \( x \in \Omega_{3R} + h \setminus \Omega_{3R} \), where \( x^* \in \Omega_{3R} \setminus \Omega_{3R} - h \) is the reflection of the point \( x \) with respect to the hyperplane. Moreover, the function \( \partial_t u_\varepsilon \) is odd. Hence, the integrand is an odd function and it follows that \( J_{13} + J_{14} = 0 \).

Next, we consider \( J_{23} \) and \( J_{24} \). We find

\[
J_{23} = -\frac{1}{|h|} \int_0^T \int_{\Omega_{3R} + h \setminus \Omega_{3R}} \nabla u_\varepsilon \cdot \nabla (\eta^2 (1 + |D_{-h} u_\varepsilon|^{p-2}) D_{-h} u_\varepsilon)
+ \frac{1}{|h|} \int_0^T \int_{\Omega_{3R} \setminus (\Omega_{3R} + h)} \nabla u_\varepsilon \cdot \nabla (\eta^2 (1 + |D_{h} u_\varepsilon|^{p-2}) D_{h} u_\varepsilon).
\]

The function \( u_\varepsilon \) is odd with respect to the hyperplane \( \{ x \in \mathbb{R}^d : x_1 = 0 \} \). Thus, the derivative \( \partial_1 u_\varepsilon \) is an even function, and \( \partial_k u_\varepsilon \) (\( 2 \leq k \leq d \)) are odd functions. Moreover, since \( \eta^2_{\pm h} (1 + |D_{\pm h} u_\varepsilon|^{p-2}) D_{\pm h} u_\varepsilon \) is even, its derivative
with respect to \( x_k \) is odd for \( k = 1 \) and even for \( k \neq 1 \). Hence, the integrand is an odd function and it holds that \( J_{23} + J_{24} = 0 \).

Moreover, we find
\[
J_{33} + J_{34} = -\frac{1}{|h|} \int_0^T \int_{\Omega_{3R} + h' \Omega_{3R}} u_\varepsilon \mu_\varepsilon \left(|\nabla u_\varepsilon|^q\right) \cdot \eta_{-h}(1 + |D^- h u_\varepsilon|^{p-2}) D^- h u_\varepsilon
- \frac{1}{|h|} \int_0^T \int_{\Omega_{3R} \setminus \Omega_{3R-h}} u_\varepsilon \mu_\varepsilon \left(|\nabla u_\varepsilon|^q\right) \cdot \eta_{h}(1 + |D^+ h u_\varepsilon|^{p-2}) D^+ h u_\varepsilon.
\]

Noting that \( u_\varepsilon \) is odd and all the other functions are even with respect to the hyperplane, we deduce \( J_{33} + J_{34} = 0 \).

Altogether, we conclude that the estimate (4.7) holds for all \( k \in \{1, \ldots, d\} \) and the assertion (4.9) follows.

Now we discuss the global regularity of \( u_\varepsilon \) in a smooth domain \( \Omega \).

**Proposition 1.** There are constants \( C_p, C_0, \) and \( p > 2 \) such that
\[
\sup_{0 < t \leq T} \int_0^T \int_{\Omega} |\nabla u_\varepsilon|^p + \int_0^T \int_{\Omega} |\nabla^2 u_\varepsilon|^2 \leq C_p \left( \int_0^T \int_{\Omega} |\nabla u_0|^p + \int_0^T \int_{\Omega} |f|^2 + C_0 \right).
\]

(4.10)

**Proof:** Let us cover \( \Omega \) by a finite number of balls \( B_{R_i}(P_i), i = 1, 2, \ldots, \) such that either \( B_{3R_i}(P_i) \subset \Omega \) or \( P_i \in \partial \Omega \). In the case that \( B_{3R_i}(P_i) \subset \Omega \), the proof of Lemma 1 yields
\[
\sup_{0 < t \leq T} \int_0^T \int_{\Omega_{R_i}} |\nabla u_\varepsilon|^p + \int_0^T \int_{\Omega_{R_i}} |\nabla^2 u_\varepsilon|^2 + C_R
\leq C_p \left( \int_0^T \int_{\Omega_{R_i}} |\nabla u_0|^p + \int_0^T \int_{\Omega_{R_i}} |f|^2 + C_0 \right) + \kappa C_{3R_i},
\]

(4.11)

where
\[
C_{R_i} = \int_0^T \int_{\Omega_{R_i}} \mu_\varepsilon \left(|\nabla u_\varepsilon|^q\right) \left(|\nabla u_\varepsilon|^2 + |\nabla u_\varepsilon|^p\right).
\]

In the case that \( P_i \in \partial \Omega \) and \( \partial \Omega \cap B_{3R_i}(P) = E_{d-1} \cap B_{3R_i}(P) \), where \( E_{d-1} \) is a \((d-1)\)-dimensional hyperplane, the estimate (4.11) follows from Lemma 2.

Now let us discuss the case that \( P_i \in \partial \Omega \) and \( \partial \Omega \cap B_{3R_i}(P) \) is not a hyperplane. Since \( \partial \Omega \) is smooth, there is a \( W^{2,\infty} \)-mapping \( \phi_i \) and a ball \( B_{3R_i}(\phi_i(P_i)) \) such that \( B_{3R_i}(\phi_i(P_i)) \cap \phi_i(\partial \Omega) \) is the intersection of \( B_{3R_i}(\phi_i(P_i)) \).
and a \((d-1)\)-dimensional hyperplane. Let \(\hat{x} = \phi_i(x)\). The function \(\hat{u}_\varepsilon(\hat{x}) := u_\varepsilon(\phi_i^{-1}(\hat{x}))\) is the weak solution of
\[
\partial_t \hat{u}_\varepsilon - \hat{\Delta} \hat{u}_\varepsilon = H_\varepsilon(\hat{u}_\varepsilon, \hat{\nabla} \hat{u}_\varepsilon) + \hat{f} \quad \text{in} \quad \hat{\Omega} \times (0, T],
\]
\[
\hat{u}_\varepsilon = 0 \quad \text{on} \quad \partial \hat{\Omega} \times (0, T],
\]
\[
\hat{u}_\varepsilon(\cdot, 0) = \hat{u}_0 \quad \text{in} \quad \hat{\Omega},
\]
where \(\hat{\nabla} = M \nabla\), \(\hat{\Delta} = \hat{\partial}_l \hat{\partial}_l\), and \(M\) has the components \(m_{jk} = \partial_j \phi^k_i\), where \(\phi^k_i\) is the \(k\)-th component of \(\phi_i\). Arguing as above, applying the substitution rule for integrals, and noting that \(M\) is positive definite, we obtain an analogous of estimate (4.11).

We now cover \(\Omega\) by a finite number of appropriate sets \(\phi_i^{-1}(B_{3R_i}(\phi_i(P_i)))\), \(i = 1, 2, \ldots\), and proceed as above. Choosing \(\kappa > 0\) sufficiently small yields the assertion.

We conclude this section with the proof of the main result for domains with a smooth boundary.

**Proof of Theorem 1:** Estimate (4.10) holds for a number \(p\) sufficiently large. Let us extract the \(p\)-th root and send \(p \to \infty\). It follows that there is a constant \(C\), independent of \(\varepsilon\), such that
\[
\|\nabla u_\varepsilon\|_{L^\infty(0, T; L^\infty(\Omega))} \leq C.
\]
Now we take \(\lim_{\varepsilon \to 0}\). We extract a subsequence, again denoted by \(u_\varepsilon\), such that \(u_\varepsilon \to u\) uniformly and
\[
\nabla u_\varepsilon \rightharpoonup^* \nabla u \quad \text{in} \quad L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^N)).
\]
Due to standard arguments, we deduce that the weak limit \(u\) satisfies the primal system (1.1)–(1.3) in the weak sense.

Since \(\nabla u\) is a \(L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^N))\)-function we may proceed as above and obtain an analogous of estimate (4.10) for \(u\). Thus, there is a constant \(C_p\), depending only on \(p\), such that
\[
\|\nabla^2 u\|_{L^2(0, T; L^2(\Omega))} \leq C_p,
\]
for \(p\) sufficiently large. Hence, \(\Delta u\) is a \(L^2(0, T; L^2(\Omega; \mathbb{R}^N))\)-function. Noting that the Hamiltonian and \(f\) are bounded, it follows that
\[
\partial_t u \in L^2(0, T; L^2(\Omega; \mathbb{R}^N))
\]
as well.
5. Generalisation to Lipschitzian domains

Our difference quotient technique can be applied to a more general class of Lipschitzian domains, such as convex polyhedrons; cf. [17, 18]. More precisely, let us assume that \( \Omega \) can be mapped in a smooth way onto a convex polyhedron. We make the following assumptions.

\((\Omega 1)\) \( \Omega \subset \mathbb{R}^d \) \((d \geq 2)\) is a bounded open set;

\((\Omega 2)\) for each \( P \in \partial \Omega \), there exists a mapping \( \phi \) and a ball \( B_R(\phi(P)) \) such that

- \( \phi(\Omega) \cap B_R(\phi(P)) \) is the intersection of \( B_R(\phi(P)) \) and a convex polyhedron,
- \( \phi, \phi^{-1} \in W_{loc}^{2,\infty}(\mathbb{R}^d) \) and the Jacobian of \( \phi \) is positive definite;

\((\Omega 3)\) \( \partial \Omega = \bigcup_{1 \leq k \leq M} \Gamma_k \), where \( \Gamma_k \) are open \((d-1)\)-dimensional Lipschitzian domains for \( k = 1, \ldots, M \), and \( \Gamma_i \cap \Gamma_k = \emptyset \) for \( i \neq k \);

\((\Omega 4)\) \( \partial \Gamma_k \cap \ldots \cap \partial \Gamma_j = \emptyset \), if \( j > d \) and \( k_1 < \ldots < k_j \) (that is, there are at most \( d \) adjacent faces \( \Gamma_k \)).

We now state the following corollary of Theorem 1.

**Corollary 1.** Let \( \Omega \) satisfy assumptions \((\Omega 1)\)–\((\Omega 4)\). Then there exists a weak solution \( u \) of the system (1.1)-(1.3), satisfying

\[
    u \in L^2(0,T; W^{2,2}(\Omega; \mathbb{R}^N)) \cap L^\infty(0,T; W^{1,\infty}(\Omega; \mathbb{R}^N))
\]

and

\[
    u_t \in L^2(0,T; L^2(\Omega; \mathbb{R}^N)).
\]

**Proof:** We proceed as in the proof of Theorem 1. First, let us discuss the case when \( \Omega \) is a convex polyhedron. We can cover \( \Omega \) by a finite number of balls \( B_{R_i}(P_i) \), \( i = 1,2,\ldots, \) such that either \( B_{3R_i}(P_i) \subset \Omega \) or \( P_i \in \partial \Omega \). For each \( P_i \in \partial \Omega \), there is an index set \( \Lambda_i \) such that \( \Gamma_k \cap B_{3R_i}(P_i) \neq \emptyset \), for all \( k \in \Lambda_i \) and \( \partial \Omega \cap B_{3R_i}(P_i) = \bigcup_{k \in \Lambda_i} \Gamma_k \cap B_{3R_i}(P_i) \). We suppose that \( P_i \in \bigcap_{k \in \Lambda_i} \Gamma_k \).

In the case that \( B_{3R_i}(P_i) \subset \Omega \), the proof of Lemma 1 yields constants \( C_p, C_R, C_0 \) and \( p > 2 \) such that

\[
    \sup_{0 < t \leq T} \int_0^T \int_{\Omega_{R_i}} |\nabla u_e|^p + \int_0^T \int_{\Omega_{R_i}} |\nabla^2 u_e|^2 + C_R \leq C_p \left( \int_0^T \int_{\Omega_{3R_i}} |\nabla u_0|^p + \int_0^T \int_{\Omega_{3R_i}} |\mathbf{f}|^2 + C_0 \right) + \kappa C_{3R_i},
\]

(5.1)
where
\[ C_{R_i} = \int_0^T \int_{\Omega_{R_i}} \mu_\varepsilon(|\nabla u_\varepsilon|^q) (|\nabla u_\varepsilon|^2 + |\nabla u_\varepsilon|^p). \]

In the case that \( P_i \in \partial \Omega \) and \( \partial \Omega \cap B_{3R_i} \) is contained in an \((d-1)\)-dimensional hyperplane, we can find \( d - 1 \) linearly independent vectors \( h \) parallel to \( \partial \Omega \cap B_{3R_i} \) and one that is normal to \( \partial \Omega \cap B_{3R_i} \). Then the proof of estimate (5.1) runs as in the proof of Lemma 2.

Now let us discuss the case when \( \partial \Omega \cap B_{3R_i}(P_i) \) is not contained in a hyperplane. Let \( k_0 \in \Lambda \) and \( e \in \mathbb{R}^d \) be a unit vector parallel to \((\partial \Omega \cap B_{3R_i}) \setminus \Gamma_{k_0}\) satisfying \( z + \lambda e \in \overline{\Omega} \), for all \( z \in \partial \Omega \cap B_{3R_i} \) and \( 0 < \lambda < R \). Furthermore, let \( e^* \) be the reflection of \( e \) with respect to the hyperplane containing \( \partial \Omega \cap B_{3R_i} \), and \( \Omega^*_3 \) be the reflection of \( \Omega_3 \setminus \Omega_3 + h \) with respect to the hyperplane. We now define the extensions of the functions \( u_\varepsilon \), \( f \), and \( \eta \) onto \( \Omega^*_3 \) by setting
\[ u_\varepsilon(z + \lambda e^*) := -u_\varepsilon(z + \lambda e), \]
\[ f(z + \lambda e^*) := f(z + \lambda e), \]
and \( \eta(z + \lambda e^*) := \eta(z + \lambda e). \)

We make use of the shift operator \( T^*_h v(x) := v(\psi(\lambda_0 \pm h)) \) and define
\[ D^*_h v(x) = \frac{T^*_h v(x) - v(x)}{|h|} \quad \text{and} \quad D^*_{-h} v(x) = \frac{v(x) - T^*_{-h} v(x)}{|h|}, \]
where
\[ \psi(\lambda) := \begin{cases} z + \lambda e & \text{for } \lambda \geq 0, \\ z - \lambda e^* & \text{for } \lambda < 0, \end{cases} \]
and \( x = \psi(\lambda_0) = z + \lambda_0 e \in \Omega_{3R_i}. \)

Taking the test function
\[ \varphi := -D^*_h (\eta^2_{-h}(1 + |D^*_{-h} u_\varepsilon|^{p-2})D^*_{-h} u_\varepsilon) - D^*_{-h}(\eta^2_h(1 + |D^*_{h} u_\varepsilon|^{p-2})D^*_{h} u_\varepsilon), \]
where \( \eta_{\pm h}(x) = T^*_{\pm h} \eta(x) \), and proceeding as in the proof of Lemma 2, estimate (5.1) follows.

Finally, let us consider the case that \( \Omega \) is not a convex polyhedron. Then there is a \( W^{2,\infty} \)-mappings \( \phi_i \) and a ball \( B_{3R_i}(\phi_i(P_i)) \) such that \( B_{3R_i}(\phi_i(P_i)) \cap \phi_i(\partial \Omega) \) is the intersection of \( B_{3R_i}(\phi_i(P_i)) \) and \((d-1)\)-dimensional hyperplane. Now the proof of estimate (5.1) runs as in the proof of Proposition 1.

Collecting results and arguing as in the proof of Theorem 1, we obtain the assertion.
6. Inhomogeneous Dirichlet boundary conditions

In this section we discuss the case of inhomogeneous Dirichlet boundary value conditions. The dynamical system

\[ dy = m(y, v) \, dt + \sigma(y) \, dw_t, \quad y(0) = x, \quad (6.1) \]

is modified by the \( N \) players through the controls \( v^k(t) \) \((1 \leq k \leq N)\). The \( k \)-th player chooses its own control to maximize its cost functional

\[
J^k_{x,t}(v) = \mathbb{E}^k_{x,t} \left[ \int_t^T l_k(y(\tau), v(\tau)) \, e^{-\int_{\tau}^T c_k(y(s), v(s)) \, ds} \, d\tau \right. \\
\left. + \phi_k(y(T)) e^{-\int_t^T c_k(y(s), v(s)) \, ds} \right].
\]

Let us assume that \( T \) is the exit time of \( y(t) \) from the domain \( \Omega \). Defining the payoff functions

\[
u^k(x, t) := J^k_{x,t}(v) \quad (6.2)
\]

and setting \( t = T \), we obtain the boundary value conditions \( u^k(x, T) = \phi_k(y(T)) \).

Let us discount the payoff function \( u^k(x, t) \) back to the time point \( t + \delta \). Thus, the payoff is given by \( u^k(y(t + \delta), t + \delta) \). Let \( v \) be the optimal choice maximising the cost functional. This leads to the payoff functions

\[
u^k(x, t) = \mathbb{E}^k_{x,t} \left[ \int_t^{t+\delta} l_k(y(\tau), v(\tau)) \, e^{-\int_{\tau}^{t+\delta} c_k(y(s), v(s)) \, ds} \, d\tau \\
+ u^k(y(t + \delta), t + \delta) e^{-\int_t^{t+\delta} c_k(y(s), v(s)) \, ds} \right].
\]

Subtracting \( u^k(x, t) \) and dividing by \( \delta \) we arrive at

\[
0 = \mathbb{E}^k_{x,t} \left[ \frac{1}{\delta} \int_t^{t+\delta} l_k(y(\tau), v(\tau)) \, e^{-\int_{\tau}^{t+\delta} c_k(y(s), v(s)) \, ds} \, d\tau \\
+ \frac{1}{\delta} \left( u^k(y(t + \delta), t + \delta) e^{-\int_t^{t+\delta} c_k(y(s), v(s)) \, ds} - u^k(x, t) \right) \right].
\]

The mean value theorem yields, for sufficiently smooth functions,

\[
\lim_{\delta \to 0} \frac{1}{\delta} \int_t^{t+\delta} l_k(y(\tau), v(\tau)) \, e^{-\int_{\tau}^{t+\delta} c_k(y(s), v(s)) \, ds} \, d\tau = l_k(y(t), v(t)).
\]
Moreover, due to the rules of stochastic differential calculus it holds that
\[
\lim_{\delta \to 0} \frac{1}{\delta} \left( u^k(y(t+\delta), t+\delta)e^{-\int_{t}^{t+\delta} c_k(y(s), v(s)) ds} - u^k(x, t) \right)
\]
\[= -u^k(x, t) c_k(y(t), v(t)) + \nabla u^k(x, t) \cdot m(y(t), v(t)) + Au^k(x, t) + \partial_t u^k(x, t),
\]
where \( Au^k = \sum_{i,j} a_{ij} \partial_i \partial_j u^k \) and \( a_{ij} = \frac{1}{2} \sum_k \sigma_{ik} \sigma_{kj} \). Thus, \( u = (u^1, \ldots, u^N) \) is a solution of the Hamilton-Jacobi-Bellmann equation (2.4).

Assuming that there is an exit time from the domain, the solution \( u \) satisfies an inhomogeneous Dirichlet boundary value condition.

**Remark 1.** Our difference quotient method can be applied to the case of inhomogeneous Dirichlet boundary value conditions as well. Let \( \phi^* \) be a sufficiently smooth function satisfying \( \phi^*(x, T) = \phi(x, T) \) on \( \partial \Omega \) in the sense of traces. Using the test function
\[
\varphi = -D_h(\eta^2_{-h}(1 + |D_{-h}(u_\varepsilon - \phi^*)|^{p-2})D_{-h}(u_\varepsilon - \phi^*)
\]
\[+ D_{-h}(\eta^2_{-h}(1 + |D_{-h}(u_\varepsilon - \phi^*)|^{p-2})D_{h}(u_\varepsilon - \phi^*)
\]
the proof runs as above.

**References**


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