Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–02

DECREASING DIAGRAMS AND COHERENT PRESENTATIONS

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Abstract: We show how decreasing diagrams introduced in the theory of rewriting systems can be used to prove coherence type theorems in category theory. We apply this method to describe a coherent presentation of the 0-Hecke monoid $\mathfrak{H}(\Sigma_n)$ of the symmetric group Σ_n, i.e. a presentation by generators, relations, and relations between relations.

Keywords: coherent presentation, decreasing diagrams, 0-Hecke monoid.
Math. Subject Classification (2010): 18A99,68Q42,20C08.

1. Introduction

Presentations of monoids are known to be essential in the study of actions on sets. In fact, given a monoid M with presentation $\langle X, \mathbf{r}\rangle$, to describe an action of M on a set S it is enough to give a set of endomorphisms $A_x\colon S \to S$, $x \in X$, satisfying the relations in $\mathbf{r}$.

In recent years the interest for actions of monoids on categories raised. In this case presentations have to be replaced by the so called *coherent presentations*. Suppose we are given an action of M on $\mathcal{C}$. Then we have a collection of endofunctors $F_m\colon \mathcal{C} \to \mathcal{C}$, $m \in M$, for which $F_m \circ F_n = F_{mn}$ does not hold in general. Instead, we have a collection of natural isomorphisms $\lambda_{m,n}\colon F_m \circ F_n \to F_{mn}$ such that the natural transformations

$$\begin{aligned} &F_l \circ F_m \circ F_n \xrightarrow{\lambda_{l,m}\circ F_n} F_{lm} \circ F_n \xrightarrow{\lambda_{lk,n}} F_{lmn} \\ &F_l \circ F_m \circ F_n \xrightarrow{F_l\circ\lambda_{m,n}} F_l \circ F_{mn} \xrightarrow{\lambda_{l,kn}} F_{lmn} \end{aligned} \tag{1}$$

are equal for all l, m, $n \in M$. Further, using $\left\{ \lambda_{m,n} \,\middle|\, m, n \in M \right\}$, we can construct natural isomorphisms

$$\lambda_{m_1,\dots,m_k}\colon F_{m_1} \circ \dots \circ F_{m_k} \to F_{m_1\dots m_k}$$

Received January 19, 2018.

This work was supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020, and by Programa Investigador FCT IF/00016/2013 funded by Portuguese state budget.

for all $m_1,\ldots, m_k \in M$. Now for every relation

$$r = (x_1 \ldots x_s, y_1 \ldots y_t)$$

in $\mathbf{r}$ we define the natural isomorphism $\tau_r = \lambda^{-1}_{y_1,\ldots,y_t} \lambda_{x_1,\ldots,x_s}$. It can be shown that one can reconstruct (up to an isomorphism) the action

$$\left\{ F_m,\ \lambda_{m,n} \,\middle|\, m, n \in M \right\}$$

of M on $\mathcal{C}$ from the collection $\{ F_x,\ \tau_r \,|\, x \in x,\ \tau_r \}$.

Not every collection $\{ F_x,\ \tau_r \,|\, x \in X,\ r \in \mathbf{r} \}$ of endofunctors of $\mathcal{C}$ and natural transformations between them can be obtained from an action of M on $\mathcal{C}$. The obstruction comes from the axiom (1). This obstruction can be translated into a set $\mathcal{E}$ of equations of the form

$$\tau_{r_1} \ldots \tau_{r_k} = \tau_{s_1} \ldots \tau_{s_l}$$

with $r_1, \ldots r_k$, $s_1,\ldots, s_l \in \mathbf{r}$. Such a set $\mathcal{E}$ is called a complete set of relations between relations from $\mathbf{r}$, and $\langle X, \mathbf{r}, \mathcal{E} \rangle$ is called a *coherent presentation* of M.

To find coherent presentations of some monoids, Guiraud and Malbos [3] used higher dimensional rewriting theory. Applying the same technique, Gausent, Guiraud, and Malbos obtained a coherent presentation of Artin braid groups associated with Coxeter systems [2].

In this article, we use a new approach to determine coherent presentations of monoids. Namely, we use the notion of decreasing diagrams, that was introduced by van Oostrom in [6] to obtain a sufficient condition for a locally confluent abstract reduction system to be confluent. In [4] Klop, van Oostrom, and de Vrijer gave a geometrical proof of the result of van Oostrom. In their proof they discovered that if an abstract reduction system admits enough decreasing elementary diagrams, then every reduction diagram can be patched by these elementary diagrams. Their result allows us to give a sufficient condition for $\langle X, \mathbf{r}, \mathcal{E} \rangle$ to be a coherent presentation of a monoid $\langle X, \mathbf{r} \rangle$ (see Theorem 6.3). Using this sufficient condition we establish a coherent presentation of the 0-Hecke monoid. This presentation was used in the joint work of the author with A. P. Santana [5] to exhibit an action of the 0-Hecke monoid on the category of rational modules for the quantum Borel group. Note that our coherent presentation contains the coherent presentation of the braid group obtained by Guiraud *et al.* in [2].

The paper is organised as follows. In Section 2 we collect results on abstract reduction systems used throughout the paper. In Section 3 we establish a relationship between abstract reduction systems and categories, and show how decreasing diagrams can be used to deduce the commutativity of an infinite set of diagrams, from the commutativity of a given, often finite, set of diagrams (Theorem 3.4). Section 4 contains the definition of an action of a monoid on a category (following [1]). In Section 5, we apply Theorem 3.4 to an abstract reduction system associated to a presentation of a monoid. The main result of Section 6 is Theorem 6.3, that gives a sufficient condition for $\langle X, \mathbf{r}, \mathcal{E} \rangle$ to be a coherent presentation. In Section 7 we describe a coherent presentation of the 0-Hecke monoid $\mathcal{H}(\Sigma_{n+1})$ of the symmetric group Σ_{n+1}.

2. Abstract reduction system

An *abstract reduction system* is a set A with a relation $R \subset A \times A$ on A which is called a set of *rewriting rules*. The elements of R will be sometimes depicted by $a \to b$ for $(a, b) \in R$. The sequence of elements $a_0, \ldots, a_k$ is called a *reduction path* from a_0 to a_k if $(a_{i-1}, a_i) \in R$ for all $1 \le i \le k$. If there is a reduction path from $a \in A$ to $b \in A$, we write $a \twoheadrightarrow b$.

A *reduction diagram* for $\langle A, R \rangle$ is an oriented planar graph Γ, such that:

(1) All the arrows of Γ go either from left to right or from top to bottom.
(2) Some arrows of Γ are solid and some arrows of Γ are dashed.
(3) The nodes of Γ are labeled by elements of A. The label of a node x will be denoted by $l(x) \in A$.
(4) If the nodes with labels a and b are connected by a solid arrow then $(a, b) \in R$.
(5) If two nodes are connected by a dashed arrow then they have equal labels.
(6) If from a node $x \in \Gamma$ there is a horizontal arrow to $y \in \Gamma$ and a vertical arrow to $z \in \Gamma$ then one of the two mutually exclusive possibilities holds
    (a) There is no vertex which is simultaneously strictly bellow and strictly to the right of x. In this case we say that x is an *open corner*.
    (b) There is a node $w \in \Gamma$, a vertical path from y to w in Γ and a horizontal path from z to w in Γ. These paths are called *convergence* paths. If x and y are connected by a dashed arrow, then z and w are connected by a dashed arrow as well, and the path from y to w contains just one arrow. Similarly, if x and z are connected by a dashed arrow then y and w are connected by a dashed arrow and the path from z to w contains just one arrow.

Bellow is an example of a reduction diagram

$$\begin{tikzcd}
a_{11} \arrow[r] \arrow[dd] & a_{12} \arrow[r] \arrow[d] & a_{13} \arrow[rr] \arrow[d] & & a_{15} \arrow[d] \\
& a_{22} \arrow[r, dashed] \arrow[d] & a_{22} \arrow[r] \arrow[d] & a_{24} \arrow[r] & a_{25} \\
a_{31} \arrow[r] \arrow[d, dashed] & a_{32} \arrow[r, dashed] \arrow[d, dashed] & a_{32} & & \\
a_{31} \arrow[r] & a_{32} & & & .
\end{tikzcd}$$

In the above reduction diagram there are two open corners with labels a_{22} and a_{32}, whose (matrix) coordinates are $(2, 3)$ and $(3, 2)$, respectively. The following graph does not satisfy the axioms of a reduction diagram

$$\begin{tikzcd}
a_{11} \arrow[r] \arrow[d] & a_{12} \arrow[dd] \\
a_{21} \arrow[d] & \\
a_{32} \arrow[r] & a_{33} .
\end{tikzcd}$$

In fact, the top left corner of the above graph is neither an open corner nor there are convergence paths for the horizontal arrow $a_{11} \to a_{12}$ and the vertical arrow $a_{11} \to a_{21}$.

Definition 2.1. A reduction diagram is called *complete* if it does not contain any open corners.

An example of a complete diagram is given by

$$\begin{tikzcd}
a_{11} \arrow[r] \arrow[dd] & a_{12} \arrow[r] \arrow[d] & a_{13} \arrow[rr] \arrow[d] & & a_{15} \arrow[d] \\
& a_{22} \arrow[r, dashed] \arrow[d] & a_{22} \arrow[r] \arrow[d] & a_{24} \arrow[r] \arrow[d, dashed] & a_{25} \arrow[d, dashed] \\
a_{31} \arrow[r] \arrow[d] & a_{32} \arrow[r, dashed] \arrow[d, dashed] & a_{32} \arrow[r] \arrow[d, dashed] & a_{24} \arrow[r] \arrow[d, dashed] & a_{25} \arrow[d, dashed] \\
a_{41} \arrow[r] & a_{32} \arrow[r, dashed] & a_{32} \arrow[r] & a_{24} \arrow[r] & a_{25}
\end{tikzcd}$$

Definition 2.2. An *elementary diagram* (e.d.) for $\langle A, R\rangle$ is a reduction diagram Γ such that the edges of Γ constitute the boundary of a rectangular.

Suppose Γ is an e.d. Then the top side and the left side of Γ contain just one arrow each, as otherwise the top-left corner of Γ would not satisfy the axiom (6) for a reduction diagram. Therefore there are four different types of e.d.s:

$$
\begin{tikzcd}
a \arrow[r] \arrow[d] & b \arrow[d, two heads] \\
c \arrow[r, two heads] & d
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, dashed] \arrow[d] & a \arrow[d] \\
c \arrow[r, dashed] & c
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r] \arrow[d, dashed] & b \arrow[d, dashed] \\
a \arrow[r] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, dashed] \arrow[d, dashed] & a \arrow[d, dashed] \\
a \arrow[r, dashed] & a
\end{tikzcd}
$$

where two headed arrows are used as an abbreviation of a path. The e.d.s of first type are called *proper* and the rest of e.d.s are called *improper*.

Let Γ be a non-complete reduction diagram for $\langle A, R\rangle$ and x is an open corner in Γ with the horizontal arrow to y and the vertical arrow to z. Suppose that we have an e.d. Γ' with the labels $l\,(x)$, $l\,(y)$, $l\,(z)$ at the top left, top right, and bottom left corners, respectively. Then we can glue (suitably stretched) Γ' into Γ identifying the top left corner of Γ' with x, the top right corner of Γ' with y, and the bottom left corner of Γ' with z. This process is called *adjoining* of the e.d. Γ' to Γ at x. It is obvious, that the resulting diagram is again a reduction diagram.

Definition 2.3. We say that $\langle A, R\rangle$ is *locally confluent* if for any $a \to b$, $a \to c$ there is an e.d. with labels a, b, c at the top left, top right, and bottom left corners, respectively.

Definition 2.4. A reduction diagram Γ for $\langle A, R\rangle$ is called *initial* if its edges constitute the top and the left sides of a rectangular.

Suppose that $\langle A, R\rangle$ is a locally confluent ARS. Let $\mathcal{E}$ be a family of e.d.s such that for every ordered pair $a \to b$, $a \to c$ in R there is an e.d. $E \in \mathcal{E}$ whose top arrow is $a \to b$ and left arrow is $a \to c$. In this case we say that $\mathcal{E}$ is a complete set of e.d.s. It is proved in Section 4 of [4] that the recursive process of adjoining of e.d.s from a complete set of e.d.s to any initial finite diagram Γ results in a complete diagram Γ' in at most a countable number of steps.

Recall that a *preorder* is a reflexive and transitive binary relation. Suppose now that the set R is equipped with a preorder $\succeq$. We write $r_1 \succ r_2$ if $r_1 \succeq r_2$ but not $r_2 \succeq r_1$. If $r_1 \succeq r_2$ and $r_2 \succeq r_1$ simultaneously, then we write $r_1 \sim r_2$. It is immediate that $\sim$ is an equivalence relation on R.

Definition 2.5. We say that the e.d.

$$\begin{tikzcd}
x \arrow[rrr, "u"] \arrow[ddd, "l"'] & & & y_1 \arrow[d, "r_1"] \\
& & & y_1 \arrow[d, dotted, no head] \\
& & & y_m \arrow[d, "r_m"] \\
z_1 \arrow[r, "d_1"] & z_2 \arrow[r, dotted, no head] & z_n \arrow[r, "d_n"] & w
\end{tikzcd}$$

is *decreasing* if the following two condition hold

1) there is $0 \leq j \leq n$ such that
   i) $u \sim d_j$ in the case $j \neq 0$;
   ii) $l \succ d_k$ for all $k < j$;
   iii) $l \succ d_k$ or $u \succ d_k$ for all $k > j$;

2) there is $0 \leq s \leq m$ such that
   i) $l \sim r_s$ in the case $s \neq 0$;
   ii) $u \succ r_t$ for all $t < s$;
   iii) $u \succ r_t$ or $l \succ r_t$ for all $t > s$.

More informally we require that either the reduction path $r_1, \ldots, r_m$ consists of the steps that are strictly less than l or u, or if it starts with the rules that are strictly less than u, then there is a step r_t which is equivalent to l and all other steps are strictly less than u or l.

Similarly the reduction path $d_1, \ldots, d_n$ either consists of the steps that are strictly less than l or u, or if it starts with the rules that are strictly less than l, then there is a step d_t which is equivalent to u and all other steps are strictly less than u or l.

We say that the preorder $\succeq$ is *well-founded* if for every sequence

$$r_1 \succeq r_2 \succeq \cdots \succeq r_m \succeq \ldots$$

of elements in R there is an integer n such that for all $N \geq n$ we have $r_N \sim r_n$; in other words, any decreasing sequence in $(R, \succeq)$ stabilizes.

The following theorem is a reformulation of [4, Proposition 15].

Theorem 2.6. *Suppose $\langle A, R \rangle$ is a locally confluent ARS and there is a well-founded preorder $\succeq$ on R and a complete set $\mathcal{E}$ of decreasing e.d.s. Then every process of adjoining chosen e.d.s to any initial finite diagram Γ results in a complete diagram Γ' in a finite number of steps.*

We will say that the sequence of elements

$$a_0, a_1, \ldots, a_k$$

is a zigzag in $\langle A, R\rangle$ from a_0 to a_k, if for ever $1 \leq j \leq k$, we have $(a_{j-1}, a_j) \in R$ or $(a_j, a_{j-1}) \in R$. We will denote zigzags by $a_0 \rightsquigarrow a_k$. Let $\langle A, R\rangle$ be an ARS, $\succ$ a well-founded preorder on R, and $\mathcal{E}$ a complete set of decreasing e.d.s. Suppose we are given a reduction diagram of the form

$$\begin{tikzcd}
& & a_1 \arrow[r, two heads] \arrow[d, two heads] & b_0 \\
& & b_1 \arrow[dl, dotted, no head] & \\
a_k \arrow[r, two heads] \arrow[d, two heads] & b_{k-1} & & \\
b_k & & &
\end{tikzcd} \tag{2}$$

Then applying Theorem 2.6 to the diagrams

$$\begin{tikzcd}
a_j \arrow[r, two heads] \arrow[d, two heads] & b_{j-1} \\
b_j, &
\end{tikzcd}$$

we can get in a finite number of steps the diagram

$$\begin{tikzcd}
& & & a_1 \arrow[r, two heads] \arrow[d, two heads] & b_0 \arrow[d, two heads] \\
& & & b_1 \arrow[r, two heads] \arrow[dl, dotted, no head] & c_0 \\
& a_{k-2} \arrow[r, two heads] \arrow[d, two heads] & b_{k-2} \arrow[d, two heads] & & \\
a_k \arrow[r, two heads] \arrow[d, two heads] & b_{k-1} \arrow[r, two heads] \arrow[d, two heads] & c_{k-2} & & \\
b_k \arrow[r, two heads] & c_{k-1}, & & &
\end{tikzcd} \tag{3}$$

where every square is in fact tiled into an elementary diagrams from $\mathcal{E}$. Note that the new diagram has $k-1$ open corners, that is one open corner less than (2). Now we can apply Theorem 2.6 to the diagrams

$$\begin{tikzcd} b_j \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & c_{j-1} \\ c_j & \end{tikzcd} .$$

As a result we get a new reduction diagram with $k-2$ open corners. Continuing, we get a reduction diagram

$$\begin{tikzcd} & & a_1 \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & b_0 \arrow[ddd, twoheadrightarrow] \\ & & b_1 \arrow[dl, dotted, no head] & \\ a_k \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & b_{k-1} & & \\ b_k \arrow[rrr, twoheadrightarrow] & & & z, \end{tikzcd} \tag{4}$$

whose interior is tiled by e.d.s from $\mathcal{E}$. Thus we get

Corollary 2.7. *Suppose $\langle A, R\rangle$ is an ARS, $\succeq$ a well-founded preorder on R, and $\mathcal{E}$ a complete set of decreasing e.d.s. Then any reduction diagram of the form* (2) *can be completed to a diagram of the form* (4) *in a finite number of steps.*

One of the applications of Proposition 2.6 is to prove *confluency* of an ARS. Similarly, Corollary 2.7 can be used to prove the Church-Rosser property of $\langle A, R\rangle$. In this paper we will apply Proposition 2.6 and Corollary 2.7 to prove commutativity of certain diagrams. It should be noted that Proposition 2.6 enlightens the long time observed connection between confluent ARSs and coherence results in category theory.

3. ARSs and categories

We will start by introducing notion that generalises normal forms for terminating ARS to the case when $\twoheadrightarrow$ is a well-founded but not necessarily terminating relation on A.

Definition 3.1. We say that $a \in A$ is *semi-normal* if for every path $a \twoheadrightarrow b$ in $\langle A, R\rangle$ there is a path $b \twoheadrightarrow a$ in $\langle A, R\rangle$. If $c \in A$ and there is a path $c \twoheadrightarrow a$ to a semi-normal element a, we say that a is a *semi-normal form* of c.

We will denote by $\twoheadleftarrow\!\!\!\twoheadrightarrow$ the equivalence relation on A defined by

$$a \twoheadleftarrow\!\!\!\twoheadrightarrow b \Leftrightarrow a \twoheadrightarrow b \;\&\; b \twoheadrightarrow a.$$

Suppose the preorder $\twoheadrightarrow$ on A is well-founded. Then for every element of A there is a semi-normal form.

Suppose $\langle A, R\rangle$ is a confluent system and b_1, b_2 are two semi-normal forms of $a \in A$. Since $\langle A, R\rangle$ is confluent, there is $c \in A$ and two paths $b_1 \twoheadrightarrow c$, $b_2 \twoheadrightarrow c$. As b_2 is semi-normal there is a path $c \twoheadrightarrow b_2$. Therefore we get the path $b_1 \twoheadrightarrow c \twoheadrightarrow b_2$. Since also b_1 is semi-normal, there is a path $b_2 \twoheadrightarrow b_1$. Hence $b_1 \twoheadleftarrow\!\!\!\twoheadrightarrow b_2$. Thus we get

Proposition 3.2. *Suppose $\langle A, R\rangle$ is a confluent ARS, and $\twoheadrightarrow$ is a well-founded preorder on A. Then for every element $a \in A$ there is a semi-normal form b which is unique up to equivalence with respect to $\twoheadleftarrow\!\!\!\twoheadrightarrow$.*

Remark 3.3. Proposition 3.2 is a generalization of the well-known fact that every element in a confluent terminating ARS has a unique normal form.

We will call the set $\mathrm{Attr}(a)$ of all semi-normal forms of $a \in A$ an *attractor* of a. We also will denote by $\mathrm{Attr}(A)$ the set of all semi-normal elements in A. Since $a \in \mathrm{Attr}(A)$ and $(a, b) \in R$ imply that $b \in \mathrm{Attr}(A)$, we see that the relation R can be restricted to $\mathrm{Attr}(A)$. We will denote the resulting relation on $\mathrm{Attr}(A)$ by $\mathrm{Attr}(R)$. We will sometimes denote the ARS $\big\langle \mathrm{Attr}(A), \mathrm{Attr}(R)\big\rangle$ by $\mathrm{Attr}(\langle A, R\rangle)$.

Now, let $\mathcal{C}$ be a category and $\langle A, R\rangle$ an ARS. Both $\langle A, R\rangle$ and $\mathcal{C}$ can be considered as graphs. Suppose $f\colon \langle A, R\rangle \to \mathcal{C}$ is a map of graphs. Then using the composition of morphisms in $\mathcal{C}$, we can extend f to the paths $a \twoheadrightarrow b$ in $\langle A, R\rangle$. In particular, given an empty path $a \dashrightarrow a$, we set $f(a \dashrightarrow a) = 1_{f(a)}$. Our aim is to find sufficient conditions on f that guarantee that for any two paths p, $q\colon a \twoheadrightarrow b$ one gets $f(p) = f(q)$.

Theorem 3.4. *Let $\langle A, R\rangle$ be an ARS, $\succeq$ be a well-founded preorder on R and $\mathcal{E}$ a complete set of decreasing e.d.s. Suppose that*

i) for every $E \in \mathcal{E}$ the diagram $f(E)$ is commutative;
ii) for every $b \in \mathrm{Attr}(A)$ and every path $p\colon b \twoheadrightarrow b$ the map $f(p)$ is equal to $1_{f(b)}$.

Then for any two paths p, $q\colon a \twoheadrightarrow b$ with $b \in \mathrm{Attr}(a)$, we get $f(p) = f(q)$.

Proof: By Theorem 2.6 we can construct a finite complete reduction diagram Γ in $\langle A, R\rangle$ whose upper side is p, left side is q and which is tiled by e.d.s in $\mathcal{E}$. It follows that $f(\Gamma)$ is a commutative diagram in $\mathcal{C}$. Suppose that the label of the bottom right corner of Γ is c. Then the right side of Γ gives a path $p'\colon b \twoheadrightarrow c$ and the bottom side of Γ gives a path $q'\colon b \twoheadrightarrow c$. From the commutativity of Γ we get

$$f\left(p'\right) f\left(p\right) = f\left(q'\right) f\left(q\right). \tag{5}$$

Now, since b is semi-normal, there is a path $t\colon c \twoheadrightarrow b$. By the theorem assumptions we have $f(tp') = 1_{f(b)} = f(tq')$. Thus from (5), we get

$$f(p) = f(tp')f(p) = f(t)f(p')f(p) = f(t)f(q')f(q) = f(tq')f(q) = f(q).$$

■

We will get several corollaries of Theorem 3.4.

Corollary 3.5. *Let $\langle A, R\rangle$ be an ARS, $\succeq$ be a well-founded preorder on R and $\mathcal{E}$ a complete set of decreasing e.d.s. Suppose that*

i) $\twoheadrightarrow$ is a well-founded preorder on A;
ii) for every $E \in \mathcal{E}$ the diagram $f(E)$ is commutative;
iii) for every $r \in R$ the map $f(r)$ is a monomorphism;
iv) for every $b \in \mathrm{Attr}(A)$ and every path $p\colon b \twoheadrightarrow b$ the map $f(p)$ is equal to $1_{f(b)}$.

Then for any two paths p, $q\colon a \twoheadrightarrow b$ in $\langle A, R\rangle$, we get $f(p) = f(q)$.

Proof: Note that $\mathrm{Attr}(b)$ is non-empty since $\twoheadrightarrow$ is a well-founded preorder. Let $b' \in \mathrm{Attr}(b) = \mathrm{Attr}(a)$. Then there is a path $s\colon b \twoheadrightarrow b'$. We get two composed paths sp, $sq\colon a \twoheadrightarrow b'$. We have $f(sp) = f(s)f(p)$ and $f(sq) = f(s)f(q)$. By Theorem 3.4 we obtain $f(sp) = f(sq)$. Since $f(s)$ is a monomorphism $f(s)f(p) = f(s)f(q)$ implies that $f(p) = f(q)$. ■

If $f\colon \langle A, R\rangle \to \mathcal{C}$ is such that the map $f(r)$ is invertible for every $r \in R$, then we can define, in the obvious way, a morphism $f(z)\colon f(a) \to f(b)$ for every zigzag $z\colon a \rightsquigarrow b$.

Corollary 3.6. *Let $\langle A, R\rangle$ be an ARS, $\succeq$ be a well-founded preorder on R and $\mathcal{E}$ a complete set of decreasing e.d.s. Suppose that*

i) $\twoheadrightarrow$ is a well-founded preorder on A;

ii) for every $E \in \mathcal{E}$ the diagram $f(E)$ is commutative;
iii) for every $r \in R$ the map $f(r)$ is an isomorphism;
iv) for every $b \in \mathrm{Attr}(A)$ and every path $p\colon b \twoheadrightarrow b$ the map $f(p)$ is equal to $1_{f(b)}$.

Then for any two zigzags z, $z'\colon a \rightsquigarrow b$, we get $f(z) = f(z')$.

Proof: The zigzags z and z' fit in the following reduction diagram

$$\begin{tikzcd}
 & & b \\
 & a \arrow[ur, squiggly, "z"] \arrow[dl, squiggly, "z'"'] & \\
b & &
\end{tikzcd}$$

of type (2). Now, by Corollary 2.7, there is a diagram of type

$$\begin{tikzcd}
 & & b \arrow[dd, "p"] \\
 & a \arrow[ur, squiggly, "z"] \arrow[dl, squiggly, "z'"'] & \\
b \arrow[rr, "q"] & & c,
\end{tikzcd}$$

which is tiled by diagrams in $\mathcal{E}$. Note, that by Corollary 3.5, we have $f(p) = f(q)$. Since every diagram $f(E)$, $E \in \mathcal{E}$, is commutative and every map $f(r)$, $r \in R$, is an isomorphism, we get that

$$f(z)f(z')^{-1} = f(p)^{-1}f(q) = 1_{f(b)}.$$

Therefore $f(z) = f(z')$. ■

Example 3.7. Suppose $\langle A, R\rangle$ is a terminating locally confluent ARS. By [6, Corollary 4.4] all elementary diagrams for a terminating ARS can be made decreasing. Further, for every $b \in \mathrm{Attr}(A)$ the only path $b \twoheadrightarrow b$ is the empty one. Thus we have $f(b \twoheadrightarrow b) = 1_{f(b)}$ for any map of graphs $f\colon \langle A, R\rangle \to \mathcal{C}$. Hence if f is such that $f(r)$ is an isomorphism for all $r \in R$ and $f(E)$ is commutative for all E in a complete set $\mathcal{E}$ of e.d.s, then $f(z) = f(z')$ for any two zigzags z, $z'\colon a \rightsquigarrow b$ in $\langle A, R\rangle$.

4. Actions of monoids on categories

Let $\mathcal{C}$ be a category and M a monoid with neutral element e. Following [1] we define a *(pseudo)action* $(\mathcal{F}, \lambda)$ of M on $\mathcal{C}$ as a collection of

i) endofunctors $F_a\colon \mathcal{C} \to \mathcal{C}$, $a \in M$, such that $F_e \cong Id$ via the natural isomorphism η;

ii) natural isomorphisms $\lambda_{a,b}\colon F_aF_b \to F_{ab}$, such that for all a, b, $c \in M$ the diagram

$$\begin{tikzcd} F_aF_bF_c \arrow[r, "\lambda_{a,b}F_c"] \arrow[d, "F_a\lambda_{b,c}"'] & F_{ab}F_c \arrow[d, "\lambda_{ab,c}"] \\ F_aF_{bc} \arrow[r, "\lambda_{a,bc}"] & F_{abc} \end{tikzcd}$$

commutes, and $\lambda_{e,a}$, $\lambda_{a,e}$ are induced by η.

Given an action (F, λ) on $\mathcal{C}$ and a sequence of elements $a_1, \dots, a_k$, with $k \geq 3$, we will define recursively the natural isomorphism $\lambda_{a_1,\dots,a_k}$ from $F_{a_1} \dots F_{a_k}$ to $F_{a_1\dots a_k}$ by

$$\lambda_{a_1,\dots,a_k} = \lambda_{a_1,a_2\cdots a_k} \circ F_{a_1}(\lambda_{a_2,\dots,a_k}).$$

The actions of M on $\mathcal{C}$ form a category $[M;\mathcal{C}]$, where a morphism from (F, λ) to (F', λ') is given by a collection of natural transformations $\rho_a\colon F_a \to F'_a$, $a \in M$, such that the diagrams

$$\begin{tikzcd} F_aF_b \arrow[r, "\lambda_{a,b}"] \arrow[d, "\rho_a\rho_b"'] & F_{ab} \arrow[d, "\rho_{ab}"] \\ F'_aF'_b \arrow[r, "\lambda'_{a,b}"] & F'_{ab} \end{tikzcd} \qquad \begin{tikzcd} F_e \arrow[r, "\eta"] \arrow[d, "\rho_e"'] & \mathrm{Id} \\ F'_e \arrow[ur, "\eta'"'] & \end{tikzcd}$$

commute. From a more abstract point of view, the category of actions of M on $\mathcal{C}$ is the category of pseudofunctors from the category $(*, M)$ to the 2-category Cat of categories.

Suppose $\langle X, \mathbf{r}\rangle$ is a presentation of M. Given $(F, \lambda) \in [M;\mathcal{C}]$, we get a collection of functors $F_x\colon \mathcal{C} \to \mathcal{C}$ and natural isomorphisms

$$\tau_r\colon F_{a_1} \dots F_{a_k} \xrightarrow{\lambda_{a_1,\dots,a_k}} F_{a_1\dots a_k} = F_{b_1\dots b_l} \xrightarrow{\lambda^{-1}_{b_1,\dots,b_l}} F_{b_1} \dots F_{b_l}$$

for every relation $r = (a_1 \dots a_k, b_1 \dots b_l)$ in $\mathbf{r}$. Let us denote by $[X, r; \mathcal{C}]$ the category whose objects are pairs $((F_x)_{x\in X}, (\tau_r)_{r\in\mathbf{r}})$ where F_x are endofunctors of $\mathcal{C}$ and, for every $r = (a_1 \dots a_k, b_1 \dots, b_l)$, τ_r is a natural isomorphism from $F_{a_1} \dots F_{a_k}$ to $F_{b_1} \dots F_{b_l}$. The morphisms from (F, τ) to (F', τ') in $[X, r; \mathcal{C}]$ are families of natural transformations $\rho_x\colon F_x \to F'_x$, $x \in X$, such that for all $r = (a_1 \dots a_k, b_1 \dots b_l) \in \mathbf{r}$ the diagrams

$$\begin{tikzcd} F_{a_1}\dots F_{a_k} \arrow[r, "\tau_r"] \arrow[d, "\rho_{a_1}\cdots\rho_{a_k}"'] & F_{b_1}\dots F_{b_l} \arrow[d, "\rho_{b_1}\cdots\rho_{b_l}"] \\ F'_{a_1}\dots F'_{a_k} \arrow[r, "\tau'_r"] & F'_{b_1}\dots F'_{b_l} \end{tikzcd}$$

are commutative. Then we get from the construction described above the restriction functor $\mathrm{Res}\colon [M;\mathcal{C}] \to [X,\mathbf{r};\mathcal{C}]$.

Theorem 4.1. *The functor* $\mathrm{Res}\colon [M;\mathcal{C}] \to [X,\mathbf{r};\mathcal{C}]$ *is full and faithful.*

Theorem 4.1 should be well-known. In Section 6, we reobtain it as a consequence of Corollary 3.6.

It is clear that it is easier to specify objects in $[X,\mathbf{r};\mathcal{C}]$ than objects in $[M;\mathcal{C}]$. Therefore it is important to have a description of the essential image of the functor Res. This can be done using the coherent presentations of M described in the next section.

5. Monoids and ARS

Let M be a monoid with neutral element e and $\langle X, \mathbf{r}\rangle$ a presentation of M. We will denote by X^* the set of all finite words over the alphabet X. The set X^* will be considered as a free monoid with multiplication given by concatenation of words and neutral element given by the empty word $\varnothing$. Denote by ϕ the canonical epimorphism from X^* to M. Let

$$X^*\mathbf{r}X^* := \left\{ (w_1ww_2, w_1w'w_2) \,\middle|\, w_1, w_2 \in X^*,\ (w,w') \in \mathbf{r}\right\} \subset X^* \times X^*.$$

We will sometimes write the elements of $X^*\mathbf{r}X^*$ in the form w_1rw_2 with $r \in \mathbf{r}$. Let us consider the ARS $\langle X^*, X^*\mathbf{r}X^*\rangle$. It is clear that if $w_1 \twoheadrightarrow w_2$ in $\langle X^*, X^*\mathbf{r}X^*\rangle$, then $\phi(w_1) = \phi(w_2)$.

Proposition 5.1. *Suppose* $\langle X^*, X^*\mathbf{r}X^*\rangle$ *is confluent and* $\twoheadrightarrow$ *is a well-founded preorder. Then* $\phi(u) = \phi(v)$ *if and only if* $\mathrm{Attr}(u) = \mathrm{Attr}(v)$.

Proof: The "if" part is obvious. Suppose $\phi(u) = \phi(v)$. Then there is a sequence of words

$$u = w_0, w_1, \dots, w_k = v$$

such that $(w_j, w_{j-1}) \in X^*\mathbf{r}X^*$ or $(w_{j-1}, w_j) \in X^*\mathbf{r}X^*$ for all $1 \le j \le k$. In other words we have a zigzag $u \rightsquigarrow v$ in $\langle X^*, X^*\mathbf{r}X^*\rangle$. Using the fact that

$\langle X^*, X^*\mathbf{r}X^*\rangle$ is confluent and following the same reasoning as in the proof of Corollary 2.7, we conclude that there are $w \in X^*$ and two paths $u \twoheadrightarrow w$, $v \twoheadrightarrow w$ in $\langle X^*, X^*\mathbf{r}\rangle$. Thus

$$\mathrm{Attr}(u) = \mathrm{Attr}(w) = \mathrm{Attr}(v).$$

■

We will denote by $l(w)$ the length of $w \in X^*$. If $r = (u, v) \in \mathbf{r}$, then we define $s(r) = u$ and $t(r) = v$.

Definition 5.2. A *critical pair* is a pair of elements in $X^*\mathbf{r}X^*$ of one of the forms

i) $(ur, r'v)$ with $us(r) = s(r')v$ in X^*;
ii) $(r, ur'v)$ with $s(r) = us(r')v$.

We say that a critical pair of the first type is *convergent* if there is $w \in X^*$ and there are paths $ut(r) \twoheadrightarrow w$, $t(r')v \twoheadrightarrow w$. A critical pair of the second type is called *convergent* if there is $w \in X^*$ and there are paths $t(r) \twoheadrightarrow w$, $ut(r')v \twoheadrightarrow w$ in $\langle X^*, X^*\mathbf{r}X^*\rangle$.

Given convergent critical pairs $(ur, r'v)$, $(r, ur'v)$ and convergence paths $ut(r) \twoheadrightarrow w$, $t(r')v \twoheadrightarrow w$, $t(r) \twoheadrightarrow w'$, $ut(r')v \twoheadrightarrow w'$, we define the following e.d.s in $\langle X^*, X^*\mathbf{r}X^*\rangle$

$$\begin{tikzcd} s(r')v = us(r) \arrow[r, "ur"] \arrow[d, "r'v"'] & ut(r) \arrow[d, two heads] \\ t(r')v \arrow[r, two heads] & w \end{tikzcd} \qquad \begin{tikzcd} us(r')v = s(r) \arrow[r, "r"] \arrow[d, "ur'v"'] & t(r) \arrow[d, two heads] \\ ut(r')v \arrow[r, two heads] & w'. \end{tikzcd} \tag{6}$$

We will call the e.d.s (6) a *critical e.d.s.* Let $\mathcal{Y}$ be a set of critical e.d.s. We say that $\mathcal{Y}$ is complete if for every critical pair there is at least one corresponding critical e.d. in $\mathcal{Y}$.

Let r, $r' \in \mathbf{r}$ and $w \in W$. We will denote by $\mathcal{N}(r, w, r')$ the e.d.

$$\begin{tikzcd} s(r)ws(r') \arrow[r, "rws(r')"] \arrow[d, "s(r)wr'"'] & t(r)ws(r') \arrow[d, "t(r)wr'"] \\ s(r)wt(r') \arrow[r, "rwt(r')"] & t(r)wt(r'). \end{tikzcd}$$

The e.d. $\mathcal{N}(r, w, r')$ is called a *natural e.d.* We write $\mathcal{N}$ for the set of all natural e.d.s. Given an e.d.

$$E := \begin{array}{ccc} u & \longrightarrow & v \\ \downarrow & & \downarrow\!\!\downarrow \\ w & \twoheadrightarrow & z \end{array}$$

in $\langle X^*, X^*\mathbf{r}X^*\rangle$ and words w_1, $w_2 \in X^*$, we define

$$w_1 E w_2 = \begin{array}{ccc} w_1uw_2 & \longrightarrow & w_1vw_2 \\ \downarrow & & \downarrow\!\!\downarrow \\ w_1ww_2 & \twoheadrightarrow & w_1zw_2 \end{array}, \qquad E^t = \begin{array}{ccc} u & \longrightarrow & w \\ \downarrow & & \downarrow\!\!\downarrow \\ v & \twoheadrightarrow & z \end{array}.$$

Let $\mathcal{Y}$ be a complete set of critical e.d.s. Then the set

$$\mathcal{E} = X^*\mathcal{N}X^* \sqcup X^*\mathcal{N}^tX^* \sqcup X^*\mathcal{Y}X^* \sqcup X^*\mathcal{Y}^tX^* \tag{7}$$

is a complete set of e.d.s for the ARS $\langle X^*, X^*\mathbf{r}X^*\rangle$.

We say that a preorder $\succeq$ on $X^*\mathbf{r}X^*$ is *monomial* if

1) for every ρ_1, $\rho_2 \in X^*\mathbf{r}X^*$ such that $\rho_1 \succ \rho_2$ and for every $w \in X^*$, we have
$$w\rho_1 \succ w\rho_2, \quad \rho_1 w \succ \rho_2 w;$$
2) for every ρ_1, $\rho_2 \in X^*\mathbf{r}X^*$ such that $\rho_1 \sim \rho_2$ and for every $w \in X^*$, we have
$$w\rho_1 \sim w\rho_2, \quad \rho_1 w \sim \rho_2 w.$$

Let us state for the late use

Proposition 5.3. *Let $\langle X, \mathbf{r}\rangle$ be a presentation of a monoid M, $\mathcal{Y}$ a complete set of critical e.d.s and $\mathcal{N}$ the set of all natural e.d.s. Define $\mathcal{E}$ as in (7). Suppose $\succeq$ is a monomial preorder on $X^*\mathbf{r}X^*$ such that all the e.d.s in $\mathcal{Y}$ and $\mathcal{N}$ are decreasing. Then all e.d.s in $\mathcal{E}$ are decreasing as well.*

6. Coherent presentation

Let $\langle X, \mathbf{r}\rangle$ be a presentation of a monoid M. Suppose $(F, \tau) \in [X, \mathbf{r}; \mathcal{C}]$. Denote by $\mathrm{End}(\mathcal{C})$ the category of endofunctors of $\mathcal{C}$. Define the map of graphs $f_{F,\tau}\colon \langle X^*, X^*\mathbf{r}X^*\rangle \to \mathrm{End}(\mathcal{C})$ by

$$f_{F,\tau}(x_1 \dots x_k) = F_{x_1} \dots F_{x_k}, \quad x_i \in X$$
$$f_{F,\tau}(x_1 \dots x_k r y_1 \dots y_l) = F_{x_1} \dots F_{x_k}\tau_r F_{y_1} \dots F_{y_l}, \quad x_i, y_j \in X,\ r \in \mathbf{r}.$$

We will also use the following natural abbreviations

$$F_w := f_{F,\tau}(w),\ w \in X^*;\ \ \tau_\rho := f_{F,\tau}(\rho), \rho \in X^*\mathbf{r}X^*$$

For every path p

$$w_0 \xrightarrow{r_1} w_1 \to \dots \xrightarrow{r_k} w_k$$

in $\langle X^*, X^*\mathbf{r}X^*\rangle$ we denote by τ_p the natural isomorphism $\tau_{r_k} \dots \tau_{r_1}$. If $p\colon w \twoheadrightarrow w$ is of length zero then τ_p is the identity transformation of F_w. Now, given a zigzag z

$$w_0 \overset{p_1}{\twoheadrightarrow} w_1 \overset{p_2}{\twoheadleftarrow} w_2 \twoheadrightarrow \dots w_{k-1} \overset{p_{2k}}{\twoheadleftarrow} w_{2k},$$

where some paths could be empty, we define τ_z to be the product

$$\tau_{p_1}\tau_{p_2}^{-1}\tau_{p_3}\dots\tau_{p_{2k-1}}^{-1}.$$

We say that two zigzags z_1 and z_2 are parallel if they have the same source and target. Given a set $\mathcal{Z}$ of pairs of parallel zigzags, we will define $[X, \mathbf{r}, \mathcal{Z}; \mathcal{C}]$ to be the full subcategory of $[X, \mathbf{r}; \mathcal{C}]$ with objects (F, τ) such that $\tau_p = \tau_q$ for all $(p, q) \in \mathcal{Z}$.

In what follows, if E is an e.d. then $E \in \mathcal{Z}$ means that the two paths, that one can obtain from E, constitute a pair in $\mathcal{Z}$. Thus if $E \in \mathcal{Z}$ and $(F, \tau) \in [X, \mathbf{r}, \mathcal{Z}; \mathcal{C}]$, then $f_{F,\tau}(E)$ is a commutative diagram.

Note that if E is a natural e.d. then $f_{F,\tau}(E)$ is commutative since τ_r are natural transformations. Similarly, if E is an e.d. such that $f_{F,\tau}(E)$ is commutative, then also $f_{F,\tau}(uEv)$ and $f_{F,\tau}(uE^tv)$ are commutative for all u, $v \in X^*$. Thus if $\mathcal{Y}$ is a set of critical e.d.s and $\mathcal{E}$ is defined as in (7) then

$$[X, \mathbf{r}, \mathcal{Y} \sqcup \mathcal{Z}; \mathcal{C}] = [X, \mathbf{r}, \mathcal{E} \sqcup \mathcal{Z}; \mathcal{C}] \tag{8}$$

for any collection $\mathcal{Z}$ of pairs of parallel zigzags.

Example 6.1. To avoid ambiguity we write elements in M^k as $m_1|m_2|\cdots|m_k$. Then we have a presentation $\langle M, \widetilde{\mathbf{r}}\rangle$ of M, with

$$\widetilde{\mathbf{r}} = \Big\{ \big(m_1|m_2, m_1m_2\big) \ :\ m_1, m_2 \in M \Big\} \sqcup \{(e, \varnothing)\}.$$

Every path in the resulting ARS $\langle M^*, M^*\widetilde{\mathbf{r}}M^*\rangle$ ends either at $m \in M^1$, $m \neq e$, or at $\varnothing$. Thus $\langle M^*, M^*\widetilde{\mathbf{r}}M^*\rangle$ is terminating. Every critical e.d. for

$\langle M, \widetilde{\mathbf{r}}\rangle$ has one of the following forms

$$\begin{tikzcd} m_1|m_2|m_3 \arrow[r] \arrow[d] & m_1|m_2m_3 \arrow[d] \\ m_1m_2|m_3 \arrow[r] & m_1m_2m_3 \end{tikzcd} \qquad \begin{tikzcd} m|e \arrow[r, "(m|e{,}m)"] \arrow[d, "m|(e{,}\varnothing)"'] & m \arrow[d, dashed] \\ m \arrow[r, dashed] & m \end{tikzcd} \qquad \begin{tikzcd} e|m \arrow[r, "(e|m{,}m)"] \arrow[d, "(e{,}\varnothing)|m"'] & m \arrow[d, dashed] \\ m \arrow[r, dashed] & m. \end{tikzcd}$$

Let us denote by $\mathcal{Y}$ the set of all critical e.d.s. Then, by comparing definitions, we get $[M, \widetilde{\mathbf{r}}, \mathcal{Y}; \mathcal{C}] = [M; \mathcal{C}]$. Let $\mathcal{E}$ be defined by (7). Then, using (8), we get

$$[M, \widetilde{\mathbf{r}}, \mathcal{E}; \mathcal{C}] = [M; \mathcal{C}].$$

Let $(F, \lambda) \in [M, \widetilde{\mathbf{r}}, \mathcal{E}; \mathcal{C}]$. By Example 3.7, it follows that for any two zigzags $z,z' \colon m_1|\cdots|m_k \to m_1'|\cdots|m_l'$ in $\langle M^*, M^*\widetilde{\mathbf{r}}M^*\rangle$ one has $f_{F,\lambda}(z) = f_{F,\lambda}(z')$.

Proposition 6.2. *Let $\langle X, \mathbf{r}\rangle$ be a presentation of a monoid M. For any set $\mathcal{Z}$ of pairs of parallel zigzags in $\langle X^*, X^*\mathbf{r}X^*\rangle$ the restriction functor* Res *defined in Section 4 factors via the embedding $[X, \mathbf{r}, \mathcal{Z}; \mathcal{C}] \to [X, \mathbf{r}; \mathcal{C}]$. We will denote the resulting functor $[M; \mathcal{C}] \to [X, \mathbf{r}, \mathcal{Z}; \mathcal{C}]$ by* $\mathrm{Res}_{\mathcal{P}}$.

Proof: Let $(F, \lambda) \in [M; \mathcal{C}]$ and $\mathrm{Res}(F, \lambda) = (F, \tau) \in [X, \mathbf{r}, \mathcal{E}; \mathcal{C}]$. By the definition of τ, every natural isomorphism τ_r, $r \in \mathbf{r}$, is a value of $f_{F,\lambda}$ on a suitable zigzag in the defined above ARS $\langle M^*, M^*\widetilde{\mathbf{r}}M^*\rangle$. Now let $(p, q) \in \mathcal{Z}$, with p, $q \colon u \rightsquigarrow w$. We have to prove that $\tau_p = \tau_q$. Since every τ_r can be replaced by λ_z for a suitable zigzag z in $\langle M^*, M^*\widetilde{\mathbf{r}}M^*\rangle$, we see that there are zigzags z', $z'' \colon u \rightsquigarrow w$ in $\langle M^*, M^*\widetilde{\mathbf{r}}M^*\rangle$ such that $\tau_p = \lambda_{z'}$ and $\tau_q = \lambda_{z''}$. But by Example 6.1, we have $\lambda_{z'} = \lambda_{z''}$. Thus also $\tau_p = \tau_q$. ∎

Proof of Theorem 4.1: The functor Res is faithful because every morphism $\rho \colon (F, \lambda) \to (G, \mu)$ in $[M; \mathcal{C}]$ can be uniquely reconstructed from its image $\nu = \mathrm{Res}(\rho)$ by use of the diagrams

$$\begin{tikzcd} F_{x_1} \dots F_{x_k} \arrow[r, "\lambda_{x_1,\dots,x_k}"] \arrow[d, "\nu_{x_1}\dots\nu_{x_k}"'] & F_{x_1\dots x_k} \arrow[d, "\rho_{x_1\dots x_k}"] \\ G_{x_1} \dots G_{x_k} \arrow[r, "\mu_{x_1,\dots,x_k}"] & G_{x_1\dots x_k}. \end{tikzcd} \tag{9}$$

Now we show that the functor Res is full. Denote the images of (F, λ) and of (G, μ) under Res by (F, τ) and (G, σ), respectively. Take a morphism

$\nu\colon (F,\tau) \to (G,\sigma)$ in $[X,\mathbf{r};\mathcal{C}]$. Let $m \in M$. Then we can write m as a product of elements in X, say $m = x_1 \dots x_k$. We define ρ_m by using the diagram (9)

$$\rho_m = \mu_{x_1\dots x_k} \circ \nu_{x_1} \dots \nu_{x_k} \circ \lambda^{-1}_{x_1\dots x_k}.$$

We have to check that the natural transformation ρ_m is well-defined. Suppose $y_1 \dots y_l = m$ with $y_j \in X$. As $\langle X, \mathbf{r}\rangle$ is a presentation of M, there is a zigzag $z\colon x_1 \dots x_k \to y_1 \dots y_l$ in $\langle X^*, X^*\mathbf{r}X^*\rangle$. Since ν is a morphism in $[X,\mathbf{r};\mathcal{C}]$ we get that the diagram

$$\begin{tikzcd}
F_{x_1} \dots F_{x_k} \arrow[r, "\tau_z"] \arrow[d, "\nu_{x_1}\dots\nu_{x_k}"'] & F_{y_1} \dots F_{y_l} \arrow[d, "\nu_{y_1}\dots\nu_{y_l}"] \\
G_{x_1} \dots G_{x_k} \arrow[r, "\sigma_z"] & G_{y_1} \dots G_{y_l}
\end{tikzcd}$$

commutes. As in the proof of Proposition 6.2, we can find a zigzag z' in $\langle M^*, M^*\widetilde{\mathbf{r}}M^*\rangle$ such that $\lambda_{z'} = \tau_z$ and $\mu_{z'} = \sigma_z$. Now, consider the zigzag

$$z''\colon x_1|\dots|x_k \twoheadrightarrow m \twoheadleftarrow y_1|\dots|y_l$$

in $\langle M^*, M^*\widetilde{\mathbf{r}}M^*\rangle$. Since z' and z'' have the same source and target, we get as in the proof of Proposition 6.2, that $\lambda_{z'} = \lambda_{z''}$ and $\mu_{z'} = \mu_{z''}$. Thus we have the commutative diagram

$$\begin{tikzcd}
F_{y_1} \dots F_{y_l} \arrow[r, "\lambda_{z''}"] \arrow[d, "\nu_{y_1}\dots\nu_{y_k}"'] \arrow[rr, bend left, "\lambda_{y_1,\dots,y_l}"] & F_{x_1} \dots F_{x_k} \arrow[r, "\lambda_{x_1,\dots,x_k}"] \arrow[d, "\nu_{x_1}\dots\nu_{x_k}"'] & F_{x_1\dots x_k} \arrow[d, "\rho_m"] \\
G_{y_1} \dots G_{y_l} \arrow[r, "\mu_{z''}"] \arrow[rr, bend right, "\mu_{y_1,\dots,y_l}"'] & G_{x_1} \dots G_{x_k} \arrow[r, "\mu_{x_1,\dots,x_k}"] & G_{x_1\dots x_k},
\end{tikzcd}$$

which shows that ρ_m does not depend on the choice of the presentation of m in $\langle X, \mathbf{r}\rangle$.

Now we have to check that $(\rho_m)_{m\in M}$ is a well-defined morphism in $[M;\mathcal{C}]$,i.e. that for every m_1, $m_2 \in M$ the diagram

$$\begin{tikzcd}
F_{m_1}F_{m_2} \arrow[r, "\lambda_{m_1,m_2}"] \arrow[d, "\rho_{m_1}\rho_{m_2}"'] & F_{m_1m_2} \arrow[d, "\rho_{m_1m_2}"] \\
G_{m_1}G_{m_2} \arrow[r, "\mu_{m_1,m_2}"] & G_{m_1m_2}.
\end{tikzcd}$$

commutes. Suppose $m_1 = x_1 \dots x_k$ and $m_2 = y_1 \dots y_l$ with x_i, $y_j \in X$. Then since we can use the presentation $x_1 \dots x_k y_1 \dots y_l$ of m_1m_2 to define $\rho_{m_1m_2}$, we get that in the diagram

$$\begin{tikzcd}
F_{x_1}\dots F_{x_k}F_{y_1}\dots F_{y_l} \arrow[r, "\lambda_{x_1,\dots,x_k}\lambda_{y_1,\dots,y_l}"] \arrow[d, "\nu_{x_1}\dots\nu_{x_k}\nu_{y_1}\dots\nu_{y_l}"'] \arrow[rr, bend left, "\lambda_{x_1,\dots,x_k,y_1,\dots,y_l}"] & F_{m_1}F_{m_2} \arrow[r, "\lambda_{m_1,m_2}"] \arrow[d, "\rho_{m_1}\rho_{m_2}"'] & F_{m_1m_2} \arrow[d, "\rho_{m_1m_2}"] \\
G_{x_1}\dots G_{x_k}G_{y_1}\dots G_{y_l} \arrow[r, "\mu_{x_1,\dots,x_k}\mu_{y_1,\dots,y_l}"] \arrow[rr, bend right, "\mu_{x_1,\dots,x_k,y_1,\dots,y_l}"'] & G_{m_1}G_{m_2} \arrow[r, "\mu_{m_1,m_2}"] & G_{m_1m_2},
\end{tikzcd}$$

the triangles commute by the definition of the λ's, the left rectangle commutes by the definition of ρ_{m_1} and ρ_{m_2}, and the external rectangle commutes by the definition of $\rho_{m_1m_2}$. Since all λ's are isomorphisms, we get that also the right rectangle commutes. This shows that ρ is a well-defined morphism from (F,λ) to (G,μ) in $[M;\mathcal{C}]$. ∎

We say that a set $\mathcal{Z}$ of pairs of parallel zigzags in $\langle X^*, X^*\mathbf{r}X^*\rangle$ defines a *coherent presentation* of M, if the functor $\mathrm{Res}_{\mathcal{Z}}\colon [M;\mathcal{C}] \to [X,\mathbf{r},\mathcal{Z};\mathcal{C}]$ is an equivalence of categories. The main result of this section is the following theorem.

Theorem 6.3. *Let $\langle X, \mathbf{r}\rangle$ be a presentation of a monoid M with neutral element e. Suppose that*

1) the transitive closure $\twoheadrightarrow$ of $X^*\mathbf{r}X^*$ is a well-founded preorder on X^*;
2) there exists a well-founded monomial preorder $\succeq$ on $X^*\mathbf{r}X^*$ such that all natural e.d.s are decreasing;
3) there is a complete set $\mathcal{Y}$ of critical e.d.s that are decreasing with respect to $\succeq$.

Denote by $\mathcal{L}$ the collection of all pairs $(p, \varnothing_b)$, where $b \in \mathrm{Attr}(A)$, $\varnothing_b$ is the empty path at b, and $p\colon b \twoheadrightarrow b$. Then $\mathrm{Res}_{\mathcal{Y}\sqcup\mathcal{L}}$ is an equivalence of categories.

Proof: Let $\mathcal{E}$ be the collection of e.d.s defined by (7). In view of (8), it is enough to show that

$$\mathrm{Res}_{\mathcal{E}\sqcup\mathcal{L}}\colon [M;\mathcal{C}] \to [X, \mathbf{r}, \mathcal{E}\sqcup\mathcal{L}; \mathcal{C}]$$

is an equivalence of categories. Since $\mathrm{Res}_{\mathcal{E}\sqcup\mathcal{L}}$ is fully faithful, it is enough to check that it is a dense functor. Let $(F,\tau) \in [X, \mathbf{r}, \mathcal{E}\sqcup\mathcal{L}; \mathcal{C}]$. Then

$$f_{F,\tau}\colon \langle X^*, X^*\mathbf{r}X^*\rangle \to \mathrm{End}(\mathcal{C})$$

satisfies conditions of Corollary 3.6. In particular, for any two zigzags z, $z'\colon u \rightsquigarrow v$ in $\langle X^*, X^*\mathbf{r}X^*\rangle$ we have $\tau_z = \tau_{z'}$.

Now, for every m in M we choose a presentation $x_1 \dots x_k$ of m in $\langle X, \mathbf{r}\rangle$. We will assume that if $m \in X$, then its chosen presentation is m itself. We define $G_m = F_{x_1} \dots F_{x_k}$ using the above chosen presentation. Now, let $m' \in M$ and $y_1 \dots y_l$ be its chosen presentation. Then $x_1 \dots x_k y_1 \dots y_l$ is a presentation of mm', which, in general, is different of the chosen presentation, say, $z_1 \dots z_n$ of mm'. Since $\langle X, \mathbf{r}\rangle$ is a presentation of M, we get that there is zigzag $\zeta\colon x_1 \dots x_k y_1 \dots y_l \rightsquigarrow z_1 \dots z_n$ in $\langle X^*, X^*\mathbf{r}X^*\rangle$. We define $\lambda_{m,m'} = \tau_\zeta$. As we already mentioned the resulting natural isomorphism is independent of the choice of ζ. Now, if $m'' \in M$, the compositions

$$\begin{aligned} G_m G_{m'} G_{m''} &\xrightarrow{G_m \lambda_{m',m''}} G_m G_{m'm''} \xrightarrow{\lambda_{m,m'm''}} G_{mm'm''} \\ G_m G_{m'} G_{m''} &\xrightarrow{\lambda_{m,m'} G_{m''}} G_{mm'} G_{m''} \xrightarrow{\lambda_{mm',m''}} G_{mm'm''} \end{aligned} \tag{10}$$

are equal to $\tau_{\zeta'}$ and $\tau_{\zeta''}$ for suitable parallel zigzags ζ', ζ'' in $\langle X*, X^*\mathbf{r}X^*\rangle$. Thus the natural transformations (10) are equal. Hence we get that (G,λ) is an object of $[M;\mathcal{C}]$.

We have to check that $\mathrm{Res}(G,\lambda) = (F,\tau)$. Since for every $x \in X$, we have $G_x = F_x$, we get $\mathrm{Res}(G,\lambda) = (F,\sigma)$. Thus, we have only to check that $\sigma = \tau$. For every $r = (u,v) \in \mathbf{r}$, we defined σ_r as λ_z for a suitable zigzag z in $\langle M^*, M^*\widetilde{\mathbf{r}}M^*\rangle$. Since every $\lambda_{m,m'}$ is of the form $\tau_{z'}$ for a zigzag z' in $\langle X^*, X^*\mathbf{r}X^*\rangle$, we get that there is a zigzag $\zeta\colon u \rightsquigarrow v$ in $\langle X^*, X^*\mathbf{r}X^*\rangle$ such that $\lambda_z = \tau_\zeta$. But now $\tau_\zeta = \tau_r$. This shows that $\sigma_r = \tau_r$. ■

Example 6.4. Suppose $\langle X^*, X^*\mathbf{r}X^*\rangle$ is terminating and $\mathbf{r}$ satisfies the Knuth-Bendix condition. Let $\mathcal{Y}$ be a complete set of critical e.d.s. Then, in view of Example 3.7, we get that $\mathrm{Res}_{\mathcal{Y}}$ is an equivalence of categories. Thus one of the ways to find a coherent presentation of a monoid M is to perform the Knuth-Bendix completion procedure on a given presentation. Unfortunately, this process can either not to finish or to lead to an enormous coherent presentation which is not useful for practical purposes.

7. Coherent presentation for the 0-Hecke monoid

The 0-Hecke monoid $\mathcal{H}(\Sigma_{n+1})$ is defined by the following presentation

$$X = \{T_1, \dots, T_n\}$$

$$\mathbf{r}' = \{a_i \,|\, 1 \le i \le n\} \sqcup \{\, b_{i+1} \,|\, 1 \le i \le n-1\} \sqcup \left\{\, c_{ji} \,\middle|\, 1 \le i \le j-2 \le n-2 \right\},$$

where

$$a_i = (T_iT_i, T_i), \quad b_{i+1} = (T_{i+1}T_iT_{i+1}, T_iT_{i+1}T_i), \quad c_{ji} = (T_jT_i, T_iT_j).$$

In this section we find a coherent presentation of $\mathcal{H}(\Sigma_{n+1})$ that extends the above presentation of $\mathcal{H}(\Sigma_{n+1})$. It is easy to see that $\langle X^*, X^*\mathbf{r}'X^*\rangle$ is not (locally) confluent. We denote by $\mathbf{r}''$ the set $\mathbf{r}' \sqcup \left\{\, c_{ij} \,\middle|\, i \le j-2 \right\}$, where $c_{ij} = (T_iT_j, T_jT_i)$. It is well-know that $\langle X^*, X^*\mathbf{r}''X^*\rangle$ is confluent.

In the diagrams below we will write i in place of T_i. We will also use $'$ for decrement and $\widehat{\ }$ for increment, thus for an integer k

$$k' = k-1, \quad \widehat{k} = k+1.$$

We will show that the set $\mathcal{P}$ of pairs of paths in $\langle X^*, X^*\mathbf{r}''X^*\rangle$ defined below gives a coherent presentation of $\mathcal{H}(\Sigma_{n+1})$, i.e. that

$$\mathrm{Res}_{\mathcal{P}} \colon [\mathcal{H}(\Sigma_{n+1}); \mathcal{C}] \to [X, \mathbf{r}'', \mathcal{P}; \mathcal{C}]$$

is an equivalence of categories. The set $\mathcal{P}$ consists of the pairs of paths that one obtains from the following diagrams

$$\begin{tikzcd} st \arrow[r, "c_{st}"] \arrow[rr, bend right, dashed] & ts \arrow[r, "c_{ts}"] & st \end{tikzcd} \tag{11}$$

$$\begin{tikzcd} kkk \arrow[r, "ka_k"] \arrow[d, "a_kk"'] & kk \arrow[d, "a_k"] \\ kk \arrow[r, "a_k"] & k \end{tikzcd} \tag{12}$$

$$\begin{tikzcd}
kk'kk \arrow[rr, "kk'a_k"] \arrow[d, "b_k k"'] & & kk'k \arrow[d, "b_k"] \\
k'kk'k \arrow[r, "k'b_k"] & k'k'kk' \arrow[r, "a_{k'}kk'"] & k'kk'
\end{tikzcd} \tag{13}$$

$$\begin{tikzcd}
kkk'k \arrow[r, "kb_k"] \arrow[dd, "a_k k'k"'] & kk'kk' \arrow[d, "b_k k'"] \\
& k'kk'k' \arrow[d, "k'ka_{k'}"] \\
kk'k \arrow[r, "b_k"] & k'kk'
\end{tikzcd} \tag{14}$$

$$\begin{tikzcd}
stt \arrow[rr, "sa_t"] \arrow[d, "c_{st}t"'] & & st \arrow[d, "c_{st}"] \\
tst \arrow[r, "tc_{st}"] & tts \arrow[r, "a_t"] & ts
\end{tikzcd} \tag{15}$$

$$\begin{tikzcd}
kk'kk'k \arrow[r, "kk'b_k"] \arrow[d, "b_k k'k"'] & kk'k'kk' \arrow[r, "ka_{k'}kk'"] & kk'kk' \arrow[d, "b_k k'"] \\
k'kk'k'k \arrow[d, "k'ka_{k'}k"'] & & k'kk'k' \arrow[d, "k'ka_{k'}"] \\
k'kk'k \arrow[r, "k'b_k"] & k'k'kk' \arrow[r, "a_{k'}kk'"] & k'kk'
\end{tikzcd} \tag{16}$$

$$\begin{tikzcd}
tss's \arrow[r, "tb_s"] \arrow[d, "c_{ts}s's"'] & ts'ss' \arrow[r, "c_{ts'}ss'"] & s'tss' \arrow[d, "s'c_{ts}s'"] \\
sts's \arrow[d, "sc_{ts'}s"'] & & s'sts' \arrow[d, "s'sc_{ts'}"] \\
ss'ts \arrow[r, "ss'c_{ts}"] & ss'st \arrow[r, "b_s t"] & s'ss't
\end{tikzcd} \tag{17}$$

$$\begin{tikzcd}
kji \arrow[r, "kc_{ji}"] \arrow[d, "c_{kj}i"'] & kij \arrow[r, "c_{ki}j"] & ikj \arrow[d, "ic_{kj}"] \\
jki \arrow[r, "jc_{ki}"] & jik \arrow[r, "c_{ji}k"] & ijk
\end{tikzcd} \tag{18}$$

$$\begin{tikzcd}
\widehat{k}kk'\widehat{k}k\widehat{k} \arrow[r, "\widehat{k}kk'b_{\widehat{k}}"] \arrow[d, "\widehat{k}kc_{k'\widehat{k}}k\widehat{k}"'] & \widehat{k}kk'k\widehat{k}k \arrow[r, "\widehat{k}b_k\widehat{k}k"] & \widehat{k}k'kk'\widehat{k}k \arrow[r, "c_{\widehat{k}k'}kk'\widehat{k}k"] & k'\widehat{k}kk'\widehat{k}k \arrow[d, "k'\widehat{k}kc_{k'\widehat{k}}k"] \\
\widehat{k}k\widehat{k}k'k\widehat{k} \arrow[d, "b_{\widehat{k}}k'k\widehat{k}"'] & & & k'\widehat{k}k\widehat{k}k'k \arrow[d, "k'b_{\widehat{k}}k'k"] \\
k\widehat{k}kk'k\widehat{k} \arrow[d, "k\widehat{k}b_k\widehat{k}"'] & & & k'k\widehat{k}kk'k \arrow[d, "k'k\widehat{k}b_k"] \\
k\widehat{k}k'kk'\widehat{k} \arrow[d, "kc_{\widehat{k}k'}kk'\widehat{k}"'] & & & k'k\widehat{k}k'kk' \arrow[d, "k'kc_{\widehat{k}k'}kk'"] \\
kk'\widehat{k}kk'\widehat{k} \arrow[r, "kk'\widehat{k}kc_{k'\widehat{k}}"] & kk'\widehat{k}k\widehat{k}k' \arrow[r, "kk'b_{\widehat{k}}k'"] & kk'k\widehat{k}kk' \arrow[r, "b_k\widehat{k}kk'"] & k'kk'\widehat{k}kk'.
\end{tikzcd} \tag{19}$$

The diagrams (18) and (19) were called *Tits-Zamolodchikov* 3-*cells* in [2].

We will say that a path p in $\langle X^*, X^*\mathbf{r}''X^*\rangle$ is a *c-path* if all the steps in p are of the form $X^*c_{st}X^*$ for some s, t such that $|s-t| \geq 2$.

Proposition 7.1. *Suppose* $(F,\tau) \in [X, \mathbf{r}'', \mathcal{P}; \mathcal{C}]$. *If* p, $p'\colon u \twoheadrightarrow v$ *are two c-paths in* $\langle X^*, X^*\mathbf{r}''X^*\rangle$, *then* $\tau_p = \tau_{p'}$.

Proof: Let $\mathbf{c}$ be the subset $\mathbf{r}''$ consisting of all c_{ji} with $j \geq i+2$. Using (11), we see that $\tau_{c_{ij}} = \tau_{c_{ji}}^{-1}$ for all $j \geq i+2$. Therefore, there are zigzags z, z' in $\langle X^*, X^*\mathbf{c}X^*\rangle$ such that $\tau_p = \tau_z$ and $\tau_{p'} = \tau_{z'}$.

The ARS $\langle X^*, X^*\mathbf{c}X^*\rangle$ is terminating and locally confluent, with the only critical e.d.s given by (18) with $k \geq j+2$, $j \geq i+2$. Therefore, by Example 3.7, we get that $\tau_z = \tau_{z'}$. ∎

The easiest way to prove our result would be to have a complete set of decreasing critical e.d.s for $\langle X^*, X^*\mathbf{r}''X^*\rangle$. The author does not know at the moment if this is possible. Instead, we will proceed as follows

1) first we replace $\mathbf{r}''$ with a bigger set $\mathbf{r}$ of generating rules;
2) then we define a preorder on $X^*\mathbf{r}X^*$ so that there is a complete set of decreasing critical e.d.s for $\langle X, \mathbf{r}\rangle$ and all natural e.d.s are decreasing;
3) further we show that all the chosen critical e.d.s can be subdivided into diagrams in $\mathcal{P}$;
4) finally we show that for all $(F,\tau) \in [X, \mathbf{r}, \mathcal{P}; \mathcal{C}]$ and any attractor loop $p\colon b \twoheadrightarrow b$ one has $\tau_p = 1_{F_b}$.

Let

$$\mathbf{r} = \{\, a_i \,|\, 1 \le i \le n \} \sqcup \left\{\, b_{ji} \,\middle|\, 1 \le i < j \le n \right\} \sqcup \left\{\, c_{st} \,\middle|\, |s-t| \ge 2 \right\}.$$

Here a_i, c_{st} are defined as before and

$$b_{ji} = (j \dots ij,\, j'jj' \dots i),$$

where $j \dots i$ denotes the interval of the arithmetic progression with the step -1. Note that in particular $b_{jj'} = b_j$, and therefore $\mathbf{r}'' \subset \mathbf{r}$. We define the preorder $\succeq$ on $X^*\mathbf{r}X^*$ as follows. Given r, r', we write $r \gg r'$ to indicate that $urv \succ u'r'v'$ for all u, v, u', $v' \in X^*$. Moreover, for example, $c \gg a$ will indicate that $c_{st} \gg a_i$ for all s, t, and i. We will write $\#_j u$ for the number of occurrences of j in $u \in X^*$. The preorder $\succeq$ is defined by the rules

1) for $i \le j''$, $b_{ji} \sim b_{jj'}j'' \dots i$, i.e. to compare two elements in $X^*\mathbf{r}X^*$, we first replace, if necessary, all b_{ji} by $b_{jj'}j'' \dots i$, and then proceed with the rules below;
2) we order the relations in $\mathbf{r}$ by
   i) $b \gg c \gg a$;
   ii) $b_{i+1,i} \gg b_{i,i-1}$;
   iii) $c_{ij} \gg c_{ts}$ for all $j \ge i+2$ and $t \ge s+2$;
   iv) for $k \ge j+2$ and $t \ge s+2$, $c_{kj} \gg c_{ts}$, if $k > t$ or $k = t$ and $j > s$;
   v) $\left\{\, c_{ij} \,\middle|\, j \ge i+2 \right\}$ are ordered arbitrary;
3) if words contain the same generating rule $r \in \mathbf{r}$ we proceed as follows:
   i) if $r = a_i$, then we just compare lengths of the words: the longer word is greater;
   ii) if $r = c_{ji}$ with $j \ge i+2$, then $uc_{ji}v \succ u'c_{ji}v'$ if
      a) $\sum\limits_{k \ge j} \#_k u > \sum\limits_{k \ge j} \#_k u'$;
      b) $\sum\limits_{k \ge j} \#_k u = \sum\limits_{k \ge j} \#_k u'$ and $\sum\limits_{k \le i} \#_k v > \sum\limits_{k \le i} \#_k v'$.
   iii) if $r = b_{kk'}$, then $ub_{kk'}v \succ u'b_{kk'}v'$ if

$$(\#_k uv, \#_{k'} uv, \dots, \#_1 uv) > (\#_k u'v', \#_{k'} u'v', \dots, \#_1 u'v')$$

   with respect to the lexicographical order on $\mathbb{N}^k$.

If for two words w, $w' \in X^*\mathbf{r}X^*$ we cannot conclude either $w \succ w'$ or $w' \succ w$, according to the above rules, then $w \sim w'$. It is obvious that the preorder $\succeq$ is monomial.

Proposition 7.2. *All the natural e.d.s of the ARS $\langle X^*, X^*\mathbf{r}X^*\rangle$ are decreasing with respect to the preorder $\succeq$.*

Proof: Let $r, r' \in \mathbf{r}$ and $w \in X^*$. Then $\mathcal{N}(r, w, r')$ has the top arrow $rws(r')$, the bottom arrow $rwt(r')$, the left arrow $s(r)wr'$, and the right arrow $t(r)wr'$. If $rws(r') \succeq rwt(r')$ and $s(r)wr' \succeq t(r)wr'$ then the e.d. $\mathcal{N}(r, w, r')$ is obviously decreasing.

Thus, we have to find triples (r, w, r') for which $rwt(r') \succ rws(r')$ or $t(r)wr' \succ s(r)wr'$.

First we identify triples (r, w, r') such that $rwt(r') \succ rws(r')$. If r is of type a, then $rwt(r') \succ rws(r')$ if and only if the length of $t(r')$ is greater than the length of $s(r')$. But there is no rule $r' \in \mathbf{r}$ with such property. Thus r cannot be of type a.

Now, suppose r if of type c. We have two cases: either $r = c_{ij}$ or $r = c_{ji}$, with $j \geq i + 2$. If $r = c_{ij}$ then $rws(r') \sim rwt(r')$. If $r = c_{ji}$ then $rwt(r') \succ rws(r')$ if and only if

$$\sum_{k\leq i} \#_k t(r') > \sum_{k\leq i} \#_k s(r'). \tag{20}$$

It is clear that r' can be of type a or c as for such rules we have $\#_k t(r') = \#_k s(r')$ for arbitrary k. If $r' = b_{ml}$ with $m > l$, then

$$\#_k t(r') = \begin{cases} \#_k s(r') - 1, & k = m \\ \#_k s(r') + 1, & j = m - 1 \\ \#_k s(r'), & \text{otherwise.} \end{cases} \tag{21}$$

Thus (20) holds only if $i = m-1$, i.e. $r' = b_{i+1,l}$ with $i+1 > l$. As $b \gg c$ we get that $s(r)wr' \succ rwt(r')$. Moreover, $s(r)wr' \sim t(r)wr'$ as $\#_k s(c_{ji}) = \#_k t(c_{ji})$ for all k. Thus $\mathcal{N}(r, w, r')$ is decreasing in this case.

Let us consider the case $r = b_{ji}$ for some $j > i$. Then we have $rwt(r') > rws(r')$ if and only if

$$\big(\#_j t(r'), \dots \#_1 t(r')\big) > (\#_j s(r'), \dots, \#_1 s(r')).$$

This is possible only in the case $r' = b_{j+1,k}$ for some $k < j+1$. As $b_{j+1,j} \gg b_{jj'}$, we get that $s(r)wr' \succ rwt(r')$. Moreover

$$\#_{j+1} s(r) = \#_{j+1}(j \dots ij) = 0 = \#_{j+1}(j'j \dots i) = \#_{j+1} t(r)$$
$$\#_j s(r) = \#_j(j \dots ij) = 2 > 1 = \#_j(j'j \dots i) = \#_j t(r).$$

Thus $s(r)wr' \succ t(r)wr'$ and we get that $\mathcal{N}(r, w, r')$ is decreasing.

Now we will analyze the triples (r, w, r') such that $t(r)wr' \succ s(r)wr'$. It is impossible to have r' of type a as the length of $t(r)w$ never exceeds the length of $s(r)w$. Suppose r' is of type c. Then $r' = c_{ij}$ or $r' = c_{ji}$ for some j, i such that $j > i + 1$. If $r' = c_{ij}$ then $t(r)wr' \sim s(r)wr'$. Thus we have only to consider the case $r' = c_{ji}$. In this situation $t(r)wr' \succ s(r)wr'$ if and only if

$$\sum_{k \geq j} \#_k t(r) > \sum_{k \geq j} \#_k s(r). \tag{22}$$

As for rules r'' of type a or c, the inequality $\#_k t(r'') \leq \#_k s(r'')$ holds for every k, we get that r cannot be of type a or c. If $r = b_{ml}$ for some $m > l$, then the sums in (22) are equal unless $j = m$, in which case the left hand side of (22) is less than the right hand side of (22). This shows that r' cannot have type c.

Suppose $r' = b_{ji}$ for some $j > i$. Then $t(r)wr' \succ s(r)wr'$ if and only if

$$\big(\#_j t(r), \dots, \#_1 t(r)\big) > \big(\#_j s(r), \dots, \#_1 s(r)\big)$$

with respect to the lexicographical order on $\mathbb{N}^j$. This is possible only in the case $r = b_{j+1,l}$ for some $l < j + 1$. As $b_{j+1,j} \gg b_{jj'}$, we get that $rws(r') \succ t(r)wr'$. Moreover, as $\#_{j+1} s(r') = \#_{j+1} t(r')$ and $\#_j s(r') > \#_j t(r')$, we have $rws(r') \succ rwt(r')$. This shows that $\mathcal{N}(r, w, r')$ is decreasing in this case. ■

Now we construct a complete set of decreasing critical e.d.s for the ARS $\langle X^*, X^* \mathbf{r} X^* \rangle$.

Note that there are not any rules r, $r' \in \mathbf{r}$ such that $s(r')$ is a proper subword of $s(r)$. Therefore all the cricital pairs for $\mathbf{r}$ are of the form (ru, vr') for some r, $r' \in \mathbf{r}$ and non-empty u, $v \in X^*$.

If $r = c_{ij}$ with $j \geq i + 2$, then the following commutative diagram is decreasing and resolving for this critical pair

$$\begin{tikzcd}
iju \arrow[r, "c_{ij}u"] \arrow[dd, "vr'"'] & jiu \arrow[d, "c_{ji}u"] \\
 & iju \arrow[d, "vr'"] \\
t(r) \arrow[r, dashed] & t(r).
\end{tikzcd}$$

In fact, $c_{ij} \gg c_{ji}$ implies that $c_{ij}u \succ c_{ji}u$. As $\succeq$ is monomial, we get that $vr' \sim vr'$. Similarly, if $r' = c_{ij}$ with $j \geq i+2$, the diagram

$$\begin{tikzcd}
uij \arrow[r, "uc_{ij}"] \arrow[dd, "ru"'] & uji \arrow[d, "uc_{ji}"] \\
 & uij \arrow[d, "ru"] \\
t(r) \arrow[r, dashed] & t(r).
\end{tikzcd} \tag{23}$$

is decreasing and resolves the critical pair (ru, vc_{ij}).

It is left to consider the critical pairs with r and r' equal to one of a_k, b_{ji}, $j \geq i+1$, c_{ts}, $t \geq s+2$. In the diagrams below we will abbreviate c-paths by $\overset{c^*}{\twoheadrightarrow}$. This does not create any ambiguity in view of Proposition 7.1.

We first consider all the critical pairs involving at least one a_k. The diagram (12) is a decreasing critical e.d. for the critical pair (ka_k, a_kk). For the critical pair $(a_kk' \ldots jk, kb_{kj})$, we consider the diagram

$$E(a_k, b_{kj}) = \begin{tikzcd}
kk\ldots jk \arrow[r, "kb_{kj}"] \arrow[dd, "a_kk'\ldots jk"'] & kk'k\ldots j \arrow[d, "b'_{k\ldots j}"] \\
 & k'kk'k'\ldots j \arrow[d, "k'ka_{k'}k''\ldots j"] \\
k\ldots jk \arrow[r, "b_{kj}"] & k'k\ldots j.
\end{tikzcd} \tag{24}$$

The diagram (24) is decreasing since kb_{kj} is greater than all the other arrows. Note, that for $j = k'$ the word $k'' \ldots j$ is empty and we recover (14).

For the critical pair $(b_{kj}k, k\ldots ja_k)$, we consider the diagram

$$E(b_{kj}, a_k) = \begin{tikzcd}
k\ldots jkk \arrow[rr, "k\ldots ja_k"] \arrow[d, "b_{kj}k"'] & & k\ldots jk \arrow[d, "b_{kj}"] \\
k'k\ldots jk \arrow[r, "k'b_{kj}"] & k'k'k\ldots j \arrow[r, "a_{k'}k\ldots j"] & k'k\ldots j.
\end{tikzcd} \tag{25}$$

The diagram (25) is decreasing since $b_{kj}k$ is greater then any other arrow. In the case $j = k'$, we recover the diagram (13).

All the critical pairs involving two b-rules are of the form $(b_{kj}k' \dots i, k \dots j b_{ki})$. For $j = k'$, we get

$$E(b_{kk'},b_{ki})= \begin{tikzcd}
kk'k\dots ik \arrow[rrr, "kk'b_{ki}"] \arrow[ddd, "b_{kk'}k'\dots ik"'] & & & kk'k'k\dots i \arrow[d, "ka_{k'}k\dots i"] \\
 & & & kk'k\dots i \arrow[d, "b_{kk'}k'\dots i"] \\
 & & & k'kk'k'\dots i \arrow[d, "k'ka_{k'}k''\dots i"] \\
k'kk'k'\dots ik \arrow[r, bend left, "k'ka_{k'}k''\dots ik"] & k'k\dots ik \arrow[r, "k'b_{ki}"] & k'k'k\dots i \arrow[r, "a_{k'}k\dots i"] & k'k\dots i.
\end{tikzcd} \tag{26}$$

The diagram (26) is decreasing since $b_{kk'}k' \dots ik \sim kk'b_{ki}$ dominates all the other arrows. Note that for $i = k'$ the subword $k'' \dots i$ is empty and we recover the diagram (16).

The diagram corresponding to the case $j \leq k''$ is not drawn in the rectangular form for the typographical reasons. We also write S for the word $k''' \dots jk'' \dots i$.

$$E(b_{kj},b_{ki})= \begin{tikzcd}
 & k\dots jk\dots ik \arrow[dl, "b_{kj}k'\dots ik"'] \arrow[dr, "k\dots jb_{ki}"] & \\
k'k\dots jk'\dots ik \arrow[d, "c^*"'] & & k\dots jk'k\dots i \arrow[d, "c^*"] \\
k'kk'k''k'kS \arrow[d, "k'kb_{k'k''}kS"'] & & kk'k''k'kk'S \arrow[d, "kb_{k'k''}kk'S"] \\
k'kk''k'k''kS \arrow[d, "c^*"'] & & kk''k'k''kk'S \arrow[d, "c^*"] \\
k'k''kk'kk''S \arrow[d, "k'k''b_{kk'}k''S"'] & & k''kk'kk''k'S \arrow[d, "k''b_{kk'}k''k'S"] \\
k'k''k'kk'k''S \arrow[d, "b_{k'k''}kk'k''S"'] & & k''k'kk'k''k'S \arrow[d, "k''k'kb_{k'k''}S"] \\
k''k'k''kk'k''S \arrow[rr, two heads, "c^*"] & & k''k'kk''k'k''S
\end{tikzcd} \tag{27}$$

In (27) all the arrows are dominated by the two top arrows $b_{kj}k'\ldots ik \sim k\ldots jb_{ki}$. For $j = k''$ and $i = k'$, (27) becomes (19).

Now we will consider critical pairs involving one b-rule and one c-rule of the form c_{kj} with $k \geq j+2$.

For $(c_{kj}j'\ldots ij, kb_{ji})$, $k \geq j+2$, $j \geq i+1$, we consider the diagram

$$E(c_{kj}, b_{ji}) = \begin{tikzcd} kj\ldots ij \arrow[rrr, "kb_{ji}"] \arrow[d, "c_{kj}j'\ldots ij"'] & & & kj'j\ldots i \arrow[d, "c^*"] \\ jkj'\ldots ij \arrow[r, two heads, "c^*"] & j\ldots ikj \arrow[r, "j\ldots ic_{kj}"] & j\ldots ijk \arrow[r, "b_{ji}k"] & j'j\ldots ik. \end{tikzcd} \tag{28}$$

In (28), we have $kb_{ji} \sim b_{ji}k$ and kb_{ji} dominates every arrow in the vertical c-path. Thus we have to show that $c_{kj}j'\ldots ij$ dominates every arrow in the horizontal c-path. The horizontal c-path involves the rules generated by c_{ks} with $s < j$ and at the last step the rule $j\ldots ic_{kj}$. Since, by definition, $c_{kj} \gg c_{ks}$ if $j > s$, we see that $c_{kj}j'\ldots ij$ dominates all the horizontal c-arrows except probably the last one. But it is easy to see that also

$$c_{kj}j'\ldots ij \succ j\ldots ic_{kj}.$$

Now suppose $k \geq s+2$ and $k \geq j+1$. For the critical pair $(b_{kj}s, k\ldots jc_{ks})$ we will consider several cases. If $j \geq s+2$, then we can take a diagram similar to (28)

$$E(b_{kj}, c_{ks}) = \begin{tikzcd} k\ldots jks \arrow[r, "k\ldots jc_{ks}"] \arrow[ddd, "b_{kj}s"'] & k\ldots jsk \arrow[d, "c^*"] \\ & ksk'\ldots jk \arrow[d, "c_{ks}k'\ldots jk"] \\ & sk\ldots jk \arrow[d, "sb_{kj}"] \\ k'k\ldots js \arrow[r, two heads, "c^*"] & sk'k\ldots j. \end{tikzcd} \tag{29}$$

In (29), $b_{kj}s \sim sb_{kj}$ and dominates every arrow in the horizontal c-path. We have to check that $k\ldots jc_{ks}$ dominates all arrows in the vertical c-path. This is done using that $c_{ks} \gg c_{ts}$ for all $k' \geq t \geq j$, and that $k\ldots jc_{ks} \succ c_{ks}k'\ldots jk$.

For $j = s+1$ we consider the diagram

$$E(b_{kj}, c_{kj'}) = \begin{tikzcd} k\dots jkj' \arrow[r, "k\dots jc_{kj'}"] \arrow[d, "b_{kj}j'"'] & k\dots j'k \arrow[d, "b_{kj'}"] \\ k'k\dots j' \arrow[r, dashed] & k'k\dots j'. \end{tikzcd} \tag{30}$$

The diagram (30) is decreasing since $b_{kj}j' \sim b_{kk'}k''\dots j' \sim b_{kj'}$.

For $j = s$, we consider the diagram

$$E(b_{kj}, c_{kj}) = \begin{tikzcd} k\dots jkj \arrow[r, "k\dots jc_{kj}"] \arrow[dd, "b_{kj}j"'] & k\dots jjk \arrow[d, "k\dots\widehat{j}a_jk"] \\ & k\dots jk \arrow[d, "b_{kj}"] \\ k'k\dots jj \arrow[r, "k\dots\widehat{j}a_j"] & k'k\dots j \end{tikzcd} \tag{31}$$

In (31), $b_{kj}j$ dominates all the arrows.

For $k'' \geq s \geq j+1$, we take the diagram

$$E(b_{kj}, c_{ks}) = \begin{tikzcd} k\dots jks \arrow[r, "k\dots jc_{ks}"] \arrow[ddd, "b_{kj}s"'] & k\dots jsk \arrow[d, "c^*"] \\ & k\dots\widehat{s}ks\dots js \arrow[d, "k\dots\widehat{s}kb_{sj}"] \\ & k\dots\widehat{s}ks's\dots j \arrow[d, "b_{k\widehat{s}}s's\dots j"] \\ k'k\dots\widehat{s}s\dots js \arrow[r, "k'k\dots\widehat{s}b_{sj}"] & k'k\dots\widehat{s}s's\dots j. \end{tikzcd} \tag{32}$$

Note that for $s = k''$ the subword $k''\dots\widehat{s}$ is empty. We claim that $b_{kj}s$ dominates all the arrows. This is obvious for all arrows except $b_{k\widehat{s}}s'\dots j$. We have $b_{kj}s \sim b_{kk'}k''\dots js$ and $b_{k\widehat{s}}s's\dots j \sim b_{kk'}k''\dots\widehat{s}s's\dots j$. Now for all

$s+1 \le t \le k$, we have

$$\#_t(k'' \dots js) = \begin{cases} 0, & t = k, k' \\ 1, & s+1 \le t \le k'' \end{cases}$$

$$= \#_t(k'' \dots \widehat{s} s' s \dots j).$$

Further,

$$\#_s(k'' \dots js) = 2 > 1 = \#_s(k'' \dots \widehat{s} s' s \dots j).$$

Thus

$$b_{kj} s \succ b_{k\widehat{s}} s' s \dots j.$$

It is left to consider the critical pairs involving two c-rules of the form c_{ts} with $t \ge s+2$. They are all of the type $(c_{kj} i, k c_{ji})$. The diagram (18) gives a decreasing convergence diagram for this pair as $c_{kj} i$ dominates all the arrows in (18). This is true as $c_{kj} \gg c_{ki} \gg c_{ji}$ and $c_{kj} i > i c_{kj}$ for $i < j$.

We will denote the set of pairs of paths that one obtains from the chosen critical e.d.s by $\mathcal{Y}$. We also write $\mathcal{L}$ for the set $(p, \varnothing_w)$, were $p\colon w \twoheadrightarrow w$ are the loops at semi-normal elements of $\langle X^*, X^* \mathbf{r} X^* \rangle$. By Theorem 6.3, we get that $\mathrm{Res}_{\mathcal{Y} \sqcup \mathcal{L}}$ is an equivalence of categories.

Now we identify pairs of paths coming from loops in the attractor of $\langle X^*, X^* \mathbf{r} X^* \rangle$.

Proposition 7.3. *The preorder $\twoheadrightarrow$ on $\langle X^*, X^* \mathbf{r} X^* \rangle$ is well-founded. Equivalently* $\mathrm{Attr}(w) \neq \varnothing$ *for every $w \in X^*$.*

Proof: Suppose

$$w_0 \to w_1 \to \dots \to w_k \to \dots \tag{33}$$

is an infinite sequence in $\langle X^*, X^* \mathbf{r} X^* \rangle$. Then we have

$$l(w_0) \ge l(w_1) \ge \dots \ge l(w_k) \ge \dots$$

Since the set of words of length not greater than $l = l(w_0)$ is finite, we see that there is a word w that appears in (33) infinitely many times. Let $w_k = w$ be the first appearance of w in (33). Then for all $m > k$, there is $n > m$ such that $w_n = w$. Thus we get that for all $m > k$ there are paths $w = w_k \twoheadrightarrow w_m$ and $w_m \twoheadrightarrow w_n = w$. This shows that for every $m > k$, we have $w \twoheadleftarrow\!\!\twoheadrightarrow w_m$, that is $\twoheadrightarrow$ is a well-founded preorder. ■

Proposition 7.4. *Let w be a semi-normal element in $\langle X^*, X^* \mathbf{r} X^* \rangle$ and $p\colon w \twoheadrightarrow w$ a path in $\langle X^*, X^* \mathbf{r} X^* \rangle$. Then p is a c-path.*

Proof: Define the map

$$\underline{l}\colon X^* \to \mathbb{N}^n$$
$$u \mapsto (l(u), \#_n(u), \#_{n-1}(u), \dots, \#_2(u)).$$

We will write $u \geq v$ if $\underline{l}(u) \geq \underline{l}(v)$ with respect to the lexicographical order on $\mathbb{N}^n$. If $u \geq v$ and $v \geq u$, then we write $u \equiv v$. It is obvious that $(u,v) \in X^*\mathbf{r}X^*$ implies that $u \geq v$ and $u \equiv v$ if and only if $(u,v) = w'c_{ij}w''$ for some w', $w'' \in X^*$. Thus p is a path in $\langle X^*, X^*cX^*\rangle$. ■

Let $\mathcal{L}'$ be a subset of $\mathcal{L}$ consisting of (11). Combining Proposition 7.1 and Proposition 7.4, we get

Corollary 7.5. *The functor* $\mathrm{Res}_{\mathcal{Y}\sqcup\mathcal{L}'}$ *is an equivalence of categories.*

Next we are going to relate the categories $[X, \mathbf{r}, \mathcal{Y} \cup \mathcal{L}'; \mathcal{C}]$ and $[X, \mathbf{r}'', \mathcal{P}; \mathcal{C}]$, where $\mathcal{P}$ was defined on page 21.

We have two functors $\mathrm{Res}\colon [X,\mathbf{r};\mathcal{C}] \to [X,\mathbf{r}'';\mathcal{C}]$ and $\mathrm{P}\colon [X,\mathbf{r}'';\mathcal{C}] \to [X,\mathbf{r};\mathcal{C}]$. The first functor is defined by $(F,\tau) \to (F, \mathrm{res}(\tau))$, where $\mathrm{res}(\tau)$ is the restriction of τ to $\mathbf{r}''$. The functor $\mathrm{P}\colon [X,\mathbf{r}'';\mathcal{C}] \to [X,\mathbf{r};\mathcal{C}]$ is defined by $\mathrm{P}(F,\tau) = (F,\tilde{\tau})$ where

$$\tilde{\tau}(a_i) = \tau(a_i), \quad \tilde{\tau}(c_{st}) = \tau(c_{st}), \quad \tilde{\tau}(b_{kk'}) = \tau(b_k)$$

and $\tilde{\tau}(b_{kj})$ for $j \leq k''$ is computed recursively using the relation

$$\tilde{\tau}(b_{kj}) = (\tilde{\tau}(b_{k,j+1})F_j) \circ (F_k \dots F_{j+1}\tau(c_{jk})).$$

Let $\mathcal{Z}$ be the set of pairs of paths in $\langle X^*, X^*\mathbf{r}X^*\rangle$obtained from (30). Then it is easy to see that Res and P induce mutually inverse equivalences of categories

$$\mathrm{Res}\colon [X,\mathbf{r},\mathcal{Z};\mathcal{C}] \rightleftarrows [X,\mathbf{r}'';\mathcal{C}]\colon \mathrm{P}$$

It is also clear that if $(F,\tau) \in [X, r, \mathcal{Y} \sqcup \mathcal{L}'; \mathcal{C}]$ then $\mathrm{Res}(F,\tau) \in [X,\mathbf{r}'',\mathcal{P};\mathcal{C}]$ since the images of the pairs in $\mathcal{P}$ under P appear among the pairs $\mathcal{Y} \sqcup \mathcal{L}'$.

Now we are going to show that for every $(F,\tau) \in [X,\mathbf{r}'',\mathcal{P};\mathcal{C}]$, we get $\mathrm{P}(F,\tau) \in [X,\mathbf{r},\mathcal{Y} \sqcup \mathcal{L}';\mathcal{C}]$. Let us write $(F,\tilde{\tau})$ for $\mathrm{P}(F,\tau)$. We have to show that every diagram in $\mathcal{Y} \sqcup \mathcal{L}'$ is mapped into a commutative diagram under $f_{F,\tilde{\tau}}$. For $\mathcal{L}'$ this is obvious, since $\mathcal{L}' \subset \mathcal{P}$ and $(F,\tau) \in [X,\mathbf{r}'',\mathcal{P};\mathcal{C}]$. Now, we have to check that all the diagrams (23-32) become commutative under $f_{F,\tilde{\tau}}$.

For (23) this is obvious, since $\tau(c_{st})\tau(c_{ts}) = \mathrm{id}$ for all $|s-t| \geq 2$. The commutativity of the other diagrams follows by an induction argument and the patching diagrams listed bellow. We label natural e.d.s by $\mathcal{N}$.

As we already noted before (24) for $j = k'$ becomes (14). Now, for $j \leq k''$, we have

$$\begin{tikzcd}
kk\dots jk \arrow[r, "kk\dots\widehat{\jmath}c_{jk}"] \arrow[d, "a_kk'\dots jk"'] \arrow[dr, phantom, "\mathcal{N}"] & kk\dots\widehat{\jmath}kj \arrow[r, "kb_{k\widehat{\jmath}}j"] \arrow[d, "a_kk'\dots\widehat{\jmath}kj"] \arrow[dr, phantom, "E(a_k, b_{k\widehat{\jmath}})j"] & kk'k\dots\widehat{\jmath}j \arrow[d, twoheadrightarrow] \\
k\dots jk \arrow[r, "k\dots\widehat{\jmath}c_{jk}"] & k\dots\widehat{\jmath}kj \arrow[r, twoheadrightarrow] & k'k\dots j.
\end{tikzcd} \tag{34}$$

Further (25) for $j = k'$ is (13), and for $j \leq k''$, we have the diagram

$$\begin{tikzcd}
k\dots jkk \arrow[rr, "k\dots ja_k"] \arrow[d, "k\dots\widehat{\jmath}c_{jk}k"'] \arrow[drr, phantom, "(15)"] & & k\dots jk \arrow[d, "k\dots\widehat{\jmath}c_{jk}"] \\
k\dots\widehat{\jmath}kjk \arrow[r, "k\dots\widehat{\jmath}kc_{jk}"] \arrow[d, "b_{k\widehat{\jmath}}jk"'] \arrow[dr, phantom, "\mathcal{N}"] & k\dots\widehat{\jmath}kkj \arrow[r, "k\dots\widehat{\jmath}a_kj"] \arrow[d, "b_{k\widehat{\jmath}}kj"'] \arrow[dr, phantom, "E(b_{k\widehat{\jmath}}, a_k)j"] & k\dots\widehat{\jmath}kj \arrow[d, twoheadrightarrow] \\
k'k\dots jk \arrow[r, "k'k\dots\widehat{\jmath}c_{jk}"] & k'k\dots\widehat{\jmath}kj \arrow[r, twoheadrightarrow] & k'k\dots\widehat{\jmath}j.
\end{tikzcd} \tag{35}$$

The diagram (26) for $i = k'$ is (16). For $i \leq k''$, we use the diagram

$$\begin{tikzcd}
kk'k\dots ik \arrow[r, "kk'k\dots\widehat{\imath}c_{ik}"] \arrow[d, "b_{kk'}k'\dots ik"'] \arrow[dr, phantom, "\mathcal{N}"] & kk'k\dots\widehat{\imath}ki \arrow[r, "kk'b_{k\widehat{\imath}}"] \arrow[d, "b_{kk'}k'\dots\widehat{\imath}ki"'] \arrow[dr, phantom, "E(b_{kk'}, b_{k\widehat{\imath}})i"] & kk'k'k\dots\widehat{\imath}i \arrow[d, twoheadrightarrow] \\
k'kk'k'\dots ik \arrow[r, "k'kk'k'\dots\widehat{\imath}c_{ik}"] & k'kk'k'\dots\widehat{\imath}ki \arrow[r, twoheadrightarrow] & k'k\dots\widehat{\imath}i.
\end{tikzcd} \tag{36}$$

The diagram (27) for $j = k''$ and $i = k'$ is (19). Let us first consider the case $i = k'$. Then we use the patching diagram

$$
\begin{tikzcd}
k \dots jkk'k \arrow[rr, "k \dots j b_{kk'}"] \arrow[d, "k \dots \widehat{\jmath} c_{jk} k'k"'] \arrow[drr, phantom, "(17)"] & & k \dots jk'kk' \arrow[d, "c^*"] \\
k \dots \widehat{\jmath}kjk'k \arrow[r, "c^*"] \arrow[d, "b_{k\widehat{\jmath}} jk'k"'] \arrow[dr, phantom, "\mathcal{N}"] & k \dots \widehat{\jmath}kk'kj \arrow[r, "k \dots \widehat{\jmath} b_{kj}"] \arrow[d, "b_{k\widehat{\jmath}} k'kj"'] \arrow[dr, phantom, "E(b_{k\widehat{\jmath}}, b_{kk'})j"] & k \dots \widehat{\jmath}k'kk'j \arrow[d] \\
k'k \dots \widehat{\jmath}jk'k \arrow[r, "c^*"] & k'k \dots \widehat{\jmath}k'kj \arrow[r, two heads] & k''k'kk''k'k'' \dots \widehat{\jmath}j.
\end{tikzcd}
\tag{37}
$$

For $i \leq k''$, we use

$$
\begin{tikzcd}
k \dots jk \dots ik \arrow[r, bend left, "k \dots jk \dots \widehat{\imath} c_{ik}"] \arrow[d, "b_{kj} k' \dots ik"'] \arrow[dr, phantom, "\mathcal{N}"] & k \dots jk \dots \widehat{\imath}ki \arrow[r, "k \dots j b_{k\widehat{\imath}} i"] \arrow[d, "b_{kj} k' \dots \widehat{\imath}ki"'] \arrow[dr, phantom, "E(b_{kj}, b_{k\widehat{\imath}})i"] & k \dots jk'k \dots \widehat{\imath}i \arrow[d] \\
k'k \dots jk' \dots ik \arrow[r, bend right, "k'k \dots jk' \dots \widehat{\imath} c_{ik}"'] & k'k \dots jk' \dots \widehat{\imath}ki \arrow[r, two heads] & k''k'kk''k'k'' \dots jk'' \dots \widehat{\imath}i.
\end{tikzcd}
\tag{38}
$$

The diagram (28) for $i = j'$ is (17). For $i \leq j''$, we consider the diagram

$$\begin{tikzcd}
kj\dots ij \arrow[r, bend left, "kj\dots vc_{ij}"] \arrow[d, "c_{kj}j'\dots ij"'] \arrow[dr, phantom, "\mathcal{N}"] & kj\dots vji \arrow[r, "kb_{jv}i"] \arrow[d, "c_{kj}j'\dots vji"] \arrow[dr, phantom, "E(c_{kj}, b_{jv})i"] & kj'j\dots vi \arrow[d, ->>] \\
jkj'\dots ij \arrow[r, bend right, "jkj'\dots vc_{ij}"'] & jkj'\dots vji \arrow[r, ->>] & j'j\dots vki.
\end{tikzcd} \tag{39}$$

The commutativity of $f_{F,\tilde{\tau}}(E(b_{kk'}, c_{ks}))$ for $s \leq k'''$ follows from the commutativity of the diagram obtained by application $f_{F,\tau}$ to (17). Now, we show that $f_{F,\tilde{\tau}}(E(b_{kj}, c_{ks}))$ is commutative for $s \leq k'''$ and $j \leq k''$ by induction on j, using the diagram

$$\begin{tikzcd}
k\dots jks \arrow[rr, "k\dots jc_{ks}"] \arrow[d, "k\dots\widehat{\jmath}c_{jk}s"'] \arrow[drr, phantom, "(18)"] & & k\dots jsk \arrow[d, "c^*"] \\
k\dots\widehat{\jmath}kjs \arrow[r, bend left, "k\dots\widehat{\jmath}kc_{js}"] \arrow[d, "b_{k\widehat{\jmath}}js"'] \arrow[dr, phantom, "\mathcal{N}"] & k\dots\widehat{\jmath}ksj \arrow[r, "k\dots\widehat{\jmath}c_{ks}j"] \arrow[d, "b_{k\widehat{\jmath}}sj"'] \arrow[dr, phantom, "E(b_{k\widehat{\jmath}}, c_{ks})j"] & k\dots\widehat{\jmath}skj \arrow[d, ->>] \\
k'k\dots js \arrow[r, bend right, "k'k\dots\widehat{\jmath}c_{js}"'] & k'k\dots\widehat{\jmath}sj \arrow[r, ->>] & sk'k\dots\widehat{\jmath}j.
\end{tikzcd} \tag{40}$$

The commutativity of the diagrams $f_{F,\tilde{\tau}}(E(b_{kj}, c_{kj'}))$ follows from the definition of $\tilde{\tau}$. Next we check that $f_{F,\tilde{\tau}}(E(b_{kj}, c_{kj}))$ is commutative.

$$\begin{tikzcd}
k \dots jkj \arrow[rr, "k\dots jc_{kj}"] \arrow[d, bend right, "k\dots\widehat{\jmath}c_{jk}j"'] \arrow[d, phantom, "(11)"] \arrow[drr, phantom, "(15)"] & & k \dots jjk \arrow[d, "k\dots\widehat{\jmath}a_j k"] \\
k \dots \widehat{\jmath}kjj \arrow[u, bend right, "k\dots\widehat{\jmath}c_{kj}j"'] \arrow[r, "k\dots\widehat{\jmath}ka_j"'] \arrow[d, "b_{k\widehat{\jmath}}jj"'] \arrow[dr, phantom, "\mathcal{N}"] & k \dots \widehat{\jmath}kj \arrow[r, "k\dots\widehat{\jmath}c_{kj}"] \arrow[d, "b_{k\widehat{\jmath}}j"] \arrow[dr, phantom, "E(b_{k\widehat{\jmath}}, c_{kj})"] & k \dots \widehat{\jmath}jk \arrow[d, "b_{kj}"] \\
k'k \dots \widehat{\jmath}kjj \arrow[r, "k'k\dots\widehat{\jmath}ka_j"] & k'k \dots \widehat{\jmath}j \arrow[r, dashed] & k'k \dots j
\end{tikzcd} \tag{41}$$

To prove the commutativity of $f_{F,\tilde{\tau}}(E(b_{kj}, c_{ks}))$, one notices that commutativity of $f_{F,\tilde{\tau}}(E(b_{kj}, c_{kj'})$ for all $j \le k'$, implies that for any $k'' \ge s \ge j+1$, the following diagram commutes uppon application of $f_{F,\tilde{\tau}}$ to it:

$$\begin{tikzcd}
k \dots jks \arrow[r, "c^*"] \arrow[d, "b_{kj}s"'] & k \dots \widehat{s}ks \dots js \arrow[d, "b_{k\widehat{s}}s\dots js"] \\
k'k \dots js \arrow[r, dashed] & k'k \dots js.
\end{tikzcd} \tag{42}$$

Further one uses that $\tilde{\tau}(c_{sk})\tilde{\tau}(c_{ks}) = \mathrm{id}$ and the natural e.d. with $r = b_{k\widehat{s}}$, $r' = b_{sj}$, and $w = \varnothing$.

This finishes the proof that the diagrams (12-19) give a coherent presentation for the 0-Hecke monoid $\mathcal{H}(\Sigma_{n+1})$.

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