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VECTOR CROSS PRODUCT DIFFERENTIAL AND DIFFERENCE EQUATIONS IN $\mathbb{R}^3$ AND IN $\mathbb{R}^7$

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ABSTRACT: Through a matrix approach of the 2-fold vector cross product in $\mathbb{R}^3$ and in $\mathbb{R}^7$, some vector cross product differential and difference equations are studied. Either the classical theory or convenient Drazin inverses, of elements belonging to the class of index 1 matrices, are applied.

KEYWORDS: 2-fold vector cross product, vector cross product differential equation, vector cross product difference equation.
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1. Introduction

The generalized Hurwitz Theorem asserts that, over a field of characteristic different from 2, if $\mathcal{A}$ is a finite dimensional composition algebra with identity, then its dimension is equal to $1, 2, 4$ or 8. Moreover, $\mathcal{A}$ is isomorphic either to the base field, a separable quadratic extension of the base field, a generalized quaternion algebra or a generalized octonion algebra, [5].

A well known consequence of the cited theorem is that the values of n for which the Euclidean spaces $\mathbb{R}^n$ can be equipped with a 2-fold vector cross product, satisfying the same requirements as the usual one in $\mathbb{R}^3$, are restricted to 1 (trivial case), 3 and 7. See [3] for a complete discussion on r-fold vector cross products on d-dimensional vector spaces.

The 2-fold vector cross product can be found in mathematical models of physical processes, control theory problems in particular, which involve differential equations, [6, 8]. In [6] and [7], through certain 3×3 skewsymmetric matrices, it is used in the description of spacecraft attitude control. In [6], the analogue problem in the 7-dimensional case is also considered.

The present work is devoted to vector cross product differential and difference equations in $\mathbb{R}^3$ and in $\mathbb{R}^7$.

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To begin with, definitions and results related to the subject are collected in section 2. Namely, the approach of the 2-fold vector cross product in $\mathbb{R}^3$ and in $\mathbb{R}^7$ from a matrix point of view, through the hypercomplex matrices S_u considered in [1], is recalled.

In the second place, further properties connected to the matrices S_u, which will be needed in sections 4 and 5, are established in section 3. This section continues the study of the properties of S_u started in [1], some of which are recalled in the previous section.

Thirdly, some differential equations involving the 2-fold vector cross product in $\mathbb{R}^3$ and in $\mathbb{R}^7$ are studied in section 4. Each of these ones is rewritten in matrix form and, when tractable, either the classical theory or a convenient Drazin inverse is applied.

Last but not least, discrete analogues of those vector cross product differential equations in $\mathbb{R}^3$ and in $\mathbb{R}^7$ are considered in section 5. As expected, the solution of the difference equation proceeds similarly to that of the differential equation when the classical theory does not apply.

2.Preliminaries

In what follows, let F be a field of characteristic different from 2.

Let V be a d-dimensional vector space over F, equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. A bilinear map $\times : V \times V \to V$ is a 2-*fold vector cross product* if, for any $u, v \in V$,

(i) $\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0$,

(ii) $\langle u \times v, u \times v \rangle = \begin{vmatrix} \langle u, u \rangle & \langle u, v \rangle \\ \langle v, u \rangle & \langle v, v \rangle \end{vmatrix}$, [3].

Throughout this work, $\mathbb{R}^{m \times n}$ denotes the set of all $m \times n$ real matrices. With $n = 1$, we identify $\mathbb{R}^{m \times 1}$ with $\mathbb{R}^m$. With $m = n = 1$, we identify $\mathbb{R}^{1 \times 1}$ with $\mathbb{R}$.

Consider the usual real vector space $\mathbb{R}^8$, with canonical basis $\{e_0, \ldots, e_7\}$, equipped with the multiplication $*$ given by $e_i * e_i = -e_0$ for $i \in \{1, \ldots, 7\}$, being e_0 the identity, and the below Fano plane, where the cyclic ordering of each three elements lying on the same line is shown by the arrows.

Then $\mathbb{O} = (\mathbb{R}^8, *)$ is the real (non-split) octonion algebra. Every element $\underline{x} \in \mathbb{O}$ may be represented* by

$$\underline{x} = x_0 + x, \text{ where } x_0 \in \mathbb{R} \text{ and } x = \sum_{i=1}^{7} x_i e_i \in \mathbb{R}^7$$

*The identity e_0 is usually omitted in $\underline{x} = x_0 e_0 + x$.

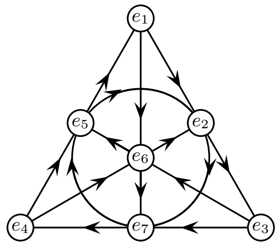

FIGURE 1. Fano plane for $\mathbb{O}$.

are, respectively, the *real part* and the *pure part* of the octonion $\underline{x}$.

The multiplication $*$ can be written in terms of the Euclidean inner product and the 2-fold vector cross product in $\mathbb{R}^7$, hereinafter denoted by $\langle\cdot,\cdot\rangle$ and $\times$, respectively. Concretely, as in [6], for any $\underline{x}, \underline{y} \in \mathbb{O}$, we have

$$\underline{x} * \underline{y} = x_0y_0 - \langle x, y\rangle + x_0y + y_0x + x \times y.$$

A similar relation may be written for the multiplication of the real (non-split) quaternion algebra $\mathbb{H} = (\mathbb{R}^4, *|_{\mathbb{R}^4})$, the Euclidean inner product $\langle\cdot,\cdot\rangle|_{\mathbb{R}^3}$ and the 2-fold vector cross product $\times|_{\mathbb{R}^3}$. For this reason, throughout the work and whenever clear from the context, the same notations $\langle\cdot,\cdot\rangle$ and $\times$ are used either in $\mathbb{R}^7$ or in $\mathbb{R}^3$.

In [1], [6] and [9], hypercomplex matrices related to the Lie algebra $(\mathbb{R}^3, \times)$ and to the Maltsev algebra $(\mathbb{R}^7, \times)$ were considered. If $u \in \mathbb{R}^7$ (respectively, $\mathbb{R}^3$), then let S_u be the matrix in $\mathbb{R}^{7\times 7}$ (respectively, $\mathbb{R}^{3\times 3}$) defined by

$$S_u x = u \times x \tag{1}$$

for any $x \in \mathbb{R}^7$ (respectively, $\mathbb{R}^3$). So, for $u = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \end{bmatrix}^T$ (respectively, $\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$), S_u is the skew-symmetric matrix

$$\left[\begin{array}{c:c} E & F \\ \hdashline G & H \end{array}\right] = \left[\begin{array}{ccc:cccc} 0 & -u_3 & u_2 & -u_5 & u_4 & -u_7 & u_6 \\ u_3 & 0 & -u_1 & -u_6 & u_7 & u_4 & -u_5 \\ -u_2 & u_1 & 0 & u_7 & u_6 & -u_5 & -u_4 \\ \hdashline u_5 & u_6 & -u_7 & 0 & -u_1 & -u_2 & u_3 \\ -u_4 & -u_7 & -u_6 & u_1 & 0 & u_3 & u_2 \\ u_7 & -u_4 & u_5 & u_2 & -u_3 & 0 & -u_1 \\ -u_6 & u_5 & u_4 & -u_3 & -u_2 & u_1 & 0 \end{array}\right] \text{(respectively, } E\text{)}.$$

Proposition 2.1. [1, 9] *Let* $n \in \{3, 7\}$, $u, v \in \mathbb{R}^n$, $\gamma \in \mathbb{R}\backslash\{0\}$ *and* $\tau, \eta \in \mathbb{R}$. *Then:*

(i) $S_{\tau u+\eta v} = \tau S_u + \eta S_v$;
(ii) $S_u v = -S_v u$;
(iii) S_u *is singular;*
(iv) $S_u^2 = uu^T - u^T u I_n$;
(v) $S_u^3 = -u^T u S_u$;
(vi) $(S_u - \gamma I_n)^{-1} = -\dfrac{1}{\gamma^2 + u^T u}\left(S_u + \gamma I_n + \dfrac{1}{\gamma} uu^T\right)$.

Let $A \in \mathbb{R}^{n\times n}$.

If A is skew-symmetric then $\mathcal{R} = \mathrm{e}^A$ is the rotation matrix, called *exponential* of A, defined by the absolutely convergent power series

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Conversely, given a rotation matrix $\mathcal{R} \in \mathbf{SO}(n)$, there exists a skew-symmetric matrix A such that $\mathcal{R} = \mathrm{e}^A$, [4].

Theorem 2.2. [6] *Let* $\underline{u} = u_0 + u \in \mathbb{O}$ *with* $\|u\| = \beta \neq 0$ *and* $t \in \mathbb{R}$. *Then*

$$\mathrm{e}^{tS_u} = \cos(\beta t) I + \frac{\sin(\beta t)}{\beta} S_u + \frac{1 - \cos(\beta t)}{\beta^2} uu^T.$$

The *index* $\mathrm{Ind}(A)$ of A is the smallest $l \in \mathbb{N}_0$ such that $R(A^l) = R(A^{l+1})$ or, equivalently, $N(A^l) = N(A^{l+1})$, where R and N stand for the column space (or range) and the nullspace, [2]. Alternatively, but equivalently, the index can be defined as the smallest $l \in \mathbb{N}_0$ such that $\mathbb{R}^n = R(A^l) \oplus N(A^l)$.

Let $\mathrm{Ind}(A) = l$. The *Drazin inverse* of A is the unique matrix $A^D \in \mathbb{R}^{n\times n}$ which satisfies

$$AA^D = A^D A,\ A^D A A^D = A^D,\ A^{l+1} A^D = A^l.$$

When $\mathrm{Ind}(A) \in \{0, 1\}$, A^D is sometimes called the *group-inverse* of A and the last equality assumes the form $AA^D A = A$. There are several methods for computing A^D, as described in [2] and references therein, some of which require all eigenvalues to be well determined.

Let $A, B \in \mathbb{R}^{n\times n}$ and $t_0 \in \mathbb{R}$. Let $f = f(t)$ be a $\mathbb{R}^n$-valued function of the real variable t. Throughout the work, $x = x(t)$ stands for an unknown $\mathbb{R}^n$-valued function of the real variable t and $\dot{x} = \dfrac{dx}{dt}$ denotes the corresponding derivative vector of x.

A vector $x_0 \in \mathbb{R}^n$ is a *consistent initial vector* for the differential equation

$$A\dot{x} + Bx = f \tag{2}$$

if the initial value problem

$$A\dot{x} + Bx = f,\ x(t_0) = x_0, \tag{3}$$

possesses at least one solution. In this case, $x(t_0) = x_0$ is said to be a *consistent initial condition*. Furthermore, (2) is called *tractable* if (3) has a unique solution for each consistent initial vector x_0, [2].

Theorem 2.3. [2] *Let $A, B \in \mathbb{R}^{n\times n}$. The homogeneous differential equation $A\dot{x} + Bx = 0$ is tractable if and only if $(\lambda A + B)^{-1}$ exists for some $\lambda \in \mathbb{R}$.*

Let $A, B \in \mathbb{R}^{n\times n}$. Let $f^{(k)} = f^{(k)}(t) \in \mathbb{R}^n$ be the k-th term of a sequence of vectors, $k = 0, 1, 2, ...$ Throughout the present work, $x^{(k)} = x^{(k)}(t) \in \mathbb{R}^n$ stands for the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, \ldots$ We assume that $x^{(0)} = x_0$ is given.

A vector $x_0 \in \mathbb{R}^n$ is a *consistent initial vector* for the difference equation

$$Ax^{(k+1)} = Bx^{(k)} + f^{(k)} \tag{4}$$

if the initial value problem

$$Ax^{(k+1)} = Bx^{(k)} + f^{(k)},\ k = 1, 2, \ldots,\ x^{(0)} = x_0, \tag{5}$$

has a solution for $x^{(k)}$. In this case, $x^{(0)} = x_0$ is said to be a *consistent initial condition*. Furthermore, (4) is called *tractable* if (5) has a unique solution for each consistent initial vector x_0, [2].

Theorem 2.4. [2] *Let $A, B \in \mathbb{R}^{n\times n}$. The homogeneous difference equation $Ax^{(k+1)} = Bx^{(k)}$ is tractable if and only if $(\lambda A + B)^{-1}$ exists for some $\lambda \in \mathbb{R}$.*

3.Matrix properties related to S_u

In this section, several properties connected to the matrices S_u are presented. The first result allows to ease the computation of their powers.

Proposition 3.1. *Let $n \in \{3, 7\}$, $u \in \mathbb{R}^n$, $\beta = \|u\|$ and $m \in \mathbb{N}$. Then*

(i) $S_u^{2m+1} = (-1)^m \beta^{2m} S_u$;
(ii) $S_u^{2m} = (-1)^{m+1}\beta^{2m-2}uu^T + (-1)^m \beta^{2m} I_n$.

Proof: By induction, owed to properties (ii), (iv) and (v) of S_u in Proposition 2.1. ■

Next, the invertibility of some matrices related to S_u is studied.

Proposition 3.2. *Let* $n \in \{3, 7\}$, $u, v \in \mathbb{R}^n$ *and* $\gamma \in \mathbb{R}$. *The matrix* $\gamma S_u + S_v$ *is singular.*

Proof: As S_u and S_v are skew-symmetric matrices, then, for any $\gamma \in \mathbb{R}$, $\gamma S_u + S_v$ is also skew-symmetric of odd order. Hence, $\det(\gamma S_u + S_v) = 0$. ■

Proposition 3.3. *Let* $n \in \{3, 7\}$, $v \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$. *The matrix* $S_v + \alpha I_n$ *is non-singular if and only if* $\alpha \neq 0$.

Proof: An easy calculation of $\det(S_v + \alpha I_n)$ leads to $\alpha(\alpha^2 + \|v\|^2)$ if $v \in \mathbb{R}^3$ and $\alpha(\alpha^2 + \|v\|^2)^3$ if $v \in \mathbb{R}^7$. In the stated conditions, $\det(S_v + \alpha I_n) = 0$ if and only if $\alpha = 0$. ■

The remaining results of this section are devoted to the indexes of S_u and certain related matrices.

Theorem 3.4. *Let* $n \in \{3, 7\}$ *and* $u \in \mathbb{R}^n \backslash \{0\}$. *Then* $\mathrm{Ind}(S_u) = 1$.

Proof: Let $u \in \mathbb{R}^3 \backslash \{0\}$. The matrix S_u has index 1 if $\mathbb{R}^3 = R(S_u) \oplus N(S_u)$.

First of all, from (iv) in Proposition 2.1, every $x \in \mathbb{R}^3$ can be written as $x = \frac{1}{\|u\|^2}(uu^T x - S_u^2 x)$. Clearly, $S_u^2 x \in R(S_u)$. By (ii) in Proposition 2.1, $uu^T x \in N(S_u)$ since $S_u(uu^T x) = (S_u u)(u^T x) = 0$.

Secondly, let $x \in R(S_u) \cap N(S_u)$. As $x \in R(S_u)$, there exists $y \in \mathbb{R}^3$ such that $x = S_u y$. In addition, $x \in N(S_u)$ which, together with (v) in Proposition 2.1, allows to write $0 = S_u^2 x = S_u^3 y = -\|u\|^2 S_u y$. Consequently, $y \in N(S_u)$, which implies $x = 0$.

A perfectly analogous reasoning provides a proof for $u \in \mathbb{R}^7 \backslash \{0\}$. ■

Lemma 3.5. *Let* $n \in \{3, 7\}$ *and* $u \in \mathbb{R}^n \backslash \{0\}$. *Then* $N(S_u) = \langle u \rangle$.

Proof: Let $n \in \{3, 7\}$ and $u \in \mathbb{R}^n \backslash \{0\}$. The inclusion $\langle u \rangle \subseteq N(S_u)$ follows from (i) and (ii) in Proposition 2.1, since, for all $\gamma \in \mathbb{R}$, $S_u(\gamma u) = 0$. As proved in [1] for $n = 7$ and in [9] for $n = 3$, the eigenvalues of S_u are 0 and $\pm\|u\|i$. Furthermore, the characteristic polynomial of S_u can be written as

$$\det(S_u - xI_n) = -x(x^2 + u^t u)^s,$$

where $s = 3$ if $n = 7$ and $s = 1$ if $n = 3$. In both cases, the eigenvalue 0 has algebraic multiplicity 1. As $0 \neq u \in N(S_u)$, the geometric multiplicity of 0 is 1. Hence, dim $N(S_u) =$ dim $\langle u \rangle = 1$. Therefore, $N(S_u) = \langle u \rangle$. ■

Theorem 3.6. *Let* $n \in \{3, 7\}$, $u, v \in \mathbb{R}^n \backslash \{0\}$ *and* $\alpha \in \mathbb{R}\backslash\{0\}$. *Then* $\operatorname{Ind}((S_v + \alpha I_n)^{-1} S_u) = 1$.

Proof: Let $n \in \{3, 7\}$, $u, v \in \mathbb{R}^n\backslash\{0\}$ and $\alpha \in \mathbb{R}\backslash\{0\}$. By Proposition 3.3, $S_v + \alpha I_n$ is non-singular. Suppose that

$$N((S_v + \alpha I_n)^{-1} S_u) \subsetneq N(((S_v + \alpha I_n)^{-1} S_u)^2).$$

Hence, there exists $x \in \mathbb{R}^n\backslash\{0\}$ such that $((S_v + \alpha I_n)^{-1}S_u)^2 x = 0$ and $(S_v + \alpha I_n)^{-1} S_u x \neq 0$. It is clear that $N((S_v + \alpha I_n)^{-1} S_u) = N(S_u)$. From Lemma 3.5, $N(S_u) = \langle u\rangle$. Thus, $(S_v + \alpha I_n)^{-1} S_u x = \delta u$ for some $\delta \in \mathbb{R}\backslash\{0\}$, that is, $S_u x = \delta(S_v u + \alpha u)$. This implies that $\delta\alpha u = u \times x - \delta v \times u$ and so, $\langle u, \delta\alpha u\rangle = \langle u, u \times x - \delta v \times u\rangle = 0$, that is, $\delta\alpha\|u\|^2 = 0$, which is a contradiction. Finally, $N(((S_v + \alpha I_n)^{-1} S_u)^0) \neq N((S_v + \alpha I_n)^{-1} S_u)$. The result is proved. ■

4. Vector cross product differential equations

In the present section, some vector cross product differential equations in $\mathbb{R}^3$ and in $\mathbb{R}^7$ are considered.

Theorem 4.1. *Let* $n \in \{3, 7\}$, $b \in \mathbb{R}^n\backslash\{0\}$ *and* $x = x(t)$ *an unknown* $\mathbb{R}^n$*-valued function of the real variable* t. *The unique solution of the vector cross product differential equation*

$$\dot{x} + b \times x = 0, \tag{6}$$

with initial condition $x(t_0) = x_0$, *is*

$$x(t) = \cos(\beta(t - t_0))x_0 - \frac{\sin(\beta(t - t_0))}{\beta} S_b x_0 + \frac{1 - \cos(\beta(t - t_0))}{\beta^2} b b^T x_0, \tag{7}$$

where $\beta = \|b\|$. *Moreover, for any* t, $\|x(t)\|^2$ *is constant.*

Proof: From (1), equation (6) assumes the form $\dot{x} + S_b x = 0$, which is a tractable equation by Theorem 2.3. In fact, from Proposition 3.3, $(\lambda I_n + S_b)^{-1}$ exists for every $\lambda \in \mathbb{R}\backslash\{0\}$. As the coefficient of the term in $\dot{x}$ is a non-singular matrix, the classical theory recalled in [2, p.171] applies to the homogeneous initial value problem $\dot{x} + S_b x = 0, x(t_0) = x_0$. Its unique solution is given by

$$x(t) = \mathrm{e}^{-(t-t_0)S_b} x_0.$$

Invoking Theorem 2.2, we obtain (7). Due to the skew-symmetry of S_b, for any t, $\|x(t)\|^2$ is constant since $\dfrac{d}{dt}(\|x\|^2) = 0$, as

$$\frac{d}{dt}\langle x, x\rangle = \langle \dot{x}, x\rangle + \langle x, \dot{x}\rangle = -(S_b x)^T x - x^T S_b x = -x^T(S_b^T + S_b)x = 0.$$

■

Theorem 4.2. *Let* $n \in \{3,7\}$, $b \in \mathbb{R}^n \backslash \{0\}$, $f = f(t)$ *a* $\mathbb{R}^n$*-valued function of the real variable* t*, continuous in some interval containing* t_0*, and* $x = x(t)$ *an unknown* $\mathbb{R}^n$*-valued function of the real variable* t*. The unique solution of the vector cross product differential equation*

$$\dot{x} + b \times x = f, \tag{8}$$

with initial condition $x(t_0) = x_0$*, is*

$$x(t) = \cos(\beta(t-t_0))x_0 - \frac{\sin(\beta(t-t_0))}{\beta}S_b x_0 + \frac{1-\cos(\beta(t-t_0))}{\beta^2}bb^T x_0 + \int_{t_0}^{t}\left(\cos(\beta(t-s)) - \frac{\sin(\beta(t-s))}{\beta}S_b + \frac{1-\cos(\beta(t-s))}{\beta^2}bb^T\right)f(s)ds, \tag{9}$$

where $\beta = \|b\|$.

Proof: Again by (1), we can rewrite equation (8) as $\dot{x} + S_b x = f$, where the coefficient of the term in $\dot{x}$ is a non-singular matrix. Thus, the classical theory applies to the inhomogeneous initial value problem $\dot{x} + S_b x = f, x(t_0) = x_0$. Its unique solution is given by

$$x(t) = \mathrm{e}^{-(t-t_0)S_b}x_0 + \int_{t_0}^{t} \mathrm{e}^{-(t-s)S_b} f(s)ds.$$

From Theorem 2.2, we obtain (9). ■

Theorem 4.3. *Let* $n \in \{3,7\}$, $a, b \in \mathbb{R}^n \backslash \{0\}$ *and* $x = x(t)$ *an unknown* $\mathbb{R}^n$*-valued function of the real variable* t*. The vector cross product differential equation*

$$a \times \dot{x} + b \times x = 0 \tag{10}$$

is not tractable.

Proof: From (1), the rewriting of equation (10) leads to $S_a\dot{x} + S_b x = 0$. By Proposition 3.2, for any $\lambda \in \mathbb{R}$, $\lambda S_a + S_b$ is a singular matrix and the result follows from Theorem 2.3. ■

Taking into account the previous result, the remaining part of the section is devoted to the study of differential equations which can be considered as perturbations of (10).

Theorem 4.4. *Let $n \in \{3,7\}$, $a, b \in \mathbb{R}^n \backslash \{0\}$, $\alpha \in \mathbb{R}\backslash\{0\}$ and $x = x(t)$ an unknown $\mathbb{R}^n$-valued function of the real variable t. A vector $x_0 \in \mathbb{R}^n$ is a consistent initial vector for the vector cross product differential equation*

$$a \times \dot{x} + b \times x + \alpha x = 0 \tag{11}$$

if and only if x_0 is of the form

$$x_0 = \hat{S}_a \hat{S}_a^D q, \tag{12}$$

for some $q \in \mathbb{R}^n$, where

$$\hat{S}_a = -\frac{1}{\alpha^2 + b^t b}\left(S_b - \alpha I_n - \frac{1}{\alpha} b b^T\right) S_a. \tag{13}$$

Moreover, if $x_0 \in \mathbb{R}^n$ is a consistent initial vector for (11)*, then the unique solution of* (11)*, with initial condition $x(t_0) = x_0$, is*

$$x(t) = \mathrm{e}^{-\hat{S}_a^D (t-t_0)} \hat{S}_a \hat{S}_a^D x_0. \tag{14}$$

Proof: According to (1), equation (11) assumes the form $S_a \dot{x} + (S_b + \alpha I_n)x = 0$ where $\alpha \in \mathbb{R}\backslash\{0\}$. Let us denote $S_b + \alpha I_n$ by B, matrix which, due to Proposition 3.3, is non-singular. Thus, $(\lambda S_a + B)^{-1}$ exists for $\lambda = 0$ and, by Theorem 2.3, $S_a \dot{x} + Bx = 0$ is a tractable equation.

Following the notation in [2], let

$$\hat{S}_{a,\lambda} = (\lambda S_a + B)^{-1} S_a \text{ and } \hat{B}_\lambda = (\lambda S_a + B)^{-1} B,$$

where $\lambda \in \mathbb{R}$ is such that $\lambda S_a + B$ is non-singular. By [2, Theorem 9.2.2, p. 174], the consistency of an initial vector for (11) and its general solution are independent of the used λ. Hence, in what follows, we drop the subscripts λ and take $\lambda = 0$.

From Theorem 3.6, $\mathrm{Ind}(\hat{S}_a) = 1$. Invoking [2, Theorem 9.2.3, p. 175], we obtain the necessary and sufficient condition $x_0 \in R(\hat{S}_a) = R(\hat{S}_a^D \hat{S}_a)$ for a vector $x_0 \in \mathbb{R}^n$ to be a consistent initial vector for (11). Since $\hat{S}_a^D \hat{S}_a = \hat{S}_a \hat{S}_a^D$, we get (12). As $\hat{S}_a = B^{-1} S_a$, then, by (vi) of Proposition 2.1, we obtain (13).

Assume now that $x_0 \in \mathbb{R}^n$ is a consistent initial vector for (11). As $\hat{B} = I_n$, once again from [2, Theorem 9.2.3], the unique solution of the homogeneous initial value problem $S_a \dot{x} + Bx = 0, x(t_0) = x_0$, is given by (14). ∎

Theorem 4.5. *Let $n \in \{3,7\}$, $a, b \in \mathbb{R}^n \backslash \{0\}$, $\alpha \in \mathbb{R}\backslash\{0\}$, $f = f(t)$ a $\mathbb{R}^n$-valued function of the real variable t, continuously differentiable around t_0, and*

$x = x(t)$ an unknown $\mathbb{R}^n$-valued function of the real variable t. A vector $x_0 \in \mathbb{R}^n$ is a consistent initial vector for the vector cross product differential equation

$$a \times \dot{x} + b \times x + \alpha x = f \tag{15}$$

if and only if x_0 is of the form

$$x_0 = (I - \hat{S}_a \hat{S}_a^D)\hat{f}(t_0) + \hat{S}_a \hat{S}_a^D q, \tag{16}$$

for some vector $q \in \mathbb{R}^n$, where

$$\hat{S}_a = -\frac{1}{\alpha^2 + b^t b}\left(S_b - \alpha I_n - \frac{1}{\alpha} b b^T\right) S_a \tag{17}$$

and

$$\hat{f} = -\frac{1}{\alpha^2 + b^t b}\left(S_b - \alpha I_n - \frac{1}{\alpha} b b^T\right) f. \tag{18}$$

Moreover, if $x_0 \in \mathbb{R}^n$ is a consistent initial vector for (15)*, then the unique solution of* (15)*, with initial condition $x(t_0) = x_0$, is*

$$x(t) = \mathrm{e}^{-\hat{S}_a^D (t-t_0)} \hat{S}_a \hat{S}_a^D x_0 + \mathrm{e}^{-\hat{S}_a^D t} \int_{t_0}^{t} \mathrm{e}^{\hat{S}_a^D s} \hat{S}_a^D \hat{f}(s)\, ds + (I - \hat{S}_a \hat{S}_a^D)\hat{f}(t). \tag{19}$$

Proof: By (1), we can rewrite equation (15) as $S_a \dot{x} + (S_b + \alpha I_n)x = f$, where $\alpha \in \mathbb{R}\backslash\{0\}$. As in the proof of Theorem 4.4, let $B = S_b + \alpha I_n$, $\hat{S}_a = B^{-1} S_a$, $\hat{B} = I_n$, $\hat{f} = B^{-1} f$.

Taking into account Theorem 3.6, $\mathrm{Ind}(\hat{S}_a) = 1$. The necessary and sufficient condition $x_0 \in \{(I - \hat{S}_a \hat{S}_a^D)\hat{f}(t_0) + R(\hat{S_a^D} \hat{S}_a)\}$ for a vector $x_0 \in \mathbb{R}^n$ to be a consistent initial vector for (15) comes from [2, Theorem 9.2.3, p. 175], which leads to (16). By (vi) of Proposition 2.1, we obtain (17) and (18).

Suppose now that $x_0 \in \mathbb{R}^n$ is a consistent initial vector for (15). Once again from [2, Theorem 9.2.3], the unique solution of the inhomogeneous initial value problem $S_a \dot{x} + Bx = f, x(t_0) = x_0$, is given by (19). ■

5. Vector cross product difference equations

In the present section, some vector cross product difference equations in $\mathbb{R}^3$ and in $\mathbb{R}^7$ are considered.

Theorem 5.1. *Let $n \in \{3, 7\}$, $b \in \mathbb{R}^n \backslash \{0\}$ and $x^{(k)} = x^{(k)}(t) \in \mathbb{R}^n$ the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, ...$ The unique solution of the vector cross product difference equation*

$$x^{(k+1)} = b \times x^{(k)}, \tag{20}$$

with initial condition $x^{(0)} = x_0$, is

$$x^{(k)} = \begin{cases} x_0, & k = 0 \\ (-1)^{\frac{k-1}{2}} \beta^{k-1} S_b x_0, & k \in \mathbb{N}, odd \\ \left((-1)^{\frac{k}{2}+1} \beta^{k-2} bb^T + (-1)^{\frac{k}{2}} \beta^k I_n\right) x_0, & k \in \mathbb{N}, even \end{cases} \tag{21}$$

where $\beta = \|b\|$.

Proof: Due to (1), equation (20) assumes the form $x^{(k+1)} = S_b x^{(k)}$, which is a tractable equation by Theorem 2.4. In fact, from Proposition 3.3, $(\lambda I_n + S_b)^{-1}$ exists for every $\lambda \in \mathbb{R}\backslash\{0\}$. Taking into account the recurrence relation, the unique solution of the homogeneous initial value problem $x^{(k+1)} = S_b x^{(k)}$, $k = 0, 1, 2, \ldots$, $x^{(0)} = x_0$, is given by

$$x^{(k)} = S_b^k x_0, \; k = 0, 1, 2, ...$$

From Proposition 3.1, we arrive at (21). ■

Theorem 5.2. *Let $n \in \{3, 7\}$, $b \in \mathbb{R}^n\backslash\{0\}$, $f^{(k)} = f^{(k)}(t) \in \mathbb{R}^n$ the k-th term of a sequence of vectors, $k = 0, 1, 2, ...$, and $x^{(k)} = x^{(k)}(t) \in \mathbb{R}^n$ the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, ...$ The unique solution of the vector cross product difference equation*

$$x^{(k+1)} = b \times x^{(k)} + f^{(k)}, \tag{22}$$

with initial condition $x^{(0)} = x_0$, is

$$x^{(k)} = \begin{cases} x_0, & k = 0 \\ (-1)^{\frac{k-1}{2}} \beta^{k-1} S_b x_0 + \displaystyle\sum_{i=0}^{k-1} S_b^{k-1-i} f^{(i)}, & k \in \mathbb{N}, odd \\ \left((-1)^{\frac{k}{2}+1} \beta^{k-2} bb^T + (-1)^{\frac{k}{2}} \beta^k I_n\right) x_0 + \displaystyle\sum_{i=0}^{k-1} S_b^{k-1-i} f^{(i)}, & k \in \mathbb{N}, even \end{cases} \tag{23}$$

where $\beta = \|b\|$.

Proof: Again by (1), equation (22) assumes the form $x^{(k+1)} = S_b x^{(k)} + f^{(k)}$. The recurrence relation allows to obtain the unique solution of the inhomogeneous initial value problem $x^{(k+1)} = S_b x^{(k)} + f^{(k)}$, $k = 0, 1, 2, \ldots$, $x^{(0)} = x_0$, given by

$$x^{(k)} = S_b^k x_0 + \sum_{i=0}^{k-1} S_b^{k-1-i} f^{(i)}, \; k = 1, 2, ... \tag{24}$$

From Proposition 3.1, we obtain (23). ■

Corollary 5.3. *Let $n \in \{3,7\}$, $b \in \mathbb{R}^n \backslash \{0\}$, $c \in \mathbb{R}^n$ and $x^{(k)} = x^{(k)}(t) \in \mathbb{R}^n$ the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, \ldots$. The unique solution of the vector cross product difference equation*

$$x^{(k+1)} = b \times x^{(k)} + c, \tag{25}$$

with initial condition $x^{(0)} = x_0$, is

$$x^{(k)} = \begin{cases} x_0, & k = 0 \\ (-1)^{\frac{k-1}{2}} \beta^{k-1} S_b x_0 + \displaystyle\sum_{i=0}^{k-1} S_b^i c, & k \in \mathbb{N}, odd \\ \left((-1)^{\frac{k}{2}+1} \beta^{k-2} b b^T + (-1)^{\frac{k}{2}} \beta^k I_n \right) x_0 + \displaystyle\sum_{i=0}^{k-1} S_b^i c, & k \in \mathbb{N}, even \end{cases} \tag{26}$$

where $\beta = \|b\|$.

Proof: A particular case of the previous result, putting c instead of the sequence $\left(f^{(k)}\right)_{k \in \mathbb{N}_0}$. ■

Remark 5.4. *Assume that all eigenvalues λ_i of S_b, which are 0 and $\pm \|b\| i$ by* [1], *satisfy $\|\lambda_i\| < 1$. Under this hypothesis, $I_n - S_b$ is invertible and*

$$\sum_{i=0}^{k-1} S_b^i = (I_n - S_b)^{-1} \left(I_n - S_b^k \right),$$

which leads to an alternative expression for the sum in (26).

Theorem 5.5. *Let $n \in \{3,7\}$, $a, b \in \mathbb{R}^n \backslash \{0\}$ and $x^{(k)} = x^{(k)}(t) \in \mathbb{R}^n$ the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, \ldots$ The vector cross product difference equation*

$$a \times x^{(k+1)} = b \times x^{(k)} \tag{27}$$

is not tractable.

Proof: From (1), the rewriting of equation (27) leads to $S_a x^{(k+1)} = S_b x^{(k)}$. From Proposition 3.2, for any $\lambda \in \mathbb{R}$, $\lambda S_a + S_b$ is a singular matrix and the result follows from Theorem 2.4. ■

Similarly to section 4, due to the previous result, perturbed versions of the difference equation (27) are now studied.

Theorem 5.6. *Let* $n \in \{3,7\}$, $a, b \in \mathbb{R}^n \backslash \{0\}$, $\alpha \in \mathbb{R}\backslash\{0\}$ *and* $x^{(k)} = x^{(k)}(t) \in \mathbb{R}^n$ *the* k*-th term of an unknown sequence of vectors,* $k = 0, 1, 2, ...$ *A vector* $x_0 \in \mathbb{R}^n$ *is a consistent initial vector for the vector cross product difference equation*

$$a \times x^{(k+1)} = b \times x^{(k)} + \alpha x^{(k)} \tag{28}$$

if and only if x_0 *is of the form*

$$x_0 = \hat{S}_a \hat{S}_a^D q, \tag{29}$$

for some $q \in \mathbb{R}^n$*, where*

$$\hat{S}_a = -\frac{1}{\alpha^2 + b^t b}\left(S_b - \alpha I_n - \frac{1}{\alpha} b b^T\right) S_a. \tag{30}$$

Moreover, if $x_0 \in \mathbb{R}^n$ *is a consistent initial vector for* (28)*, then the unique solution of* (28)*, with initial condition* $x^{(0)} = x_0$*, is*

$$x^{(k)} = \left(\hat{S}_a^D\right)^k x_0,\ k = 0, 1, 2, \ldots \tag{31}$$

Proof: From (1), equation (28) assumes the form $S_a x^{(k+1)} = B x^{(k)}$ where $B = S_b + \alpha I_n$ with $\alpha \in \mathbb{R}\backslash\{0\}$, by Proposition 3.3, is non-singular. Owed to this fact, $\lambda S_a + B$ is also a non-singular matrix if $\lambda = 0$ and, by Theorem 2.4, (28) is a tractable equation.

Following the notation in [2], let

$$\hat{S}_{a,\lambda} = (\lambda S_a + B)^{-1} S_a \text{ and } \hat{B}_\lambda = (\lambda S_a + B)^{-1} B,$$

where $\lambda \in \mathbb{R}$ is such that $\lambda S_a + B$ is non-singular. By [2, Theorem 9.2.2, p. 174], the consistency of an initial vector for (28) and its general solution are independent of the used λ. Hence, in what follows, we drop the subscripts λ and take $\lambda = 0$.

By Theorem 3.6, Ind$(\hat{S}_a) = 1$. Invoking [2, Theorem 9.3.2, p. 182-183], we get the necessary and sufficient condition $x_0 \in R(\hat{S}_a) = R(\hat{S_a^D}\hat{S}_a)$ for a vector $x_0 \in \mathbb{R}^n$ to be a consistent initial vector for (28). As $\hat{S_a^D}\hat{S}_a = \hat{S}_a\hat{S_a^D}$, we obtain (29). Since $\hat{S}_a = B^{-1} S_a$, then, by (vi) of Proposition 2.1, we arrive at (30).

Suppose now that $x_0 \in \mathbb{R}^n$ is a consistent initial vector for (28). Since $\hat{B} = I_n$, once again from [2, Theorem 9.3.2], the unique solution of the homogeneous initial value problem $S_a x^{(k+1)} = B x^{(k)}$, $k = 0, 1, \ldots$, $x^{(0)} = x_0$, is given by (31). ■

Theorem 5.7. *Let* $n \in \{3,7\}$, $a, b \in \mathbb{R}^n \backslash \{0\}$, $\alpha \in \mathbb{R}\backslash\{0\}$, $f^{(k)} = f^{(k)}(t) \in \mathbb{R}^n$ *the* k*-th term of a sequence of vectors,* $k = 0, 1, 2, ...$*, and* $x^{(k)} = x^{(k)}(t) \in \mathbb{R}^n$ *the*

k-th term of an unknown sequence of vectors, $k = 0, 1, 2, \ldots$. A vector $x_0 \in \mathbb{R}^n$ is a consistent initial vector for the vector cross product difference equation

$$a \times x^{(k+1)} = b \times x^{(k)} + \alpha x^{(k)} + f^{(k)},\ k = 0, 1, 2, \ldots, \tag{32}$$

if and only if x_0 is of the form

$$x_0 = -\left(I_n - \hat{S}_a \hat{S}_a^D\right) \hat{f}^{(0)} + \hat{S}_a \hat{S}_a^D q, \tag{33}$$

for some $q \in \mathbb{R}^n$, where

$$\hat{S}_a = -\frac{1}{\alpha^2 + b^t b}\left(S_b - \alpha I_n - \frac{1}{\alpha} b b^T\right) S_a \tag{34}$$

and

$$\hat{f}^{(k)} = -\frac{1}{\alpha^2 + b^t b}\left(S_b - \alpha I_n - \frac{1}{\alpha} b b^T\right) f^{(k)}. \tag{35}$$

Moreover, if $x_0 \in \mathbb{R}^n$ is a consistent initial vector for (32)*, then the unique solution of* (32)*, with initial condition $x^{(0)} = x_0$, is*

$$x^{(k)} = \begin{cases} x_0, & k = 0 \\ \left(\hat{S}_a^D\right)^k \hat{S}_a \hat{S}_a^D x_0 + \hat{S}_a^D \sum_{i=0}^{k-1} \left(\hat{S}_a^D\right)^{k-i-1} \hat{f}^{(i)} - \left(I_n - \hat{S}_a \hat{S}_a^D\right) \hat{f}^{(k)}, & k = 1, 2, \ldots \end{cases} \tag{36}$$

Proof: The rewriting of equation (32) leads to $S_a x^{(k+1)} = B x^{(k)} + f^{(k)}$, where $B = S_b + \alpha I_n$ with $\alpha \in \mathbb{R} \backslash \{0\}$, since we have (1). As in the proof of Theorem 5.6, let $\hat{S}_a = B^{-1} S_a$, $\hat{B} = I_n$, $\hat{f}^{(k)} = B^{-1} f^{(k)}$.

From Theorem 3.6, $\mathrm{Ind}(\hat{S}_a) = 1$. The necessary and sufficient condition $x_0 \in \{-(I_n - \hat{S}_a \hat{S}_a^D)\hat{f}^{(0)} + R(\hat{S}_a^D \hat{S}_a)\}$ for a vector $x_0 \in \mathbb{R}^n$ to be a consistent initial vector for (32) comes from [2, Theorem 9.3.2, p. 182-183]. Thus, we obtain (33). By (vi) of Proposition 2.1, we get (34) and (35).

Assume now that $x_0 \in \mathbb{R}^n$ is a consistent initial vector for (32). Once again from [2, Theorem 9.3.2], the unique solution of the inhomogeneous initial value problem $S_a x^{(k+1)} = B x^{(k)} + f^{(k)}$, $k = 0, 1, 2, \ldots$, $x^{(0)} = x_0$, is given by (36). ■

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