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GEVREY WELL POSEDNESS OF THE GENERALIZED GOURSAT-DARBOUX PROBLEM FOR A LINEAR PDE

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ABSTRACT: We consider the generalized Goursat-Darboux problem for a third order linear PDE with real coefficients. Our purpose is to find necessary conditions for the problem to be well-posed in the Gevrey classes Γ^s with $s > 1$. It is proved that there exists some critical index s_0 such that if the Goursat-Darboux problem is well posed in Γ^s for $s > s_0$ then some conditions should be imposed on the coefficients of the derivatives with respect to one of the variables. In order to prove our results, we first construct an explicit solution of a family of problems with data depending on a parameter $\eta > 0$ and then we obtain an asymptotic representation of a solution as η tends to infinity.

KEYWORDS: Goursat-Darboux problems, Gevrey classes, asymptotic solutions.
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1. Introduction

The simplest generalized Goursat-Darboux problem for a third order linear PDE with real constant coefficients in the classes of Gevrey functions was studied in [8]. Given an open set $\Omega \subseteq \mathbf{R^{3+m}}$, neighborhood of origin, the problem is defined on Ω by

$$\begin{cases} \partial_t\partial_x\partial_y u(t,x,y,z) = \displaystyle\sum_{0\le|j|\le 3} A_j\partial_z^j u(t,x,y,z) \\ u(0,x,y,z) = f_1(x,y,z) \\ u(t,0,y,z) = f_2(t,y,z) \\ u(t,x,0,z) = f_3(t,x,z) \end{cases} \tag{1.1}$$

where initial data satisfy necessary compatibility conditions

$$\begin{cases} f_1(0,y,z) = f_2(0,y,z) \\ f_1(x,0,z) = f_3(0,x,z) \\ f_2(t,0,z) = f_3(t,0,z) \\ f_1(0,0,z) = f_2(0,0,z) = f_3(0,0,z)\,. \end{cases} \tag{1.2}$$

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Let us begin by introducing the Gevrey classes [5] and the concept of the well posed problem in the sense of Hadamard [6].

Definition 1.1. (Gevrey classes)
Let $s > 1$ be a real number and Ω be an open subset of $\mathbf{R^n}$. The Gevrey class of index s on Ω, $\Gamma^s(\Omega)$, is the space of the all functions $f \in C^\infty(\Omega)$ such that for every compact $K \subset \Omega$ there exist constants $C > 0$ and $L > 0$ satisfying

$$\sup_{x\in K} \mid \partial^\alpha f(x) \mid \leq CL^{|\alpha|}\alpha!^s \,, \text{for all multi-index } \alpha. \tag{1.3}$$

We choose a topology for $\Gamma^s(\Omega)$ in according to Rodino [11].

Definition 1.2. (Problem well-posed in the Gevrey classes)
Let $s > 1$ be a real number and Ω be an open subset of $\mathbf{R^n}$, neighborhood of origin. We say that the problem (1.1)-(1.2) is $\Gamma^s(\Omega)$ well-posed on Ω if there exists a neighborhood $\mathcal{U} \subset \Omega$ such that

- For every data $f_i \in \Gamma^s(\Omega \cap \Sigma_i)$, $i = 1, 2, 3$, the problem (1.1)-(1.2) has a solution $u \in \Gamma^s(\mathcal{U})$ and it is unique;
- It depends continuously on the data. It means that for every compact $K \subset \Omega$ and every constant $L > 0$ there exist compacts K_i and constants $L_i > 0$, $i = 1, 2, 3$, and $C > 0$ such that

$$\|u\|^s_{L,K} \leq C \left(\|f_1\|^s_{L_1,K_1} + \|f_2\|^s_{L_2,K_2} + \|f_3\|^s_{L_3,K_3} \right) . \tag{1.4}$$

We are now interested in the so-called case I [2], for a more general class of PDEs. Our goal is to find necessary conditions for the problem to be well-posed in the Gevrey classes We will try to find some critical index s_0 such that if the generalized Goursat-Darboux problem is well posed in Γ^s for $s > s_0$ then some conditions should be imposed on the coefficients of the derivatives with respect to one of the variables.

2. Formulation of the generalized Goursat-Darboux problem

For simplicity we suppose $m = 1$ but the formulation and solvability of our problem can be generalized to $m > 1$. Let $\Omega \subseteq \mathbf{R^4}$ be an open set, neighborhood of origin and let

$$P_i(\partial_z) = D_{2,i}\partial_z^2 + D_{1,i}\partial_z \,,\ i = 1, 2, 3 \wedge Q(\partial_z) = E_3\partial_z^3 + E_2\partial_z^2 + E_1\partial_z + E_0 \tag{2.1}$$

be a differential operators with real constant coefficients.
We consider the following generalized Goursat-Darboux problem on Ω:

$$\begin{cases} \partial_t\partial_x\partial_y u(t,x,y,z) = \left(P_1(\partial_z)\partial_t + P_2(\partial_z)\partial_x + P_3(\partial_z)\partial_y + Q(\partial_z)\right) u(t,x,y,z) \\ u(0,x,y,z) = f_1(x,y,z) \\ u(t,0,y,z) = f_2(t,y,z) \\ u(t,x,0,z) = f_3(t,x,z) \end{cases} \tag{2.2}$$

where the initial data satisfy the necessary compatibility conditions (1.2) on three characteristic hyperplanes $t = 0$, $x = 0$ and $y = 0$.
It was showed in [2] that if the problem (2.2)-(1.2) is locally C^∞ well-posed in the neighborhood of origin then the coefficients of the derivatives with respect to z are zero. So we expect stronger results in the Gevrey framework.
From now on we suppose that the problem (2.2) is Γ^s well-posed on Ω. As we have done in [8] the problem (2.2) can be reduced to the Cauchy problem following ideas of Bronshtein [1]. By linearity, if $u(t,x,y,z)$ is a solution of the problem (2.2) on Ω then

$$v(t,x,y,z) = u(t,x,y,z) + u(x,y,t,z) + u(y,t,x,z) \tag{2.3}$$

is a solution of the corresponding problem on $\Omega' \subset \Omega$

$$\begin{cases} \partial_t\partial_x\partial_y v(t,x,y,z) = \left(P(\partial_z)\partial_t + P(\partial_z)\partial_x + P(\partial_z)\partial_y + Q(\partial_z)\right) v(t,x,y,z) \\ v(0,x,y,z) = f_1(x,y,z) + f_3(x,y,z) + f_2(y,x,z) \\ v(t,0,y,z) = f_2(t,y,z) + f_1(y,t,z) + f_3(y,t,z) \\ v(t,x,0,z) = f_3(t,x,z) + f_2(x,t,z) + f_1(t,x,z) \,. \end{cases} \tag{2.4}$$

where

$$P(\partial_z) = \frac{1}{3}\left(P_1(\partial_z) + P_2(\partial_z) + P_3(\partial_z)\right) \,. \tag{2.5}$$

We then reduce the number of the independent variables by setting $t = x = y$. For every parameter $\eta > 0$, taking

$$v(0,x,y,z) = v(t,0,y,z) = v(t,x,0,z) = e^{i\eta z}$$

we are looking for a unique solution depending continuously on the data. If v_η is the solution of the problem on Ω'

$$\begin{cases} \partial_t\partial_x\partial_y v(t,x,y,z) = \left(P(\partial_z)\partial_t + P(\partial_z)\partial_x + P(\partial_z)\partial_y + Q(\partial_z)\right) v(t,x,y,z) \\ v(0,x,y,z) = v(t,0,y,z) = v(t,x,0,z) = e^{i\eta z} \end{cases} \tag{2.6}$$

then $w_\eta(r,z) = v_\eta(r,r,r,z)$ is the solution of the Cauchy problem on $\tilde{\Omega} \subseteq \mathbf{R^2}$

$$\left\{ \begin{array}{rcl} \partial_r^3 w(r,z) & = & 27\left((D_2\partial_z^2 + D_1\partial_z)\partial_r + (E_3\partial_z^3 + E_2\partial_z^2 + E_1\partial_z + E_0)\right) w(r,z) \\ w(0,z) & = & e^{i\eta z}, \end{array} \right. \tag{2.7}$$

where $D_j = \dfrac{1}{3}(D_{j,1} + D_{j,2} + D_{j,3})$, $j = 1, 2$. We remark that there are two arbitrary data $\partial_r w(0,z)$ and $\partial_r^2 w(0,z)$.

3. Solving the Cauchy problem

If the Cauchy problem (2.7) is well-posed in the Gevrey classes then necessarily

$$E_3^2 - 4D_2^3 \geq 0 \tag{3.1}$$

by applying the Lax-Mizohata theorem [9].
We determine a unique solution of the problem (2.7) in the form $w_\eta(r,z) = m_\eta(r)e^{i\eta z}$, hence $m_\eta(r)$ is the solution of the initial value problem

$$\left\{ \begin{array}{l} m'''(r) = 27(-D_2\eta^2 + iD_1\eta)m'(r) + 27(-E_3 i\eta^3 - E_2\eta^2 + iE_1\eta + E_0)m(r) \\ m(0) = 1 \\ m'(0) = \alpha \\ m''(0) = \beta \end{array} \right. \tag{3.2}$$

where α and β are unknown. In order to solve the corresponding linear ODE we use its characteristic equation

$$\lambda^3 + p(\eta)\lambda + q(\eta) = 0\,, \tag{3.3}$$

$p(\eta) = -27(-D_2\eta^2 + iD_1\eta)$ and $q(\eta) = -27(-E_3 i\eta^3 - E_2\eta^2 + iE_1\eta + E_0)$. That equation has a solution ζ_η which is given by $\zeta_\eta = z_\eta + \omega_\eta$. To obtain ζ_η we proceed in three steps (Vieta's method):

(1) Find $A_\eta \neq 0$ such that $A_\eta^2 = \Delta_\eta = \left(\frac{q(\eta)}{2}\right)^2 + \left(\frac{p(\eta)}{3}\right)^3$;
(2) Find a solution $z_\eta \neq 0$ of the equation $z^3 = -\frac{q(\eta)}{2} + A_\eta$ by de Moivre's formula;
(3) Calculate $\omega_\eta = -\dfrac{p(\eta)}{3z_\eta}$.

The other two solutions are $\zeta_\eta = \gamma z_\eta + \overline{\gamma}\omega_\eta$ and $\zeta_\eta = \overline{\gamma} z_\eta + \gamma\omega_\eta$.

Lemma 3.1. *Let γ and $\overline{\gamma}$ be conjugate complex roots of unity. If $\zeta_\eta = z_\eta + \omega_\eta$, $\zeta_\eta \neq 0$, is a solution of (3.3) then the solution of the problem (3.2) is given by*

$$\begin{aligned} m_\eta(r) = \tfrac{1}{3}(1 + a_\eta + b_\eta)e^{(z_\eta+\omega_\eta)r} + \tfrac{1}{3}\left(1 + \overline{\gamma}a_\eta + \gamma b_\eta\right)e^{(\gamma z_\eta + \overline{\gamma}\omega_\eta)r} + \\ + \tfrac{1}{3}\left(1 + \gamma a_\eta + \overline{\gamma} b_\eta\right)e^{(\overline{\gamma} z_\eta + \gamma\omega_\eta)r} \end{aligned} \tag{3.4}$$

where

$$a_\eta = \frac{\alpha z_\eta^2 - (\beta + 2p(\eta)/3)\omega_\eta}{z_\eta^3 - \omega_\eta^3} \quad \wedge \quad b_\eta = \frac{-\alpha\omega_\eta^2 + (\beta + 2p(\eta)/3)z_\eta}{z_\eta^3 - \omega_\eta^3}\,. \tag{3.5}$$

If ζ_η is a real root of the (3.3) we simplify (3.4) by using the Euler's formula.

Theorem 3.1 (Characteristic equation with one real root). If $\zeta_\eta = z_\eta + \omega_\eta \in \mathbf{R} - \{0\}$ and $\kappa_\eta = z_\eta - \omega_\eta \in \mathbf{R} - \{0\}$ then

$$\begin{aligned} m_\eta(r) = \tfrac{1}{3}(1 - c_\eta)e^{\zeta_\eta r} + \tfrac{1}{3}\left(2 + c_\eta\right)\cos\left(\sqrt{3}\kappa_\eta r/2\right)e^{-\zeta_\eta r/2} + \\ + \tfrac{\sqrt{3}}{3} d_\eta \sin\left(\sqrt{3}\kappa_\eta r/2\right)e^{-\zeta_\eta r/2} \end{aligned} \tag{3.6}$$

where

$$c_\eta = -\frac{\alpha\zeta_\eta + \beta + 2p(\eta)/3}{\zeta_\eta^2 + p(\eta)/3} = -a_\eta - b_\eta$$

and

$$d_\eta = \frac{-i[\alpha(\zeta_\eta^2 + \kappa_\eta^2)/2 - (\beta + 2p(\eta)/3)\zeta_\eta]}{(\zeta_\eta^2 + p(\eta)/3)\kappa_\eta} = -i(a_\eta - b_\eta)\,.$$

If ζ_η is a pure imaginary root of the (3.3), (3.4) can be written in a simpler expression.

Theorem 3.2 (Characteristic equation with a pure imaginary root). If $\zeta_\eta = z_\eta + \omega_\eta = -iY_\eta$ and $\kappa_\eta = z_\eta - \omega_\eta = -iX_\eta$ with $X_\eta \in \mathbf{R} - \{0\}$ and $Y_\eta \in \mathbf{R} - \{0\}$ then

$$\begin{aligned} m_\eta(r) = \tfrac{1}{3}\left[(2 + c_\eta)\cosh\left(\sqrt{3}X_\eta r/2\right) + \sqrt{3}d_\eta \sinh\left(\sqrt{3}X_\eta r/2\right)\right]e^{iY_\eta r/2} + \\ + \tfrac{1}{3}(1 - c_\eta)e^{-iY_\eta r} \end{aligned} \tag{3.7}$$

where $c_\eta = -a_\eta - b_\eta$ and $d_\eta = -i(a_\eta - b_\eta)$.

4. Results

In the next asymptotic estimates we use big O, little o and $\sim$ symbols to compare the growth of functions [10].

Definition 4.1. Let f and g be complex functions of the real variable η, $\eta > 0$. As $\eta \to \infty$, we say that

- **(i):** f and g are asymptotically equal, $f(\eta) \sim g(\eta)$, if $\lim_{\eta\to\infty} \frac{f(\eta)}{g(\eta)} = 1$;
- **(ii):** f is of order not exceeding g, $f(\eta) = O(g(\eta))$, if there exists a constant k such that $|f(\eta)| \leq k|g(\eta)|$ for all $\eta > 0$;
- **(iii):** f is of order less than g, $f(\eta) = o(g(\eta))$, if $\lim_{\eta\to\infty} \frac{f(\eta)}{g(\eta)} = 0$.

In previous works ([2], [3], [7]) an explicit solution of the generalized Goursat-Darboux problem involves a hypergeometric function of several variables. However some difficulties for obtaining asymptotic representations for these kind of functions were point out in the paper [4].
In our work we have a linear combination of complex exponential functions as solution of the Cauchy problem. In the next propositions, we provide asymptotic representations, as η tends to infinity, for the absolute value of complex functions m_η on a compact, which depends on η.

Theorem 4.1. If $p(\eta) = 0$, $q(\eta) = -27E_1\eta i$, $E_1 \neq 0$, and $s > 3$ then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{\sqrt[3]{|E_1|}\eta^{1/s}} \tag{4.1}$$

as η tends to infinity.

Theorem 4.2. If $q(\eta) = O(\eta)$, $p(\eta) = -27D_1\eta i$, $D_1 \neq 0$, and $s > 2$ then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{\sqrt[3]{|D_1|}\eta^{1/s}} \tag{4.2}$$

as η tends to infinity.

Theorem 4.3. If $p(\eta) = O(\eta)$, $q(\eta) = 27E_2\eta^2$, $E_2 \neq 0$, and $s > 3/2$ then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{\sqrt[3]{|E_2|}\eta^{1/s}} \tag{4.3}$$

as η tends to infinity.

Theorem 4.4. Let $p(\eta) = 27D_2\eta^2 + O(\eta)$, $q(\eta) = 27E_3 i\eta^3 + O(\eta^2)$ such that $E_3^2 - 4D_2^3 > 0$.

(i): If $D_2 \neq 0$ and $s > 1$ then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{\mid \sqrt[3]{\rho^2} - 9D_2 \mid \eta^{1/s}} \tag{4.4}$$

as η tends to infinity, where $\rho = \frac{27}{2}\left(\sqrt{E_3^2 - 4D_2^3} + E_3\right) \neq 0$;

(ii): If $D_2 = 0 \wedge E_3 \neq 0$ and $s > 1$ then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{\sqrt[3]{|E_3|}\eta^{1/s}} \tag{4.5}$$

as η tends to infinity.

In the proofs of the propositions, our approach is based on asymptotic analysis of the initial data in order to have only one exponential function as dominant term, that is, when one exponential function tends to infinity and the others tend to zero.
Finally we present main results.

Theorem 4.2. *If the problem (2.2)-(1.2) is* Γ^s *well-posed on* Ω *then*

(i)::

$$s > 1 \quad \Rightarrow \quad 27E_3^2 = 4(D_{2,1} + D_{2,2} + D_{2,3})^3\,; \tag{4.6}$$

(ii)::

$$s > \frac{3}{2} \quad \Rightarrow \quad E_2 = 0\,; \tag{4.7}$$

(iii)::

$$s > 2 \quad \Rightarrow \quad D_{1,1} + D_{1,2} + D_{1,3} = 0\,; \tag{4.8}$$

(iv)::

$$s > 3 \quad \Rightarrow \quad E_1 = 0\,. \tag{4.9}$$

Proof: We suppose that the problem (2.2)-(1.2) is Γ^s well-posed on Ω with $s > 1$. Then for every $\eta > 0$ the corresponding problem (2.6) has a unique solution v_η on Ω'.
On the one hand, we determine *a prior* an estimate for the Gevrey norm of v_η, an upper bound, from the initial data, $\|e^{i\eta z}\|^s_{L,K}$, for every compact $K \subset \Omega \subseteq \mathbf{R^{3+m}}$ and every constant $L > 0$. The partial derivatives of $e^{i\eta z}$

with respect to multi-index (l, k, j, α), such that $l \neq 0$ or $k \neq 0$ or $j \neq 0$, are zero. Otherwise, it is clear that

$$\partial_z^\alpha(e^{i\eta z}) = (i\eta)^{|\alpha|}e^{i\eta z}\,,$$

it follows that

$$\sup_{(t,x,y,z)\in K} \mid \partial^\alpha(e^{i\eta z}) \mid = \eta^{|\alpha|}\,.$$

Using $\mid \alpha \mid ! \leq m^{|\alpha|}\alpha!$ and $\mid \alpha \mid^{|\alpha|} \leq e^{|\alpha|} \mid \alpha \mid !$ we get

$$\|e^{i\eta z}\|^s_{L,K} \leq \sup_\alpha \left(\mid \alpha \mid^{-s|\alpha|} L^{-|\alpha|}(m^s e^s \eta)^{|\alpha|}\right)\,.$$

Since the supremum is equal to $e^{smL^{-1/s}\eta^{1/s}}$ there exist constants $c_1 = smL^{-1/s}$ and $C > 0$ such that

$$\|v_\eta\|^s_{L,K} \leq C\|e^{i\eta z}\|^s_{L,K} \leq Ce^{c_1\eta^{1/s}} \tag{4.10}$$

for every $\eta > 0$. It is a condition for stability of solution.
On the other hand, let's see that if we suppose, $E_3^2 - 4D_2^3 > 0$ in (i), $E_2 \neq 0$ in (ii), $D_1 \neq 0$ in (iii), $E_1 \neq 0$ in (iv), then we obtain a contradiction with (4.10). By using previous propositions we construct an asymptotic representation of a solution as η tends to infinity. For every neighborhood of the origin $\mathcal{O}$ there exist a compact K_η, $K_\eta \subset \mathcal{O}$, and constants $C > 0$ and $c_2 > 0$ such that

$$\sup_{r\in K_\eta} \mid v_\eta(r, r, r, z) \mid \sim Ce^{c_2\eta^{1/s}} \tag{4.11}$$

Notice that $K_\eta \subset \mathcal{O}$ only if $s_0 = 1$ in (i), $s_0 = 3/2$ in (ii), $s_0 = 2$ in (iii) and $s_0 = 3$ in (iv). We have

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid = \sup_{r\in K_\eta} \mid w_\eta(r, z) \mid = \sup_{r\in K_\eta} \mid v_\eta(r, r, r, z) \mid$$

and

$$\|v_\eta\|^s_{L,K_\eta} > \sup_{r\in K_\eta} \mid v_\eta(r, r, r, z) \mid\,,$$

for all $L > 0$. We can choose L with $L > \left(\dfrac{sm}{c_2}\right)^s$ such that

$$\|v_\eta\|^s_{L,K_\eta} > Ce^{c_2\eta^{1/s}} \tag{4.12}$$

as η tends to infinity. We conclude that (4.12) contradicts (4.10) because of $c_2 > c_1$.

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