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TRACES AND EXTENSIONS OF GENERALIZED SMOOTHNESS MORREY SPACES ON DOMAINS

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Abstract: We study traces on the boundary of generalized smoothness Morrey spaces on C^k domains Ω. These spaces are equipped with three parameters s, p, q and a function parameter φ. Our results remain valid for the usual Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\Omega)$, Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\Omega)$, and Triebel-Lizorkin type spaces $F^{s,\tau}_{p,q}(\Omega)$, which are all included in our scales as special cases. Moreover, to complete our investigation, we also study traces of Besov-type spaces $B^{s,\tau}_{p,q}(\Omega)$, which are not covered by our approach and give some applications in terms of *a priori* estimates generalizing results from [Bar05].

Keywords: generalized smoothness Morrey spaces, Besov-type spaces, traces, extension operator, lift operator, quarkonial decomposition, smooth domains.
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1. Introduction

The concept of generalized Besov-Morrey spaces $\mathcal{N}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ with parameters $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, and a function $\varphi : (0,\infty) \to (0,\infty)$ was developed in [NNS16], where the authors studied several properties of these spaces and obtained results concerning traces on hyperplanes. One immediate and obvious advantage of these generalized scales of spaces is, that for different choices of the function parameter φ, one recovers a lot of well-known (scales of) spaces as special cases for which the obtained results immediately follow. For a detailed overview we refer to the discussion in [NNS16, Appendix A].
Thinking about possible applications to PDEs, it is highly desirable to investigate not only spaces defined on $\mathbb{R}^n$ but also corresponding spaces on domains and have trace results on their boundaries as well. This will be the focus of the present paper.

Let us briefly sketch the history of the scales of spaces we wish to consider in the sequel and why they are important to study. Generalized Morrey spaces were

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introduced by Nakai in [Nak94] and defined as

$$\mathcal{M}_p^{\varphi}(\mathbb{R}^n) := \left\{ f \in L_p^{\mathrm{loc}}(\mathbb{R}^n) : \ \|f \mid \mathcal{M}_p^{\varphi}(\mathbb{R}^n)\| < \infty \right\},$$

where

$$\|f \mid \mathcal{M}_p^{\varphi}(\mathbb{R}^n)\| := \sup_{Q \in \mathcal{Q}} \varphi\big(\ell(Q)\big) \left(\frac{1}{|Q|} \int_Q |f(y)|^p \, dy \right)^{\frac{1}{p}}$$

and $\mathcal{Q}$ denotes the set of all dyadic cubes Q with side length $\ell(Q)$. In particular, one recovers the Morrey spaces $\mathcal{M}_{u,p}(\mathbb{R}^n)$, $0 < p \leq u < \infty$, if $\varphi(t) = t^{n/u}$ and the Lebesgue spaces $L_p(\mathbb{R}^n)$ when $\varphi(t) = t^{n/p}$. It should be mentioned that the passage from the classical Morrey spaces to the generalized Morrey spaces is not a mere question of generalization. These spaces naturally emerge when one considers the limiting case of the Sobolev embedding, cf. [SW13, EGNS14, NNS16]. To be more precise, in [SW13, Thm. 5.1] is was shown that for $1 < p < u < \infty$ there exists a constant $C_{u,p}$ such that for $\varphi(t) = (1 + t^n)^{1/u} \left(\log(e + t^{-n})\right)^{-1}$ it holds that

$$\|f|\mathcal{M}_1^{\varphi}(\mathbb{R}^n)\| \leq C_{u,p} \|(1-\Delta)^{n/2u} f|\mathcal{M}_{u,p}(\mathbb{R}^n)\| \qquad \text{for all} \quad f \in \mathcal{M}_1^{\varphi}(\mathbb{R}^n).$$

This is a generalization of Sobolev's embedding since for $u = p$ the right hand side becomes $\|f|H_u^{n/u}(\mathbb{R}^n)\|$, showing the importance of generalized Morrey spaces in the limiting situation when $s = \frac{n}{u}$.

The generalized Besov-Morrey space $\mathcal{N}^s_{\mathcal{M}_p^{\varphi},q}(\mathbb{R}^n)$ is then defined with the help of $\mathcal{M}_p^{\varphi}(\mathbb{R}^n)$ to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f \mid \mathcal{N}^s_{\mathcal{M}_p^{\varphi},q}(\mathbb{R}^n)\big\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}(\mu_j \mathcal{F} f) \mid \mathcal{M}_p^{\varphi}(\mathbb{R}^n)\big\|^q \right)^{1/q} < \infty \tag{1.1}$$

with the usual modification if $q = \infty$. Similar for the generalized Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{\mathcal{M}_p^{\varphi},q}(\mathbb{R}^n)$, where we interchange the order in which the ℓ_q- and $\mathcal{M}_p^{\varphi}$-norms are taken. Concerning the functions μ_j appearing in (1.1) and further details we refer to Definition 2.3.

These scales of spaces cover many well-known function spaces for suitable choices of the function φ. For $\varphi(t) = t^{n/u}$ we get the Besov-Morrey spaces

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^n) = \mathcal{N}^s_{\mathcal{M}_p^{\varphi},q}(\mathbb{R}^n), \qquad \varphi(t) = t^{n/u},$$

which were introduced in [KY94] by Kozono and Yamazaki and used by them and by Mazzucato [Maz03] to study Navier-Stokes equations and the corresponding Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$, which were originally introduced in [TX05] by Tang and Xu, where the authors established the Morrey version of Fefferman-Stein vector-valued inequality. Additionally, if $p = u$, the both scales include the classical Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ as special cases. Furthermore, we also recover the Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ defined in [YSY10], when $0 \le \tau < 1/p$, as $F^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$ with $u = \frac{p}{1-p\tau}$. Note that the corresponding Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ are not covered by our approach. In particular, by [YSY10, Cor. 3.3, p. 64] we have

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^n) \hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^n) \quad \text{with} \quad u = \frac{p}{1-p\tau}, \tag{1.2}$$

and the embedding is proper if $\tau > 0$ and $q < \infty$. In this paper we now define generalized Besov-Morrey spaces $\mathcal{N}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$ and Triebel-Lizorkin Morrey spaces $\mathcal{E}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$ on C^k domains $\Omega \subset \mathbb{R}^n$ via restriction of their counterparts on $\mathbb{R}^n$. Our main result concerning traces on the boundary can be formulated as follows. Let $n \ge 2$,

$$s > \frac{1}{p} + (n-1)\left(\frac{1}{\min(1,p)} - 1\right) \tag{1.3}$$

and put $\varphi^*(t) := \varphi(t)\, t^{-1/p}$. Then, subject to some further restriction on φ, we have

$$\operatorname{Tr}_{\partial\Omega}\mathcal{N}^s_{\mathcal{M}^\varphi_p,q}(\Omega) = \mathcal{N}^{s-\frac{1}{p}}_{\mathcal{M}^{\varphi^*}_p,q}(\partial\Omega), \tag{1.4}$$

where we require the smoothness k of the domain Ω to be large enough, cf. Theorem 3.8. Similar for the generalized Triebel-Lizorkin Morrey spaces $\mathcal{E}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$. These results generalize corresponding trace results on hyperplanes from [NNS16, Thm. 5.1].

Concerning the spaces $B^{s,\tau}_{p,q}(\Omega)$, which by (1.2) are not covered by our scale of generalized Besov-Morrey spaces, in Theorem 3.20 we show that for $0 \le \tau \le \frac{1}{p}$, $n \ge 2$, and (1.3), we have

$$\operatorname{Tr}_{\partial\Omega} B^{s,\tau}_{p,q}(\Omega) = B^{s-\frac{1}{p},\frac{n\tau}{n-1}}_{p,q}(\partial\Omega),$$

where again the smoothness k of the domain Ω has to be large enough. These results are in good agreement with the corresponding trace results for the spaces $F^{s,\tau}_{p,q}(\Omega)$, which are covered by the results from Theorem 3.8. For the proof we develop in Theorem 3.12 a quarkonial decomposition for the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$, which is of independent interest. We apply the trace results for the Besov type spaces in order to obtain some *a priori* estimates for solutions of elliptic boundary value problem, which extend the results from [Bar05].

The paper is organized as follows. In Section 2, we introduce the generalized smoothness Morrey spaces on domains we want to study and provide some important properties needed for our later investigations. In Section 3, we obtain trace results on the boundary. For this a delicate construction of a lift and an extension operator is needed. Finally, Section 4, gives some applications to PDEs.

2. Preliminaries

We start by collecting some general notation used throughout the paper.

As usual, we denote by $\mathbb{N}$ the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^n$, $n \in \mathbb{N}$, the n-dimensional real Euclidean space with $|x|$, for $x \in \mathbb{R}^n$, denoting the Euclidean norm of x and $\langle x \rangle := \sqrt{1+|x|^2}$. Moreover, $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n, x' \in \mathbb{R}^{n-1}, x_n > 0\}$ stands for the half-space. By $\mathbb{Z}^n$ we denote the lattice of all points in $\mathbb{R}^n$ with integer components. Let $\mathbb{N}^n_0$, where $n \in \mathbb{N}$, be the set of all multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$ and $|\alpha| := \sum_{j=1}^n \alpha_j$. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$, then we put $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. For $a \in \mathbb{R}$, let $[a] := \max\{k \in \mathbb{Z} : k \leq a\}$ and $a_+ := \max(a, 0)$. We denote by c a generic positive constant which is independent of the main parameters, but its value may change from line to line. The expression $A \lesssim B$ means that $A \leq c\,B$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded.

If E is a measurable subset of $\mathbb{R}^n$, we denote by χ_E its characteristic function and by $|E|$ its Lebesgue measure. By $\operatorname{supp} f$ we denote the support of the function f.

Let $\mathrm{BC}(\mathbb{R}^n)$ be the space of all bounded continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ and $\mathrm{BUC}(\mathbb{R}^n)$ be the set of those functions that are bounded and uniformly continuous. Both spaces are Banach spaces endowed with the norm $\|f \mid L_\infty(\mathbb{R}^n)\|$. For $k \in \mathbb{N}_0$, we denote by $\mathrm{BC}^k(\mathbb{R}^n)$ the space of all functions $f \in \mathrm{BC}(\mathbb{R}^n)$ such that $D^\alpha f \in \mathrm{BC}(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \leq k$, endowed with the norm $\sum_{|\alpha| \leq k} \|D^\alpha f \mid L_\infty(\mathbb{R}^n)\|$. Let $\nu \in \mathbb{R}$, then $H^\nu_2(\mathbb{R}^n)$ denotes the fractional Sobolev spaces, i.e.,

$$H^\nu_2(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \ \|f|H^\nu_2(\mathbb{R}^n)\| := \|(1-\Delta)^{\nu/2} f|L_2(\mathbb{R}^n)\| < \infty\}.$$

For each cube $Q \subset \mathbb{R}^n$ we denote its centre by x_Q, its side length by $\ell(Q)$, and, for $a \in (0,\infty)$, we denote by aQ the cube concentric with Q having the side length $a\ell(Q)$. For $x \in \mathbb{R}^n$ and $r \in (0,\infty)$ we denote by $Q(x,r)$ the compact cube centred at x with side length r, whose sides are parallel to the axes of coordinates. We write simply $Q(r) = Q(0,r)$ when $x = 0$.

The following is our convention for dyadic cubes. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, denote by Q_{jk} the dyadic cube $2^{-j}([0,1)^n + k)$ and $x_{Q_{jk}}$ its lower left corner. Let $\mathcal{Q} := \{Q_{jk} : j \in \mathbb{Z},\, k \in \mathbb{Z}^n\}$, $\mathcal{Q}^* := \{Q \in \mathcal{Q} : \ell(Q) \leq 1\}$ and $j_Q := -\log_2 \ell(Q)$ for all $Q \in \mathcal{Q}$.

Moreover, for $p, q \in (0,\infty]$, put

$$\sigma_p := n\left(\frac{1}{\min(1,p)} - 1\right), \qquad \sigma_{p,q} := n\left(\frac{1}{\min(1,p,q)} - 1\right).$$

Recall first that the classical Morrey space $\mathcal{M}_{u,p}(\mathbb{R}^n)$, $0 < p \leq u < \infty$, is defined to be the set of all locally p-integrable functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f \mid \mathcal{M}_{u,p}(\mathbb{R}^n)\| := \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{u}-\frac{1}{p}} \left(\int_Q |f(y)|^p \mathrm{d}y\right)^{\frac{1}{p}} < \infty\,. \tag{2.1}$$

In this paper we consider generalized Morrey spaces according to the following definition.

Definition 2.1. *Let $0 < p < \infty$ and $\varphi : (0,\infty) \to (0,\infty)$ be a function. Then $\mathcal{M}_p^{\varphi}(\mathbb{R}^n)$ is the set of all locally p-integrable functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which*

$$\|f \mid \mathcal{M}_p^{\varphi}(\mathbb{R}^n)\| := \sup_{Q \in \mathcal{Q}} \varphi\big(\ell(Q)\big) \left(\frac{1}{|Q|}\int_Q |f(y)|^p \, dy\right)^{\frac{1}{p}} < \infty\,.$$

Remark 2.2. *The above definition goes back to [Nak94]. When $\varphi(t) := t^{\frac{n}{u}}$ for $t > 0$ and $0 < p \leq u < \infty$ then $\mathcal{M}_p^{\varphi}(\mathbb{R}^n)$ coincides with $\mathcal{M}_{u,p}(\mathbb{R}^n)$, which in turn recovers the Lebesgue space $L_p(\mathbb{R}^n)$ when $u = p$. Another example of particular interest is the case of $\varphi(t) := (1+t^n)^{\frac{1}{u}}\big(\log(e+t^{-n})\big)^{-1}$, which arises naturally in the target space when studying embeddings of Sobolev-Morrey spaces in the critical case, cf. [SW13, Thm. 5.1].*

Observe that in the quasi-norm (2.1), in the proper Morrey case of $p < u$, as $\ell(Q)$ increases the integral also increases while the remaining term $|Q|^{\frac{1}{u}-\frac{1}{p}} = \ell(Q)^{\frac{n}{u}-\frac{n}{p}}$ decreases. If we want to keep this feature in the generalized Morrey case it is natural to consider functions φ in the class $\mathcal{G}_p$, $0 < p < \infty$, where $\mathcal{G}_p$ is set of all nondecreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $\varphi(t)t^{-n/p}$ is a nonincreasing function, in this context we also refer to [Nak00].

Let $\mathcal{S}(\mathbb{R}^n)$ be the set of all Schwartz functions on $\mathbb{R}^n$, endowed with the usual topology, and denote by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, namely, the space of all bounded linear functionals on $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak $*$-topology. For all $f \in \mathcal{S}(\mathbb{R}^n)$ or $f \in \mathcal{S}'(\mathbb{R}^n)$, we use $\mathcal{F}f$ to denote its Fourier transform, and $\mathcal{F}^{-1}f$ for its inverse.

Now let us define the generalized Besov-Morrey spaces and the generalized Triebel-Lizorkin-Morrey spaces introduced in [NNS16].

Definition 2.3. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. Let $\mu_0, \mu \in \mathcal{S}(\mathbb{R}^n)$ be nonnegative compactly supported functions satisfying*

$$\mu_0(x) > 0 \quad \textit{if} \quad x \in Q(2),$$

$$0 \notin \operatorname{supp} \mu \quad \textit{and} \quad \mu(x) > 0 \quad \textit{if} \quad x \in Q(2) \setminus Q(1).$$

For $j \in \mathbb{N}$, let $\mu_j(x) := \mu(2^{-j}x)$, $x \in \mathbb{R}^n$.

(i) *The generalized Besov-Morrey space $\mathcal{N}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\big\| f \mid \mathcal{N}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\big\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}(\mu_j \mathcal{F} f) \mid \mathcal{M}^\varphi_p(\mathbb{R}^n)\big\|^q \right)^{1/q} < \infty$$

with the usual modification made in case of $q = \infty$.

(ii) *When $q < \infty$, assume that there exist $C, \varepsilon > 0$ such that*

$$\frac{t^\varepsilon}{\varphi(t)} \le C \frac{r^\varepsilon}{\varphi(r)} \quad \textit{if} \quad t \ge r. \tag{2.2}$$

The generalized Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \mid \mathcal{E}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\big\| := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}(\mu_j \mathcal{F} f)(\cdot)|^q \right)^{1/q} \mid \mathcal{M}^\varphi_p(\mathbb{R}^n) \right\| < \infty$$

with the usual modification made in case of $q = \infty$.

Convention. We shall adopt the nowadays usual custom to write $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}$ instead of $\mathcal{N}^s_{\mathcal{M}^\varphi_p,q}$ or $\mathcal{E}^s_{\mathcal{M}^\varphi_p,q}$, respectively, when both scales of spaces are meant simultaneously in some context, assuming always that there exist $C, \varepsilon > 0$ such that (2.2) holds, when $q < \infty$ and $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}$ denotes $\mathcal{E}^s_{\mathcal{M}^\varphi_p,q}$.

Remark 2.4. *The above spaces have been introduced in [NNS16]. There the authors have proved that those spaces are independent of the choice of the functions μ_0 and μ considered in the definition, as different choices lead to equivalent quasi-norms, cf. [NNS16, Theorem 1.4]. When $\varphi(t) := t^{\frac{n}{u}}$ for $t > 0$ and $0 < p \le u < \infty$ then*

$$\mathcal{N}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n) = \mathcal{N}^s_{u,p,q}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{E}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n) = \mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$$

are the usual Besov-Morrey and Triebel-Lizorkin-Morrey spaces, which are studied in [YSY10] or in the recent survey papers by Sickel [Si12, Si13]. We remark that, in this particular case, the additional condition (2.2) on φ required in Definition 2.3(ii) for the generalized Triebel-Lizorkin-Morrey spaces is automatically fulfilled, as there always exist $0 < \varepsilon < n/u$. Of course, we can recover the classical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and the classical Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ for any $0 < p < \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$, since

$$B^s_{p,q}(\mathbb{R}^n) = \mathcal{N}^s_{p,p,q}(\mathbb{R}^n) \quad \text{and} \quad F^s_{p,q}(\mathbb{R}^n) = \mathcal{E}^s_{p,p,q}(\mathbb{R}^n).$$

Furthermore, we also recover the Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ defined in [YSY10], when $0 \le \tau < 1/p$ as

$$F^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathcal{E}^s_{u,p,q}(\mathbb{R}^n) \quad \text{with} \quad u = \frac{p}{1 - p\tau}, \tag{2.3}$$

for any $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$, cf. [YSY10, Cor. 3.3, p. 63]. Note that the corresponding Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$, see page 29 for the definition, are not covered by our approach. In particular, by [YSY10, Cor. 3.3, p. 64] we have

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^n) \hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^n) \quad \text{with} \quad u = \frac{p}{1 - p\tau}, \tag{2.4}$$

and the embedding is proper if $\tau > 0$ and $q < \infty$. However, if $\tau = 0$ or $q = \infty$ then both spaces coincide. Besides the elementary embeddings

$$\mathcal{A}^{s+\varepsilon}_{\mathcal{M}^\varphi_p,q_1}(\mathbb{R}^n) \hookrightarrow \mathcal{A}^s_{\mathcal{M}^\varphi_p,q_2}(\mathbb{R}^n), \quad \varepsilon > 0,$$

and

$$\mathcal{A}^s_{\mathcal{M}^\varphi_p,q_1}(\mathbb{R}^n) \hookrightarrow \mathcal{A}^s_{\mathcal{M}^\varphi_p,q_2}(\mathbb{R}^n), \quad q_1 \le q_2,$$

cf. [NNS16, Prop. 3.3], we can also easily prove that

$$\mathcal{N}^s_{\mathcal{M}^\varphi_p,\min\{p,q\}}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^s_{\mathcal{M}^\varphi_p,\infty}(\mathbb{R}^n).$$

It is known that

$$\mathcal{E}^0_{u,p,2}(\mathbb{R}^n) = \mathcal{M}_{u,p}(\mathbb{R}^n) \quad \text{for} \quad 1 < p \le u < \infty,$$

cf. [Maz03, Prop. 4.1]. The following is the counterpart for the generalized Triebel-Lizorkin-Morrey spaces.

Proposition 2.5. *Let $1 < p < \infty$ and $\varphi \in \mathcal{G}_p$ satisfy* (2.2)*. If φ is strictly increasing, then*

$$\mathcal{E}^0_{\mathcal{M}^\varphi_p,2}(\mathbb{R}^n) = \mathcal{M}^\varphi_p(\mathbb{R}^n).$$

Proof : This is a consequence of Theorem 3.12 and Proposition 3.18 of [YZY15] letting $\phi(x,r) := \varphi(r)^{-1} r^{\frac{n}{p}}$ for any $x \in \mathbb{R}^n$ and $r > 0$. □

The following result is a direct consequence of Theorem 2.19 from [NNS16] and plays a crucial role.

Theorem 2.6. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $\varphi \in \mathcal{G}_p$, and assume*

$$\nu > \frac{n}{\min(1,p,q)} + \frac{n}{2}.$$

Suppose that for each $j \in \mathbb{N}$ we are given a compact set K_j of $\mathbb{R}^n$ with diameter d_j, $H_j \in \mathcal{S}(\mathbb{R}^n)$ and $f_j \in \mathcal{M}^\varphi_p(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathcal{F} f_j \subset K_j$.

(i) *The inequality*

$$\|2^{js}\mathcal{F}^{-1}(H_j \mathcal{F} f_j) \mid \mathcal{M}^\varphi_p(\mathbb{R}^n)\| \lesssim \left(\sup_{k\in\mathbb{N}_0} \|H_k(d_k\cdot) \mid H^\nu_2(\mathbb{R}^n)\|\right) \|2^{js} f_j \mid \mathcal{M}^\varphi_p(\mathbb{R}^n)\|$$

holds for all $j \in \mathbb{N}$.

(ii) *Assume* (2.2) *in addition when $q < \infty$. If the collection of measurable functions $\{f_j\}_{j=1}^\infty$ satisfies*

$$\left\| \left(\sum_{j=1}^\infty 2^{jsq} |f_j|^q \right)^{1/q} \mid \mathcal{M}^\varphi_p(\mathbb{R}^n) \right\| < \infty,$$

then we have

$$\left\| \left(\sum_{j=1}^\infty 2^{jsq} |\mathcal{F}^{-1}(H_j \mathcal{F} f_j)|^q \right)^{1/q} \mid \mathcal{M}^\varphi_p(\mathbb{R}^n) \right\|$$
$$\lesssim \left(\sup_{k\in\mathbb{N}_0} \|H_k(d_k\cdot) \mid H^\nu_2(\mathbb{R}^n)\|\right) \left\| \left(\sum_{j=1}^\infty 2^{jsq} |f_j|^q \right)^{1/q} \mid \mathcal{M}^\varphi_p(\mathbb{R}^n) \right\|.$$

The atomic decomposition. An important tool in our later considerations is the characterization of the generalized Besov-Morrey and Triebel-Lizorkin-Morrey spaces by means of atomic decompositions. We follow [NNS16] and start by defining the appropriate sequence spaces and atoms.

Definition 2.7. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$.*

(i) *The generalized Besov-Morrey sequence space $\mathbf{n}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ is the set of all doubly indexed sequences $\lambda := \{\lambda_{jm}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C}$ for which the quasi-norm*

$$\|\lambda \mid \mathbf{n}^s_{\mathcal{M}^\varphi_p,q}\| := \left(\sum_{j=1}^{\infty} 2^{jsq} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_{jm}\chi_{Q_{jm}} \,\Big|\, \mathcal{M}^\varphi_p(\mathbb{R}^n)\right\|^q\right)^{1/q}$$

is finite (with the usual modification if $q = \infty$).

(ii) *Assume in addition* (2.2) *when $q < \infty$. The generalized Triebel-Lizorkin-Morrey sequence space $\mathbf{e}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ is the set of all doubly indexed sequences $\lambda := \{\lambda_{jm}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C}$ for which the quasi-norm*

$$\|\lambda|\mathbf{e}^s_{\mathcal{M}^\varphi_p,q}\| := \left\| \left\{ \sum_{j=1}^{\infty} 2^{jsq} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|\chi_{Q_{jm}}\right)^q \right\}^{1/q} \,\Big|\, \mathcal{M}^\varphi_p(\mathbb{R}^n)\right\|$$

is finite (with the usual modification if $q = \infty$).

Convention. We adopt the same custom to write $\mathbf{a}^s_{\mathcal{M}^\varphi_p,q}$ instead of $\mathbf{n}^s_{\mathcal{M}^\varphi_p,q}$ or $\mathbf{e}^s_{\mathcal{M}^\varphi_p,q}$, for convenience, when both scales are meant simultaneously, assuming always that there exist $C, \varepsilon > 0$ such that (2.2) holds, when $q < \infty$ and $\mathbf{a} = \mathbf{e}$.

Definition 2.8. *Let $L \in \mathbb{N}_0 \cup \{-1\}$, $K \in \mathbb{N}_0$, and $d > 1$. A C^K-function $a : \mathbb{R}^n \to \mathbb{C}$ is said to be a (K, L, d)-atom centered at Q_{jm}, where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if*

$$2^{-j|\alpha|}|D^\alpha a(x)| \leq \chi_{dQ_{jm}}(x) \tag{2.5}$$

for all $x \in \mathbb{R}^n$ and for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq K$, and when for $j \in \mathbb{N}$ it holds

$$\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \tag{2.6}$$

for all $\beta \in \mathbb{N}^n_0$ with $|\beta| \leq L$ when $L \geq 0$. In the sequel we write a_{jm} instead of a if the atom is located at Q_{jm}, i.e., $\operatorname{supp} a_{jm} \subset dQ_{jm}$.

The following coincides with [NNS16, Thm. 4.4, Thm. 4.5], cf. also [NNS16, Rmk. 4.3].

Theorem 2.9. *Let* $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, *and* $\varphi \in \mathcal{G}_p$. *Assume in addition that* φ *satisfies* (2.2) *when* $q < \infty$ *and* $\mathcal{A} = \mathcal{E}$. *Let also* $d > 1$, $L \in \mathbb{N}_0 \cup \{-1\}$ *and* $K \in \mathbb{N}_0$ *be such that*

$$K \geq [1+s]_+ \quad \text{and} \quad L \geq \begin{cases} \max(-1, [\sigma_p - s]), & \text{if } \mathcal{A} = \mathcal{N}, \\ \max(-1, [\sigma_{pq} - s]), & \text{if } \mathcal{A} = \mathcal{E}. \end{cases}$$

(i) *Let* $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$. *Then there exists a family* $\{a_{jm}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *of* (K, L, d)*-atoms and a sequence* $\lambda = \{\lambda_{jm}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n} \in \mathbf{a}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ *such that*

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm} \qquad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)$$

and

$$\|\lambda \mid \mathbf{a}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \lesssim \|f \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|.$$

(ii) *Let* $\{a_{jm}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ *be a family of* (K, L, d)*-atoms and* $\lambda = \{\lambda_{jm}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n} \in \mathbf{a}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$. *Then*

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ *and belongs to* $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$. *Furthermore*

$$\|f \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \lesssim \|\lambda \mid \mathbf{a}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|.$$

The quarkonial decomposition. The consideration of special atoms, so-called quarks, and subatomic or quarkonial decompositions goes back to [Tri97]. For the quarkonial decomposition for the spaces $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ we follow [NNS16].

Throughout this section the function $\theta \in \mathcal{S}(\mathbb{R}^n)$ is fixed so that it has compact support and $\{\theta(\cdot - m)\}_{m\in\mathbb{Z}^n}$ forms a partition of unity:

$$\sum_{m\in\mathbb{Z}^n} \theta(x-m) = 1 \quad \text{for} \quad x \in \mathbb{R}^n \tag{2.7}$$

and, for some $R > 0$,

$$\operatorname{supp} \theta \subset 2^R Q_{00}. \tag{2.8}$$

Remark 2.10. *In [NNS16] the authors used* $\operatorname{supp}\theta \subset Q(2^R)$ *instead of* (2.8). *We changed the notation since this does not fit with the cubes used in that paper with left corner* $2^{-j}m$; *in particular, cf. [NNS16, formula (4.27)].*

Definition 2.11. *Let $\beta \in \mathbb{N}_0^n$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then the function θ^β and the quark $(\beta qu)_{\nu m}$ are defined by*

$$\theta^\beta(x) := x^\beta \theta(x) \quad \text{and} \quad (\beta qu)_{\nu m}(x) := \theta^\beta(2^\nu x - m) \quad \text{for} \quad x \in \mathbb{R}^n.$$

The following coincides with [NNS16, Thm. 4.18].

Theorem 2.12. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. Assume in addition that φ satisfies (2.2) when $q < \infty$ and $\mathcal{A} = \mathcal{E}$. Suppose further that*

$$s > \begin{cases} \sigma_p, & \text{if } \mathcal{A} = \mathcal{N}, \\ \sigma_{p,q}, & \text{if } \mathcal{A} = \mathcal{E}, \end{cases}$$

and let ρ be such that $\rho > R$, where R is a constant as in (2.8).

(i) *Let $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$. Then there exists a triply indexed complex sequence*

$$\lambda := \{\lambda^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0^n, \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$$

such that

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m}\, (\beta qu)_{\nu m}, \tag{2.9}$$

convergence being in $\mathcal{S}'(\mathbb{R}^n)$, and

$$\|\lambda \mid \mathbf{a}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|_\rho := \sup_{\beta \in \mathbb{N}_0^n} 2^{\rho|\beta|} \, \|\lambda^\beta \mid \mathbf{a}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \lesssim \|f \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|.$$

The numbers $\lambda^\beta_{\nu m}$ depend continuously and linearly on f.

(ii) *If $\lambda := \{\lambda^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0^n, \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ satisfies $\|\lambda \mid \mathbf{a}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|_\rho < \infty$, then*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m}\, (\beta qu)_{\nu m} \tag{2.10}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and belongs to $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$. Furthermore

$$\|f \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \lesssim \|\lambda \mid \mathbf{a}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|_\rho.$$

Embeddings into BUC($\mathbb{R}^n$). The next result was proved in [NNS16, Lem. 3.4] and will be used in Theorem 3.7 for the construction of a suitable extension operator.

Proposition 2.13. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. Assume in addition that φ satisfies (2.2) when $q < \infty$ and $\mathcal{A} = \mathcal{E}$. If $s > 0$ is such that*

$$\sum_{j=1}^{\infty} \frac{1}{2^{sj}\varphi(2^{-j})} < \infty, \tag{2.11}$$

then

$$\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow \mathrm{BUC}(\mathbb{R}^n).$$

Remark 2.14. *Since $\varphi \in \mathcal{G}_p$, it is clear that (2.11) is satisfied when $s > \frac{n}{p}$. Moreover, (2.11) is also necessary in order to have $\mathcal{N}_{\mathcal{M}^\varphi_p,q,\infty}(\mathbb{R}^n) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^n)$, cf. [NNS16, Rmk. 3.5].*

2.1. Generalized smoothness Morrey spaces on domains. Let $\Omega \subset \mathbb{R}^n$ be a domain. Recall that domain always stands for open set. Furthermore, $\Gamma = \partial\Omega$ denotes the boundary of Ω. We define generalized smoothness Morrey spaces on domains in the usual way by restriction. Recall that $\mathcal{D}'(\Omega)$ is the collection of all complex-valued distributions on Ω. If $g \in \mathcal{S}'(\mathbb{R}^n)$ then the restriction of g to Ω is an element of $\mathcal{D}'(\Omega)$, which will be denoted by $g\big|_\Omega$.

Definition 2.15. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. Additionally assume that φ satisfies (2.2) when $q < \infty$ and $\mathcal{A} = \mathcal{E}$. The space $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$ is defined as the restriction of the corresponding space $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ to Ω, quasi-normed by*

$$\|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega)\| := \inf \|g|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ with $g\big|_\Omega = f$ in the sense of $\mathcal{D}'(\Omega)$.

Since in the sequel we will deal with so-called C^k domains $\Omega \subset \mathbb{R}^n$ and traces on their boundary, we give a precise definition.

Definition 2.16. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then Ω is said to be a C^k domain, $k \in \mathbb{N} \cup \{\infty\}$, if there exist N open balls $K_1, \ldots, K_N$ such that*

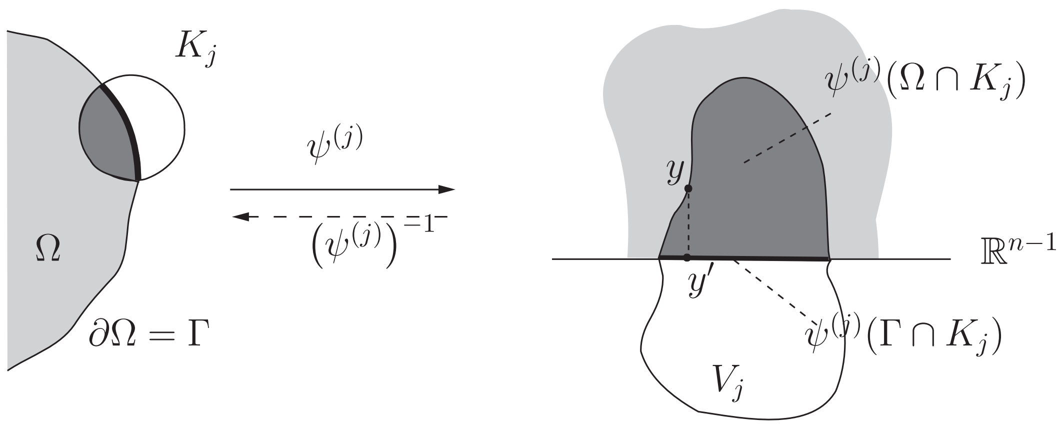
$$\bigcup_{j=1}^{N} K_j \supset \Gamma \qquad \textit{and} \qquad K_j \cap \Gamma \neq \emptyset \quad \textit{if} \quad j = 1, \ldots, N,$$

with the following property: for every ball K_j there are diffeomorphic C^k maps $\psi^{(j)}$ such that

$$\psi^{(j)} : K_j \to V_j, \quad j = 1, \dots, N,$$

where

$$\psi^{(j)}(K_j \cap \Omega) \subset \mathbb{R}^n_+, \qquad \psi^{(j)}(K_j \cap \Gamma) \subset \mathbb{R}^{n-1}.$$



Remark 2.17. The C^k maps $\psi^{(j)}$ can be extended outside K_j in such a way that the extended vector functions (denoted by $\psi^{(j)}$ as well) yield diffeomorphic mappings from $\mathbb{R}^n$ onto itself (k-diffeomorphisms). Note that our understanding of a C^k diffeomorphism implies that the inverse $\left(\psi^{(j)}\right)^{-1}$ is also a C^k map. In [NNS16, Sect. 5.3] a diffeomorphism with this property was called *regular*. We do not make this distinction.

Resolution of unity. Let K_j with $j = 1, \dots, N$ be the same balls as in Definition 2.16. Let K_0 be an inner domain with $\overline{K}_0 \subset \Omega$, so that

$$\Gamma \subset \bigcup_{j=1}^{N} K_j \qquad \text{and} \qquad \overline{\Omega} \subset K_0 \cup \left(\bigcup_{j=1}^{N} K_j \right).$$

Let $\{\varphi_j\}_{j=0}^N$ be a related resolution of unity of $\overline{\Omega}$, i.e., φ_j are nonnegative functions with

$$\varphi_j \in \mathcal{D}(K_j) \qquad \text{and} \qquad \sum_{j=0}^{N} \varphi_j(x) = 1 \tag{2.12}$$

in a neighbourhood of $\overline{\Omega}$. Obviously, the restriction of φ_j to Γ is a resolution of unity with respect to Γ. Now we can decompose $f \in L_p(\Omega)$ such that

$$f(x) = \varphi_0(x) f(x) + \sum_{j=1}^{N} \varphi_j(x) f(x), \qquad x \in \Omega,$$

where the term $\varphi_0 f$ can be extended outside of Ω by zero.

Generalized smoothness Morrey spaces on the boundary. We consider the boundary $\partial\Omega = \Gamma$ of a bounded C^k domain Ω. Then $\mathcal{D}'(\Gamma)$ stands for the distributions on the compact C^k manifold Γ.

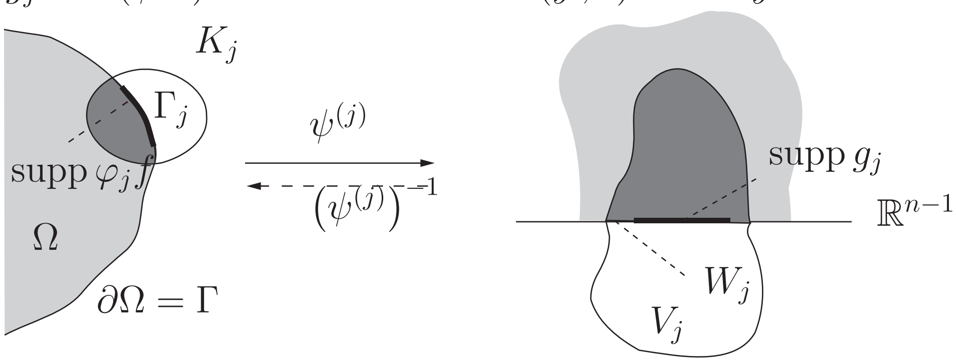
We require the introduction of Besov-Morrey and Triebel-Lizorkin-Morrey spaces on Γ. We rely on the resolution of unity according to (2.12) and the local diffeomorphisms $\psi^{(j)}$ mapping $\Gamma_j = \Gamma \cap K_j$ onto $W_j = \psi^{(j)}(\Gamma_j)$, recall Definition 2.16. We define

$$g_j(y) := (\varphi_j f) \circ (\psi^{(j)})^{-1}(y), \qquad j = 1, \dots, N,$$

which restricted to $y = (y', 0) \in W_j$,

$$g_j(y') = (\varphi_j f) \circ (\psi^{(j)})^{-1}(y'), \qquad j = 1, \dots, N, \quad f \in D'(\Gamma),$$

makes sense. This results in distributions $g_j \in \mathcal{D}'(W_j)$ with compact supports in the $(n-1)$-dimensional C^k domain W_j. We do not distinguish notationally between g_j and $(\psi^{(j)})^{-1}$ as distributions of $(y', 0)$ and of y'.



Our constructions enable us to define the generalized smoothness Morrey spaces on the boundary Γ as follows.

Definition 2.18. *Let $n \geq 2$, and let Ω be a bounded C^k domain in $\mathbb{R}^n$ with boundary Γ, and φ_j, $\psi^{(j)}$, W_j be as above. Assume $0 < p < \infty$, $0 < q \leq \infty$,*

$s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. Additionally assume that φ satisfies (2.2) *when $q < \infty$ and $\mathcal{A} = \mathcal{E}$. Then we introduce*

$$\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Gamma) := \{f \in \mathcal{D}'(\Gamma) : g_j \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(W_j),\ j = 1, \dots, N\},$$

equipped with the quasi-norm

$$\|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Gamma)\| := \sum_{j=1}^N \|g_j|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(W_j)\|.$$

Remark 2.19. The spaces $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Gamma)$ turn out to be independent of the particular choice of the covering $\{K_j\}_{j=1}^N$, the resolution of unity $\{\varphi_j\}_{j=1}^N$ and the local diffeomorphisms $\{\psi^{(j)}\}_{j=1}^N$ (the proof is similar to the proof of [Tri83, Prop. 3.2.3(ii)], making use of Theorem 2.20 and Proposition 2.21 below).
Note that we could furthermore replace W_j in the definition of the norm above by $\mathbb{R}^{n-1}$ if we extend g_j outside W_j with zero, i.e.,

$$\|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Gamma)\| \sim \sum_{j=1}^N \|g_j|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^{n-1})\|.$$

2.2. Diffeomorphisms and multipliers.

The following theorem about diffeomorphisms and pointwise multiplication can be found in [NNS16, Thm. 5.4, Thm. 5.5].

Theorem 2.20. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. Additionally assume that φ satisfies* (2.2) *when $q < \infty$ and $\mathcal{A} = \mathcal{E}$. Moreover, let*

$$k > s > \begin{cases} \sigma_p, & \text{if } \mathcal{A} = \mathcal{N}, \\ \sigma_{p,q}, & \text{if } \mathcal{A} = \mathcal{E}. \end{cases}$$

(i) *For all $g \in \mathrm{BC}^k(\mathbb{R}^n)$ we have that*

$$f \to gf$$

is a linear and bounded operator from $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ into itself, i.e., there exists a positive constant $C(k)$ such that

$$\|gf|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \le C(k)\|g|\mathrm{BC}^k(\mathbb{R}^n)\| \cdot \|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|.$$

(ii) *For all k-diffeomorphisms ψ we have that*

$$f \to f \circ \psi$$

is a linear and bounded operator from $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ into itself. In particular, we have for some positive constant $C(\psi)$,

$$\|f \circ \psi|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \le C(\psi)\|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|.$$

For later purposes we establish an equivalent norm for $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$.

Proposition 2.21. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\varphi \in \mathcal{G}_p$. Additionally assume that φ satisfies (2.2) when $q < \infty$ and $\mathcal{A} = \mathcal{E}$. Furthermore, let $\Omega \subset \mathbb{R}^n$ be a bounded C^k domain with*

$$k > s > \begin{cases} \sigma_p, & \text{if } \mathcal{A} = \mathcal{N}, \\ \sigma_{p,q}, & \text{if } \mathcal{A} = \mathcal{E}. \end{cases}$$

Then

$$\|\varphi_0 f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| + \sum_{j=1}^{N} \|(\varphi_j f)(\psi^{(j)^{-1}}(\cdot))|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)\| \tag{2.13}$$

is an equivalent quasi-norm in $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$, where we extended $\varphi_0 f$ by zero outside K_0 and $(\varphi_j f)(\psi^{(j)^{-1}}(\cdot))$ by zero from $\psi^{(j)}(K_j \cap \Omega)$ to $\mathbb{R}^n_+$ for $j = 1, \ldots, N$.

The proof is the same as for the classical case, cf. [Tri83, Prop. 3.2.3] relying now on Theorem 2.20.

3. Traces on the boundary of C^k domains Ω

The trace operator plays an important role when dealing with existence and uniqueness of solutions of boundary value problems on domains.

Assume that $n \ge 2$. If $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we put $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ and in this case we might also write $x = (x', x_n)$. We explain our understanding of the trace operator on the hyperplane $\{(x', 0) : x' \in \mathbb{R}^{n-1}\}$ of $\mathbb{R}^n$, interpreted as $\mathbb{R}^{n-1}$, in the context of the scales of (generalized) Besov-Morrey and Triebel-Lizorkin-Morrey spaces.

If f is a smooth function, e.g., $f \in \mathcal{S}(\mathbb{R}^n)$, it makes sense to define the restriction of f pointwise on the hyperplane and define the trace operator $\mathrm{Tr}_{\mathbb{R}^{n-1}}$ by

$$(\mathrm{Tr}_{\mathbb{R}^{n-1}} f)(x') := f(x', 0), \quad x' \in \mathbb{R}^{n-1}. \tag{3.1}$$

Let $X(\mathbb{R}^n) := \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ denote a generalized smoothness Morrey space according to Definition 2.3. In order to define the trace for $f \in X(\mathbb{R}^n)$ we use the atomic decomposition from Theorem 2.9, i.e.,

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \tag{3.2}$$

and define $\mathrm{Tr}_{\mathbb{R}^{n-1}} f$ by

$$\mathrm{Tr}_{\mathbb{R}^{n-1}} f := \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}(\cdot', 0) \right). \tag{3.3}$$

This definition makes sense since the proof of the atomic decomposition from Theorem 2.9 reveals that $\lambda_{jm} a_{jm}$ is obtained canonically from $f \in X(\mathbb{R}^n)$, meaning there exists a continuous linear operator I_{jm} from $X(\mathbb{R}^n)$ into the space of L_∞-functions with compact support such that $I_{jm}(f) := \lambda_{jm} a_{jm}$.

If $f \in \mathcal{S}(\mathbb{R}^n)$ then (3.3) actually coincides with (3.1). This can be seen as follows: the limit in (3.2) takes place in $BUC(\mathbb{R}^n)$ since $f \in B^{\varepsilon}_{\infty,\infty}(\mathbb{R}^n) \hookrightarrow BUC(\mathbb{R}^n)$ for all $\varepsilon \in (0,1)$. But this in turn implies pointwise convergence since

$$\lim_{J \to \infty} \left(\sup_{x \in \mathbb{R}^n} \left| f(x) - \sum_{j=0}^{J} \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}(x) \right| \right) = 0,$$

demonstrating that (3.3) is well-defined.

Remark 3.1. *In order to explain the understanding of the trace operator in some function space $X(\mathbb{R}^n)$ one usually uses the fact that $\mathcal{S}(\mathbb{R}^n)$ is dense in $X(\mathbb{R}^n)$, since then the trace is completely defined by density arguments. However, this is not available when dealing with generalized Besov-Morrey and Triebel-Lizorkin-Morrey spaces, which is why we have to rely on the atomic decomposition techniques as described from above. This approach was also used in [NNS16, Saw10].*

Below we now recall the trace results on hyperplanes, which were obtained in [NNS16, Thms. 5.1, 5.3].

Theorem 3.2. *Let $n \geq 2$. Let $0 < p < \infty$, $0 < q \leq \infty$, and $\varphi \in \mathcal{G}_p$. Additionally assume that φ satisfies (2.2) when $q < \infty$ and $\mathcal{A} = \mathcal{E}$. Define s^* and φ^* by*

$$s^* := s - \frac{1}{p} \qquad \textit{and} \qquad \varphi^*(t) := \varphi(t)\, t^{-1/p}, \quad t > 0.$$

Assume that

$$s > \frac{1}{p} + (n-1)\cdot \begin{cases} \left(\frac{1}{\min(1,p)} - 1\right), & \text{if } \mathcal{A} = \mathcal{N}, \\ \left(\frac{1}{\min(1,p,q)} - 1\right), & \text{if } \mathcal{A} = \mathcal{E}. \end{cases}$$

and that φ^* *is increasing and satisfies*

$$\sum_{j=0}^{\infty} \frac{1}{\varphi^*(2^j s)} \lesssim \frac{1}{\varphi^*(s)}, \quad 0 < s \leq 1.$$

Then $\mathrm{Tr}_{\mathbb{R}^{n-1}}$ *is a bounded linear operator from* $\mathcal{A}^s_{\mathcal{M}^{\varphi}_p,q}(\mathbb{R}^n)$ *onto* $\mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1})$*,*

$$\mathrm{Tr}_{\mathbb{R}^{n-1}}\mathcal{A}^s_{\mathcal{M}^{\varphi}_p,q}(\mathbb{R}^n) = \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1}) \qquad \text{where} \quad r = \begin{cases} q, & \text{if } \mathcal{A} = \mathcal{N}, \\ p, & \text{if } \mathcal{A} = \mathcal{E}. \end{cases}$$

3.1. Lift operator. In this section we use the family of lift operators $\{J_\sigma\}_{\sigma\in\mathbb{R}}$ due to Franke and Runst [FR95], which goes back to Triebel [Tri78], and extend it to our setting such that it has the following properties:

- J_σ is an isomorphism from $\mathcal{A}^s_{\mathcal{M}^{\varphi}_p,q}(\mathbb{R}^n)$ to $\mathcal{A}^{s-\sigma}_{\mathcal{M}^{\varphi}_p,q}(\mathbb{R}^n)$.
- J_σ and $J_{-\sigma}$ are inverse to each other.
- If $f \in \mathcal{S}'(\mathbb{R}^n)$ is supported on $\mathbb{R}^{n-1} \times (-\infty, 0]$, so is $J_\sigma f$.

Note that by [NNS16, Prop. 3.2] the lift operators $(1-\Delta)^{\sigma/2}$ also satisfy the above conditions except the third one, which is crucial for us.

We start with the construction of a function on which our family of operators is build on. Let $\eta \in \mathcal{S}(\mathbb{R})$ be a positive real-valued function with $\operatorname{supp}\eta \subset [-2,-1]$ and $\int_{\mathbb{R}} \eta(x)\mathrm{d}x = 2$. For any $0 \leq \varepsilon \ll 1$, we define a holomorphic function ψ_ε on $\mathbb{C}$ by

$$\psi_\varepsilon(z) := \int_{-\infty}^{0} \eta(t) e^{-i\varepsilon t z}\mathrm{d}t - iz.$$

Let $\mathbb{H} := \{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\}$ and $\overline{\mathbb{H}} := \{z \in \mathbb{C} : \ \mathrm{Im}(z) \geq 0\}$. Furthermore, we consider the domain $\Omega = \{z \in \mathbb{C} : \ |z| > 4, \mathrm{Re}(z) > 0\}$. If $z \in \mathbb{C}$ satisfies $|z| > 4$ and $\mathrm{Im}(z) > 0$, it follows that $-iz \in \Omega$. Hence, we see that

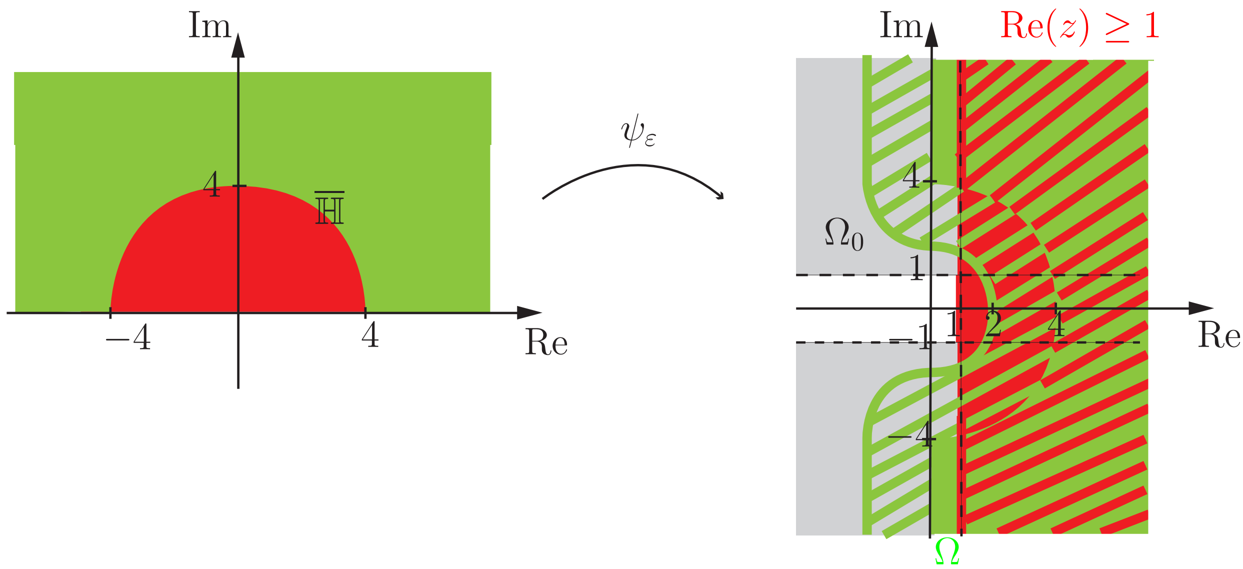
$$\mathrm{dist}(\psi_\varepsilon(z), \Omega) \leq |\psi_\varepsilon(z) + iz| = \left| \int_{\infty}^{0} \eta(t) e^{-i\varepsilon t z}\mathrm{d}t \right| < 2. \tag{3.4}$$

If $z \in \mathbb{C}$ satisfies $|z| \leq 4$ and $\mathrm{Im}(z) \geq 0$, then we have $\mathrm{Re}(\psi_0(z)) = 2 + \mathrm{Im}(z)$. Thus, for any $0 < \varepsilon \ll 1$, we obtain

$$\mathrm{Re}(\psi_\varepsilon(z)) = \int_{-\infty}^{0} \eta(t) e^{\varepsilon t \mathrm{Im}(z)} \cos(\varepsilon t \mathrm{Re}(z)) \mathrm{d}t + \mathrm{Im}(z) \geq \frac{3}{2},$$

since the integrand is continuous. If $\varepsilon > 0$ is a sufficiently small number, we see that ψ_ε maps $\overline{\mathbb{H}}$ to

$$\Omega_0 := \{z \in \mathbb{C} : \ \mathrm{Re}(z) > 1\} \cup \{z \in \mathbb{C} : \ |\mathrm{Im}(z)| > 1\}.$$



Below fix a small $\varepsilon > 0$. We select a branch-cut of $\log$ on $\mathbb{C} \setminus (-\infty, 0]$ such that $\log 1 = 0$. Then we define $z^a = \exp(a \log z)$ for $z \in \mathbb{C} \setminus (-\infty, 0]$. For any $\sigma \in \mathbb{R}$, we define the function $\Phi^{(\sigma)} : \mathbb{R}^{n-1} \times \overline{\mathbb{H}} \to \mathbb{C}$ by

$$\Phi^{(\sigma)}(x', z_n) := \left(\langle x' \rangle \psi_\varepsilon \left(\frac{z_n}{\langle x' \rangle} \right) \right)^\sigma, \quad z_n \in \overline{\mathbb{H}},$$

which is well-defined for $\sigma \in \mathbb{R}$. Here we put $\langle x' \rangle = \sqrt{1 + x_1^2 + \ldots + x_{n-1}^2}$. Setting $\Phi^{(1)} = \Phi$ we have the following lemma clarifying the behaviour of Φ with respect to differentiation, cf. [Saw10, L. 4.3].

Lemma 3.3. *For any multi-index $\alpha \in \mathbb{N}_0^n$, there exists a constant $c_\alpha > 0$ such that*

$$|\partial^\alpha \Phi(x', z_n)| \leq c_\alpha \left(\langle x' \rangle + |z_n| \right)^{1-|\alpha|} \tag{3.5}$$

for all $(x', z_n) \in \mathbb{R}^{n-1} \times \overline{\mathbb{H}}$. Furthermore, we can even arrange that c_0 satisfies

$$c_0^{-1} \left(\langle x' \rangle + |z_n| \right) \leq |\Phi(x', z_n)| \leq c_0 \left(\langle x' \rangle + |z_n| \right) \tag{3.6}$$

for all $(x', z_n) \in \mathbb{R}^{n-1} \times \overline{\mathbb{H}}$.

We will also use the same symbol $\Phi^{(\sigma)}$ for $\Phi^{(\sigma)}\big|_{\mathbb{R}^{n-1}\times\mathbb{R}}$. Then by Theorem 2.6 and Lemma 3.3, we derive the following proposition.

Proposition 3.4. *Let* $0 < p < \infty$, $0 < q \leq \infty$, *and* $\varphi \in \mathcal{G}_p$. *Additionally assume that* φ *satisfies* (2.2) *when* $q < \infty$ *and* $\mathcal{A} = \mathcal{E}$. *Then for any* $\sigma \in \mathbb{R}$ *we have the following properties:*

(i) $J_\sigma := \mathcal{F}^{-1}[\Phi^{(\sigma)}\mathcal{F}]$ *is a linear isomorphism between* $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ *and* $\mathcal{A}^{s-\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$.

(ii) $J_{-\sigma}$ *is the inverse operator of* J_σ.

(iii) *For any* $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$, *we have* $\|J_\sigma f|\mathcal{A}^{s-\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \sim \|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|$.

Proof : Let μ_0, μ be functions as in Definition 2.3, and let $R \in \mathbb{N}$ be such that

$$\operatorname{supp}\mu_0 \subset \{x \in \mathbb{R}^n : |x| \leq 2^R\} \quad \text{and} \quad \operatorname{supp}\mu \subset \{x \in \mathbb{R}^n : 2^{-R} \leq |x| \leq 2^R\}.$$

We consider the case $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n) = \mathcal{E}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ as the other case follows in an analogous way. We have

$$\big\|J_\sigma f \mid \mathcal{E}^{s-\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\big\| = \left\|\left(\sum_{j=0}^{\infty} 2^{j(s-\sigma)q}|\mathcal{F}^{-1}(\mu_j\Phi^{(\sigma)}\mathcal{F}f)(\cdot)|^q\right)^{1/q} \mid \mathcal{M}^\varphi_p(\mathbb{R}^n)\right\|.$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\phi(x) := 1 \quad \text{if} \quad 2^{-R} \leq |x| \leq 2^R \quad \text{and} \quad \operatorname{supp}\phi \subset \{x \in \mathbb{R}^n : 2^{-R-1} \leq |x| \leq 2^{R+1}\}.$$

Then

$$\mathcal{F}^{-1}(\mu_j\Phi^{(\sigma)}\mathcal{F}f) = \mathcal{F}^{-1}(\Phi^{(\sigma)}\phi(2^{-j}\cdot)\mu_j\mathcal{F}f)\,, \quad j \in \mathbb{N}.$$

Applying Theorem 2.6 with $\nu \in \mathbb{N}$ such that $\nu > \frac{n}{\min(1,p,q)} + \frac{n}{2}$ and

$$H_j(x) := 2^{-\sigma j}\Phi^{(\sigma)}(x)\phi(2^{-j}x), \quad j \in \mathbb{N},$$

we obtain

$$\begin{aligned}
&\left\|\left(\sum_{j=1}^{\infty} 2^{j(s-\sigma)q}|\mathcal{F}^{-1}(\mu_j\Phi^{(\sigma)}\mathcal{F}f)(\cdot)|^q\right)^{1/q} \mid \mathcal{M}^\varphi_p(\mathbb{R}^n)\right\| \\
&\lesssim \left(\sup_{k\in\mathbb{N}} \|H_k(2^{k+R+1}\cdot) \mid H^\nu_2(\mathbb{R}^n)\|\right)\left\|\left(\sum_{j=1}^{\infty} 2^{jsq}|\mathcal{F}^{-1}(\mu_j\mathcal{F}f)(\cdot)|^q\right)^{1/q} \mid \mathcal{M}^\varphi_p(\mathbb{R}^n)\right\| \\
&\lesssim \|f \mid \mathcal{E}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|,
\end{aligned}$$

where we used Lemma 3.3 to estimate the first term in the last inequality. The term corresponding to $j = 0$ can be dealt with in a similar way, so that we arrive at

$$\left\| J_\sigma f \mid \mathcal{E}^{s-\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\right\| \lesssim \|f \mid \mathcal{E}^{s}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|.$$

The proof is completed by observing that $J_\sigma J_{-\sigma} f = f$. □

For the support of $J_\sigma f$ we have the following result. A proof may be found in [Saw10, Prop. 4.6].

Proposition 3.5. *If $f \in \mathcal{S}'(\mathbb{R}^n)$ is supported in $\mathbb{R}^{n-1} \times (-\infty, 0]$, then so is $J_\sigma f$.*

We have seen that our family of lift operators satisfies all the required properties stated at the beginning of this section. This now enables us to prove the following corollary, which will be used in the next section in order to construct a suitable extension operator.

Corollary 3.6. *Let $0 < p < \infty$, $0 < q \leq \infty$, and $\varphi \in \mathcal{G}_p$. Additionally assume that φ satisfies (2.2) when $q < \infty$ and $\mathcal{A} = \mathcal{E}$.*

- **(i)** *Let $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$. Then $J_\sigma f := (J_\sigma g)\big|_{\mathbb{R}^n_+}$ does not depend on the choice of the representative $g \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ of f.*
- **(ii)** *J_σ is an isomorphism from $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$ to $\mathcal{A}^{s-\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$. Furthermore, $J_{-\sigma}$ is the inverse of J_σ.*

P r o o f : We will prove (i). Let $g_1, g_2 \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ satisfy $f = g_1\big|_{\mathbb{R}^n_+} = g_2\big|_{\mathbb{R}^n_+}$. Then we have

$$(J_\sigma g_1)\big|_{\mathbb{R}^n_+} - (J_\sigma g_2)\big|_{\mathbb{R}^n_+} = (J_\sigma(g_1 - g_2))\big|_{\mathbb{R}^n_+} = 0,$$

by linearity of J_σ, the fact that $(g_1 - g_2)\big|_{\mathbb{R}^n_+} = 0$ and Proposition 3.5. Thus, we obtain

$$(J_\sigma g_1)\big|_{\mathbb{R}^n_+} = (J_\sigma g_2)\big|_{\mathbb{R}^n_+},$$

which means that $J_\sigma f$ does not depend on $g \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ satisfying $g\big|_{\mathbb{R}^n_+} = f$. Assertion (ii) follows immediately from the properties of J_σ as an operator on $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$. □

3.2. Extension operator for $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$. Having constructed the lift operator J_σ, we are now able to establish the following extension theorem. We shall deal with the following set in the sequel:

$$R(N) := \left\{(p,q,s):\ \frac{1}{N} \le p < \infty,\ \frac{1}{N} \le q \le \infty,\ |s| < N\right\}.$$

Theorem 3.7. *Let $0 < p < \infty$, $0 < q \le \infty$, and $\varphi \in \mathcal{G}_p$. Additionally assume that φ satisfies* (2.2) *when $q < \infty$ and $\mathcal{A} = \mathcal{E}$. Then for $N \in \mathbb{N}$, there exists an extension operator Ext_N,*

$$\mathrm{Ext}_N : \bigcup_{(p,q,s)\in R(N)} \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+) \longrightarrow \bigcup_{(p,q,s)\in R(N)} \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n),$$

that satisfies the properties: if $(p,q,s) \in R(N)$ then the restriction $\mathrm{Ext}_N\big|_{\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)}$ is a continuous mapping from $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$ to $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ satisfying $\mathrm{Ext}_N f\big|_{\mathbb{R}^n_+} = f$ for all $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$.

Proof : Step 1. We start with the general set up. Details may be found in [Eva98, Ch. 5]. Let $M \in \mathbb{N}$ be large enough. We define $\lambda_1, \dots, \lambda_M$ so that

$$\sum_{j=0}^{M} (-j)^l \lambda_j = \delta_{0,l}, \tag{3.7}$$

for all $l = 0, \dots, M$. Here $\delta_{0,l}$ denotes the Kronecker-symbol, i.e., $\delta_{0,0} = 1$ and $\delta_{0,l} = 0$ for $l \ge 1$, and it is assumed that $0^l = \delta_{0,l}$. The determinant of this system of linear equations is a constant multiple of the Vandermonde determinant $\{j^i\}_{i,j=0,1,\dots,M}$, which is never 0. Therefore, the unknowns $\lambda_1, \dots, \lambda_M$ are determined uniquely by (3.7). Given a function $f : \mathbb{R}^{n-1} \times [0,\infty) \to \mathbb{C}$, we define

$$f^*(x) := \begin{cases} f(x), & \text{if } x_n \ge 0, \\ \sum_{j=0}^{M} \lambda_j f(x', -jx_n), & \text{if } x_n \le 0. \end{cases}$$

By definition of the λ_j we see that $\sum_{j=0}^{M} \lambda_j f(x',0) = f(x',0)$. Furthermore, if f is in BC^M and defined in a neighbourhood of $\mathbb{R}^{n-1} \times [0,\infty)$, then $f^* : \mathbb{R}^n \to \mathbb{C}$ also belongs to $\mathrm{BC}^M(\mathbb{R}^n)$.

Step 2. We define Ext_N for $s > \frac{n}{p}$. By Proposition 2.13 and Remark 2.14, in this case we have

$$\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{BUC}(\mathbb{R}^n). \tag{3.8}$$

We assume now that

$$\frac{1}{N} \le p < \infty, \quad \frac{1}{N} \le q \le \infty, \quad \frac{n}{p} < s \le N. \tag{3.9}$$

Let $M \gg (n+1)N$ be large enough ($M > s$ for all s in (3.9)), where M is the integer from Step 1. Given $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$, we pick a representative $g \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ such that

$$f = g\big|_{\mathbb{R}^n_+}, \quad \|g|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \le 2\|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)\|.$$

Taking the quarkonial decomposition of g,

$$g = \sum_{\beta \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{jm} (\beta q u)_{jm},$$

the coefficients satisfy

$$\|\lambda|a^s_{\mathcal{M}^\varphi_p,q}\|_\rho \le c\|g|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \le c'\|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)\|$$

with $\rho > R$, cf. Theorem 2.12. Since (3.8) holds, we see that

$$g^* := \sum_{\beta \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{jm} (\beta q u)^*_{jm}$$

does not depend on the particular choice of the representative g. Define

$$\mathrm{Ext}_N f := \sum_{\beta \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{jm} (\beta q u)^*_{jm}$$

and its β-partial sum

$$\mathrm{Ext}^\beta_N f := \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{jm} (\beta q u)^*_{jm}.$$

Although the sum defining $\mathrm{Ext}_N f$ is not a quarkonial decomposition, we are still able to regard $2^{-(R+\varepsilon)|\beta|}\mathrm{Ext}^\beta_N f$ as an atomic decomposition if $0 < \varepsilon < \rho - R$. Putting $\delta := -R - \varepsilon + \rho > 0$ we have

$$\begin{aligned}
\left\|\mathrm{Ext}^\beta_N f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\right\| &\le 2^{(R+\varepsilon)|\beta|}\|\lambda^\beta|a^s_{\mathcal{M}^\varphi_p,q}\| \\
&\le 2^{(R+\varepsilon)|\beta|}2^{-\rho|\beta|}\|\lambda|a^s_{\mathcal{M}^\varphi_p,q}\|_\rho \\
&= 2^{-\delta|\beta|}\|\lambda|a^s_{\mathcal{M}^\varphi_p,q}\|_\rho \lesssim 2^{-\delta|\beta|}\|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)\|.
\end{aligned}$$

With this for $\varkappa := \min(1, p, q)$ we compute

$$\begin{aligned}\|\mathrm{Ext}_N f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|^\varkappa &= \Big\|\sum_\beta \mathrm{Ext}^\beta_N f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\Big\|^\varkappa \\ &\le \sum_\beta \|\mathrm{Ext}^\beta_N f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\|^\varkappa \\ &\lesssim \sum_\beta 2^{-\delta|\beta|\varkappa}\|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)\|^\varkappa \lesssim \|f|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)\|^\varkappa.\end{aligned}$$

Thus, we see that Ext_N is a continuous mapping with the desired properties.
Step 3. We deal with the construction of Ext_N in general. For $(p, q, s) \in R(N)$, choose $\sigma \in \mathbb{R}$ and $L \in \mathbb{N}$ large enough so that

$$\frac{n}{p} \le nN < -N + \sigma < s + \sigma < N + \sigma < L.$$

Hence, $s + \sigma$ satisfies the assumptions of Step 2, so that $\mathrm{Ext}_L\big|\mathcal{A}^{s+\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$ is a continuous mapping from $\mathcal{A}^{s+\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$ to $\mathcal{A}^{s+\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ and $\mathrm{Ext}_L f\big|_{\mathbb{R}^n_+} = f$ for all $f \in \mathcal{A}^{s+\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$. Since $J_{-\sigma}$ maps $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$ to $\mathcal{A}^{s+\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$ and $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ to $\mathcal{A}^{s+\sigma}_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ continuously, the following composite mapping

$$\mathrm{Ext}_N := J_\sigma \circ \mathrm{Ext}_L \circ J_{-\sigma} : \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+) \to \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$$

makes sense.

We verify that $\mathrm{Ext}_N f\big|_{\mathbb{R}^n_+} = f$ for $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$. For this we pick a smooth test function $\psi \in \mathcal{D}(\mathbb{R}^n_+)$ and denote by $E\psi$ its extension to $\mathcal{S}(\mathbb{R}^n)$ obtained by setting $\psi(x) \equiv 0$ outside $\mathbb{R}^n_+$. We see that

$$\begin{aligned}\langle \mathrm{Ext}_N f|_{\mathbb{R}^n_+}, \psi\rangle = \langle \mathrm{Ext}_N f, E\psi\rangle &= \langle \mathrm{Ext}_L J_{-\sigma} f, \mathcal{F}[\Phi^{(\sigma)}\mathcal{F}^{-1}E\psi]\rangle \\ &= \langle J_{-\sigma} f, \mathcal{F}[\Phi^{(\sigma)}\mathcal{F}^{-1}E\psi]\big|_{\mathbb{R}^n_+}\rangle\end{aligned}$$

from the property of Ext_L and Proposition 3.5. We further obtain for a representative $g \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$ of f,

$$\langle \mathrm{Ext}_N f|_{\mathbb{R}^n_+}, \psi\rangle = \langle J_{-\sigma} g, \mathcal{F}[\Phi^{(\sigma)}\mathcal{F}^{-1}E\psi]\rangle = \langle g, E\psi\rangle = \langle f, \psi\rangle.$$

Therefore, $\mathrm{Ext}_N f\big|_{\mathbb{R}^n_+} = f$ for all $f \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)$ and the proof is finished. □

3.3. Traces for the spaces $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$. Now we are able to state and proof our main theorem concerning traces of the generalized smoothness Morrey spaces on domains.

Theorem 3.8. *Let $n \geq 2$. Let $0 < p < \infty$, $0 < q \leq \infty$, and $\varphi \in \mathcal{G}_p$. Additionally assume that φ satisfies (2.2) when $q < \infty$ and $\mathcal{A} = \mathcal{E}$. Furthermore, let $\Omega \subset \mathbb{R}^n$, with boundary Γ, be a C^k domain for k large enough. Define s^* and φ^* by*

$$s^* := s - \frac{1}{p} \qquad \textit{and} \qquad \varphi^*(t) := \varphi(t)\, t^{-1/p}, \quad t > 0.$$

Assume that

$$s > \frac{1}{p} + (n-1) \cdot \begin{cases} \left(\frac{1}{\min(1,p)} - 1\right), & \textit{if } \mathcal{A} = \mathcal{N}, \\ \left(\frac{1}{\min(1,p,q)} - 1\right), & \textit{if } \mathcal{A} = \mathcal{E}, \end{cases} \tag{3.10}$$

and that φ^* is increasing and satisfies

$$\sum_{j=0}^{\infty} \frac{1}{\varphi^*(2^j s)} \lesssim \frac{1}{\varphi^*(s)}, \quad 0 < s \leq 1.$$

Then Tr_Γ is a linear and bounded operator from $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$ onto $\mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma)$,

$$\mathrm{Tr}_\Gamma \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega) = \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma) \qquad \textit{where} \quad r = \begin{cases} q, & \textit{if } \mathcal{A} = \mathcal{N}, \\ p, & \textit{if } \mathcal{A} = \mathcal{E}. \end{cases}$$

Proof : Our understanding of the trace operator on Γ is as follows. If f is smooth, using the partition of unity from (2.12) we can write

$$\mathrm{Tr}_\Gamma f = \sum_{j=1}^{N} \varphi_j \left(\mathrm{Tr}_\Gamma f\right).$$

Note that the term with φ_0 is unimportant because only the values of f near the boundary are of interest. Locally we see that for $x \in K_j \cap \Omega$ with $\psi^{(j)}(x) = y \in V_j \cap \mathbb{R}^n_+$, we have

$$\begin{aligned} \varphi_j \mathrm{Tr}_\Gamma f(x) = \varphi_j \mathrm{Tr}_\Gamma f \circ (\psi^{(j)})^{-1} \circ \psi^{(j)}(x) =& \varphi_j \mathrm{Tr}_\Gamma f \circ (\psi^{(j)})^{-1}(y) \\ =& \mathrm{Tr}_{\mathbb{R}^{n-1}} g_j(y) = g_j(y', 0), \end{aligned}$$

where in the second step we extended $g_j := \varphi_j f \circ (\psi^{(j)})^{-1}$ by zero outside $V_j \cap \mathbb{R}^n_+$ for $\mathrm{Tr}_{\mathbb{R}^{n-1}}$ to make sense (concerning the notation we refer to Definition 2.16).

Thus, the trace is well defined for smooth f. For general f we use the fact that $\mathrm{Tr}_{\mathbb{R}^{n-1}} g_j$ can be understood as explained at the beginning of Section 3, since $g_j \in \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$. Therefore, the trace makes sense in this case as well and by the definition of the spaces on the boundary Γ, we have that $\mathrm{Tr}_\Gamma f \in \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma)$, if $\mathrm{Tr}_{\mathbb{R}^{n-1}} g_j \in \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1})$ for all $j = 1, \dots, N$.

Step 1. We wish to prove in this step that

$$\mathrm{Tr}_\Gamma \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega) \subset \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma). \tag{3.11}$$

According to Theorem 3.7 there exists a bounded extension operator

$$\mathrm{Ext} : \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+) \longrightarrow \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)$$

with

$$\|\mathrm{Ext} f | \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \sim \|f | \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)\|.$$

In particular, for the trace operator $\mathrm{Tr}_{\mathbb{R}^{n-1}}$ we see that

$$\mathrm{Tr}_{\mathbb{R}^{n-1}}(\mathrm{Ext}\, h)(x) = \mathrm{Tr}_{\mathbb{R}^{n-1}} h(x) = h(x', 0),$$

whenever the pointwise trace for h makes sense. Using Theorem 3.2 we have

$$\begin{aligned} \|\mathrm{Tr}_{\mathbb{R}^{n-1}} h \mid \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1})\| &= \|\mathrm{Tr}_{\mathbb{R}^{n-1}}(\mathrm{Ext}\, h) \mid \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1})\| \\ &\le c\|\mathrm{Ext}\, h \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| \sim \|h \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)\|, \end{aligned} \tag{3.12}$$

which shows that

$$\mathrm{Tr}_{\mathbb{R}^{n-1}} \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+) \subset \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1}). \tag{3.13}$$

With this we calculate

$$\begin{aligned} \|\mathrm{Tr}_\Gamma f \mid \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma)\| &= \sum_{j=1}^N \|\mathrm{Tr}_{\mathbb{R}^{n-1}} g_j \mid \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1})\| \\ &= \sum_{j=1}^N \|\varphi_j f \circ (\psi^{(j)})^{-1}(\cdot, 0) \mid \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1})\| \\ &\le \sum_{j=1}^N \|\varphi_j f \circ (\psi^{(j)})^{-1} \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n_+)\| \\ &\le c\|f \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega)\|, \end{aligned} \tag{3.14}$$

where in the third step we used (3.12) and the last step is a consequence of Proposition 2.21. In fact the calculations in (3.14) show that our problem (3.11) reduces to (3.13).

Step 2. In order to see that the trace operator Tr_Γ is onto $\mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma)$, we establish the existence of a bounded extension operator

$$\widetilde{\mathrm{Ex}} : \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma) \longrightarrow \mathcal{A}^{s}_{\mathcal{M}^{\varphi}_p,q}(\Omega), \qquad \widetilde{\mathrm{Ex}}\, g\big|_\Gamma = g,$$

such that for $g \in \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma)$ we have

$$\|\widetilde{\mathrm{Ex}}\, g \mid \mathcal{A}^{s}_{\mathcal{M}^{\varphi}_p,q}(\Omega)\| \leq c\|g \mid \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma)\|.$$

We choose functions $\eta_j \in D(\mathbb{R}^n)$, $j = 1, \ldots, N$ with

$$\operatorname{supp} \eta_j \subset K_j, \qquad \eta_j = 1, \quad \text{if} \quad x \in \operatorname{supp} \varphi_j.$$

Put

$$\widetilde{\mathrm{Ex}}\, g(x) := \sum_{j=1}^{N} \eta_j(x) \cdot \mathrm{Ex}\left((\varphi_j g)(\psi^{(j)})^{-1}(\cdot, 0) \right) \left(\psi^{(j)}(x) \right), \qquad x \in \Omega,$$

where

$$\mathrm{Ex} : \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1}) \longrightarrow \mathcal{A}^{s}_{\mathcal{M}^{\varphi}_p,q}(\mathbb{R}^n)$$

stands for the extension operator from Theorem 3.2, cf. [NNS16, Thm. 5.3]. In particular, our construction can be extended from Ω to $\mathbb{R}^n$ by putting $\eta_j(x) \cdot$

$\mathrm{Ex}\,(\dots)\,\big(\psi^{(j)}(x)\big)=0$ outside K_j. This yields

$$\begin{aligned}
\big\|\widetilde{\mathrm{Ex}}\,g \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega)\big\| &= \inf\Big\{\|h|\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\| : h\in\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n),\ h\big|_\Omega=\widetilde{\mathrm{Ex}}\,g\Big\}\\
&\le \Big\|\sum_{j=1}^N \eta_j(\cdot)\mathrm{Ex}\,\big((\varphi_j g)(\psi^{(j)})^{-1}(\cdot,0)\big)(\psi^{(j)}(\cdot)) \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\Big\|\\
&\sim \sum_{j=1}^N \Big\|\eta_j(\cdot)\mathrm{Ex}\,\big((\varphi_j g)(\psi^{(j)})^{-1}(\cdot,0)\big)(\psi^{(j)}(\cdot)) \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\Big\|\\
&\sim \sum_{j=1}^N \Big\|\mathrm{Ex}\,\big((\varphi_j g)(\psi^{(j)})^{-1}(\cdot,0)\big) \mid \mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\mathbb{R}^n)\Big\|\\
&\le c\sum_{j=1}^N \Big\|(\varphi_j g)(\psi^{(j)})^{-1}(\cdot,0) \mid \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\mathbb{R}^{n-1})\Big\|\\
&= c\big\|g \mid \mathcal{A}^{s^*}_{\mathcal{M}^{\varphi^*}_p,r}(\Gamma)\big\|,
\end{aligned}$$

where in the 4th step we used Theorem 2.20(i), (ii), since $\psi^{(j)}$ is a k-diffeomorphism from $\mathbb{R}^n$ onto itself, and $\eta_j\in D(\mathbb{R}^n)$ if we put $\eta_j(x)=0$ whenever $x\in\mathbb{R}^n\setminus K_j$. This completes the proof.

□

Remark 3.9. *Note that the above proof relies on the available diffeomorphism property, cf. Theorem 2.20(ii), and this assertion does in general not apply to variable exponent spaces. In the latter case, the values of the exponent depend on the point of the domain and therefore a diffeomorphism assertion like the one referred to above cannot be expected in the context of these variable exponent spaces, unless a strict condition is imposed. In this regard we refer for instance to [Gon17, Theorem 6.5.6].*

Furthermore, it would also be interesting to clarify the trace in the limiting case, which corresponds in the classical case to $s=1/p$, see e.g. [Tri97]. This study will be postponed to future work.

3.4. Traces for the Besov-type spaces $B^{s,\tau}_{p,q}(\Omega)$. We now turn our attention to the Besov-type spaces $B^{s,\tau}_{p,q}(\Omega)$, which by (2.4) are related but not included in the scale of generalized Besov-Morrey spaces $\mathcal{N}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$ considered so far. Our aim is to obtain trace results on C^k domains similar to Theorem 3.8 for these spaces. We briefly sketch the main ideas needed in this context. Our understanding of the trace operator is in the same spirit as explained in Section 3, we also refer to [YSY10, p. 164] in this context. A close inspection of the proof of Theorem 3.8 reveals that the method we used carries over to other function spaces if we can find substitutes for the following assertions:

(A) Multipliers and diffeomorphisms according to Theorem 2.20 (which in turn lead to an equivalent quasi-norm as in Proposition 2.21).
(B) An extension operator according to Theorem 3.7.
(C) Results for traces on hyperplanes (and extension operators) as stated in Theorem 3.2.

Let $s,\tau\in\mathbb{R}$ and $0<p,q\leq\infty$. The inhomogeneous Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\,|\,B^{s,\tau}_{p,q}(\mathbb{R}^n)\| := \sup_{Q\in\mathcal{Q}}\frac{1}{|Q|^\tau}\left(\sum_{j=\max(j_Q,0)}\left[\int_Q\left(2^{js}|\mathcal{F}^{-1}[\mu_j(\xi)\mathcal{F}f(\xi)](x)|\right)^p\,\mathrm{d}x\right]^{q/p}\right)^{1/q} \tag{3.15}$$

is finite, where the functions μ_j are as in Definition 2.3. In this case it follows from [YSY10, Cor. 2.1] that the definition of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ is independent of the choice of μ_j.

Corresponding spaces $B^{s,\tau}_{p,q}(\Omega)$ on domains $\Omega\subset\mathbb{R}^n$ are defined via restriction as in Definition 2.15, whereas on the boundary $\Gamma=\partial\Omega$ we use Definition 2.18 and obtain the spaces via localization and pull-back onto $\mathbb{R}^{n-1}$ with the help of suitable diffeomorphisms (recall Definition 2.16).

The quarkonial decomposition. In this subsection we establish the quarkonial decomposition for spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$. Recall the definition of quarks given in Definition 2.11, with θ satisfying (2.7) and (2.8). We start by defining the corresponding sequence spaces and provide an auxiliary lemma.

Definition 3.10. *Let $0<p\leq\infty$, $0<q\leq\infty$, $s\in\mathbb{R}$, and $\tau\geq 0$. The Besov-type sequence space $b^{s,\tau}_{p,q}(\mathbb{R}^n)$ is the set of all doubly indexed sequences*

$\lambda := \{\lambda_{j,m}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C}$ *for which the quasi-norm*

$$\|\lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\| := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Bigg(\sum_{j=\max(j_P,0)} 2^{j(s-\frac{n}{p})q}\Big(\sum_{m\in\mathbb{Z}^n: Q_{jm}\subset P} |\lambda_{j,m}|^p\Big)^{q/p}\Bigg)^{1/q}$$

is finite (with the usual modification if $p=\infty$ *or* $q=\infty$*).*

Lemma 3.11. *Let* $0<p\le\infty$, $0<q\le\infty$, $s\in\mathbb{R}$, *and* $\tau\ge 0$. *There exists a positive constant* c *such that*

$$\|\lambda^l \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\| \le c\,\langle l\rangle^{n\tau} \|\lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|$$

for all $\lambda=\{\lambda_{j,m}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ *and all* $l\in\mathbb{Z}^n$, *where* $\lambda^l := \{\lambda_{j,m+l}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}$.

Proof : Let $P\in\mathcal{Q}$, $P=Q_{\nu,m}$ for some $\nu\in\mathbb{Z}$ and $k\in\mathbb{Z}^n$. Denoting by $x_P=(x_P^1,\cdots,x_P^n)$ the center of the cube P, if a cube $Q_{jm}=2^{-j}m+[0,2^{-j})^n$ is contained in P, then

$$j\ge\nu \quad\text{and}\quad |x_i+2^{-j}m_i-x_P^i|\le 2^{-\nu-1} \quad\text{for all}\quad x_i\in[0,2^{-j}),\ i\in\{1,\cdots,n\}.$$

Then, for a point in the cube $Q_{j,m+l}=2^{-j}(m+l)+[0,2^{-j})^n$, we have

$$|2^{-j}(m_i+l_i)+x_i-x_P^i|\le|2^{-j}m_i+x_i-x_P^i|+2^{-j}|l_i|\le 2^{-\nu-1}+2^{-\nu}|l_i|\le 2\langle l\rangle 2^{-\nu-1},$$

and hence $Q_{j,m+l}\subset 2\langle l\rangle P$.
Let $r\in\mathbb{N}$ be such that $2^r\le 2\langle l\rangle<2^{r+1}$ and put $P^*:=2^{r+1}P$. Then we have

$$\begin{aligned}
&\frac{1}{|P|^\tau}\Bigg(\sum_{j=\max(j_P,0)} 2^{j(s-\frac{n}{p})q}\Big(\sum_{m\in\mathbb{Z}^n: Q_{jm}\subset P} |\lambda_{j,m+l}|^p\Big)^{q/p}\Bigg)^{1/q}\\
&\quad\le \frac{|P^*|^\tau}{|P|^\tau}\frac{1}{|P^*|^\tau}\Bigg(\sum_{j=\max(j_P,0)} 2^{j(s-\frac{n}{p})q}\Big(\sum_{m\in\mathbb{Z}^n: Q_{j,m+l}\subset P^*} |\lambda_{j,m+l}|^p\Big)^{q/p}\Bigg)^{1/q}\\
&\quad= 2^{(r+1)n\tau}\frac{1}{|P^*|^\tau}\Bigg(\sum_{j=\max(j_{P^*},0)} 2^{j(s-\frac{n}{p})q}\Big(\sum_{m\in\mathbb{Z}^n: Q_{jm}\subset P^*} |\lambda_{j,m}|^p\Big)^{q/p}\Bigg)^{1/q}\\
&\quad\le 4^{n\tau}\langle l\rangle^{n\tau}\|\lambda\mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|.
\end{aligned}$$

By taking the supremum over all $P\in\mathcal{Q}$ we arrive at the desired inequality. □

With this we now obtain the following quarkonial decomposition for our Besov-type spaces.

Theorem 3.12. *Let $0 < p < \infty$, $0 < q \le \infty$, $0 \le \tau \le \frac{1}{p}$, and $s > \sigma_p$. Let ρ be such that $\rho > R$ where R is a constant as in* (2.8)*.*

(i) *If $f \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ then there exists a triply indexed complex sequence*

$$\lambda := \{\lambda^\beta_{\nu m}\}_{\beta\in\mathbb{N}^n_0,\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$$

such that

$$f = \sum_{\beta\in\mathbb{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{\nu m}\,(\beta qu)_{\nu m}$$

convergence being in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|\lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|_\rho := \sup_{\beta\in\mathbb{N}^n_0} 2^{\rho|\beta|}\,\|\lambda^\beta \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\| \lesssim \|f \mid B^{s,\tau}_{p,q}(\mathbb{R}^n)\|.$$

The numbers $\lambda^\beta_{\nu m}$ depend continuously and linearly on f.

(ii) *If $\lambda := \{\lambda^\beta_{\nu m}\}_{\beta\in\mathbb{N}^n_0,\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ satisfies $\|\lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|_\rho < \infty$, then*

$$f = \sum_{\beta\in\mathbb{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{\nu m}\,(\beta qu)_{\nu m} \tag{3.16}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^n)$. Furthermore,

$$\|f \mid B^{s,\tau}_{p,q}(\mathbb{R}^n)\| \lesssim \|\lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|_\rho.$$

Proof : We start by proving (i) and follow the proof presented in [Tri97, 14.15] in the context of classical Besov spaces. Let $(\phi_j)_{j\in\mathbb{N}_0}$ be a dyadic partition of unity such that

$$\phi_0(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \operatorname{supp}\phi_0 \subset \{x\in\mathbb{R}^n : |x| \le 2\}.$$

For $\nu \in \mathbb{N}$ we put

$$\phi_\nu(x) := \phi_0(2^{-\nu}x) - \phi_0(2^{-\nu+1}x), \quad x\in\mathbb{R}^n.$$

Then

$$\sum_{\nu=0}^{\infty} \phi_\nu(x) = 1, \quad x\in\mathbb{R}^n,$$

and, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, it follows that

$$f = \sum_{\nu=0}^{\infty} \mathcal{F}^{-1}(\phi_\nu \mathcal{F} f) \qquad \text{with convergence in} \quad \mathcal{S}'(\mathbb{R}^n).$$

Let $\kappa \in \mathcal{S}(\mathbb{R}^n)$ be such that $\kappa(x) = 1$ if $|x| \leq 2$ and $\operatorname{supp} \kappa \subset \pi Q(0)$. For $(\nu, k) \in \mathbb{N}_0 \times \mathbb{Z}^n$, let $\Lambda_{\nu,k} := c[\mathcal{F}^{-1}(\phi_\nu \mathcal{F} f)](2^{-\nu} k)$. Then we have, for any $x \in \mathbb{R}^n$,

$$[\mathcal{F}^{-1}(\phi_\nu \mathcal{F} f)](x) = \sum_{k\in\mathbb{Z}^n} \Lambda_{\nu,k} \, (\mathcal{F}^{-1}\kappa)(2^\nu x - k)$$
$$= \sum_{k\in\mathbb{Z}^n} \Lambda_{\nu,k} \sum_{m\in\mathbb{Z}^n} (\mathcal{F}^{-1}\kappa)(2^\nu x - k) \, \theta(2^{\nu+\rho} x - m),$$

where the last equality is due to $\sum_{m\in\mathbb{Z}^n} \theta(x - m) = 1$ for all $x \in \mathbb{R}^n$. Expanding $(\mathcal{F}^{-1}\kappa)(2^\nu \cdot - k)$ in a Taylor series at the point $2^{-(\nu+\rho)} m$, we obtain

$$(\mathcal{F}^{-1}\kappa)(2^\nu x - k) = \sum_{\beta\in\mathbb{N}_0^n} \frac{2^{\nu|\beta|}}{\beta!} [\partial^\beta(\mathcal{F}^{-1}\kappa)](2^{-\rho} m - k) \, (x - 2^{-(\nu+\rho)} m)^\beta$$

thus,

$$[\mathcal{F}^{-1}(\phi_\nu \mathcal{F} f)](x) = \sum_{k\in\mathbb{Z}^n} \Lambda_{\nu,k} \sum_{m\in\mathbb{Z}^n} \sum_{\beta\in\mathbb{N}_0^n} \frac{2^{-\rho|\beta|}}{\beta!} [\partial^\beta(\mathcal{F}^{-1}\kappa)](2^{-\rho} m - k) \, \theta^\beta(2^{\nu+\rho} x - m)$$

and hence

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \sum_{\beta\in\mathbb{N}_0^n} \theta^\beta(2^{\nu+\rho} x - m) \sum_{k\in\mathbb{Z}^n} \Lambda_{\nu,k} \frac{2^{-\rho|\beta|}}{\beta!} [\partial^\beta(\mathcal{F}^{-1}\kappa)](2^{-\rho} m - k)$$
$$= \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \sum_{\beta\in\mathbb{N}_0^n} \lambda^\beta_{\nu+\rho,m} \, (\beta qu)_{\nu+\rho,m}$$

with

$$\lambda^\beta_{\nu+\rho,m} := \frac{2^{-\rho|\beta|}}{\beta!} \sum_{k\in\mathbb{Z}^n} [\partial^\beta(\mathcal{F}^{-1}\kappa)](2^{-\rho} m - k) \, \Lambda_{\nu,k}.$$

As a consequence of the Paley-Wiener-Schwartz theorem and iterative application of Cauchy's representation formula one can prove that

$$|\partial^\beta(\mathcal{F}^{-1}\kappa)(x)| \leq c(\eta) \, \beta! \, \langle x \rangle^{-\eta} \qquad \text{for any } \eta > 0,$$

where $c(\eta)$ is a positive constant independent of $x \in \mathbb{R}^n$ and of the multi-index $\beta \in \mathbb{N}_0^n$. Then, for $l \in \mathbb{Z}^n$ and l_0 a lattice point in $[0, 2^\rho)^n$, we obtain

$$|\lambda^\beta_{\nu+\rho, 2^\rho l + l_0}| \lesssim 2^{-\rho|\beta|} \sum_{k\in\mathbb{Z}^n} \langle l - k \rangle^{-\eta} \, |\Lambda_{\nu,k}| = 2^{-\rho|\beta|} \sum_{k\in\mathbb{Z}^n} \langle k \rangle^{-\eta} |\Lambda_{\nu,l+k}|.$$

For each $k \in \mathbb{Z}^n$ let $\Lambda^k := \{|\Lambda_{\nu,l+k}|\}_{\nu\in\mathbb{N}_0,l\in\mathbb{Z}^n}$. By $\mathbb{Z}^n = 2^\rho\mathbb{Z}^n + [0,2^\rho)^n$ we see that with $d := \min(1,p,q)$ and by Lemma 3.11, we have

$$\begin{aligned}\|\lambda^\beta \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\| &\lesssim 2^{-\rho|\beta|}\Big\| \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{-\eta}\Lambda^k \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\Big\| \\ &\lesssim 2^{-\rho|\beta|}\Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{-\eta d}\|\Lambda^k \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|^d\Big)^{1/d} \\ &\lesssim 2^{-\rho|\beta|}\Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(n\tau-\eta)d}\|\Lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|^d\Big)^{1/d}.\end{aligned}$$

Hence, choosing η large enough such that $(n\tau-\eta)d < -1$, it follows that

$$\|\lambda^\beta \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|_\rho \lesssim \|\Lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|.$$

Finally we have to prove that $\|\Lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\| \lesssim \|f \mid B^{s,\tau}_{p,q}(\mathbb{R}^n)\|$. Note that, for $\nu \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$ and any $y \in Q_{\nu k}$, we have

$$\begin{aligned}|\Lambda_{\nu,k}| = |[\mathcal{F}^{-1}(\phi_\nu\mathcal{F}f)](2^{-\nu}k)| &= (1+2^\nu|y-2^{-\nu}k|)^a\frac{|[\mathcal{F}^{-1}(\phi_\nu\mathcal{F}f)](2^{-\nu}k)|}{(1+2^\nu|y-2^{-\nu}k|)^a} \\ &\lesssim (\phi^*_\nu f)_a(y),\end{aligned}$$

where $a > \frac{n}{p}$ and $(\phi^*_\nu f)_a$ are the Peetre's maximal functions defined by

$$(\phi^*_\nu f)_a(x) := \sup_{y\in\mathbb{R}^n} \frac{|[\mathcal{F}^{-1}(\phi_\nu\mathcal{F}f)](y)|}{(1+2^\nu|x-y|)^a}, \quad x\in\mathbb{R}^n.$$

Then

$$\begin{aligned}\|\Lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\| &= \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{j=\max(j_P,0)} 2^{jsq}\Big(\sum_{m\in\mathbb{Z}^n:Q_{jm}\subset P}\int_{Q_{jm}} |\Lambda_{j,m}|^p\,dy\Big)^{q/p}\Big)^{1/q} \\ &\le \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{j=\max(j_P,0)} 2^{jsq}\Big(\sum_{m\in\mathbb{Z}^n:Q_{jm}\subset P}\int_{Q_{jm}} (\phi^*_j f)_a(y)^p\,dy\Big)^{q/p}\Big)^{1/q} \\ &\le \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{j=\max(j_P,0)} 2^{jsq}\Big(\int_P (\phi^*_j f)_a(y)^p\,dy\Big)^{q/p}\Big)^{1/q} \\ &\lesssim \|f \mid B^{s,\tau}_{p,q}(\mathbb{R}^n)\|.\end{aligned}$$

The last step is justified by the equivalent characterization of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$, cf. [LSUYY12, Thm. 3.6] (in the homogeneous case) and [YSY10, Lem. 4.1].

Now we prove (ii). We decompose the representation (3.16) as

$$f = \sum_{\beta \in \mathbb{N}_0^n} f^\beta$$

with

$$f^\beta := \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} \, (\beta q u)_{\nu m}.$$

Note that for all $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and $\beta \in \mathbb{N}_0^n$ it holds

$$\operatorname{supp} (\beta q u)_{\nu m} \subset 2^R Q_{\nu m}$$

and

$$|\partial^\alpha (\beta q u)_{\nu m}(x)| \lesssim 2^{|\alpha|\nu + (R+\varepsilon)|\beta|}, \quad x \in \mathbb{R}^n,$$

for any $\varepsilon > 0$. Applying [YSY10, Thm. 3.3] we can conclude that $f^\beta \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and

$$\|f^\beta \mid B^{s,\tau}_{p,q}(\mathbb{R}^n)\| \leq c_1 \, 2^{(R+\varepsilon)|\beta|} \, \|\lambda^\beta \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|,$$

where $c_1 > 0$ is independent of β. So, with $0 < \varepsilon < \rho - R$,

$$\|f^\beta \mid B^{s,\tau}_{p,q}(\mathbb{R}^n)\| \leq c_1 \, 2^{(R+\varepsilon-\rho)|\beta|} \sup_{\beta \in \mathbb{N}_0^n} 2^{\rho|\beta|} \, \|\lambda^\beta \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|,$$

and applying the d-triangle inequality, where $d := \min(1, p, q)$, we get

$$\begin{aligned} \Big\| \sum_{\beta \in \mathbb{N}_0^n} f^\beta \mid B^{s,\tau}_{p,q}(\mathbb{R}^n) \Big\| &\leq \Big(\sum_{\beta \in \mathbb{N}_0^n} \|f^\beta \mid B^{s,\tau}_{p,q}(\mathbb{R}^n)\|^d \Big)^{1/d} \\ &\leq c_1 \Big(\sum_{\beta \in \mathbb{N}_0^n} 2^{(R+\varepsilon-\rho)d|\beta|} \Big)^{1/d} \sup_{\beta \in \mathbb{N}_0^n} 2^{\rho|\beta|} \, \|\lambda^\beta \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\| \\ &\leq c_2 \, \|\lambda \mid b^{s,\tau}_{p,q}(\mathbb{R}^n)\|_\rho. \end{aligned}$$

□

Remark 3.13. *The restriction* $0 \leq \tau \leq \frac{1}{p}$ *in Theorem* 3.12 *can be replaced by* $0 \leq \tau < \tau_{s,p}$ *with* $\tau_{s,p}$ *defined as in [YSY10, formula (1.6)], which follows from the atomic decomposition theorem for the Besov-type spaces, cf. [YSY10, Thm. 3.3], we rely on in the proof. For simplicity we restrict ourselves to* $\tau \leq \frac{1}{p}$ *here, since in [YY13] the remarkable result was proven that*

$$B^{s,\tau}_{p,q}(\mathbb{R}^n) = B^{s+(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^n)$$

whenever $\tau > \frac{1}{p}$ or $\tau = \frac{1}{p}$ and $q = \infty$. Hence, concerning traces, only $\tau \in [0, 1/p]$ is of interest.

We now collect available substitutes for (A), (B), and (C) in order to establish the trace results with the methods from Theorem 3.8.

Concerning (A), in terms of diffeomorphisms and multipliers we have the following results.

Theorem 3.14. *[YSY10, Thm. 6.1] Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$, and $0 \leq \tau \leq \frac{1}{p}$.*

(i) *If $k \in \mathbb{N}$ is sufficiently large, then for all $g \in C^k(\mathbb{R}^n)$ we have that*
$$f \to gf$$
is a linear and bounded operator from $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ into itself, i.e., there exists a positive constant $C(k)$ such that
$$\|gf|B^{s,\tau}_{p,q}(\mathbb{R}^n)\| \leq C(k)\|g|C^k(\mathbb{R}^n)\| \cdot \|f|B^{s,\tau}_{p,q}(\mathbb{R}^n)\|.$$

(ii) *If $k \in \mathbb{N}$ is sufficiently large, then for all k-diffeomorphisms ψ we have that*
$$f \to f \circ \psi$$
is a linear and bounded operator from $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ onto itself.

Concerning (B), with the help of the quarkonial decomposition from Theorem 3.12 we can construct the following the extension operator.

Theorem 3.15. *Let $0 < p < \infty$, $0 < q \leq \infty$, and $0 \leq \tau \leq \frac{1}{p}$. Then for $N \in \mathbb{N}$, there exists an extension operator Ext_N,*
$$\mathrm{Ext}_N : \bigcup_{(p,q,s) \in R(N)} B^{s,\tau}_{p,q}(\mathbb{R}^n_+) \longrightarrow \bigcup_{(p,q,s) \in R(N)} B^{s,\tau}_{p,q}(\mathbb{R}^n),$$
that satisfies the properties: if $(p, q, s) \in R(N)$ then the restriction $\mathrm{Ext}_N\big|_{B^{s,\tau}_{p,q}(\mathbb{R}^n_+)}$ is a continuous mapping from $B^{s,\tau}_{p,q}(\mathbb{R}^n_+)$ to $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ satisfying $\mathrm{Ext}_N f\big|_{\mathbb{R}^n_+} = f$ for all $f \in B^{s,\tau}_{p,q}(\mathbb{R}^n_+)$.

P r o o f : The proof follows along the same lines as the proof of Theorem 3.7. We use the same construction to obtain an extended function f^* as described in Step 1 of the proof. In Step 2 instead of (3.8) we now make use of [YHM15, Prop. 4.1], i.e.,
$$B^{s,\tau}_{p,q}(\mathbb{R}^n) \hookrightarrow BUC(\mathbb{R}^n), \quad \text{if, and only if,} \quad s > n\left(\frac{1}{p} - \tau\right).$$

In particular, this enables us to define Ext_N for $s > \frac{n}{p}$ with the help of the quarkonial decomposition from Theorem 3.12. Finally, in Step 3 of Theorem 3.7 we note that the lift operator from Corollary 3.6 can also be generalized to the Besov-type spaces $B^{s,\tau}_{p,q}$. □

Remark 3.16. *Our results on the extension operator generalize [YSY10, Thm. 6.11] to the case when $p \leq 1$.*

An assertion for traces on hyperplanes as required in (C) is also available.

Theorem 3.17. *[YSY10, Thm. 6.8] Let $n \geq 2$, $0 < p, q \leq \infty$, $0 \leq \tau \leq \frac{1}{p}$, and*

$$s > \frac{1}{p} + (n-1)\left(\frac{1}{\min(1,p)} - 1\right). \tag{3.17}$$

Then $\mathrm{Tr}_{\mathbb{R}^{n-1}}$ is a linear and bounded operator from $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ onto $B^{s-\frac{1}{p},\frac{n\tau}{n-1}}_{p,q}(\mathbb{R}^{n-1})$,

$$\mathrm{Tr}_{\mathbb{R}^{n-1}} B^{s,\tau}_{p,q}(\mathbb{R}^n) = B^{s-\frac{1}{p},\frac{n\tau}{n-1}}_{p,q}(\mathbb{R}^{n-1}).$$

Remark 3.18. The proof in [YSY10, Thm. 6.8] also establishes the existence of a linear and bounded extension operator $\widetilde{\mathrm{Ex}} : B^{s-\frac{1}{p},\frac{n\tau}{n-1}}_{p,q}(\mathbb{R}^{n-1}) \to B^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Remark 3.19. In Theorem 3.14(i),(ii) we will require

$$k \geq \max([s + n\tau + 1], 0). \tag{3.18}$$

This follows from a closer look at the proof of [YSY10, Thm. 6.1], whenever $s > \sigma_p$. This is always the case for us since with the restriction (3.17) on s from Theorem 3.17 we have

$$\begin{aligned} \sigma_p &= n\left(\frac{1}{\min(1,p)} - 1\right) \\ &= \frac{1}{p} + (n-1)\left(\frac{1}{\min(1,p)} - 1\right) \underbrace{- \frac{1}{p} + \frac{1}{\min(1,p)} - 1}_{<0} \\ &< \frac{1}{p} + (n-1)\left(\frac{1}{\min(1,p)} - 1\right) < s. \end{aligned}$$

The restriction (3.18) on k comes from the atomic decomposition of the Besov-type spaces as established in [YSY10, Th. 3.3]. In this book we have the same restriction for the $F^{s,\tau}_{p,q}$ scale. The fact that τ comes into play in (3.18) is a little confusing. Note that for the atomic decomposition of the spaces $\mathcal{E}^s_{u,p,q}$ we only need $k \geq \max\{[s+1], 0\}$, cf. Theorem 2.9, which is independent of τ. By the coincidence $\mathcal{E}^s_{u,p,q} = F^{s,\tau}_{p,q}$ with $u = \frac{p}{1-p\tau}$ and $0 \leq \tau < \frac{1}{p}$, the dependence on τ in (3.18) can be removed for the $F^{s,\tau}_{p,q}$ scale. This was also noted in [NNS16, Thms. 4.4, 4.5] (mentioned in the proof given there). This raises the question whether in (3.18) the dependence on τ can be removed for $B^{s,\tau}_{p,q}$ as well.

Putting together our substitues for (A)-(C), we obtain the following result concerning traces on C^k domains for Besov-type spaces.

Theorem 3.20. *Let $n \geq 2$, $0 \leq p < \infty$, $0 < q \leq \infty$, $0 \leq \tau \leq \frac{1}{p}$, and*

$$s > \frac{1}{p} + (n-1)\left(\frac{1}{\min(1,p)} - 1\right). \tag{3.19}$$

Furthermore, let $\Omega \subset \mathbb{R}^n$ be a bounded C^k domain with boundary Γ, where

$$k \geq [s + n\tau + 1].$$

Then Tr_Γ is a linear and bounded operator from $B^{s,\tau}_{p,q}(\Omega)$ onto $B^{s-\frac{1}{p},\frac{n\tau}{n-1}}_{p,q}(\Gamma)$,

$$\mathrm{Tr}_\Gamma B^{s,\tau}_{p,q}(\Omega) = B^{s-\frac{1}{p},\frac{n\tau}{n-1}}_{p,q}(\Gamma).$$

4. Applications: *A priori* estimates for solutions of elliptic boundary value problems

In [Bar05] the author obtains *a priori* estimates for solutions of elliptic boundary value problems in the spaces $\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $\lambda \geq 0$, $1 \leq p < \infty$, which are linked with our Besov-type spaces via

$$\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n) = B^{s,\frac{\lambda}{np}}_{p,p}(\mathbb{R}^n).$$

His estimates are based on trace results for the respective spaces on hyperplanes. With the help of our trace results from Theorem 3.20 we are now able to to improve the a priori estimates from [Bar05, Thm. 1.8] to C^k domains.

We consider the following elliptic Dirichlet problem:

$$\begin{cases} Lu = f & \text{on } \Omega, \\ \mathrm{Tr}_\Gamma u = g & \text{on } \Gamma = \partial\Omega, \end{cases}$$

where L is a differential operator of second order with smooth coefficients in $\overline{\Omega}$, i.e.,

$$L = \sum_{|\alpha| \leq 2} a_\alpha(x) D_x^\alpha,$$

which is properly elliptic. By this we mean that the following conditions are satisfied:

(H1) For any $x \in \overline{\Omega}$,

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \neq 0, \qquad \xi \in \mathbb{R}^n \setminus \{0\}.$$

(H2) For any $x \in \Gamma$, $\xi_x \in \mathbb{R}^n \setminus \{0\}$ tangent to Γ at x, the polynomial in the complex variable z,

$$P(z) = \sum_{|\alpha|=2} a_\alpha(x)(\xi_x + z\nu_x)^\alpha,$$

has exactly one root with positive imaginary part (and therefore exactly one root lying in the lower half plane). Here ν_x denotes the inward unit normal vector to the boundary Γ at x.

In this setting [Bar05, Thm. 1.8] can be generalized (and reformulated) in terms of our Besov-type spaces as follows.

Theorem 4.1. *Let $s > 0$, $1 \leq p < \infty$, and $0 \leq \tau \leq \frac{1}{p}$. Furthermore, let $\Omega \subset \mathbb{R}^n$ be a bounded C^k domain with boundary Γ, where*

$$k \geq [s + n\tau + 1].$$

Then, under the hypotheses (H1) and (H2), there is a constant $C > 0$ such that

$$\|u|B^{s+2,\tau}_{p,p}(\Omega)\| \leq C \left(\|Lu|B^{s,\tau}_{p,p}(\Omega)\| + \|\mathrm{Tr}_\Gamma u|B^{s+2-\frac{1}{p},\frac{n\tau}{n-1}}_{p,p}(\Gamma)\| + \|u|B^{s+1,\tau}_{p,p}(\Omega)\| \right)$$

holds for any $u \in B^{s+2,\tau}_{p,p}(\Omega)$.

Remark 4.2. *It would be interesting to study whether the* a priori *estimates from [Bar05] can be generalized to the spaces $B^{s,\tau}_{p,q}(\Omega)$ when $p \neq q$ or even to the spaces $\mathcal{A}^s_{\mathcal{M}^\varphi_p,q}(\Omega)$ with the help of our trace results obtained in Theorem* 3.8. *This will be investigated in a forthcoming paper.*

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