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# PURE ROLLING MOTION OF PSEUDO-RIEMANNIAN MANIFOLDS: AN EXTRINSIC PERSPECTIVE

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ABSTRACT: This paper is devoted to rolling motions of one differentiable manifold over another of equal dimension, subject to no-slip and no-twist constraints, when this motion occurs inside an ambient space which is a pseudo-Riemannian manifold. We first introduce a definition of rolling map which generalizes the classical definition of Sharpe [17], from Euclidean submanifolds to pseudo-Riemannian submanifolds. We also present essential properties of rolling and make the connection between rolling motions and parallel transport of vectors along curves. After presenting the general framework, we analyse the particular rolling of hyperquadrics in pseudo-Euclidean spaces. The central theme is the rolling of a pseudo-hyperbolic space over the affine space associated with the tangent space at a point. Rolling of a pseudo-hyperbolic space on another and rolling of pseudo-spheres are equally treated. The kinematic equations of these rolling motions will be presented, as well as the corresponding explicit solutions for two specific cases.

KEYWORDS: pseudo-Riemannian manifolds, rolling maps, no-slip, no-twist, hyperquadrics, kinematic equations, pseudo-hyperbolic space, pseudo-sphere, affine tangent space, geodesics, parallel transport.
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## 1. Introduction

The rolling motion of a rigid body over a surface is a very common situation. That can be described by rotations and translations, under some restrictions on velocity and the assumption that the rolling object is always in contact with the stationary surface. If there are no "slips" or "twists" the rolling is sometimes referred as "pure rolling". The most classical example is the rolling of a sphere on a tangent plane. This case is well studied in the literature, in part due to diverse applications in the engineering areas, but also due to the easy visualization of what happens in Euclidean space $I\!R^3$ (Jurdjevic [7]).

When going to other environmental spaces, it is possible to generalize the notion of pure rolling keeping the main ideas. Of course, this generalization is theoretically more challenging and for higher dimensions the geometric intuition may be lost, but it may also have interesting practical applications in areas such as robotics and computer vision.

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Theoretical works devoted to the study of rolling manifolds are relatively recent. After the formal definition of rolling map, introduced by Sharpe [17] in 1996 for submanifolds of Euclidean spaces, a number of papers have been devoted to the rolling of certain specific manifolds. For instance, in Hüper and Leite [4] the kinematics equations for the rolling of the special orthogonal group $\mathrm{SO}(n)$ and Grassmann manifolds, over the affine tangent space at a point, were deduced. This article also has important properties of the corresponding rolling maps. In Hüper *et al.* [5] some of the results contained in [4] were generalized to submanifolds of an arbitrary Riemannian manifold. The particular case of the rolling of an ellipsoid, embedded in a space provided with a metric resulting from a deformation of the Euclidean metric, appears in Krzysztof and Leite [10].

In the case of non-Riemannian manifolds, the work presented in 2008 by Jurdjevic and Zimmerman [8] was the first attempt to extend results from Euclidean space to pseudo-Riemannian manifolds. More recent results in this regard exist in Korolko and Leite [9] for the Lorentzian sphere, in Marques and Leite [13] for pseudo-hyperbolic spaces, in Crouch and Leite [2] for pseudo-orthogonal groups, and in Marques and Leite [14] for symplectic groups.

The approach used in all the publications so far mentioned is from an extrinsic point of view, since it is always assumed that the two rolling manifolds are embedded in a third environment manifold. However, work has also emerged where the approach is purely intrinsic, namely Godoy *et al.* [3] and Chitour and Kokkonen [1] for Riemannian case, and Markina and Leite [12] for the pseudo-Riemannian case. Also with this type of approach, the case of rolling manifolds with different dimensions is studied in Mortada *et al.* [15].

Among the possible applications of rolling manifolds we can find the generation of interpolation problems in non-Euclidean spaces. Algorithms that use rolling motions to produce $C^2$-smooth interpolating curves appear, for example, in [4]. A demonstration of the practical utility of this type of interpolation is found in the work of Shen *et al.* [18], about path planning of a robot.

The present paper focuses on the study of rolling motions occurring within pseudo-Riemannian manifolds, and involving submanifolds with equal dimension. The main goal is to make a theoretical approach to this type of rolling, but also present some results for the case of an important family of pseudo-Riemannian hypersurfaces, the hyperquadrics.

This article is organized in the following way. Aiming to make this work as self-contained as possible, in Section 2 we compile the main auxiliary concepts that will be used later and also fix some terminology and notations.

Section 3 begins with the formalization of the concept of rolling motion of one manifold over another, when both are embedded in a pseudo-Riemannian manifold, through the definition of rolling map. The particular case in which the environment manifold is $\mathbb{R}^n_\kappa$, the pseudo-Euclidean space of dimension $n$ and index $\kappa$, allows to observe at once that this definition of rolling map generalizes the definition presented in [13], as well as the definition of Sharpe [17]. We present three basic properties, which are essential for deriving new rolling motions from others previously known and simpler. We will also analyse the close relationship between rolling without twist and parallel vector transport.

In Section 4, we discuss the rolling of the hyperquadrics of pseudo-Euclidean spaces, namely the pseudo-hyperbolic space $H^n_\kappa(r)$ and the pseudo-sphere $S^n_\kappa(r)$. The central subject is the rolling of $H^n_\kappa(r)$ over the affine space associated with the tangent space at a point. The rolling of a pseudo-hyperbolic space on another and the rolling of pseudo-spheres will be achieved later using properties introduced in Section 3. We present the kinematic equations of these rolling motions, as well as their solutions for some simple cases.

## 2. Background

We review here the main concepts about pseudo-Riemannian manifolds that will be used throughout the paper and refer to [16] for more details. It should be noted that in some literature, in particular in [16], the term "pseudo" is replaced by "semi".

**2.1. Pseudo-Riemannian Manifolds.** Let $V$ be a finite dimensional vector space. A symmetric bilinear form $\langle .,. \rangle : V \times V \longrightarrow \mathbb{R}$ is said *nondegenerate* if $\langle u, v \rangle = 0$, for all $v \in V$, implies that $u = 0$. A *scalar product* on $V$ is a nondegenerate symmetric bilinear form. An *inner product* is a positive definite scalar product.

Assume that $V$ is equipped with a scalar product $\langle .,. \rangle$. Then, $V$ is said to be a *scalar product space*. The *norm* of a vector $v \in V$ is defined by $\|v\| = \sqrt{|\langle v, v \rangle|}$. If $\{e_1, \cdots, e_n\}$ is an (arbitrary) orthonormal basis of $V$ and $\varepsilon_j = \langle e_j, e_j \rangle = \pm 1$, the number $\kappa$ of negative signs in the list $(\varepsilon_1, \cdots, \varepsilon_n)$ is the *index* of $\langle .,. \rangle$, also called *index* of $V$. Occurs $\kappa = 0$ if and only if $\langle .,. \rangle$ is an inner product.

**Example 2.1.** *With $0 \leq \kappa \leq n$, the following formula*

$$\langle (u_1, \cdots, u_n), (v_1, \cdots, v_n) \rangle = -\sum_{i=1}^{\kappa} u_i v_i + \sum_{i=\kappa+1}^{n} u_i v_i, \tag{1}$$

*defines a scalar product on the vector space $V = I\!R^n$, having index $\kappa$.*

A *pseudo-Riemannian metric* $\Phi$ on a differentiable manifold $M$ assigns to each point $p \in M$ a scalar product $\langle .,. \rangle_p$ on the tangent space $T_pM$, so that: $i$) for any smooth vector fields $X$ and $Y$ whose domains have a non-empty intersection $\mathcal{U}$, the mapping $p \in \mathcal{U} \rightsquigarrow \langle X_p, Y_p \rangle_p$ is smooth; $ii$) all scalar products $\langle .,. \rangle_p$ have the same index.

A differentiable manifold equipped with a pseudo-Riemannian metric is called a *pseudo-Riemannian manifold*. The common value $\kappa$ of the indices of the scalar products is called the *index* of $M$. We have $0 \leq \kappa \leq \dim(M)$. If (and only if) $\kappa = 0$, that is, when each $\langle .,. \rangle_p$ is an inner product on $T_pM$, the prefix "pseudo" is removed and it is said, therefore, that $M$ is a *Riemannian manifold* and that it is equipped with a *Riemannian metric*. We write $\langle X_p, Y_p \rangle$ to designate $\langle X_p, Y_p \rangle_p$.

Since on a pseudo-Riemannian manifold the scalar product on each tangent space may not be a definite bilinear form, the following classification will be convenient. A tangent vector $v$ to a pseudo-Riemannian manifold is said to be: $i$) *spacelike* if $\langle v, v \rangle > 0$ or $v = 0$; $ii$) *lightlike* (or null) if $\langle v, v \rangle = 0$ and $v \neq 0$; $iii$) *timelike* if $\langle v, v \rangle < 0$. The category into which a given tangent vector falls is called its *causal character*.

In the context of this article, a specially important case of pseudo-Riemannian manifolds are those that can be built from vector spaces. "Environments" where the study of rolling motions without slipping or twisting become easier are of this type. Before addressing this case, it is worth remembering that if $(\mathcal{U}, \varphi = (x_1, \cdots, x_n))$ is a coordinate chart on a differentiable manifold $M$ and $p \in \mathcal{U}$, the images by the differential map $d\varphi^{-1}$ (at $\varphi(p)$) of the vectors of standard base of $T_{\varphi(p)} I\!R^n$ constitute a base of $T_pM$. We shall denote the vectors of this base by $\dfrac{\partial}{\partial x_i}\Big|_p$.

We know that any finite dimensional vector space $V$ is a differentiable manifold, in a natural way. If we fix any one ordered base $u_1, \cdots, u_n$ in $V$ and we take in the corresponding isomorphism $\varphi : \alpha_1 u_1 + \cdots + \alpha_n u_n \rightsquigarrow (\alpha_1, \cdots, \alpha_n)$, the open sets of the topology of $V$ are the inverse images by $\varphi$ of the open sets of $\mathbb{R}^n$ (equipped with the Euclidean topology) and the differentiable structure of $V$ is the maximal atlas which contains the coordinate chart $(V, \varphi)$. Moreover, there is a natural identification (independent of the fixed base) of each tangent space $T_pV$ with $V$ itself, obtained by

$$v \in V \quad \longleftrightarrow \quad v_p = x_1(v)\frac{\partial}{\partial x_1}\Big|_p + \cdots + x_n(v)\frac{\partial}{\partial x_n}\Big|_p \in T_pV, \tag{2}$$

where $x_1, \cdots, x_n$ are the coordinates defined by any base of $V$.

Then, if $\langle \cdot, \cdot \rangle$ is a scalar product on a vector space $V$, one can equip the differentiable manifold $V$ with the pseudo-Riemannian metric given on each $T_p(V)$ by $\langle u_p, v_p \rangle := \langle u, v \rangle$, where $u_p \leftrightarrow u$ and $v_p \leftrightarrow v$, making $V$ a pseudo-Riemannian manifold. In the particular case where $V = \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the scalar product defined in (1), the pseudo-Riemannian manifold so formed is called a *pseudo-Euclidean space* (of dimension $n$ and index $\kappa$), and will be denoted by $\mathbb{R}^n_\kappa$.

To conclude this section, we will now address the concept of isometry, which is an extension of the usual notion of isometry in Euclidean space. In this perspective, it corresponds to the "rigid motions". Let $M_1$ and $M_2$ be two pseudo-Riemannian manifolds with metrics $\Phi_{M_1}$ and $\Phi_{M_2}$, respectively. An *isometry* from $M_1$ to $M_2$ is a diffeomorphism $\phi : M_1 \longrightarrow M_2$ that "preserves the metrics", i.e., such that $\phi^*(\Phi_{M_2}) = \Phi_{M_1}$, where $\phi^*$ is the pullback. Explicitly, $\phi : M_1 \longrightarrow M_2$ is an isometry if and only if $\langle d\phi(u_p), d\phi(v_p) \rangle = \langle u_p, v_p \rangle$, $\forall u_p, v_p \in T_pM_1$ and $\forall p \in M_1$. The set of the isometries of the shape $\phi : M \longrightarrow M$ is a group, called the *isometry group* of $M$. If we replace the relationship $\phi^*(\Phi_{M_2}) = \Phi_{M_1}$ by $\phi^*(\Phi_{M_2}) = \mu\Phi_{M_1}$, with $\mu \in \mathbb{R} \setminus \{0\}$ (constant), we obtain the concept *homothety of coefficient* $\mu$. That is, an isometry is just a homothety with $\mu = 1$. Another particular case that we will also be interested in corresponds to $\mu = -1$, whereby we now say that $\phi$ is an *anti-isometry*.

**2.2. Parallel Transport and Geodesics.** Hereafter, $M$ denotes a pseudo-Riemannian manifold and $\gamma : I \longrightarrow M$ a smooth curve.

The set of all smooth vector fields along $\gamma$ will be denoted by $\mathfrak{X}(\gamma)$. By convention, the operator

$$\begin{array}{rcl} \dfrac{D}{dt} : \mathfrak{X}(\gamma) & \longrightarrow & \mathfrak{X}(\gamma) \\ V & \rightsquigarrow & V' = \dfrac{DV}{dt} \end{array}$$

will indicates the covariant derivative resulting from the Levi-Civita connection of $M$. Recall that in the particular case in which $M = I\!R^n_\kappa$, if $x_1, \cdots, x_n$ denote the natural coordinates and $V(t) = \sum V_i(t) \frac{\partial}{\partial x_i}\big|_{\gamma(t)}$, we have $V'(t) = \sum \dot{V}_i(t) \frac{\partial}{\partial x_i}\big|_{\gamma(t)}$, that is, the covariant derivative coincides (through the identification (2)) with the usual derivative.

A smooth vector field $V$ along $\gamma$ is said to be *parallel* if $V' = 0$. The fundamental fact about parallel fields is that for any $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$, there is a unique parallel vector field $V$ along $\gamma$ such that $V(t_0) = v_0$. This field is said the *(tangent) parallel transport of* $v_0$ *along* $\gamma$.

A curve in Euclidean space is a straight line if and only if its acceleration is identically zero. It is this characterization that is here taken into account in the definition of geodesic. The *acceleration* of $\gamma$ is the vector field along $\gamma$ defined by $\gamma'' = \frac{D\gamma'}{dt}$, where $\gamma'$ denotes the velocity field. The curve $\gamma$ is said to be a *geodesic* if its acceleration is zero. We also say that a curve is a *broken geodesic* if there is a partition of its domain such that the corresponding restrictions are geodesics.

For each $p \in M$ and each $v \in T_pM$ there is a unique geodesic $\gamma : I \longrightarrow M$ whose domain is as large as possible, i.e., it is not a segment of a geodesic with a greater domain, called *maximal geodesic*, such that $\gamma(0) = p$ and $\gamma'(0) = v$. This maximal geodesic is often simply called *the geodesic with initial point* $p$ *and initial velocity* $v$. A geodesic is said to be *spacelike* [*timelike/lightlike*] when its initial velocity vector (and therefore any other vector in its velocity field) is spacelike [timelike/lightlike].

Now consider that $M$ is a pseudo-Riemannian submanifold of another manifold $\overline{M}$. That is, the metric on $M$ results from applying the scalar product of $T_p\overline{M}$ to each pair of vectors of $T_pM$, for all $p \in M$. Accordingly, we have

$$T_p\overline{M} = T_pM \oplus (T_pM)^{\perp}, \tag{3}$$

where $(T_pM)^\perp = \left\{w \in T_p\overline{M} : \langle v, w\rangle = 0, \forall v \in T_pM\right\}$. The set $(T_pM)^\perp$ is called the *orthogonal complement* of $T_pM$, and its vectors are said to be *normal* to $M$.

When $V$ is a smooth vector field (tangent to $M$) along $\gamma$, the *Gauss formula* states the following: $\dot{V}(t) = V'(t) \oplus \Pi(\gamma'(t), V(t))$, where $\dot{V} = \frac{\overline{D}V}{dt}$ is the covariant derivative in $\overline{M}$, $V' = \frac{DV}{dt}$ is the covariant derivative in $M$ and $\Pi$ is the second fundamental form of $M \subseteq \overline{M}$. Thus, by immediate consequence of the previous formula, the covariant derivative $\frac{DV}{dt}$ is, for every $t$, the projection in $T_{\gamma(t)}M$ of the covariant derivative $\frac{\overline{D}V}{dt}$.

Let us now turn to the concept of normal parallel transport. Assume that $W$ is a smooth vector field along $\gamma$ always normal to $M$, that is, such that $W(t) \in (T_{\gamma(t)}M)^\perp$ for all $t \in I$. The *normal covariant derivative* $\frac{D^\perp W}{dt}$ is defined as the (vector field along $\gamma$ given by the) normal component of the covariant derivative $\frac{\overline{D}W}{dt}$, resulting from the decomposition (3). It is said that $W$ is *normal parallel* if $\frac{D^\perp W}{dt} = 0$. Similarly to what was previously mentioned for tangent vectors, any $w_0 \in (T_{\gamma(t_0)}M)^\perp$ may be extended, in a unique way, to a parallel normal vector field $W$ along the curve $\gamma$. This field $W$ is said to be the *normal parallel transport* of $w_0$ along $\gamma$.

The notions of (tangent) parallel transport and normal parallel transport can be expanded to the case where the curves are "only" piecewise smooth, as follow. Let us now consider that $\gamma : [a, b] \longrightarrow M$ is a piecewise smooth curve, that is, let us assume that there is a finite partition $a = t_0 < t_1 < \cdots < t_r = b$ such that each restriction of $\gamma$ to the subintervals $[t_{i-1}, t_i]$ is a smooth curve. Then, given any vector $v_a \in T_{\gamma(a)}M$ $[w_a \in (T_{\gamma(a)}M)^\perp]$, there is a unique continuous field $V$ $[W]$ defined in $[a, b]$ of tangent [normal] vectors to $M$ along the curve $\gamma$, such that its restriction to each subinterval $]t_{i-1}, t_i[$ is a parallel [normal parallel] field and $V(a) = v_a$ $[W(a) = w_a]$. This vector field is called the *parallel* [*normal parallel*] transport of $v_a$ $[w_a]$ along the piecewise smooth curve $\gamma$, and consists of the parallel [normal parallel] transport of $v_a$ $[w_a]$ along the first smooth segment of $\gamma$, when $a \leq t \leq t_1$, afterwards in the parallel [normal parallel] transport of $V(t_1)$ $[W(t_1)]$ along the second smooth segment of $\gamma$, when $t_1 \leq t \leq t_2$, and so on.

**2.3. Orientability.** It is said that two ordered bases $\{b_1, b_2, \cdots, b_n\}$ and $\left\{\widehat{b}_1, \widehat{b}_2, \cdots, \widehat{b}_n\right\}$ of a vector space $V$ have the *same orientation* if $\det(A) > 0$, where $A = (a_{ij})$ is the (unique) non-singular matrix such that $\widehat{b}_i = \sum_{j=1}^{n} a_{ij} b_j$ $(i = 1, \cdots n)$; they have *opposite orientation* if $\det(A) < 0$. It is easy to check that "having the same orientation" is an equivalence relation on the set of all bases of $V$ and that there are exactly two equivalence classes, called *orientations* of $V$. The orientation that contains the base $\{b_1, b_2, \cdots, b_n\}$ will be represented by $[b_1, b_2, \cdots, b_n]$.

Let us consider $M$ as a differentiable manifold. For $(\mathcal{U}, \varphi = (x_1, \cdots, x_n))$ a coordinate chart on $M$, with $p \in \mathcal{U}$, denote $\lambda_\varphi(p) := \left[\frac{\partial}{\partial x_1}\big|_p, \cdots, \frac{\partial}{\partial x_n}\big|_p\right]$. Then, an *orientation* $\lambda$ of $M$ is a correspondence that for each point $p \in M$ associates an orientation $\lambda(p)$ of $T_pM$, which is *smooth* in the sense that for each point of $M$ there is a coordinate mapping $\varphi$ such that $\lambda = \lambda_\varphi$ on some neighborhood of that point. $M$ is said to be *orientable* if there exists an orientation of $M$. For example, $\mathbb{R}^n$ is orientable and a possible orientation is $\lambda_\varphi$ where $\varphi$ is the identity mapping (natural coordinates). This is the usual orientation of $\mathbb{R}^n$. If $\lambda$ is an orientation of $M$, then so is $-\lambda$, which assigns to each point $p$ the opposite orientation of $T_pM$. If $M$ is connected then $\pm\lambda$ are its only two orientations.

When $\phi : M_1 \longrightarrow M_2$ is a local diffeomorphism and $p \in M_1$, it is easy to see that the correspondence $\widehat{\phi}$ given by $\widehat{\phi}\left([b_1, \cdots, b_n]\right) := [d\phi(b_1), \cdots, d\phi(b_n)]$ is a well-defined one-to-one correspondence from the orientations of $T_pM_1$ to the orientations of $T_{\phi(p)}M_2$. Under these conditions, if $M_1$ and $M_2$ are oriented by $\lambda_{M_1}$ and $\lambda_{M_2}$, respectively, it is said that: *i)* $\phi$ *preserves orientation* if $\widehat{\phi}(\lambda_{M_1}(p)) = \lambda_{M_2}(\phi(p))$, for all $p \in M_1$; *ii)* $\phi$ *reverses orientation* if $\widehat{\phi}(\lambda_{M_1}(p)) = -\lambda_{M_2}(\phi(p))$, for all $p \in M_1$. In the particular case in which $M_1 = M_2 =: M$ and it is connected, the fact that $\phi$ preserves or reverses orientation is independent of how $M$ is oriented. Thus, whenever we simply write that a local diffeomorphism $\phi : M \longrightarrow M$, with $M$ orientable and conntected, *preserves* [*reverses*] *orientation*, it will mean that $\phi$ preserves [reverses] orientation with respect to either of the two possible orientations of $M$.

Basic examples of transformations that preserve orientation are translations and rotations around one point on the plane $M = \mathbb{R}^2$. Inversely, reflections on an axis reverse orientation. In the following proposition we address the case of linear isomorphisms. The proof is quite simple.

**Proposition 2.1.** *Let* $\phi : I\!\!R^n \longrightarrow I\!\!R^n$ *be defined by* $\phi(x) = Rx$, *with* $R$ *belonging to the general linear group* $GL(n)$. *Then,* $\phi$ *preserves orientation if and only if* $\det(R) > 0$.

**2.4. Pseudo-Orthogonal Groups.** $J_\kappa$ will designate the diagonal matrix whose main diagonal entries are $\varepsilon_1 = \cdots = \varepsilon_\kappa = -1$ and $\varepsilon_{\kappa+1} = \cdots = \varepsilon_n = 1$, for $0 \leq \kappa \leq n$. Therefore, we have $J_\kappa^{-1} = J_\kappa^\top = J_\kappa$. Associated with this matrix, define

$$\mathrm{O}_\kappa(n) = \left\{R \in GL(n) : R^{-1} = J_\kappa R^\top J_\kappa\right\}.$$

$\mathrm{O}_\kappa(n)$ is a closed algebraic subgroup of $GL(n)$ and, hence, is itself a Lie group, known as *pseudo-orthogonal group*. Clearly, any matrix of $\mathrm{O}_\kappa(n)$ has determinant equal to $\pm 1$ . The Lie subgroup $\mathrm{SO}_\kappa(n) := \{R \in \mathrm{O}_\kappa(n) : \det(R) = 1\}$ is called the *special pseudo-orthogonal group*.

Each isometry of the pseudo-Euclidean space $I\!\!R^n_\kappa$, for $0 \leq \kappa \leq n$, has a unique expression as follows

$$\begin{array}{rccc} T_{(R,s)}: & I\!\!R^n_\kappa & \longrightarrow & I\!\!R^n_\kappa \\ & x & \rightsquigarrow & Rx + s \end{array},$$

with $R \in \mathrm{O}_\kappa(n)$ and $s \in I\!\!R^n$. We also have $T_{(R_2,s_1)} \circ T_{(R_2,s_1)} = T_{(R_2R_1, R_2s_1+s_2)}$. For this reason, we identify the isometry group of $I\!\!R^n_\kappa$ with the semi-direct product $\mathrm{O}_\kappa(n) \rtimes I\!\!R^n := \{(R,s) : R \in \mathrm{O}_\kappa(n), s \in I\!\!R^n\}$, having the group operation defined by $(R_2,s_2)\circ(R_1,s_1) := (R_2R_1, R_2s_1+s_2)$. We also use the obvious identification among the subgroup of the linear isometries $T_{(R,0)} : x \rightsquigarrow Rx$ and the (multiplicative) group $\mathrm{O}_\kappa(n)$. Thus, as an immediate consequence of Proposition 2.1, the subgroup of linear isometries of $I\!\!R^n_\kappa$ which preserve orientation is $\mathrm{SO}_\kappa(n)$.

When $\kappa = 0$ or $\kappa = n$, the pseudo-orthogonal group $\mathrm{O}_\kappa(n)$ reduces to the orthogonal group $\mathrm{O}(n) := \left\{R \in GL(n) : R^{-1} = R^\top\right\}$. This group is the disjoint union of the special orthogonal group $\mathrm{SO}(n) := \{R \in \mathrm{O}(n) : \det(R) = 1\}$ with the set $\{R \in \mathrm{O}(n) : \det(R) = -1\}$. $\mathrm{SO}(n)$ is a connected Lie subgroup. When $0 < \kappa < n$, considering each matrix of $\mathrm{O}_\kappa(n)$ decomposed as $R = \left[\begin{array}{c|c} R_1 & R_2 \\ \hline R_3 & R_4 \end{array}\right]$, with $R_1$ of order $\kappa$ and $R_4$ of order $n-\kappa$, we have that $\mathrm{O}_\kappa(n)$ decomposes into the following 4 disjoint sets, indexed to the signs of the determinants $\det(R_1)$ and $\det(R_4)$: $\mathrm{O}^{++}_\kappa(n)$, $\mathrm{O}^{+-}_\kappa(n)$, $\mathrm{O}^{-+}_\kappa(n)$, $\mathrm{O}^{--}_\kappa(n)$. It can be prove that $\mathrm{O}^{++}_\kappa(n) \cup \mathrm{O}^{--}_\kappa(n) = \mathrm{SO}_\kappa(n)$, and we also have that $\mathrm{O}^{++}_\kappa(n)$ is a connected set, unlike $\mathrm{SO}_\kappa(n)$. From now on, this connected component, which contains the identity $I_n$, and is a Lie subgroup of $\mathrm{O}_\kappa(n)$, will be denoted by $\mathrm{SO}^{\mathrm{I}}_\kappa(n)$.

Let us now consider $0 \leq \kappa \leq n$. We define $\mathrm{SO}^{\mathrm{I}}_0(n) = \mathrm{SO}^{\mathrm{I}}_n(n) := \mathrm{SO}(n)$. Since translations in $\mathbb{R}^n_\kappa$ preserve orientation, it results from previous considerations that the maximal connected Lie subgroup of the group of isometries of $\mathbb{R}^n_\kappa$ which preserve orientation is:

$$\mathrm{SO}^{\mathrm{I}}_\kappa(n) \rtimes \mathbb{R}^n := \{(R,s) : R \in \mathrm{O}_\kappa(n), \det(R_1) > 0, \det(R_4) > 0, s \in \mathbb{R}^n\},$$

where $R = \left[\begin{array}{c|c} R_1 & R_2 \\ \hline R_3 & R_4 \end{array}\right]$, with $R_1$ of order $\kappa$ and $R_4$ of order $n-\kappa$, assuming $R = R_4$ and $R = R_1$ when $\kappa = 0$ and $\kappa = n$.

For $0 \leq \kappa \leq n$, the Lie algebra $\mathfrak{o}_\kappa(n)$ of $\mathrm{O}_\kappa(n)$ is the subalgebra of $\mathfrak{gl}(n)$ formed by all matrices $S$ such that $S^\top = -J_\kappa S J_\kappa$. Since they have common neighborhoods of the identity $I_n$, the groups $\mathrm{O}_\kappa(n)$, $\mathrm{SO}_\kappa(n)$ and $\mathrm{SO}^{\mathrm{I}}_\kappa(n)$ have the same Lie algebra, i.e., $\mathfrak{o}_\kappa(n) = \mathfrak{so}_\kappa(n) = \mathfrak{so}^{\mathrm{I}}_\kappa(n)$.

## 3. Rolling Maps for Pseudo-Riemannian Manifolds

**3.1. Definition of Rolling Map without Slipping or Twisting.** The definition that we introduce in this section is the main definition of this article. It formalizes the concept of "pure" rolling motion of a pseudo-Riemannian manifold (the rolling moving) over another (the stationary manifold) of equal dimension, assuming an extrinsic approach. This definition generalize that of a rolling map without slipping or twisting for submanifolds of the Euclidean space found in [17].

We start by fixing some auxiliary notations. Let $\overline{M}$ be a connected and orientable pseudo-Riemannian manifold. The isometry group of $\overline{M}$ is denoted by $\mathrm{Isom}(\overline{M})$ and their subgroup of the isometries that preserve orientation by $\mathrm{Isom}^+(\overline{M})$. It is well known that $\mathrm{Isom}(\overline{M})$ has a Lie group structure and acts smoothly on $\overline{M}$. The symbol $*$ will represent the natural action of $\mathrm{Isom}(\overline{M})$ on $\overline{M}$, i.e., $* : \mathrm{Isom}(\overline{M}) \times \overline{M} \longrightarrow \overline{M}$ is defined by $f * p := f(p)$. With $p \in \overline{M}$ fixed, the corresponding orbital mapping will be denoted by $\zeta_p$, i.e., $\zeta_p : \mathrm{Isom}(\overline{M}) \longrightarrow \overline{M}$ is defined by $\zeta_p(f) := f * p$.

**Definition 3.1.** *Let $M_1$ and $M_2$ be two pseudo-Riemannian submanifolds of $\overline{M}$, having equal dimension and index. A* rolling map without slipping or twisting of $M_1$ over $M_2$ *is a piecewise smooth curve*

$$\begin{aligned} g : [0,\tau] &\longrightarrow \overline{G} \\ t &\rightsquigarrow g(t), \end{aligned} \tag{4}$$

*where* $\overline{G}$ *is a connected subgroup of* $\mathrm{Isom}^+(\overline{M})$*, satisfying the following properties 1, 2 and 3.*

(1) *Rolling condition. There is a piecewise smooth curve* $\alpha : [0,\tau] \longrightarrow M_1$*, smooth at all times* $t$ *where* $g$ *is smooth, such that for every* $t \in [0,\tau]$*:*
   (a) $\alpha_{\mathrm{dev}}(t) := g(t) * \alpha(t) \in M_2$,
   (b) $T_{\alpha_{\mathrm{dev}}(t)}(g(t) * M_1) = T_{\alpha_{\mathrm{dev}}(t)} M_2$.

(2) *No-slip condition. There is a partition* $0 = t_0 < t_1 < \cdots < t_r = \tau$ *such that* $g$ *is smooth in all subintervals* $]t_{i-1}, t_i[$ *and for each value* $t$ *of these subintervals we have:*

$$(\zeta_{\alpha(t)} \circ g)'(t) = (\sigma \rightsquigarrow g(\sigma) * \alpha(t))'(t) = 0. \tag{5}$$

(3) *No-twist condition. There is a partition* $0 = t_0 < t_1 < \cdots < t_r = \tau$ *such that* $g$ *is smooth in all subintervals* $]t_{i-1}, t_i[$ *and the following items (a) and (b) are verified. For each* $v \in T_{\alpha_{\mathrm{dev}}(t)}\overline{M}$*,* $X^v$ *denotes the vector field along curve* $\sigma \in [0,\tau] \rightsquigarrow g(\sigma) * \alpha(t) \in \overline{M}$ *defined by*

$$X^v(\sigma) := d\left(g(\sigma) \circ g(t)^{-1}\right)(v). \tag{6}$$

*Then,* $\forall t \in \cup_{i=1}^r ]t_{i-1}, t_i[$*, we have:*

$$(a) \;\; \textit{(tangential part)} \quad \forall v \in T_{\alpha_{\mathrm{dev}}(t)} M_2,\ \dot{X}^v(t) \in (T_{\alpha_{\mathrm{dev}}(t)} M_2)^{\perp}; \tag{7}$$

$$(b) \;\; \textit{(normal part)} \quad \forall v \in \left(T_{\alpha_{\mathrm{dev}}(t)} M_2\right)^{\perp},\ \dot{X}^v(t) \in T_{\alpha_{\mathrm{dev}}(t)} M_2, \tag{8}$$

*where* $\dot{X}^v(t) = \dfrac{\overline{D}X^v}{d\sigma}(t)$ *denotes the covariant derivative on* $\overline{M}$*, determined by the respective Levi-Civita connection.*

The curve $\alpha$ is called the *rolling curve*, while the curve $\alpha_{\mathrm{dev}} : [0,\tau] \longrightarrow M_2$ defined by $\alpha_{\mathrm{dev}}(t) := g(t) * \alpha(t)$ is said to be the *development of* $\alpha$ *in* $M_2$.*

The following remark will be useful.

*Whenever $t \rightsquigarrow g(t) \in \mathrm{Isom}(\overline{M})$ and $t \rightsquigarrow \alpha(t) \in \overline{M}$ are smooth, $t \rightsquigarrow (g(t), \alpha(t)) \in \mathrm{Isom}(\overline{M}) \times \overline{M}$ is smooth. Therefore, the curve $t \rightsquigarrow \alpha_{\mathrm{dev}}(t) = g(t) * \alpha(t)$ is smooth on the intervals where $g$ is smooth.

**Remark 3.1.** *Since* $g(t) : \overline{M} \longrightarrow \overline{M}$ *is an isometry, for all* $t \in [0, \tau]$*, we have* $dg(t)(T_{\alpha(t)}M_1) = T_{g(t)*\alpha(t)}g(t)*M_1$, $dg(t)\left((T_{\alpha(t)}M_1)^{\perp}\right) = \left(T_{g(t)*\alpha(t)}g(t)*M_1\right)^{\perp}$. *Therefore, from the rolling condition results the following:*

(1) $dg(t)(T_{\alpha(t)}M_1) = T_{\alpha_{\text{dev}}(t)}M_2$

(2) $dg(t)\left(\left(T_{\alpha(t)}M_1\right)^{\perp}\right) = (T_{\alpha_{\text{dev}}(t)}M_2)^{\perp}$.

The next proposition introduces a reformulation of the no-slip condition, stating that it is equivalent to saying that the differential of $g(t)$, at point $\alpha(t)$, for almost all $t$, transforms the velocity vector $\alpha'(t)$ of the rolling curve into the velocity vector $\alpha'_{\text{dev}}(t)$ of the development curve.

**Proposition 3.1.** *Under the conditions of Definition 3.1,*

$$(5) \iff \alpha'_{\text{dev}}(t) = dg(t)(\alpha'(t)).$$

*Proof*: Let us consider the curve $\gamma : [0, \tau] \longrightarrow \text{Isom}(\overline{M}) \times \overline{M}$ defined by $\gamma(t) = (g(t), \alpha(t))$. Since $\gamma$ is a curve in a product of differentiable manifolds, at any instant $t$ we have

$$\gamma'(t) = (\sigma \rightsquigarrow (g(\sigma), \alpha(t)))'(t) + (\sigma \rightsquigarrow (g(t), \alpha(\sigma)))'(t).$$

Thus, the properties of the differential mapping allow us to write the following equalities:

$$\begin{aligned}
\alpha'_{\text{dev}}(t) &= (* \circ \gamma)'(t) \\
&= d*(\gamma'(t)) \\
&= d*\left((\sigma \rightsquigarrow (g(\sigma), \alpha(t)))'(t)\right) + d*\left((\sigma \rightsquigarrow (g(t), \alpha(\sigma)))'(t)\right) \\
&= (\sigma \rightsquigarrow g(\sigma) * \alpha(t))'(t) + (\sigma \rightsquigarrow g(t) * \alpha(\sigma))'(t) \\
&= (\sigma \rightsquigarrow g(\sigma) * \alpha(t))'(t) + dg(t)(\alpha'(t)).
\end{aligned}$$

Therefore, $(\sigma \rightsquigarrow g(\sigma) * \alpha(t))'(t) = 0$ if and only if $\alpha'_{\text{dev}}(t) = dg(t)(\alpha'(t))$. ■

**3.2. Interpretation of the Definition of Rolling Map in $\overline{M} = \mathbb{R}^n$.** In order to explain the conditions of Definition 3.1, we will follow the reasoning used in [17]. The rolling map $g$ associates an isometry with each value $t \in [0, \tau]$. So, with "successive" transformations $g(t)$ we can conceive that each point makes a certain "movement" in $\mathbb{R}^n$, describing a continuous trajectory with the positions held over the course of time $t$. Specifically, under the "effect" of the mapping $g$, each "mobile point" travels along its trajectory from the corresponding initial position $g(0) * p$ to the end position $g(\tau) * p$, so that at the instant $t$ it "passes" into position $g(t) * p$ with velocity $(\zeta_p \circ g)'(t)$, if $g$ is smooth at that instant. Thus,

(1) the rolling condition says that $M_1$ moves so as to be tangent to $M_2$, at each instant $t$, at point the $\alpha_{\rm dev}(t)$;
(2) when item $(a)$ of the rolling condition is verified, the no-slip condition says that for almost all $t$, that is, except for when at most a finite number of values $t$, the mobile point which at the initial instant occupies the position corresponding to $\alpha(t) \in M_1$ describes a movement with a stop ("smooth"), at instant $t$, when it reaches position $\alpha_{\rm dev}(t) \in M_2$. That is, this condition means that the linear velocity at the point of contact is zero. (See Figure 1).

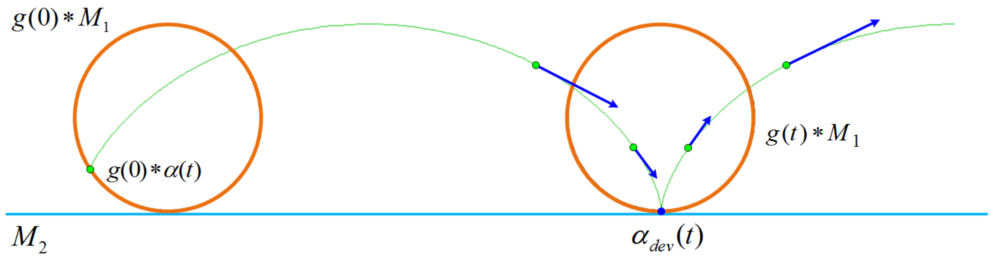


FIGURE 1. Velocity vector of a mobile point in no-slip rolling.

Assuming the rolling condition, for each $v \in T_{\alpha_{\rm dev}(t)}M_2$ $[(T_{\alpha_{\rm dev}(t)}M_2)^\perp]$, the vector $dg(t)^{-1}(v)$ belongs to $T_{\alpha(t)}M_1$ $[(T_{\alpha(t)}M_1)^\perp]$ and the vector field $X^v$ expresses the "transport" of this vector by rolling, describing it as "stuck" to the manifold in motion and taking it to coincide with $v$ at instant $t$. Furthermore, in $I\!R^n$ the covariant derivative $\dot{X}^v(t) = \frac{\overline{D}X^v}{d\sigma}(t)$ is the usual derivative. So, in this context,

(3) the tangential [normal] part of no-twist condition says that, for almost all $t$, the transport velocity of each tangent [normal] vector to $M_1$ at $\alpha(t)$ has no component in the tangential [normal] direction at instant $t$. (See Figure 2.)

**3.3. Rolling Maps in $\overline{M} = I\!R^n_\kappa$.** In the particular case of $\overline{M} = I\!R^n_\kappa$, for $0 \leq \kappa \leq n$, we know that $\overline{G} = \mathrm{SO}^{\mathrm{I}}_\kappa(n) \rtimes I\!R^n$ is a connected Lie subgroup of the group of isometries of $\overline{M}$ that preserve orientation. Thus, a piecewise

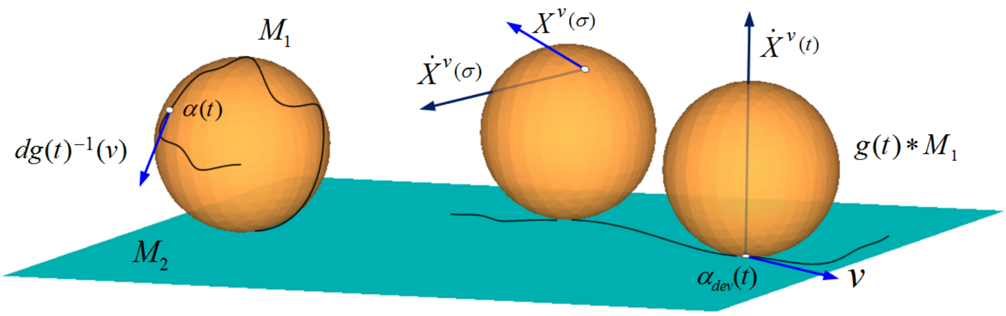


FIGURE 2. No-twist condition (tangential part) in rolling of the bidimensional sphere.

smooth curve

$$\begin{aligned} g : [0,\tau] &\longrightarrow \mathrm{SO}^{\mathrm{I}}_{\kappa}(n) \rtimes I\!R^n \\ t &\rightsquigarrow (R(t), s(t)), \end{aligned} \tag{9}$$

is a rolling map if the corresponding properties 1, 2 and 3 of Definition 3.1 are satisfied, with the natural action $*$ of the isometries given by

$$(R,s) * p = Rp + s, \ \ \forall (R,s) \in \mathrm{O}_{\kappa}(n) \rtimes I\!R^n, \ \forall p \in I\!R^n_{\kappa}.$$

Next we rewrite the relationships (5) - (8), adapting them to this particular case. Let $t \rightsquigarrow g(t) = (R(t), s(t))$ be a piecewise smooth curve in $\mathrm{SO}^{\mathrm{I}}_{\kappa}(n) \rtimes I\!R^n$ and $v$ a tangent vector to $I\!R^n_{\kappa}$, arbitrary. Then, with the usual identification of each $T_p I\!R^n$ with $I\!R^n$, we can easily verify that: *i*) in open intervals where $g$ is smooth, $(\zeta_p \circ g)'(t) = \dot{R}(t)p + \dot{s}(t)$; *ii*) $dg(\sigma) : I\!R^n \longrightarrow I\!R^n$ is defined by $dg(\sigma)(\eta) = R(\sigma)\eta$ and, consequently, $X^v(\sigma) = R(\sigma)R^{-1}(t)v$.

Thus, in this situation, equation (5) of the no-slip condition is reduced to:

$$\dot{R}(t)\alpha(t) + \dot{s}(t) = 0; \tag{10}$$

and, since in $I\!R^n_{\kappa}$ the covariant derivative coincides with the usual derivative, relationships (7) and (8) of the no-twist condition are reduced to:

$$\text{(tangencial part)} \qquad \forall v \in T_{\alpha_{\mathrm{dev}}(t)}M_2,\ \dot{R}(t)R^{-1}(t)v \in (T_{\alpha_{\mathrm{dev}}(t)}M_2)^{\perp}; \tag{11}$$

$$\text{(normal part)} \qquad \forall v \in \left(T_{\alpha_{\mathrm{dev}}(t)}M_2\right)^{\perp},\ \dot{R}(t)R^{-1}(t)v \in T_{\alpha_{\mathrm{dev}}(t)}M_2. \tag{12}$$

**3.4. Properties of Rolling Motions.** In this section we present three basic properties of rolling motions without slipping or twisting. The first two are limited to cases where the manifold environment $\overline{M}$ results from a scalar product space.

Let us begin by introducing an auxiliary result, needed for some proofs of later results.

**Lemma 3.1.** *Let us suppose that $\overline{M}$ is (a pseudo-Riemannian manifold constructed from) a scalar product space, and that the curves $\sigma \in I \rightsquigarrow \alpha(\sigma) \in \overline{M}$ and $\sigma \in I \rightsquigarrow g(\sigma) \in \mathrm{Isom}(\overline{M})$ are smooth. If $V : I \longrightarrow T\overline{M}$ is a smooth vector field such that $V(\sigma) \in T_{g(\sigma)*\alpha(\sigma)}\overline{M}$, $\forall \sigma \in I$, and $Y : I \longrightarrow T\overline{M}$ is the corresponding vector field defined by $Y(\sigma) := dg(\sigma)^{-1}(V(\sigma)) \in T_{\alpha(\sigma)}\overline{M}$, then, for every $t$ fixed in $I$, we have*

$$\frac{\overline{D}V}{d\sigma}(t) = \frac{\overline{D}}{d\sigma}\Big(dg(\sigma)\big(Y(t)\big)\Big)(t) + dg(t)\Big(\frac{\overline{D}Y}{d\sigma}(t)\Big). \tag{13}$$

(See Figure 3, where $\overline{M} = I\!R^3_\kappa$ is considered.)

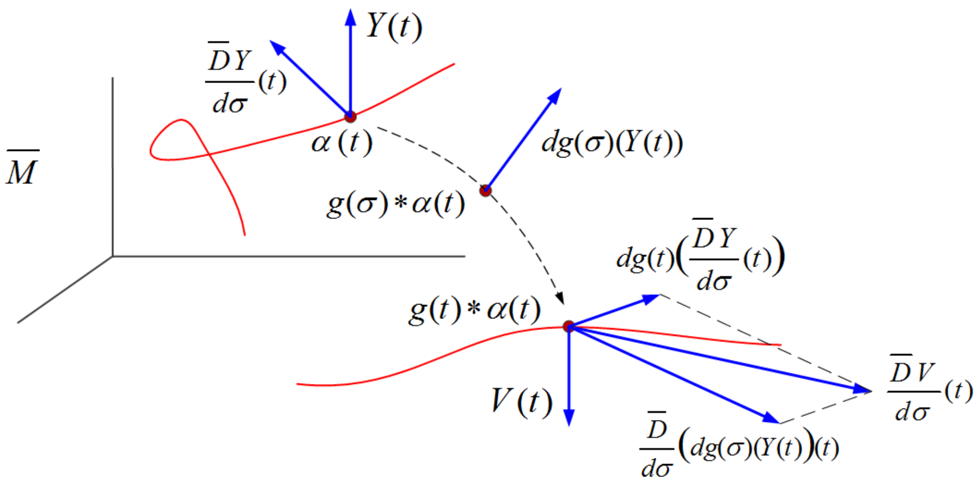


FIGURE 3. Auxiliary property of the covariant derivative.

*Proof*: Assume that $\overline{M}$ results from a scalar product space with dimension $n$ and index $\kappa$. Therefore, $\overline{M}$ is isometric to $I\!R^n_\kappa$ (See [16], p. 59). Let $\varphi : \overline{M} \longrightarrow I\!R^n_\kappa$ be the existing isometry, and let us consider that $\sigma \in I \rightsquigarrow \varphi \circ g(\sigma) \circ \varphi^{-1} \in \mathrm{Isom}(I\!R^n_\kappa)$ verifies $\left(\varphi \circ g(\sigma) \circ \varphi^{-1}\right)(x) = R(\sigma)x + s(\sigma)$, with $R(\sigma) \in \mathrm{O}_\kappa(n)$ and $s(\sigma) \in I\!R^n$.

The following equalities result from the fact that any isometry preserves the covariant derivative, as well as the fact that $d\left(\varphi \circ g(\sigma) \circ \varphi^{-1}\right)(\eta) = R(\sigma)\eta$, $\forall \eta \in T_p(I\!R^n)$. The "dot" denotes the covariant derivative in $I\!R^n_\kappa$, which, through the natural identification of each $T_pI\!R^n$ with $I\!R^n$, is the usual derivative in $I\!R^n$.

Effectively, we have:

$$
\begin{aligned}
\frac{\overline{D}V}{d\sigma}(t) &= d\varphi^{-1}\Big(\left(d\varphi(V)\right)^{\cdot}(t)\Big)\\
&= d\varphi^{-1}\Big(\left(d\left(\varphi\circ g(\sigma)\circ\varphi^{-1}\right)\left(d\varphi(Y)\right)\right)^{\cdot}(t)\Big)\\
&= d\varphi^{-1}\Big(\left(R(\sigma)\, d\varphi(Y)\right)^{\cdot}(t)\Big)\\
&= d\varphi^{-1}\Big(\left(R(\sigma)\, d\varphi(Y(t))\right)^{\cdot}(t) + \left(R(t)\, d\varphi(Y)\right)^{\cdot}(t)\Big)\\
&= d\varphi^{-1}\Big(\left(d\varphi\left(dg(\sigma)(Y(t))\right)\right)^{\cdot}(t)\Big) + d\varphi^{-1}\Big(\left(d\varphi\left(dg(t)(Y)\right)\right)^{\cdot}(t)\Big)\\
&= \frac{\overline{D}}{d\sigma}\Big(\, dg(\sigma)\big(Y(t)\big)\Big)(t) + \frac{\overline{D}}{d\sigma}\Big(\, dg(t)\big(Y\big)\Big)(t)\\
&= \frac{\overline{D}}{d\sigma}\Big(\, dg(\sigma)\big(Y(t)\big)\Big)(t) + dg(t)\Big(\frac{\overline{D}Y}{d\sigma}(t)\Big)
\end{aligned}
$$

■

The first property of rolling motions, described in the next proposition, concerns the "composition" of simultaneous rolling motions.

**Proposition 3.2.** *(*Transitivity of rolling motions*) Let $M_1$, $M_2$ and $M_3$ be pseudo-Riemannian submanifolds of $\overline{M}$, a manifold constructed from a scalar product space. Suppose that the following hold:*

(i) *$g_1 : [0,\tau] \longrightarrow \overline{G}$ is a rolling map of $M_1$ over $M_2$, with rolling curve $\alpha_1$ and development curve $\alpha_2$.*
(ii) *$g_2 : [0,\tau] \longrightarrow \overline{G}$ is a rolling map of $M_2$ over $M_3$, having $\alpha_2$ as rolling curve and development curve $\alpha_3$.*

*Then, $g_2 \circ g_1 : [0,\tau] \longrightarrow \overline{G}$, defined by $(g_2 \circ g_1)(t) = g_2(t) \circ g_1(t)$, is a rolling map of $M_1$ over $M_3$, having $\alpha_1$ as rolling curve and $\alpha_3$ as development curve.*

*Proof*: The proof consists in confirming the veracity of conditions (1), (2) and (3) of Definition 3.1. The no-slip and no-twist conditions are verified at the intervals obtained by "overlapping" the partitions guaranteed for $g_1$ and $g_2$. We have the following:

(1) Verifying the rolling condition. Let $t \in [0, \tau]$ be arbitrary. From the hypothesis, we get $(g_2 \circ g_1)(t) * \alpha_1(t) = \alpha_3(t) \in M_3$ and

$$\begin{aligned} T_{\alpha_3(t)}(g_2 \circ g_1)(t) * M_1 &= d\,g_2(t)\left(T_{\alpha_2(t)} g_1(t) * M_1\right) \\ &= d\,g_2(t)\left(T_{\alpha_2(t)} M_2\right) \\ &= T_{\alpha_3(t)} g_2(t) * M_2 \\ &= T_{\alpha_3(t)} M_3. \end{aligned}$$

(2) Verifying the no-slip condition. From hypotheses *i*) and *ii*), we have $\alpha_2'(t) = d\,g_1(t)(\alpha_1'(t))$ and $\alpha_3'(t) = d\,g_2(t)(\alpha_2'(t))$. Therefore

$$\begin{aligned} d\left(g_2(t) \circ g_1(t)\right)\left(\alpha_1'(t)\right) &= d\,g_2(t)\left(d\,g_1(t)(\alpha_1'(t))\right) \\ &= d\,g_2(t)\left(\alpha_2'(t)\right) \\ &= \alpha_3'(t). \end{aligned}$$

(3) Verifying the no-twist condition. We only address the tangential part of this condition, since the proof of the normal part is similar.

With $j = 1, 2$, the assumptions assure us that, $\forall v \in T_{\alpha_{j+1}(t)} M_{j+1}$, $\dot{X}_j^v(t) \in (T_{\alpha_{j+1}(t)} M_{j+1})^\perp$, where $X_j^v(\sigma) := d\left(g_j(\sigma) \circ g_j(t)^{-1}\right)(v)$. We need now to show that, $\forall v \in T_{\alpha_3(t)} M_3$, $\dot{X}^v(t) \in (T_{\alpha_3(t)} M_3)^\perp$, where

$$\begin{aligned} X^v(\sigma) &:= d\left(\left(g_2(\sigma) \circ g_1(\sigma)\right) \circ \left(g_2(t) \circ g_1(t)\right)^{-1}\right)(v) \\ &= \left(d\,g_2(\sigma) \circ d\,g_1(\sigma) \circ d\,g_1(t)^{-1} \circ d\,g_2(t)^{-1}\right)(v) \\ &= d\,g_2(\sigma)\left(X_1^{d\,g_2(t)^{-1}(v)}(\sigma)\right). \end{aligned}$$

However, fixing any $v \in T_{\alpha_3(t)} \overline{M}$, if from Lemma 3.1 we consider the particular case in which $\alpha(\sigma) = g_1(\sigma) * \alpha_1(t)$, $g(\sigma) = g_2(\sigma)$ and $V(\sigma) = X^v(\sigma)$, equality (13) is reduced to

$$\dot{X}^v(t) = \dot{X}_2^v(t) + d\,g_2(t)\left(\dot{X}_1^{d\,g_2(t)^{-1}(v)}(t)\right). \tag{14}$$

Hence, $\dot{X}^v(t) \in \left((T_{\alpha_3(t)} M_3)^\perp + d\,g_2(t)\left((T_{\alpha_2(t)} M_2)^\perp\right)\right) = (T_{\alpha_3(t)} M_3)^\perp$, $\forall v \in T_{\alpha_3(t)} M_3$.

(See Figure 4.) ∎

The "reciprocity" of the rolling motions is addressed in the following proposition.

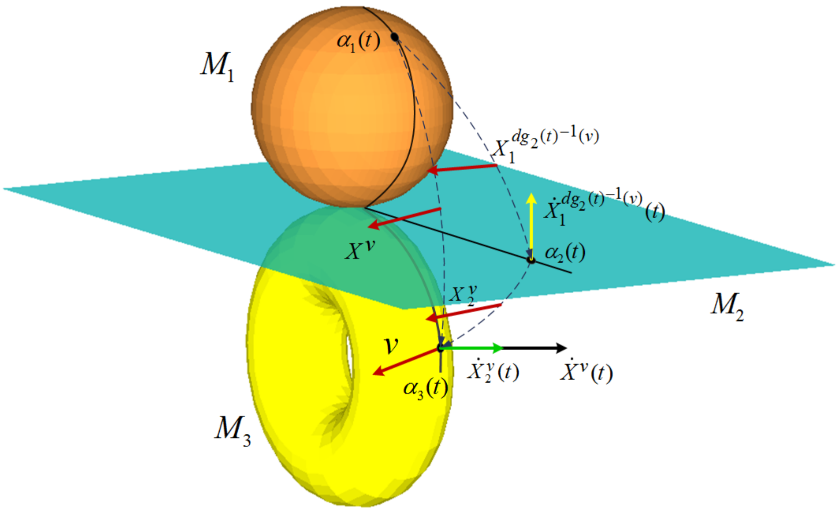


FIGURE 4. Transitivity of rolling motions.

**Proposition 3.3.** *(*Symmetry of rolling motions*) Consider that $M_1$ and $M_2$ are pseudo-Riemannian submanifolds of $\overline{M}$, a manifold constructed from a scalar product space. Let $g : [0,\tau] \longrightarrow \overline{G}$ be a rolling map of $M_1$ over $M_2$, with a rolling curve $\alpha_1$ and a development curve $\alpha_2$. Then, $g^{-1} : [0,\tau] \longrightarrow \overline{G}$, defined by $g^{-1}(t) = g(t)^{-1}$, is a rolling map of $M_2$ over $M_1$, having $\alpha_2$ as rolling curve and $\alpha_1$ as development curve.*

*Proof*: Confirmation of the no-slip and no-twist conditions is naturally carried out with the same partition of the interval $[0,\tau]$ that the hypothesis guarantees regarding $g$. We have the following:

(1) Verifying the rolling condition. $g^{-1}(t) * \alpha_2(t) = \alpha_1(t) \in M_1$, obviously. The hypothesis assures the equality $T_{\alpha_2(t)}g(t) * M_1 = T_{\alpha_2(t)}M_2$, therefore

$$\begin{aligned} T_{g^{-1}(t)*\alpha_2(t)}g^{-1}(t) * M_2 &= d\,g^{-1}(t)\left(T_{\alpha_2(t)}M_2\right) \\ &= d\,g^{-1}(t)\left(T_{\alpha_2(t)}g(t) * M_1\right) \\ &= T_{g^{-1}(t)*\alpha_2(t)}M_1. \end{aligned}$$

(2) Verifying of the no-slip condition. From $\alpha_2'(t) = d\,g(t)\left(\alpha_1'(t)\right)$, verified by hypothesis, we can immediately write $d\,g^{-1}(t)\left(\alpha_2'(t)\right) = \alpha_1'(t)$.

(3) Verifying the no-twist condition. With regards to the tangential part, we need to show that, $\forall v \in T_{\alpha_1(t)}M_1$ , $\dot{X}_{-1}^v(t) \in (T_{\alpha_1(t)}M_1)^{\perp}$, with $X_{-1}^v(\sigma) := d\left(g(\sigma)^{-1} \circ g(t)\right)(v)$. The hypothesis assures that, $\forall v \in$

$T_{\alpha_2(t)}M_2$, $\dot{X}^v(t) \in (T_{\alpha_2(t)}M_2)^\perp$, being $X^v(\sigma)$ as in (6). Furthermore, fixing any $v \in T_{\alpha_1(t)}\overline{M}$, if in Lemma 3.1 we consider $\alpha(\sigma) = g(\sigma)*\alpha_1(t)$, $g(\sigma) = g(\sigma)^{-1}$ and $V(\sigma) = v$, equality (13) allows us to conclude the following

$$\dot{X}^v_{-1}(t) = -\,dg(t)^{-1}\left(\dot{X}^{dg(t)(v)}(t)\right). \tag{15}$$

Therefore, we have $\dot{X}^v_{-1}(t) \in \left(-\,dg(t)^{-1}\left((T_{\alpha_2(t)}M_2)^\perp\right)\right) = (T_{\alpha_1(t)}M_1)^\perp$, $\forall v \in T_{\alpha_1(t)}M_1$.

The proof regarding the normal part is carried out with the obvious adaptations.

■

**Remark 3.2.** *The above propositions enable us, in particular, to decrease the study of the rolling motions between submanifolds of $\mathbb{R}^n_\kappa$, under the condition that they have equal dimension and index and are tangent to each other, to the case in which one of the two is the* affine tangent space *to the other manifold at a point (arbitrary), which is defined for any $M$ and $p \in M$ by*

$$T_p^{\mathrm{aff}}M := \{p + v : v \in T_pM\}.$$

*Indeed, considering $M_1$ and $M_2$ tangent at a point $p_0$, that is, such that $T_{p_0}^{\mathrm{aff}}M_1 = T_{p_0}^{\mathrm{aff}}M_2 =: N$, if we know how to roll $M_1$ and $M_2$ over $N$, by symmetry we will also know how to roll $N$ over $M_2$; therefore, the rolling of $M_1$ over $M_2$ may then be obtained by transitivity. The importance of this reasoning is due to the fact that rolling motions over an affine tangent space are, from the outset, easier to describe.*

From any known rolling motion, we can deduce a rolling motion between the corresponding manifolds transformed by homothety. This is what we will show below.

**Proposition 3.4.** *Let $\overline{M}_1$ and $\overline{M}_2$ be pseudo-Riemannian manifolds, which are connected, orientable and with equal dimension. Assume that:*

(i) *$M_1$ and $M_2$ are two pseudo-Riemannian submanifolds of $\overline{M}_1$;*
(ii) *$g : [0,\tau] \longrightarrow \overline{G}_1$ is a rolling map of $M_1$ over $M_2$, with rolling curve $\alpha_1$ and development curve $\alpha_2$;*
(iii) *$\phi : \overline{M}_1 \longrightarrow \overline{M}_2$ is a (fixed) homothety.*

*Then $\phi \circ g \circ \phi^{-1} : [0,\tau] \longrightarrow \overline{G}_2 = \{\phi \circ f \circ \phi^{-1} : f \in \overline{G}_1\}$, defined by $(\phi \circ g \circ \phi^{-1})(t) = \phi \circ g(t) \circ \phi^{-1}$, is a rolling map of $\phi(M_1)$ over $\phi(M_2)$, having*

$\phi\circ\alpha_1 : t \rightsquigarrow \phi(\alpha_1(t))$ *as rolling curve and* $\phi\circ\alpha_2 : t \rightsquigarrow \phi(\alpha_2(t))$ *as development curve.*

*Proof*: We note first that each mapping $\phi \circ f \circ \phi^{-1} \in \overline{G}_2$ is an isometry on $\overline{M}_2$ and that it preserves orientation. In fact, since if $\phi$ has coefficient $\mu$ then $\phi^{-1}$ is a homothety with coefficient $\mu^{-1}$, we can write the following: $\forall p \in M_2$, $\forall u_p, v_p \in T_pM_2$,

$$\begin{aligned}\langle d(\phi\circ f\circ\phi^{-1})(u_p), d(\phi\circ f\circ\phi^{-1})(v_p)\rangle &= \langle d\phi(df(d\phi^{-1}(u_p))), d\phi(df(d\phi^{-1}(v_p)))\rangle \\ &= \mu\langle df(d\phi^{-1}(u_p)), df(d\phi^{-1}(v_p))\rangle \\ &= \mu\langle d\phi^{-1}(u_p), d\phi^{-1}(v_p)\rangle \\ &= \langle u_p, v_p\rangle.\end{aligned}$$

In addition, with oriented connected manifolds, we can prove that a diffeomorphism preserves orientation if and only if its inverse mapping preserves orientation, and also that the composition of two diffeomorphisms reverses orientation if and only if one preserves and the other reverses orientation. Therefore, we immediately conclude that $\phi \circ f \circ \phi^{-1}$ preserves orientation.

We will now confirm the rolling, no-slip and no-twist conditions. With regards the last two, we take the partition of $[0, \tau]$ that the hypothesis assures for $g$. We have the following:

(1) Verifying the rolling condition. The hypothesis allow us to write clearly $(\phi \circ g \circ \phi^{-1})(t) * (\phi \circ \alpha_1)(t) = (\phi \circ \alpha_2)(t) \in \phi(M_2)$, and

$$\begin{aligned}T_{(\phi\circ\alpha_2)(t)}(\phi \circ g \circ \phi^{-1})(t) * (\phi(M_1)) &= T_{(\phi\circ\alpha_2)(t)}(\phi \circ g(t))(M_1) \\ &= d\phi\left(T_{\alpha_2(t)}g(t) * M_1\right) \\ &= d\phi\left(T_{\alpha_2(t)}M_2\right) \\ &= T_{(\phi\circ\alpha_2)(t)}\phi(M_2).\end{aligned}$$

(2) Verifying the no-slip condition. The hypothesis assures the equality $(\zeta_{\alpha(t)} \circ g)'(t) = 0$. We need to show that $\left(\zeta_{\phi(\alpha(t))} \circ \phi \circ g \circ \phi^{-1}\right)'(t) = 0$. But, this immediately results from the following

$$\begin{aligned}\left(\zeta_{\phi(\alpha(t))} \circ \phi \circ g \circ \phi^{-1}\right)'(t) &= \left(\sigma \rightsquigarrow (\phi \circ g \circ \phi^{-1})(\sigma) * \phi(\alpha(t))\right)'(t) \\ &= \left(\sigma \rightsquigarrow (\phi \circ g(\sigma))(\alpha(t))\right)'(t) \\ &= d\phi\left((\sigma \rightsquigarrow g(\sigma) * \alpha(t))'(t)\right) \\ &= d\phi\left((\zeta_{\alpha(t)} \circ g)'(t)\right).\end{aligned}$$

(3) Verifying the no-twist condition. We once again only present the proof regarding the tangential part, for the same reason as before.

By the hypothesis, we have $\forall v \in T_{\alpha_2(t)}M_2$, $\dot{X}^v(t) \in (T_{\alpha_2(t)}M_2)^\perp$, being $X^v(\sigma)$ as in (6). We now need to assure that, $\forall v \in T_{(\phi\circ\alpha_2)(t)}\phi(M_2)$, $\dot{X}^v_\phi(t) \in (T_{(\phi\circ\alpha_2)(t)}\phi(M_2))^\perp$, where

$$\begin{aligned} X^v_\phi(\sigma) &= d\left((\phi \circ g(\sigma) \circ \phi^{-1}) \circ (\phi \circ g(t) \circ \phi^{-1})^{-1}\right)(v) \\ &= d\left(\phi \circ g(\sigma) \circ g(t)^{-1} \circ \phi^{-1}\right)(v) \\ &= d\phi\left(d(g(\sigma) \circ g(t)^{-1})(d\phi^{-1}(v)\right) \\ &= d\phi\left(X^{d\phi^{-1}(v)}(\sigma)\right). \end{aligned}$$

Let us arbitrarily take $v \in T_{(\phi\circ\alpha_2)(t)}\phi(M_2)$. Because $\phi$ is a homothety we have

$$\dot{X}^v_\phi(\sigma) = d\phi\Big(\dot{X}^{d\phi^{-1}(v)}(\sigma)\Big).^\dagger \tag{16}$$

Therefore, since $d\phi^{-1}(v) \in T_{\alpha_2(t)}M_2$, the hypothesis allow us to conclude that $\dot{X}^v_\phi(t) \in d\phi\left(\left(T_{\alpha_2(t)}M_2\right)^\perp\right) = (T_{(\phi\circ\alpha_2)(t)}\phi(M_2))^\perp$, as intended.

■

**Remark 3.3.** *The reason for Propositions 3.2 and 3.3 being limited to rolling motions wherein the environment manifold $\overline{M}$ results from a scalar product space is related to the fact that only in these manifolds can we prove equalities (14) and (15), which we presented as corollaries of Lemma 3.1. The analogous equality to these shown in the proof of Proposition 3.4, equality (16), is easily established for general manifolds. Only this allows us to release the last proposition from the limitation of the first propositions.*

**3.5. Rolling *versus* Parallel Transport.** Here we address the close relationship between the concept of rolling motion without twisting and the concept of parallel vector field along a curve. The results from this section are restricted to situations where the environment manifold $\overline{M}$ originates from a scalar product space, such as $\mathbb{R}^n_\kappa$. The reason for this limitation relates (only) to the fact that we will employ Lemma 3.1. The main result is as follows:

$^\dagger$Homotheties preserve Levi-Civita connections, so they preserve the induced covariant derivative on a curve.

**Proposition 3.5.** *When $\overline{M}$ is (a pseudo-Riemannian manifold constructed from) a scalar product space, if the rolling condition is satisfied, the points (a) and (b) of the no-twist condition in Definition 3.1 are equivalent to the following:*

(a') *(tangential part) At each interval $]t_{i-1},t_i[$, a vector field $Y$ is parallel along the curve $\alpha$ if and only if $V(\sigma)=dg(\sigma)\big(Y(\sigma)\big)$ defines a parallel vector field along $\alpha_{\rm dev}$.*[‡]

(b') *(normal part) At each interval $]t_{i-1},t_i[$, a vector field $Y$ is normal parallel along the curve $\alpha$ if and only if $V(\sigma)=dg(\sigma)\big(Y(\sigma)\big)$ defines a normal parallel vector field along $\alpha_{\rm dev}$.*

*Proof*: We only present the proof of the equivalence (a) $\Longleftrightarrow$ (a'), due to the fact that the proof for the normal part is entirely similar.

((a) $\Longrightarrow$ (a'))

Let us suppose that (a) is satisfied. We arbitrarily take an instant (fixed) $t\in\,]t_{i-1},t_i[$, a smooth field $Y$ of vectors tangent to $M_1$ along $\alpha$ and the corresponding field $V$, defined by $V(\sigma)=dg(\sigma)\big(Y(\sigma)\big)$.

Firstly, from (a) results

$$\frac{\overline{D}}{d\sigma}\Big(\,dg(\sigma)\big(Y(t)\big)\Big)(t)\Big[=\frac{\overline{D}X^{V(t)}}{d\sigma}(t)\Big]\in\big(T_{\alpha_{\rm dev}(t)}M_2\big)^{\perp}.$$

Therefore, Lemma 3.1 allows us to write

$$dg(t)\Big(\frac{\overline{D}Y}{d\sigma}(t)\Big)\in\big(T_{\alpha_{\rm dev}(t)}M_2\big)^{\perp}\ \text{ iff }\ \frac{\overline{D}V}{d\sigma}(t)\in\big(T_{\alpha_{\rm dev}(t)}M_2\big)^{\perp}. \tag{17}$$

On the other hand, we know that $dg(t)\Big(\,\big(T_{\alpha(t)}M_1\big)^{\perp}\Big)=\Big(T_{\alpha_{\rm dev}(t)}M_2\Big)^{\perp}$. Hence, because $dg(t):T_{\alpha(t)}\overline{M}\longrightarrow T_{\alpha_{\rm dev}(t)}\overline{M}$ is an isomorphism, we also have

$$dg(t)\Big(\frac{\overline{D}Y}{d\sigma}(t)\Big)\in\big(T_{\alpha_{\rm dev}(t)}M_2\big)^{\perp}\ \text{ iff }\ \frac{\overline{D}Y}{d\sigma}(t)\in\big(T_{\alpha(t)}M_1\big)^{\perp}. \tag{18}$$

Finally, from (17) and (18) we now immediately deduce, by transitivity, the following:

$$\frac{\overline{D}Y}{d\sigma}(t)\in\big(T_{\alpha(t)}M_1\big)^{\perp}\ \text{ iff }\ \frac{\overline{D}V}{d\sigma}(t)\in\big(T_{\alpha_{\rm dev}(t)}M_2\big)^{\perp},$$

so we may conclude the equivalence stated in (a').

[‡]It is understood that $Y$ is parallel as a vector field on $M_1$ and that $V$ is parallel as a vector field on $M_2$.

((a) $\Longleftarrow$ (a'))

Let us suppose that (a') is satisfied, and we arbitrarily take an instant $t$ (fixed) in any subinterval $]t_{i-1}, t_i[$ and a vector $v \in T_{\alpha_{\text{dev}}(t)} M_2$.

Let $V$ be the parallel transport of $v$ along the restriction of $\alpha_{\text{dev}}$ at the interval $]t_{i-1}, t_i[$, satisfying $V(t) = v$. Then, we have $\frac{\overline{D}V}{d\sigma}(t) \in (T_{\alpha_{\text{dev}}(t)} M_2)^{\perp}$.

On the other hand, from (a') we can conclude that $Y(\sigma) = dg(\sigma)^{-1}\big(V(\sigma)\big)$ defines a parallel vector field along the restriction of $\alpha$ at the interval $]t_{i-1}, t_i[$. Therefore, $\frac{\overline{D}Y}{d\sigma}(t) \in (T_{\alpha(t)} M_1)^{\perp}$. Consequently, $dg(t)\Big(\frac{\overline{D}Y}{d\sigma}(t)\Big) \in (T_{\alpha_{\text{dev}}(t)} M_2)^{\perp}$.

Thus, if we consider $X^v$ defined as in (6), from (13) one obtains $\frac{\overline{D}X^v}{d\sigma}(t) \in \big(T_{\alpha_{\text{dev}}(t)} M_2\big)^{\perp}$, that is, condition (a) is satisfied. ∎

This reformulation of the no-twist condition also appears in [12] (Proposition 1, p. 5), for rolling motions in $I\!R^n_\kappa$. However, the argument made by the authors is substantially different, since the definition of rolling map without slipping or twisting that they used, although equivalent, is enunciated differently from what is presented in this article.

**Remark 3.4.** *Within the context in which the above proposition can be applied, a particular consequence of the no-twist condition is as follows: $Y$ is the parallel [normal parallel] transport of $Y_0 = Y(0) \in T_{\alpha(0)} M_1$ $[(T_{\alpha(0)} M_1)^{\perp}]$ along $\alpha$ if and only if $V(\sigma) = dg(\sigma)\big(Y(\sigma)\big)$ defines the parallel [normal parallel] transport of $V_0 = V(0) \in T_{\alpha_{\text{dev}}(0)} M_2$ $[(T_{\alpha_{\text{dev}}(0)} M_2)^{\perp}]$ along $\alpha_{\text{dev}}$.*

We also note that in the particular case where $\overline{M} = I\!R^n_\kappa$ and $g(0) = (I_n, 0)$, we find the explicit formula $Y(\sigma) = R^{-1}(\sigma)V(\sigma)$ to express the parallel transport $Y$ of a vector along the rolling curve, due to the parallel transport $V$ of that vector along the development curve. This may be of special interest when the parallel transport along the development curve is seen as simpler to describe than the parallel transport along the rolling curve, such as for example in the rolling motion of a spherical surface on a plane

The affinity of the no-twist condition with parallel vector fields along curves, also allows us to relate geodesic unions between the two rolling manifolds.

**Proposition 3.6.** *Let us consider that $g$, as defined in (4), is a rolling map without slipping or twisting of $M_1$ over $M_2$, with $\overline{M}$ (a pseudo-Riemannian manifold constructed from) a scalar product space. Then, the rolling curve*

*$\alpha$ is a broken geodesic in $M_1$ if and only if the corresponding development curve $\alpha_{\text{dev}}$ is a broken geodesic in $M_2$.*

*Proof*: Evidently, a geodesic can be characterized as a smooth curve whose velocity field is parallel. We also know that no-slip condition (5) is equivalent to having $\alpha'_{\text{dev}}(t) = dg(t)(\alpha'(t))$.

Let $\alpha$ be a broken geodesic and let $0 = t_0 < t_1 < \cdots < t_r = \tau$ be a partition of the interval $[0, \tau]$, which meets the requirements of no-slip and no-twist conditions, such that each restriction of $\alpha$ to the subintervals $[t_{i-1}, t_i]$ is a geodesic. Then, in each of these intervals, the field defined by $Y(\sigma) = \alpha'(\sigma)$ is parallel along $\alpha$. Consequently, the field given by $V(\sigma) = \alpha'_{\text{dev}}(\sigma)$ is parallel along $\alpha_{\text{dev}}$ in the same intervals, therefore $\alpha_{\text{dev}}$ is also a broken geodesic.

With the same reasoning we could also show that $\alpha$ is a geodesic broken whenever $\alpha_{\text{dev}}$ is; hence, the demonstration is complete. ■

**Remark 3.5.** *Several of the results of Section 3 are limited to the case where the environment manifold is obtained from a vector space, just because we use an auxiliary result, the Lemma 3.1, whose demonstration, for the moment, we can only present in this particular case. If it is possible to extend the Lemma 3.1 to other manifolds, then all the results of this section will automatically be valid in those environment manifolds.*

# 4. Rolling Hyperquadratics in a Pseudo-Euclidean Space

In this section we will study rolling motions without slipping or twisting of an important family of pseudo-Riemannian hypersurfaces, embedded in pseudo-Euclidean spaces. This family consists of the pseudo-hyperbolic space $H^n_\kappa(r)$ and the pseudo-sphere $S^n_\kappa(r)$, which will be defined in (19) and (20) below. Our main concern is the rolling motion of $H^n_\kappa(r)$ over its affine tangent space at a point $p_0$, that is, over the affine space associated with $T_{p_0}H^n_\kappa(r)$, defined by

$$T^{\text{aff}}_{p_0}H^n_\kappa(r) := \{p_0 + v : v \in T_{p_0}H^n_\kappa(r)\}\,.$$

Knowledge of this rolling motion will then allow us to deduce some others rolling motions, at the expense of properties previously introduced. Rolling motions with the pseudo-sphere $S^n_\kappa(r)$ will only be addressed at the end, by means of a convenient homothety that imports the information obtained for the case of $H^n_\kappa(r)$.

For the sake of completeness we include here some results already presented in [13].

**4.1. Hyperquadratics.** Let $q : I\!R^{n+1}_{\kappa} \longrightarrow I\!R$ be defined by $q(u) = \langle u, u\rangle$. For $r > 0$ and $\epsilon = \pm 1$, it is known that $Q = q^{-1}(\epsilon r^2)$ is a pseudo-Riemannian hypersurface of $I\!R^{n+1}_{\kappa}$, having index $\kappa$ or $\kappa - 1$ according to whether $\epsilon$ is 1 or $-1$, respectively. These hypersurfaces $Q$ are called the *hyperquadratics* (with a centre) of $I\!R^{n+1}_{\kappa}$. Depending on the value $\epsilon$ that is considered, the following definition assigns a specific name to the corresponding hyperquadratic.

**Definition 4.1.** *Let $n \geq 1$ and $0 \leq \kappa \leq n$. Then:*

(1) *the* pseudo-hyperbolic space *of radius $r > 0$ in $I\!R^{n+1}_{\kappa+1}$ is*

$$H^n_\kappa(r) := \left\{p \in I\!R^{n+1}_{\kappa+1} : \langle p, p\rangle = -r^2\right\}; \tag{19}$$

(2) *the* pseudo-sphere *of radius $r > 0$ in $I\!R^{n+1}_{\kappa}$ is*

$$S^n_\kappa(r) := \left\{p \in I\!R^{n+1}_{\kappa} : \langle p, p\rangle = r^2\right\}. \tag{20}$$

(See the appendix, which contains the surfaces for $n = 2$ and $\kappa = 0, 1, 2$.)

**Remark 4.1.** *$S^n_\kappa(r)$ and $H^n_\kappa(r)$ both have dimension $n$ and index $\kappa$. Note, nevertheless, that $S^n_\kappa(r) \subset I\!R^{n+1}_{\kappa}$ and $H^n_\kappa(r) \subset I\!R^{n+1}_{\kappa+1}$. In addition, $S^n_0(r)$ and $H^n_0(r)$ are Riemannianan manifolds, since they have a zero index.*

$H^n_\kappa(r)$ is connected whenever $\kappa \geq 1$, but when $\kappa = 0$ this hyperquadratic consists of two connected components: the *upper sheet*, which contains the point $(r, 0, \cdots, 0)$, and the *lower sheet*, which contains $(-r, 0, \cdots, 0)$. However, these two components can be identified projectively and we will only deal with one of them. Thus, hereafter we will assume that $H^n_0(r)$ designates the corresponding upper sheet. A similar situation happens with the pseudo-sphere $S^n_\kappa(r)$, and we will assume that $S^n_n(r)$ is the connected component which contains $(0, 0, \cdots, r)$.

The hyperquadratics defined in (19) and (20) are “centred” at the origin. However, these definitions can be trivially adapted so that the centre becomes any other point $c$, replacing $\langle p, p\rangle$ with $\langle p - c, p - c\rangle$. Evidently, the resulting regions are also pseudo-Riemannian hypersurfaces of the corresponding pseudo-Euclidean space.

Given that the analysis of rolling motions of pseudo-spheres will be performed based on the study of rolling motions with pseudo-hyperbolic spaces, the preliminary results that we present below focus only on $H^n_\kappa(r)$.

We first note that $H^n_\kappa(r)$ and $T^{\mathrm{aff}}_{p_0} H^n_\kappa(r)$ are both immersed in the pseudo-Riemannian manifold $I\!R^{n+1}_{\kappa+1}$. As usual, we will make use of the identifications that enable us to alternate freely between points of the manifold

$\mathbb{R}^{n+1}_{\kappa+1}$, tangent vectors of each $T_p(\mathbb{R}^{n+1}_{\kappa+1})$ and vectors of the vector space $\mathbb{R}^{n+1}$ equipped with the scalar product defined by $\langle u, v\rangle = u^\top J_{\kappa+1} v$, where $J_{\kappa+1} = \mathrm{diag}(-I_{\kappa+1}, I_{n-\kappa})$.

The proof of part of the following proposition is in [13]. The remnants are quite simple.

**Proposition 4.1.** *$\forall\, 0 \leq \kappa \leq n$, $\forall p_0 \in H^n_\kappa(r)$, we have:*

(1) $T_{p_0} H^n_\kappa(r) = \left\{v \in \mathbb{R}^{n+1} : v = \Omega p_0,\ \Omega \in \mathfrak{so}_{\kappa+1}(n+1)\right\}$;
(2) $(T_{p_0} H^n_\kappa(r))^\perp = \mathbb{R} p_0$;
(3) $T^{\mathrm{aff}}_{p_0} H^n_\kappa(r) = \left\{p \in \mathbb{R}^{n+1}_{\kappa+1} : p = p_0 + \Omega p_0,\ \Omega \in \mathfrak{so}_{\kappa+1}(n+1)\right\}$;
(4) $T^{\mathrm{aff}}_{p_0} H^n_\kappa(r) \cap H^n_\kappa(r) = \{p_0 + \Omega p_0 : \Omega \in \mathfrak{so}_{\kappa+1}(n+1), \langle \Omega p_0, \Omega p_0\rangle = 0\}$;
(5) $\forall v \in T_{p_0}\mathbb{R}^{n+1}_{\kappa+1},\ v \in T_{p_0} H^n_\kappa(r) \Leftrightarrow \langle v, p_0\rangle = 0$;
(6) $\forall q \in T^{\mathrm{aff}}_{p_0} H^n_\kappa(r),\ T_q\left(T^{\mathrm{aff}}_{p_0} H^n_\kappa(r)\right) = T_{p_0} H^n_\kappa(r)$.

(See Figure 5).

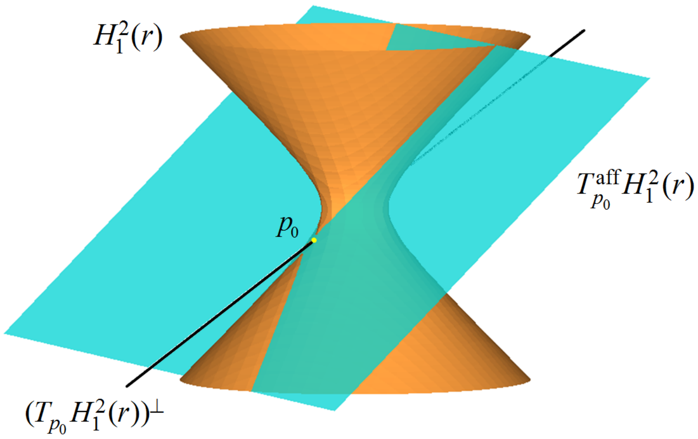


FIGURE 5. Affine tangent space and orthogonal complement.

We remark that $p_0$ is the unique point of the intersection of $T^{\mathrm{aff}}_{p_0} H^n_\kappa(r)$ and $H^n_\kappa(r)$ if and only if $\kappa = 0$ or $\kappa = n$. Thus, contrary to classic rolling motion of an Euclidean sphere on a plane, for $0 < \kappa < n$ there are many points of contact between $H^n_\kappa(r)$ and $T^{\mathrm{aff}}_{p_0} H^n_\kappa(r)$.

We also note that, $\forall R \in \mathrm{SO}^{\mathrm{I}}_{\kappa+1}(n+1)$, we have $\{Rp : p \in H^n_\kappa(r)\} = H^n_\kappa(r)$. Therefore, the "rotational parte" of a rolling map maintains $H^n_\kappa(r)$ invariant, while a set of points. However, the points move "inside" the hypersurface.

The rolling motions that this article best describe correspond to the case where the rolling curve is a geodesic in $H^n_\kappa(r)$. These geodesics can be written explicitly. In the next proposition the equations of the three possible types of geodesics in $H^n_\kappa(r)$ will be presented.

**Proposition 4.2.** (See [13].) *Let $p_0 \in H^n_\kappa(r)$ and $v \in T_{p_0}H^n_\kappa(r)$. We have the following:*

(1) *If $v$ is spacelike and $\|v\| = r$, then $t \rightsquigarrow \gamma(t) = p_0 \cosh(t) + v \sinh(t)$ defines the unique spacelike geodesic in $H^n_\kappa(r)$ which starts at $p_0$ with velocity $v$;*
(2) *If $v$ is timelike and $\|v\| = r$, then $t \rightsquigarrow \gamma(t) = p_0 \cos(t) + v \sin(t)$ defines the unique timelike geodesic in $H^n_\kappa(r)$ which starts at $p_0$ with velocity $v$;*
(3) *If $v$ is lightlike, then $t \rightsquigarrow \gamma(t) = p_0 + vt$ defines the unique lightlike geodesic in $H^n_\kappa(r)$ which starts at $p_0$ with velocity $v$.*

### 4.2. Rolling $H^n_\kappa(r)$ over $T^{\mathrm{aff}}_{p_0}H^n_\kappa(r)$.

The following theorem shows the equations that describe the "translational" and "rotational" velocities, determined by a "control", of the rolling motion of $H^n_\kappa(r)$ over $T^{\mathrm{aff}}_{p_0}H^n_\kappa(r)$, where $p_0$ is an arbitrary point. Therefore, equations (21) are the *kinematic equations* for this rolling motion.

**Theorem 4.1.** (See [13].) *Let $p_0$ be a point of $H^n_\kappa(r)$ and $t \in [0,\tau] \rightsquigarrow u(t) \in \mathbb{R}^{n+1}_{\kappa+1}$ a piecewise smooth mapping such that $\langle u(t), p_0\rangle = 0$.*[§] *If $t \in [0,\tau] \rightsquigarrow (R(t), s(t)) \in \mathrm{SO}^{\mathrm{I}}_{\kappa+1}(n+1) \rtimes \mathbb{R}^{n+1}$ is the piecewise smooth curve which in each open interval where $u$ is smooth verifies the following system*

$$\begin{cases} \dot{s}(t) = r^2 u(t) \\ \dot{R}(t) = R(t)\left(-u(t)p_0^\top + p_0 u^\top(t)\right) J_{\kappa+1} \end{cases} \tag{21}$$

*and satisfies a given initial condition $(R(0), s(0)) = (R_0, s_0)$, with $s_0$ belonging to $T_{p_0}H^n_\kappa(r)$, then $g(t) = (R^{-1}(t), s(t))$ defines a rolling map of $H^n_\kappa(r)$ over $T^{\mathrm{aff}}_{p_0}H^n_\kappa(r)$ without slipping or twisting, with the rolling curve $t \rightsquigarrow \alpha(t) = R(t)p_0$ and the development curve $t \rightsquigarrow \alpha_{\mathrm{dev}}(t) = p_0 + s(t)$.*

**Remark 4.2.** *In the particular case where $p_0 = (r, 0, \cdots, 0)$, we must have $u(t) = \begin{bmatrix} 0 & u_2(t) & \cdots & u_{n+1}(t) \end{bmatrix}^\top$ and the second equation in (21) reduces to*

$$\dot{R}(t) = R(t)\left(\sum_{i=2}^{\kappa+1} r u_i(t)(E_{i1} - E_{1i}) + \sum_{i=\kappa+2}^{n+1} r u_i(t)(E_{i1} + E_{1i})\right),$$

[§] $u$ is a *piecewise smooth mapping* if there is a partition $0 = t_0 < t_1 < \cdots < t_r = \tau$ such that each restriction $u|_{]t_{i-1},t_i[}$ can be extended by a smooth curve defined in $[t_{i-1}, t_i]$.

*where $E_{ij}$ is the matrix of order $n+1$ with the $(i,j)$-entry equal to $1$ and the others all zero.*

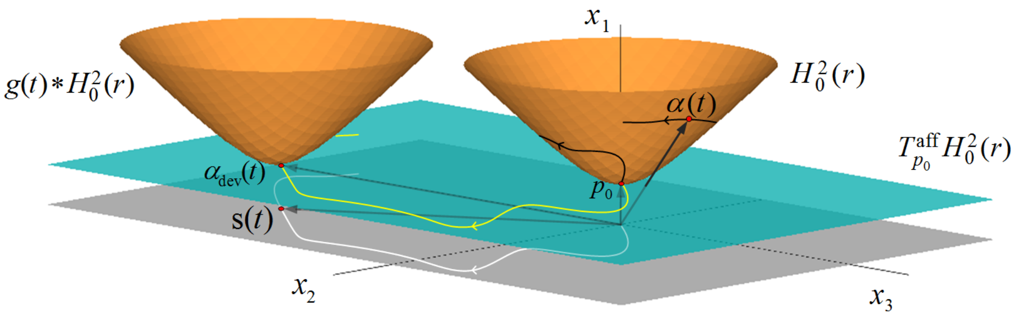


FIGURE 6. Rolling $H_0^2(r)$ over $T_{p_0}^{\text{aff}}H_0^2(r)$, with $p_0=(r,0,0)$.

The kinematic equations (21) can be solved explicitly, in two special cases, which follow. This allows us to get explicitly the corresponding rolling map, the rolling curve and its development.

**Case I:** In this situation the mapping $u$ is constant and $p_0$ is an arbitrary point.

**Proposition 4.3.** *When $u(t)=u\in \mathbb{R}_{\kappa+1}^{n+1}$ is a constant vector satisfying $\langle u,p_0\rangle=0$, the solution of the kinematic equations (21), with the initial condition $(R(0),s(0))=(R_0,s_0)$, is given by*

$$R(t)=R_0e^{tA},\qquad s(t)=s_0+r^2ut,$$

*with $A=\left(-up_0^\top+p_0u^\top\right)J_{\kappa+1}$.*

*Moreover, the rolling curve $\alpha$ and its development $\alpha_{\text{dev}}$ are geodesics in $H_\kappa^n(r)$ and $T_{p_0}^{\text{aff}}H_\kappa^n(r)$, respectively, having the same causal character as the vector $u$.*

The proof of the above proposition is in [13]. We remark that this proof allows to know the three possible equations of the rolling curve, corresponding to the causal character of $u$, which are shown in the table below. The development curve is always given by $\alpha_{\text{dev}}(t)=p_0+s_0+r^2ut$. (See Table 1.)

**Case II:** The idea of choosing the mapping $u$ which we will consider in this second case was obtained from [6], where the author showed that when the (Euclidean) sphere $S^2$ rolls over the plane tangent to its south pole, the

| causal character of $u$ | rolling curve |
|---|---|
| spacelike | $\alpha(t)=R_0p_0\cosh(r\|u\|t)+\frac{rR_0u}{\|u\|}\sinh(r\|u\|t)$ |
| timelike | $\alpha(t)=R_0p_0\cos(r\|u\|t)+\frac{rR_0u}{\|u\|}\sin(r\|u\|t)$ |
| lightlike | $\alpha(t)=R_0p_0+r^2R_0ut$ |

TABLE 1. Equation of the rolling curve, in case I.

kinematic equations can be solved explicitly if the development curve is a circle.

Let us consider that $p_0=(r,0,\cdots,0)\in I\!R^{n+1}$. Consequently, we have $u(t)=(0,u_2(t),\cdots,u_{n+1}(t))\in I\!R^{n+1}$ and $s(t)=(s_1,s_2(t),\cdots,s_{n+1}(t))\in I\!R^{n+1}$, with $s_1$ constant. However, for convenience, we will identify $u(t)$ with $(u_2(t),\cdots,u_{n+1}(t))\in I\!R^n$ and $s(t)$ with $(s_2(t),\cdots,s_{n+1}(t))\in I\!R^n$.

**Proposition 4.4.** *Under the above conventions, when $u(t)=e^{-tB}Bc$ with $B\in\mathfrak{so}_\kappa(n)$ and $c\in I\!R^n$, the solution of the kinematic equations (21) with initial condition $(R(0),s(0))=(R_0,\overline{s}_0)$, where $\overline{s}_0=(s_1,s_0)\in I\!R\times I\!R^n$, is given by*

$$R(t)=R_0e^{t\widetilde{A}}Q(t),\qquad s(t)=r^2(I_n-e^{-tB})c+s_0,$$

*where* $\widetilde{A}=\left[\begin{array}{c|c}0&(rBc)^\top\\\hline -rBc&0\end{array}\right]J_{\kappa+1}-\left[\begin{array}{c|c}0&0\\\hline 0&B\end{array}\right]$ *and* $Q(t)=\left[\begin{array}{c|c}1&0\\\hline 0&e^{tB}\end{array}\right]$.

*Proof*: The expression of $s(t)$ readily results from the first kinematic equation and the initial condition.

In order to solve the second kinematic equation, $\dot{R}(t)=R(t)A(t)$, we will consider the change of variable $R\rightsquigarrow\widetilde{R}$ defined by $R(t)=\widetilde{R}(t)Q(t)$. After substitution we obtain

$$\dot{\widetilde{R}}(t)=\widetilde{R}(t)\left(Q(t)A(t)Q^{-1}(t)-\dot{Q}(t)Q^{-1}(t)\right). \tag{22}$$

But

$$\dot{Q}(t)Q^{-1}(t)=\left[\begin{array}{c|c}0&0\\\hline 0&Be^{tB}\end{array}\right]\left[\begin{array}{c|c}1&0\\\hline 0&e^{-tB}\end{array}\right]=\left[\begin{array}{c|c}0&0\\\hline 0&B\end{array}\right]$$

and

$$
\begin{aligned}
Q(t)A(t)Q^{-1}(t) &= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & e^{tB} \end{array}\right]\left[\begin{array}{c|c} 0 & (re^{-tB}Bc)^\top \\ \hline -re^{-tB}Bc & 0 \end{array}\right] J_{\kappa+1} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & J_\kappa (e^{tB})^\top J_\kappa \end{array}\right] \\
&= \left[\begin{array}{c|c} 0 & r(Bc)^\top (e^{-tB})^\top \\ \hline -rBc & 0 \end{array}\right] J_{\kappa+1} J_{\kappa+1} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & (e^{tB})^\top \end{array}\right] J_{\kappa+1} \\
&= \left[\begin{array}{c|c} 0 & (rBc)^\top \\ \hline -rBc & 0 \end{array}\right] J_{\kappa+1},
\end{aligned}
$$

therefore, equation (22) is reduced to $\dot{\widetilde{R}}(t) = \widetilde{R}(t)\widetilde{A}$. As this equation is a differential equation on $SO_{\kappa+1}(n+1)$ with $\widetilde{A}$ constant and belonging to Lie algebra $\mathfrak{so}_{\kappa+1}(n+1)$, its solution satisfying $\widetilde{R}(0) = R_0$ is $\widetilde{R}(t) = R_0 e^{t\widetilde{A}}$. Thus, we can conclude that $R(t) = R_0 e^{t\widetilde{A}} Q(t)$. ■

**Remark 4.3.** *We have* $\langle s(t) - (r^2 c + s_0), s(t) - (r^2 c + s_0)\rangle = r^4 \langle c, c\rangle$, *where* $\langle \cdot, \cdot \rangle$ *is the scalar product of* $\mathbb{R}^n_\kappa$. *Consequently, with* $c \neq 0$, *the following holds:*

(1) *If* $c$ *is spacelike, then* $s(t)$ *belongs to the pseudo-sphere of* $\mathbb{R}^n_\kappa$ *with centre at point* $r^2 c + s_0$ *and radius* $\|r^2 c\|$.
(2) *If* $c$ *is timelike, then* $s(t)$ *belongs to the pseudo-hyperbolic space of* $\mathbb{R}^n_\kappa$ *with centre at point* $r^2 c + s_0$ *and radius* $\|r^2 c\|$.
(3) *If* $c$ *is lightlike, then* $s(t)$ *belongs to the pseudo-cone of* $\mathbb{R}^n_\kappa$ *with centre at point* $r^2 c + s_0$.¶

From the above remark, it follows, in particular, that $H^2_0(r)$ (upper sheet of the two-sheet hyperboloid) and $H^2_2(r)$ (spherical surface) roll over a circumference, while $H^2_1(r)$ (hyperboloid of one sheet) rolls either over a straight line or over a hyperbola. (See Table 2 and Figure 7.)

**4.3. Rolling a Pseudo-Hyperbolic Space not Centred on Origin.** In the previous section we dealt with the rolling motion of $H^n_\kappa(r)$, which is a hyperquadratic "centred" at the origin, over an affine tangent space. We now want to extend this rolling motion to the more general situation in which the centre can be any other point.

¶The pseudo-cone in $\mathbb{R}^n_\kappa$ with centre $c$ is defined by $C^n_\kappa(c) := \{p \in \mathbb{R}^n_\kappa : \langle p - c, p - c\rangle = 0\}$.

| hyperquadratic | curve $t \rightsquigarrow s(t)$, when $r=1$, $c=(c_1,c_2)$, $s_0=0$ |
|---|---|
| $H_0^2(r)$ | $(x_2-c_1)^2+(x_3-c_2)^2=c_1^2+c_2^2$ |
| $H_1^2(r)$ | $x_3=\pm x_2+(c_2\mp c_1)$ if $c_1=c_2$ or $c_1=-c_2$<br>$(x_2-c_1)^2-(x_3-c_2)^2=c_1^2-c_2^2$ if $c_1\neq \pm c_2$ |
| $H_2^2(r)$ | $(x_2-c_1)^2+(x_3-c_2)^2=c_1^2+c_2^2$ |

TABLE 2. Equations of the curves which contain $s(t)$, on plane $x_2ox_3$.

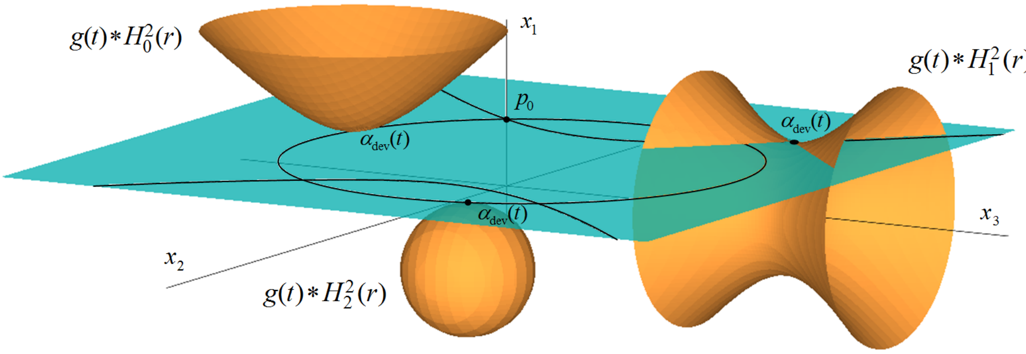


FIGURE 7. Rolling of $H_0^2(r)$, $H_1^2(r)$ and $H_2^2(r)$, in case II.

The *pseudo-hyperbolic space of radius* $r>0$ *and center* $c$ *in* $\mathbb{R}^{n+1}_{\kappa+1}$ is the pseudo-Riemannian submanifold of dimension $n$ and index $\kappa$ defined by

$$H^n_\kappa(r,c) := \left\{p\in \mathbb{R}^{n+1}_{\kappa+1} : \langle p-c, p-c\rangle = -r^2\right\}.$$

As was done for $H^n_0(r)$, we shall assume that $H^n_0(r,c)$ is the corresponding upper sheet.

Since $H^n_\kappa(r,c) = H^n_\kappa(r)+c$, that is, $H^n_\kappa(r,c)$ is a translation of $H^n_\kappa(r)$, the analysis of the rolling motion of $H^n_\kappa(r,c)$ over any its affine tangent space can be easily carried out from Theorem 4.1, through Proposition 3.4. Indeed, taking $\phi=(I_{n+1},c)$, Proposition 3.4 ensures that if $g(t)=(R^{-1}(t),s(t))$ is a rolling map of $H^n_\kappa(r)$ over $T^{\rm aff}_{p_0}H^n_\kappa(r)$, with rolling curve $\alpha$ and development curve $\alpha_{\rm dev}$, then

$$\begin{aligned}\widetilde{g}(t) &= \left(\widetilde{R}^{-1}(t),\widetilde{s}(t)\right)\\ &:= \phi\circ g(t)\circ\phi^{-1}\\ &= \left(R^{-1}(t), -R^{-1}(t)c+s(t)+c\right)\end{aligned}$$

is a rolling map of $H^n_\kappa(r,c)$ over $\phi\left(T^{\rm aff}_{p_0}H^n_\kappa(r)\right) = T^{\rm aff}_{p_0+c}H^n_\kappa(r,c)$, with rolling curve $\widetilde{\alpha}(t) = \alpha(t) + c$ and development curve $\widetilde{\alpha}_{\rm dev}(t) = \alpha_{\rm dev}(t) + c$.

Thus, if we fix a point $p_0 \in H^n_\kappa(r)$, a mapping $t \rightsquigarrow u(t) \in I\!R^{n+1}_{\kappa+1}$ such that $\langle u(t), p_0\rangle = 0$ ("control function") and a certain initial condition, then, by application of Theorem 4.1, a rolling map $t \rightsquigarrow \widetilde{g}(t) = \left(\widetilde{R}^{-1}(t), \widetilde{s}(t)\right)$ of $H^n_\kappa(r,c)$ over $T^{\rm aff}_{p_0+c}H^n_\kappa(r,c)$ is also determined. The kinematic equations corresponding to this rolling motion are immediately deduced from equations (21) and can be written in the following form:

$$\begin{cases} \dot{\widetilde{s}}(t) = \left(-u(t)p_0^\top + p_0u^\top(t)\right) J_{\kappa+1}\widetilde{R}^{-1}(t)c + r^2u(t) \\ \dot{\widetilde{R}}(t) = \widetilde{R}(t)\left(-u(t)p_0^\top + p_0u^\top(t)\right) J_{\kappa+1}. \end{cases}$$

Furthermore, $\widetilde{\alpha}(t) = \widetilde{R}(t)p_0 + c$ and $\widetilde{\alpha}_{\rm dev}(t) = p_0 + \widetilde{s}(t) + \widetilde{R}^{-1}(t)c$.

**4.4. Rolling a Pseudo-Hyperbolic Space over Another.** Our goal in this section is to describe the rolling motion, in $\overline{M} = I\!R^{n+1}_{\kappa+1}$, of a pseudo-hyperbolic space (centred at the origin) over another pseudo-hyperbolic space (not centred at the origin), tangent to the first. We will use a similar argument to the one appearing in [11] for the rolling motion of an Euclidean sphere over another, implementing the reasoning mentioned in Remark 3.2. That is, we will reach our goal at the expense of the kinematic equations for the rolling motion of such submanifolds over an affine tangent space, along with the properties of symmetry and transitivity contained in Section 3.4.

With $r_1, r_2 \in I\!R^+$, consider $p_0 = (r_1, 0, \cdots, 0), q_0 = (r_2, 0, \cdots, 0) \in I\!R^{n+1}_{\kappa+1}$ and let $\eta = p_0 - q_0$. Let us denote $M_1 = H^n_\kappa(r_1)$, $N = T^{\rm aff}_{p_0}H^n_\kappa(r_1)$ and $M_2 = H^n_\kappa(r_2,\eta)$. Since $T_{p_0}H^n_\kappa(r_2,\eta) = T_{q_0}H^n_\kappa(r_2) = T_{p_0}H^n_\kappa(r_1)$, the affine tangent space to $M_2$ at $p_0$ obviously coincides with $N$.

We know explicitly how to roll $M_1$ and $M_2$ over $N$. Consequently, by symmetry (Proposition 3.3), we also know how to roll $N$ over $M_2$. Therefore, by transitivity (Proposition 3.2), we will be able to describe the rolling motion of $M_1$ over $M_2$.

With regard to rolling motion of $M_1$ over $N$, we know that after fixing a piecewise smooth mapping $t \rightsquigarrow u(t) \in I\!R^{n+1}_{\kappa+1}$ such that $\langle u(t), p_0\rangle = 0$, if $R_1(t)$ and $s_1(t)$ constitute the solution-curve of the problem

$$\begin{cases} \dot{s}_1(t) = r_1^2u(t) \\ \dot{R}_1(t) = R_1(t)\left(-u(t)p_0^\top + p_0u^\top(t)\right) J_{\kappa+1} \end{cases}; \quad R_1(0) = I_{n+1},\ s_1(0) = 0,$$

then $g_1(t) = (R_1^{-1}(t), s_1(t))$ defines a rolling map, with rolling curve given by $\alpha_1(t) = R_1(t)p_0$ and development curve given by $\alpha_{1_{\rm dev}}(t) = p_0 + s_1(t)$.

With regard to rolling motion of $M_2$ over $N$, we know that after fixing a piecewise smooth mapping $t \rightsquigarrow \widehat{u}(t) \in I\!R^{n+1}_{\kappa+1}$ such that $\langle \widehat{u}(t), q_0\rangle = 0$, if $R_2(t)$ and $s_2(t)$ form the solution-curve of the problem

$$\begin{cases} \dot{s}_2(t) = \left(-\widehat{u}(t)q_0^\top + q_0\widehat{u}^\top(t)\right) J_{\kappa+1}R_2^{-1}(t)\eta + r_2^2\widehat{u}(t) \\ \dot{R}_2(t) = R_2(t)\left(-\widehat{u}(t)q_0^\top + q_0\widehat{u}^\top(t)\right) J_{\kappa+1} \end{cases}; R_2(0)=I_{n+1},\ s_2(0)=0,$$

then $g_2(t) = (R_2^{-1}(t), s_2(t))$ defines a rolling map, having rolling curve given by $\alpha_2(t) = R_2(t)q_0 + \eta$ and its development by $\alpha_{2_{\rm dev}}(t) = q_0 + s_2(t) + R_2^{-1}(t)\eta$. Hence, Proposition 3.3 ensures that $g_2^{-1}(t) = (R_2(t), -R_2(t)s_2(t))$ is a rolling map of $N$ over $M_2$, with rolling curve $\alpha_{2_{\rm dev}}$ and development curve $\alpha_2$.

Thus, under the condition of $u$ and $\widehat{u}$ being such that $\alpha_{1_{\rm dev}} = \alpha_{2_{\rm dev}}$, with the previous rolling maps $g_1$ and $g_2$, Proposition 3.2 allows us to conclude that

$$\begin{aligned} g_3(t) &= g_2^{-1}(t) \circ g_1(t) \\ &= \left(R_2(t)R_1^{-1}(t), R_2(t)(s_1(t) - s_2(t))\right) \end{aligned}$$

defines a rolling map of $M_1$ over $M_2$, with rolling curve $\alpha_1$ and development curve $\alpha_2$. In order to be able to establish the kinematic equations for this rolling motion, we still need to see the relationship that must be fulfilled between $u(t)$ and $\widehat{u}(t)$ so that $\alpha_{1_{\rm dev}}(t) = \alpha_{2_{\rm dev}}(t)$. But, with direct calculations on the assumed conditions, we can deduce the following equivalences:

$$\begin{aligned} &\alpha_{1_{\rm dev}}(t) = \alpha_{2_{\rm dev}}(t) \\ \Leftrightarrow & s_2(t) - s_1(t) = \eta - R_2^{-1}(t)\eta \\ \Leftrightarrow & \dot{s}_2(t) - \dot{s}_1(t) = R_2^{-1}(t)\dot{R}_2(t)R_2^{-1}(t)\eta \\ \Leftrightarrow & r_2^2\widehat{u}(t) = r_1^2 u(t). \end{aligned}$$

In conclusion, taking into account the particular structure of $p_0$ and the fact that $q_0 = \dfrac{r_2}{r_1}p_0$, we can establish the following result.

**Theorem 4.2.** *Consider $r_1, r_2 \in I\!R^+$, $p_0 = (r_1, 0, \cdots, 0), q_0 = (r_2, 0, \cdots, 0)$ and $\eta = p_0 - q_0$. Let $t \in [0, \tau] \rightsquigarrow u(t) = (0, u_2(t), \cdots, u_{n+1}(t)) \in I\!R^{n+1}$ be a* *piecewise smooth mapping and let us define*

$$U(t) = \sum_{i=2}^{\kappa+1} u_i(t)(E_{i1} - E_{1i}) + \sum_{i=\kappa+2}^{n+1} u_i(t)(E_{i1} + E_{1i}),$$

*where $E_{ij}$ denotes the matrix of order $n+1$ with the entry $(i,j)$ equal to 1 and the others all zero. If $(R_1(t), R_2(t), s_1(t), s_2(t))$ constitutes the solution-curve of the system*

$$\begin{cases} \dot{s}_1(t) = r_1^2 u(t) \\ \dot{s}_2(t) = \dfrac{r_1^2}{r_2} U(t) R_2^{-1}(t)\eta + r_1^2 u(t) \\ \dot{R}_1(t) = r_1 R_1(t) U(t) \\ \dot{R}_2(t) = \dfrac{r_1^2}{r_2} R_2(t) U(t) \end{cases} \tag{23}$$

*with initial condition $(R_1(0), R_2(0), s_1(0), s_2(0)) = (I_{n+1}, I_{n+1}, 0, 0)$, then $t \rightsquigarrow g(t) = \big(R_2(t)R_1^{-1}(t), R_2(t)(s_1(t) - s_2(t))\big) \in \mathrm{SO}^{\mathrm{I}}_{\kappa+1}(n+1) \rtimes I\!R^{n+1}$ is a rolling map of $H^n_\kappa(r_1)$ over $H^n_\kappa(r_2, \eta)$, without slipping or twisting, with rolling curve given by $\alpha(t) = R_1(t)p_0$ and development curve given by $\alpha_{\mathrm{dev}}(t) = R_2(t)q_0 + \eta$. (See Figure 8).*

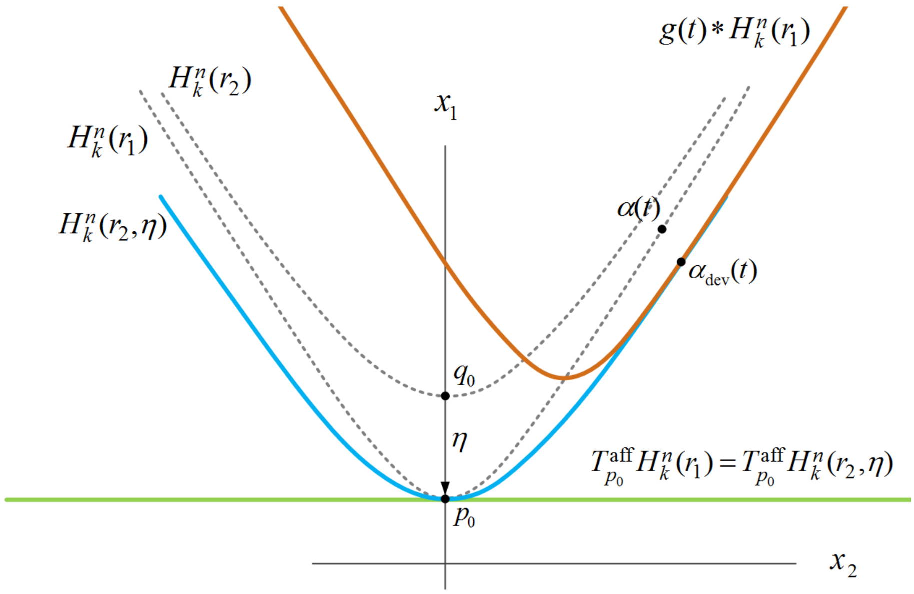


FIGURE 8. Rolling $H^n_\kappa(r_1)$ over $H^n_\kappa(r_2,\eta)$, with $n = 1$ and $\kappa = 0$.

**4.5. Rolling a Pseudo-Sphere $S^n_\kappa(r)$.** In what follows we will address the rolling motion of pseudo-spheres, using the knowledge available for the rolling motion of pseudo-hyperbolic spaces. The fundamental idea is the introduction of an anti-isometric transformation which will make a connection between the rolling motions of the two hyperquadratics.

Throughout this section $\phi : \mathbb{R}^{n+1}_{n-\kappa+1} \longrightarrow \mathbb{R}^{n+1}_\kappa$ will designate the mapping defined by

$$\phi(x) = Qx, \qquad \text{with } Q = \left[\begin{array}{c|c} 0 & I_\kappa \\ \hline I_{n-\kappa+1} & 0 \end{array}\right]. \tag{24}$$

Explicitly, $\phi(x_1, \cdots, x_{n+1}) = (x_{n-\kappa+2}, \cdots, x_{n+1}, x_1 \cdots, x_{n-\kappa+1})$. Therefore, considering the scalar product defined by $\langle u, v\rangle_J = u^\top J v$, confirmation of the following equality is immediate,

$$\langle \phi(x), \phi(x)\rangle_{J_\kappa} = -\sum_{i=n-\kappa+2}^{n+1} x_i^2 + \sum_{i=1}^{n-\kappa+1} x_i^2 = -\langle x, x\rangle_{J_{n-\kappa+1}}.$$

As a consequence of this formula, we have that $\phi$ is an anti-isometry and $\phi\left(H^n_{n-\kappa}(r)\right) = S^n_\kappa(r)$. That is, $\phi$ transforms the pseudo-hyperbolic space $H^n_{n-\kappa}(r)$ anti-isometrically into the pseudo-sphere $S^n_\kappa(r)$.

The transformation $\phi$ is a path that, roughly speaking, allows us to transfer much of our geometric knowledge about pseudo-hyperbolic spaces to the corresponding study with pseudo-spheres. This results from the fact that $\phi$ is a homothety. In fact, since homotheties preserve Levi-Civita connections, they preserve all geometric notions that depend solely on the Levi-Civita connection, such as geodesics and parallel transport. Nevertheless, as $\phi$ is a homothety with negative coefficient, the causal caracter is reversed, that is: $v$ is timelike $\Rightarrow d\phi(v)$ is spacelike; $v$ is spacelike $\Rightarrow d\phi(v)$ is timelike (also finding: $v$ is lightlike $\Rightarrow d\phi(v)$ is lightlike).

In particular, the equations of the geodesics in $S^n_\kappa(r)$, with an initial point and an initial velocity vector previously established, can be readily written from Proposition 4.2. Just exchange the words spacelike and timelike with each other, keeping the equations.

As in the case of pseudo-hyperbolic space, in addition to the rolling motion of $S^n_\kappa(r)$ over the affine tangent space at a point, the rolling with a pseudo-sphere not centred at the origin and the rolling of a pseudo-sphere over another naturally also makes sense. However, here we will only worry about the first rolling motion, since the others can clearly be obtained from this with the exposed reasoning for $H^n_\kappa(r)$.

Arbitrarily taking a point $q_0 \in H^n_{n-\kappa}(r)$ and a piecewise smooth mapping $t \rightsquigarrow \widetilde{u}(t) \in \mathbb{R}^{n+1}_{n-\kappa+1}$ such that $\langle \widetilde{u}(t), q_0 \rangle = 0$, from Theorem 4.1 we know that if $\widetilde{R}(t)$ and $\widetilde{s}(t)$ form the solution-curve to the problem

$$\begin{cases} \dot{\widetilde{s}}(t) = r^2 \widetilde{u}(t) \\ \dot{\widetilde{R}}(t) = \widetilde{R}(t) \left( -\widetilde{u}(t) q_0^\top + q_0 \widetilde{u}^\top(t) \right) J_{n-\kappa+1} \end{cases} ; \quad \widetilde{R}(0) = \widetilde{R}_0, \ \widetilde{s}(0) = \widetilde{s}_0, \tag{25}$$

where $\widetilde{R}_0 \in \mathrm{SO}^{\mathrm{I}}_{n-\kappa+1}(n+1)$ and $\widetilde{s}_0 \in T_{q_0} H^n_{n-\kappa}(r)$, then $\widetilde{g}(t) = (\widetilde{R}^{-1}(t), \widetilde{s}(t))$ defines a rolling map of $H^n_{n-\kappa}(r)$ over $T^{\mathrm{aff}}_{q_0} H^n_{n-\kappa}(r)$, with rolling and development curves given by $\widetilde{\alpha}(t) = \widetilde{R}(t) q_0$ and $\widetilde{\alpha}_{\mathrm{dev}}(t) = q_0 + \widetilde{s}(t)$. Therefore, Proposition 3.4 allows us to conclude that

$$g(t) = \left( R^{-1}(t), s(t) \right) := \left( Q \widetilde{R}^{-1}(t) Q^{-1}, Q \widetilde{s}(t) \right) \tag{26}$$

defines a rolling map of $S^n_\kappa(r)$ over $\phi\left( T^{\mathrm{aff}}_{q_0} H^n_{n-\kappa}(r) \right) = T^{\mathrm{aff}}_{Qq_0} S^n_\kappa(r)$, verifying $R(0) = Q \widetilde{R}_0 Q^{-1} \in \mathrm{SO}^{\mathrm{I}}_\kappa(n+1)$ and $s(0) = Q \widetilde{s}_0 \in T_{Qq_0} S^n_\kappa(r)$, with the rolling curve given by $\alpha(t) = Q \widetilde{\alpha}(t) = R(t) Q q_0$ and the development curve given by $\alpha_{\mathrm{dev}}(t) = Q \widetilde{\alpha}_{\mathrm{dev}}(t) = Q q_0 + s(t)$.

In which regards the kinematic equations of this rolling motion, starting from (25) and (26) we obtain:

$$\dot{s}(t) = Q \dot{\widetilde{s}}(t) = r^2 Q \widetilde{u}(t)$$

and

$$\begin{aligned} \dot{R}(t) &= Q \dot{\widetilde{R}}(t) Q^{-1} \\ &= R(t) Q \left( -\widetilde{u}(t) q_0^\top + q_0 \widetilde{u}^\top(t) \right) J_{n-\kappa+1} Q^{-1} \\ &= R(t) \left( -Q \widetilde{u}(t) q_0^\top Q^{-1} + Q q_0 \widetilde{u}^\top(t) Q^{-1} \right) Q J_{n-\kappa+1} Q^{-1} \\ &= R(t) \left( -Q \widetilde{u}(t) (Q q_0)^\top + Q q_0 (Q \widetilde{u}(t))^\top \right) (-J_\kappa) . \end{aligned}$$

Therefore, if we define $p_0 = Q q_0$ and $u(t) = Q \widetilde{u}(t)$, the expressions of the velocities of $s$ and $R$ may be written as shown in system (27) below.

We also observe that $\langle \widetilde{u}(t), q_0 \rangle = 0$ in $\mathbb{R}^{n+1}_{n-\kappa+1}$ if and only if $\langle u(t), p_0 \rangle = 0$ in $\mathbb{R}^{n+1}_\kappa$, because $\langle u(t), p_0 \rangle_{J_\kappa} = \langle \phi(\widetilde{u}(t)), \phi(q_0) \rangle_{J_\kappa} = - \langle \widetilde{u}(t), q_0 \rangle_{J_{n-\kappa+1}}$

In short, we have the following:

**Theorem 4.3.** *Let $p_0$ be a point of $S^n_\kappa(r)$ and $t \in [0,\tau] \rightsquigarrow u(t) \in I\!R^{n+1}_\kappa$ a piecewise smooth mapping such that $\langle u(t), p_0\rangle = 0$. If $t \in [0,\tau] \rightsquigarrow (R(t), s(t)) \in \mathrm{SO}^{\mathrm{I}}_\kappa(n+1) \rtimes I\!R^{n+1}$ is the piecewise smooth curve which, in each open interval where $u$ is smooth, verifies the following system*

$$\begin{cases} \dot{s}(t) = r^2 u(t) \\ \dot{R}(t) = R(t)\left(u(t)p_0^\top - p_0 u^\top(t)\right) J_\kappa \end{cases} \tag{27}$$

*and satisfies a given initial condition $(R(0), s(0)) = (R_0, s_0)$, with $s_0$ belonging to $T_{p_0}S^n_\kappa(r)$, then $g(t) = (R^{-1}(t), s(t))$ defines a rolling map of $S^n_\kappa(r)$ over $T^{\mathrm{aff}}_{p_0}S^n_\kappa(r)$, without slipping or twisting, with the rolling curve given by $\alpha(t) = R(t)p_0$ and the development curve given by $\alpha_{\mathrm{dev}}(t) = p_0 + s(t)$.*

This theorem is in accordance with Theorem 4.1 in [9], where the rolling motion of the Lorentzian sphere $S^n_1(1)$ was presented.

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# Appendix

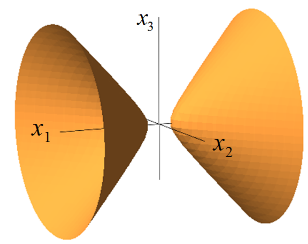
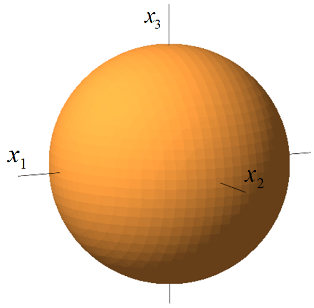
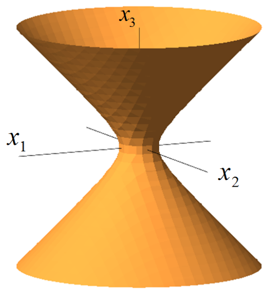
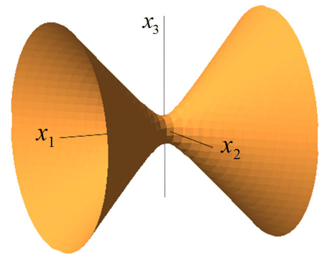
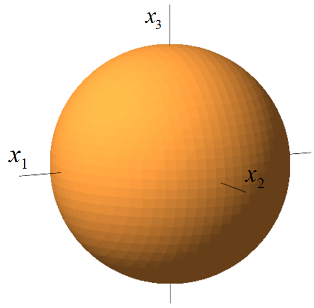
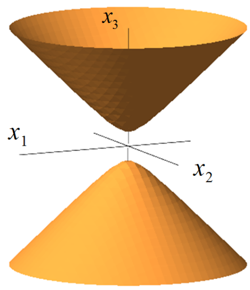
| | $H^n_\kappa(r) \subset I\!\!R^{n+1}_{\kappa+1}$ | $S^n_\kappa(r) \subset I\!\!R^{n+1}_\kappa$ |
|---|---|---|
| $n = 2$<br>$\kappa = 0$ |  |  |
| $n = 2$<br>$\kappa = 1$ |  |  |
| $n = 2$<br>$\kappa = 2$ |  |  |

TABLE 3. Hyperquadrics

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