Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–21

REGULARITY OF THE VANISHING IDEAL OVER A BIPARTITE NESTED EAR DECOMPOSITION

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ABSTRACT: We study the Castelnuovo–Mumford regularity of the vanishing ideal over a bipartite graph endowed with a decomposition of its edge set. We prove that, under certain conditions, the regularity of the vanishing ideal over a bipartite graph obtained from a graph by attaching a path of length ℓ increases by $\lfloor \frac{\ell}{2} \rfloor (q-2)$, where q is the order of the field of coefficients. We use this result to show that the regularity of the vanishing ideal over a bipartite graph, G, endowed with a weak nested ear decomposition is equal to

$$\tfrac{|V_G|+\epsilon-3}{2}(q-2),$$

where ϵ is the number of even length ears and pendant edges of the decomposition. As a corollary, we show that for bipartite graph the number of even length ears in a nested ear decomposition starting from a vertex is constant.

KEYWORDS: Castelnuovo–Mumford regularity, Binomial ideal, ear decomposition.
MATH. SUBJECT CLASSIFICATION (2000): 13F20 (primary); 14G15, 11T55, 05E40, 05C70.

1. Introduction

Given G, a simple graph, and K, a finite field, $K[E_G]$ denotes the polynomial ring with coefficients in K, the variables of which are in one-to-one correspondence with the edges of the graph. The vanishing ideal over G is a binomial ideal of $K[E_G]$, denoted here by $I_q(G)$, given as the vanishing ideal of the projective toric subset parameterized by E_G. They were defined by Renteria, Simis and Villarreal in [16], with a view towards applications to the theory of linear codes and hence the presence of a finite field. The aim of this work is to continue the study of the Castelnuovo–Mumford regularity of these ideals. Originally this invariant is related to the error-correcting performance of the linear codes involved, however, here, we wish to regard it strictly from the point of view of the link between commutative algebra and graph theory provided by this construction.

Received June 6, 2018.
This work was partially supported by Centro de Matemática da Universidade de Coimbra UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

This idea has been used for other classes of ideals, that one can associate to a graph, and the existing results point to interesting combinatorial invariants. For instance, the Castelnuovo–Mumford regularity of the *edge ideal* of graph is bounded below by the induced matching number and above by the co-chordal cover number (cf. [7], [8, Lemma 2.2] and [19]). There are also partial results for the *toric ideal* of a graph (cf. [18, 1]) and for the *binomial edge ideal* of a graph (cf. [2, 9, 12]).

The fact that, for the vanishing ideal over a graph, the quotient of $K[E_G]$ by $I_q(G)$ is a Cohen–Macaulay graded ring of dimension one explains why we know relatively more about this invariant in this case than in the cases of edge, toric or binomial edge ideals. The Castelnuovo–Mumford regularity of the vanishing ideal over a graph has been computed for many classes of graphs, including trees, cycles (cf. [14]), complete graphs (cf. [6]), complete bipartite graphs (cf. [4]), complete multipartite graphs (cf. [13]) and, more recently, parallel compositions of paths (cf. [11]). Additionally we know that, in the bipartite case, the regularities of the vanishing ideals over the members of the block decomposition of a graph completely determine the regularity of the vanishing ideal over the graph (see Proposition 2.10, below).

In this work we establish a formula for the Castelnuovo–Mumford regularity of the vanishing ideals over graphs in the class of bipartite graphs endowed with certain decomposition of its edge set into paths. The simplest case of a such a decomposition, a so-called *ear decomposition*, is a partition of the edge set of G into subgraphs $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ such that $\mathcal{P}_0$ is a vertex and, for all $1 \leq i \leq r$, the path has its end-vertices in $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{i-1}$ and *none* of its inner vertices in this union. Ear decompositions play a central role in graph connectivity as, by Whitney's theorem, a graph is 2-vertex-connected if and only if it is endowed with an *open* ear decomposition (one in which every $\mathcal{P}_i$ with $i > 1$ has distinct end-vertices). In [3], Eppstein introduces the notion of *nested* ear decomposition, a special case of ear decomposition in which, firstly, the paths $\mathcal{P}_i$ are forced to have end-vertices in a (same) P_j, for some $j < i$, and, secondly, a *nesting* condition is to be satisfied for two paths having their end-vertices in a same $\mathcal{P}_j$ (see Definition 4.1).

By Theorem 4.4, below, it follows that the Castelnuovo–Mumford of the quotient of $K[E_G]$ by $I_q(G)$, when G is endowed with a nested ear decomposition, is given by:

$$\tfrac{|V_G|+\epsilon-3}{2}(q-2) \tag{1}$$

where ϵ is the number of paths of even length in the decomposition and q is the order of the field K. As a corollary, we deduce that the number of even length paths in any nested ear decomposition of a bipartite graph, that starts from a vertex, is constant (cf. Corollary 4.5).

This article is organized as follows. In the next section we set up the notation used throughout and define the vanishing ideal over a graph. We recall several characterizations of this ideal which allow a direct definition without mentioning the projective toric subset parameterized by the edges of the graph. We also recall the Artinian reduction technique, which is the main tool in the computation of the regularity (cf. Proposition 2.5). After reviewing some known values of the regularity (cf. Table 1) we go through the existing results bounding the regularity in terms of combinatorial data on the graph, among these, the bound from the independence number of the graph (cf. Proposition 2.6). Other results reviewed in this section include the upper bounds obtained from a spanning subgraph and from an edge cover and two other results, one relating the regularity with the block decomposition and another relating it with the leaves of the graph. Sections 3 and 4 contain the main results of this work. In Section 3, we investigate the contribution to the regularity of the vanishing ideal over a graph obtained from another graph by attaching to it a path by its end-vertices. Theorem 3.4 states that, under some conditions, the regularity increases by $\lfloor \frac{\ell}{2} \rfloor (q-2)$, where ℓ is the length of the path attached and q the cardinality of K. In Section 4, we use the previous result to establish the regularity of a bipartite graph endowed with a *weak* nested ear decomposition. This notion is a slight generalization of the notion of nested ear decomposition and arises naturally in the context of the proof of Theorem 4.4. Its distinctive feature is that one allows the existence of pendant edges in the decomposition. Theorem 4.4 expresses the regularity of a bipartite graph endowed with a weak nested ear decomposition by the formula (1), where, now, ϵ is the number of even length paths and pendant edges. Corollary 4.5, stating that the number of even length paths in a nested ear decomposition of a graph is constant, is then a direct consequence of this formula. As an application of Theorem 4.4 we finish by producing a family of graphs with regularities arbitrarily larger than the lower bound given by their independence numbers.

2. Preliminaries

The graphs considered in this work are assumed to be simple graphs (finite, undirected, loopless and without multiple edges). Additionally, we will assume throughout that no isolated vertices occur. To simplify the notation, we assume that the vertex set, V_G, is a subset of $\mathbb{N}$.

2.1. The vanishing ideal over a graph. We will denote by K a finite field of order $q > 2$. Given a graph G, we consider a polynomial ring with coefficients in K the variables of which are in bijection with the edges of G and denote it by $K[E_G]$. A variable in $K[E_G]$ corresponding to and edge $\{i, j\} \in E_G$ will be denoted by t_{ij}, which is the abbreviated form of $t_{\{i,j\}}$. Given a non-negative integer valued function on the edge set, $\alpha \in \mathbb{N}^{E_G}$, the monomial $\mathbf{t}^\alpha \in K[E_G]$ is, by definition,

$$\mathbf{t}^\alpha = \prod_{\{i,j\}\in E_G} t_{ij}^{\alpha\{i,j\}}.$$

We say that $\mathbf{t}^\alpha$ is supported on the edges of a subgraph $H \subset G$ if

$$\alpha\{i, j\} \neq 0 \iff \{i, j\} \in E_H.$$

Consider $\mathbb{P}^{|E_G|-1}$, the projective space over K with coordinate ring $K[E_G]$ and let $\mathbb{P}^{|V_G|-1}$, be the projective space with coordinate ring $K[x_i : i \in V_G]$. The ring homomorphism $\varphi\colon K[E_G] \to K[x_i : i \in V_G]$ given by:

$$t_{ij} \mapsto x_i x_j \tag{2}$$

defines a rational map $\varphi^\sharp\colon \mathbb{P}^{|V_G|-1} \to \mathbb{P}^{|E_G|-1}$, the restriction of which to the projective torus, $\mathbb{T}^{|V_G|-1}$, the subset of projective space of points with every coordinate a nonzero scalar, is a regular map.

Definition 2.1. The projective toric subset parameterized by G is the subset of $\mathbb{P}^{|E_G|-1}$ defined by:

$$X = \varphi^\sharp(\mathbb{T}^{|V_G|-1}) \subset \mathbb{P}^{|E_G|-1}.$$

The vanishing ideal of X is denoted by $I_q(G) \subset K[E_G]$.

We note that $I_q(G)$ can be defined directly from G, without reference to X, as the ideal generated by the homogeneous polynomials $f \in K[E_G]$ which vanish after substitution of each variable t_{ij} by $a_i a_j$, for all $a_i \in K^*$, with $i \in V_G$. For this reason we refer to $I_q(G)$ as the *vanishing ideal over G*.

The ideal $I_q(G)$ was defined in [16]. Being a vanishing ideal, it is automatically a radical, graded ideal. We also know that $I_q(G)$ has a binomial generating set. The fact that $I_q(G)$ contains the vanishing ideal of the torus over the finite field K, which is given by

$$I_q = \left(t_{ij}^{q-1} - t_{kl}^{q-1} \,:\, {\scriptstyle \{i,j\},\{k,l\}} \in E_G\right), \tag{3}$$

implies that the height of $I_q(G)$ is $|E_G|-1$ and hence the quotient $K[E_G]/I_q(G)$ is a one-dimensional graded ring. Additionally, since any monomial in $K[E_G]$ is a regular element in this quotient (since no variable vanishes on the torus), we deduce that $K[E_G]/I_q(G)$ is Cohen–Macaulay. We refer the reader to [16, Theorem 2.1] for complete proofs of these statements.

The ideal $I_q(G)$ can be related to the toric ideal of G, i.e., the ideal $P_G \subset K[E_G]$ given by $P_G = \ker\varphi$, where $\varphi\colon K[E_G] \to K[x_i : i \in V_G]$ is the map defined by (2). It can be shown (see [16, Theorem 2.5]) that

$$I_q(G) = (P_G + I_q) : (\mathbf{t}^*)^\infty, \tag{4}$$

where I_q is the vanishing ideal of the torus, given in (3), and by $\mathbf{t}^*$ we denote the product of all variables of the polynomial ring $K[E_G]$,

$$\mathbf{t}^* = \prod_{\{i,j\}\in E_G} t_{ij}.$$

The relation with the toric ideal (4) reinforces the idea, already expressed above, that $I_q(G)$ can be defined without any reference to the projective toric subset X parameterized by E_G. Yet another way to do this is by a characterization of the set homogeneous binomials of $I_q(G)$, achieved by the following proposition. The proof of this result can be found in [13, Lemma 2.3].

Proposition 2.2. *Let $\mathbf{t}^\nu - \mathbf{t}^\mu \in K[E_G]$ be a homogeneous binomial. Then $\mathbf{t}^\nu - \mathbf{t}^\mu$ belongs to $I_q(G)$ if and only if, for all $i \in V_G$,*

$$\sum_{k\in N_G(i)} \nu\{i,k\} \equiv \sum_{k\in N_G(i)} \mu\{i,k\} \pmod{q-1}, \tag{5}$$

where $N_G(\cdot)$ denotes the set of neighbors of a vertex.

With this characterization of $I_q(G)$ by means of a generating set of homogeneous binomials satisfying (5), the following relation between the ideal $I_q(G)$ and the vanishing ideal over a subgraph of G is easy to prove.

Corollary 2.3. *Let H be a subgraph of G. Then, under the inclusion of polynomial rings $K[E_H] \subset K[E_G]$, we have $I_q(H) = I_q(G) \cap K[E_H]$.*

Despite the multiple characterizations of $I_q(G)$, a complete classification of the subgraphs of G that support binomials of a minimal binomial generating set of $I_q(G)$ is still lacking, for general G; in contrast with the case of the toric ideal P_G in which the binomials in a minimal generating set are in one-to-one correspondence with the closed even walks on the graph.

2.2. Castelnuovo–Mumford regularity. Recall that if S is a polynomial ring and M is any graded S-module, the Castelnuovo–Mumford regularity of M is, by definition,

$$\operatorname{reg} M = \max_{i,j} \{j - i : \beta_{ij} \neq 0\},$$

where β_{ij} are the graded Betti numbers of M. The Castelnuovo–Mumford regularity of $K[E_G]/I_q(G)$ is thus an integer we can associate to any simple graph without isolated vertices.

Definition 2.4. Let G be a simple graph without isolated vertices and K a finite field. We define the Castelnuovo–Mumford regularity of G over the field K to be the regularity of the quotient $K[E_G]/I_q(G)$ and we denote it by $\operatorname{reg} G$.

Since $K[E_G]/I_q(G)$ is a Cohen–Macaulay one-dimensional graded ring, its regularity coincides with its *index of regularity*, i.e., the least integer from which the value of the Hilbert function equals the value of the Hilbert Polynomial (cf. [18, Proposition 4.2.3]). Additionally, given that any monomial $\mathbf{t}^\delta \in K[E_G]$ is a regular element of $K[E_G]/I_q(G)$ and the quotient of $K[E_G]$ by the extended ideal, $(I_q(G), \mathbf{t}^\delta)$, is a zero-dimensional graded ring with index of regularity equal to $\operatorname{reg} G + \deg \mathbf{t}^\delta$, we get:

$$\operatorname{reg} G = \min \left\{ i : \dim_K \big(K[E_G]/(I_q(G), \mathbf{t}^\delta) \big)_i = 0 \right\} - \deg \mathbf{t}^\delta.$$

The idea of taking the Artinian quotient $K[E_G]/(I_q(G), \mathbf{t}^\delta)$ to compute $\operatorname{reg} G$ is the main ingredient in the proof of the next proposition, which will be used several times in this article. See [11, Propositions 2.2 and 2.3] for a proof.

Proposition 2.5. *Let G be a graph, $\mathbf{t}^\delta \in K[E_G]$ a monomial and d a positive integer. Then $\operatorname{reg} G \leq d - \deg(\mathbf{t}^\delta)$ if and only if for every monomial $\mathbf{t}^\nu$ of degree d there exists $\mathbf{t}^\mu$, of degree d, such that $\mathbf{t}^\delta \mid \mathbf{t}^\mu$ and $\mathbf{t}^\nu - \mathbf{t}^\mu \in I_q(G)$.*

2.3. Graph invariants and the regularity. Table 1 contains the values of the Castelnuovo–Mumford regularity of several families of graphs. The simplest cases are those of a tree and an odd cycle. These are the simplest

Graph	$\operatorname{reg} G$
Tree	$(\lvert V_G\rvert - 2)(q-2)$
Odd cycle	$(\lvert V_G\rvert - 1)(q-2)$
Even cycle	$\frac{\lvert V_G\rvert-2}{2}(q-2)$
Complete graph $\mathcal{K}_n$, $n \geq 4$	$\lceil (n-1)(q-2)/2 \rceil$
Complete bipartite graph $\mathcal{K}_{a,b}$	$(\max\{a,b\} - 1)(q-2)$

Table 1.

cases because, in both, X, the projective toric subset parameterized by E_G, coincides with the torus (cf. [16, Corollary 3.8]) and therefore the ideal $I_q(G)$ is equal to the vanishing ideal of the torus, I_q, given in (3). The fact that I_q is a complete intersection enables the straightforward computation of the regularity. In [14] the case of an even cycle was dealt with. The cases of the complete graph and of the complete bipartite graph were studied in [6] and [4], respectively.

By now, there are many ways to produce estimates for the Castelnuovo–Mumford regularity of a particular graph using combinatorial invariants of the graph. We begin by mentioning the lower bound obtained from the vertex independence number of the graph.

Proposition 2.6 ([11, Proposition 2.7])**.** *If $V \subset V_G$ is a set of r independent vertices, such that the edge set of $G - V$ is not empty, then* $\operatorname{reg} G \geq r(q-2)$.

Since G has no isolated vertices, it follows from this result that

$$\operatorname{reg} G \geq (\alpha(G) - 1)(q-2), \tag{6}$$

where $\alpha(G)$ is the vertex independence number of G. However, as can be easily seen by the values of the regularity of Table 1, this bound is not sharp if G is non-bipartite or, even for a bipartite graph, if it fails to be 2-connected. As an application of Theorem 4.4, we shall give an infinite family of 2-connected bipartite graphs for which the bound (6) is not sharp. (See Example 4.6.)

The operation of vertex identification also yields lower bounds for the Castelnuovo–Mumford regularity of G. The next result was proved in [11, Proposition 2.5].

Proposition 2.7. *Let v_1 and v_2 be two nonadjacent vertices of G and let H be the simple graph obtained after identifying v_1 with v_2. Then* $\operatorname{reg} G \geq \operatorname{reg} H$.

Note that the identification of two vertices can create multiple edges. By simple graph, in the statement, we refer to the graph obtained after the removal of all multiple edges created.

Bounds for the Castelnuovo–Mumford regularity of a graph can also be obtained from its subgraphs. The next result follows from [17, Lemma 2.13].

Proposition 2.8. *Let H be a spanning subgraph of G which is non-bipartite if G is non-bipartite. Then* $\operatorname{reg} G \leq \operatorname{reg} H$.

Reversing the roles of G and H, this result can also be used to produce lower bounds of the regularity. For instance, if G is bipartite and spans a $\mathcal{K}_{a,b}$ then

$$\operatorname{reg} G \geq (\max\{a,b\}-1)(q-2)$$

and if G is non-bipartite with $|V_G| \geq 4$, then

$$\operatorname{reg} G \geq \operatorname{reg} \mathcal{K}_{|V_G|} = \left\lceil \tfrac{(|V_G|-1)(q-2)}{2} \right\rceil.$$

Another way to obtain upper bounds for the regularity of a graph is by using a decomposition of G into two subgraphs with, at least, one edge in common.

Proposition 2.9 ([11, Proposition 2.6])**.** *If H_1 and H_2 are two subgraphs of G with a common edge and $G = H_1 \cup H_2$ then*

$$\operatorname{reg} G \leq \operatorname{reg} H_1 + \operatorname{reg} H_2.$$

A graph is said 2-vertex-connected (or, simply, 2-connected) if $|V_G| \geq 3$ and $G - v$ is connected for every $v \in V_G$. Any graph can be decomposed into a set of edge disjoint subgraphs consisting of either isolated vertices, single edges (called bridges) or maximal 2-connected subgraphs. This decomposition is called the block decomposition of the graph. In [15] the relation between the regularity of a bipartite graph and the regularities of the members of its block decomposition was described.

Proposition 2.10 ([15, Theorem 7.4])**.** *Let G be a simple bipartite graph without isolated vertices and let $G = H_1 \cup \cdots \cup H_m$ be the block decomposition of G, then*

$$\operatorname{reg} G = \textstyle\sum_{k=1}^{m} \operatorname{reg} H_i + (m-1)(q-2). \tag{7}$$

The previous result does not hold if we drop the bipartite assumption. The graph in Figure 1 is a counterexample. We used *Macaulay2*, [10], to compute its Castelnuovo–Mumford for some values of the order of the ground field. For $q \in \{3, 4, 5, 7, 8, 9, 11, 13, 16\}$, the regularity is given by the formula $\lceil 5(q-2)/2 \rceil$. On the other hand, its block decomposition has three blocks; two triangles, of regularity $2(q-2)$, and a cut-edge, of regularity zero. Using this in formula (7) yields $6(q-2)$.

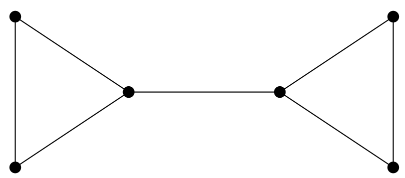

FIGURE 1.

The next proposition gives an additive formula for the regularity of G with respect to its leaves which holds for both bipartite and non-bipartite graphs.

Proposition 2.11 ([11, Proposition 2.4])**.** *If $v_1, \ldots, v_r$ are vertices of degree one and $G^\flat$ is the graph defined by $G^\flat = G - \{v_1, \ldots, v_r\}$ then*

$$\operatorname{reg} G = \operatorname{reg} G^\flat + r(q-2).$$

Proposition 2.10 motivates the study of the regularity of a general 2-connected bipartite graph. In view of Whitney's structure theorem for 2-connected graphs (see Section 4) one is naturally drawn to the problem of assessing the change produced in the regularity when we attach a path to a graph. We will explore this idea in the next two sections.

3. Ears and Regularity

The aim of this section is to provide a relation between the Castelnuovo–Mumford regularities of a graph and of the graph obtained by attaching a path by its end-vertices. The main theorem of this section, Theorem 3.4, states that the addition of such a path increases the regularity by $\lfloor \frac{\ell}{2} \rfloor (q-2)$, where ℓ is the length of the path. In this result, we assume that G is bipartite and that the end-vertices of the path are identified with two vertices of the graph which, in turn, are connected in the graph by a path the inner vertices of which have degree two. Both assumptions are necessary (see Examples 3.5 and 3.6). Proposition 3.2 addresses a special case in which we can afford to drop the bipartite assumption.

By a path, $\mathcal{P} \subset G$, we mean a subgraph of G endowed with an order of its vertices, $v_0, v_1, \ldots, v_\ell$, where $\ell > 0$, such that $v_1, \ldots, v_\ell$ are ℓ distinct vertices and $E_{\mathcal{P}}$ consists of the ℓ distinct edges $\{v_i, v_{i+1}\}$, $i = 0, \ldots, \ell - 1$. If $v_0 = v_\ell$, $\mathcal{P}$ is also called a cycle. However, since we are assuming that the edges $\{v_i, v_{i+1}\}$ are distinct, the case $\ell = 2$ and $v_0 = v_2$ is not allowed. The *inner vertices* of $\mathcal{P}$ are $v_1, \ldots, v_{\ell-1}$ and the *end-vertices* of $\mathcal{P}$ are v_0 and v_ℓ. The set of inner vertices of $\mathcal{P}$ will be denoted by $\mathcal{P}^\circ \subset V_G$. The number of edges in $\mathcal{P}$ is called the length of $\mathcal{P}$ and will be denoted by $\ell(\mathcal{P})$.

Definition 3.1. A path $\mathcal{P} \subset G$ is called an *ear* of G if all inner vertices of $\mathcal{P}$ have degree two in G. If the end-vertices of $\mathcal{P}$ are distinct, $\mathcal{P}$ is called an open ear if they coincide, $\mathcal{P}$ is called a pending cycle.

Proposition 3.2. *Let $\mathcal{P} \subset G$ be an ear of G, of length $\ell > 1$. Assume either:*

(i) *ℓ is odd and the end-vertices of $\mathcal{P}$ are distinct and adjacent in G or*
(ii) *ℓ is even and the end-vertices of $\mathcal{P}$ coincide.*

Denote the graph $G - \mathcal{P}^\circ$ by $G^\flat$ and assume that $G^\flat$ has no isolated vertices. Then

$$\operatorname{reg} G = \operatorname{reg}(G^\flat) + \lfloor \tfrac{\ell}{2} \rfloor (q-2). \tag{8}$$

Proof: Note that since $\ell > 1$ and, when ℓ is even $\ell \geq 4$, we get $\ell \geq 3$. Without loss of generality, we may assume that $\mathcal{P}$ is the path in G given by $(1, \ldots, \ell+1)$, if ℓ is odd or $(1, \ldots, \ell, 1)$ if ℓ is even. (See Figure 2.) In both cases, it follows that G contains an even cycle the vertex set of which coincides with $V_{\mathcal{P}}$. Since the generators of P_G, the toric ideal of G, are given by the closed even walks on G, using the relation between P_G and $I_q(G)$,

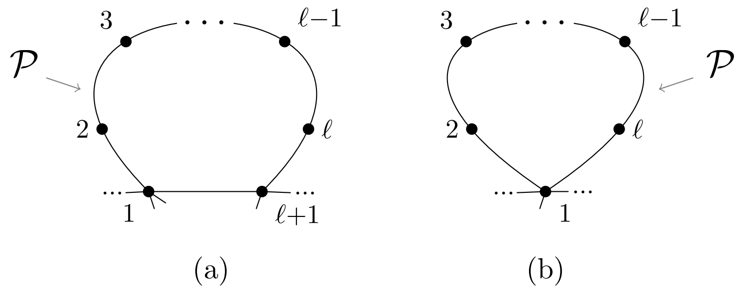

FIGURE 2.

expressed in (4), it follows that

$$\begin{cases} t_{12}t_{34}\cdots t_{\ell(\ell+1)} - t_{23}t_{45}\cdots t_{(\ell+1)1} \in I_q(G), & \text{if } \ell \text{ is odd, or} \\ t_{12}t_{34}\cdots t_{(\ell-1)\ell} - t_{23}t_{45}\cdots t_{\ell 1} \in I_q(G), & \text{if } \ell \text{ is even.} \end{cases} \tag{9}$$

Let us start by showing that

$$\operatorname{reg} G \geq \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell}{2} \right\rfloor (q-2)$$

Fix $t_{kl} \in E_{G^\flat}$. By Proposition 2.5, applied to the graph $G^\flat$, we deduce that there exists $\mathbf{t}^\alpha \in K[E_{G^\flat}]$, of degree $\operatorname{reg}(G^\flat)$, for which no monomial $\mathbf{t}^\beta \in K[E_{G^\flat}]$ divisible by t_{kl} is such that $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I_q(G^\flat)$.

Let $\mathbf{t}^\nu \in K[E_G]$ be the monomial of degree $\operatorname{reg}(G^\flat) + \left\lfloor \tfrac{\ell}{2} \right\rfloor (q-2)$ given by:

$$\begin{cases} \mathbf{t}^\nu = \mathbf{t}^\alpha (t_{23}t_{45}\cdots t_{(\ell-1)\ell})^{q-2}, & \text{if } \ell \text{ is odd, or} \\ \mathbf{t}^\nu = \mathbf{t}^\alpha (t_{23}t_{45}\cdots t_{\ell 1})^{q-2}, & \text{if } \ell \text{ is even.} \end{cases}$$

Suppose there exists $\mathbf{t}^\mu \in K[E_G]$, with $\mu\{k,l\} > 0$, such that

$$\mathbf{t}^\nu - \mathbf{t}^\mu \in I_q(G). \tag{10}$$

Modifying appropriately, with the use of $t_{ij}^{q-1} - t_{kl}^{q-1} \in I_q(G)$, we may assume that $0 \leq \mu\{i,i+1\} \leq q-2$, for all $i = 1, \ldots, \ell-1$, and $0 \leq \mu\{\ell,\ell+1\} \leq q-2$, if ℓ is odd, or $0 \leq \mu\{1,\ell\} \leq q-2$, if ℓ is even. In other words, we may assume that the variables along the path $\mathcal{P}$ appear in $\mathbf{t}^\mu$ raised to powers not greater than $q-2$. Then, evaluating the congruences of Proposition 2.2 at the vertices $2, \ldots, \ell$, we get, if ℓ is odd,

$$\mu\{i-1,i\} + \mu\{i,i+1\} \equiv q-2, \ \forall_{i \in \{2,\ldots,\ell\}}$$

or, if ℓ is even,

$$\mu_{\{i-1,i\}}+\mu_{\{i,i+1\}}\equiv q-2,\ \forall_{i\in\{2,\dots,\ell-1\}},\quad \text{and}\quad \mu_{\{\ell-1,\ell\}}+\mu_{\{\ell,1\}}\equiv q-2,$$

where all congruences are modulo $q-1$. We deduce that there exist

$$a,b\in\{0,\dots,q-2\},$$

with $a+b\equiv q-2$ such that, if ℓ is odd,

$$\begin{cases}\mu_{\{1,2\}}=\mu_{\{3,4\}}=\cdots=\mu_{\{\ell,\ell+1\}}=a\\ \mu_{\{2,3\}}=\mu_{\{4,5\}}=\cdots=\mu_{\{\ell-1,\ell\}}=b\end{cases}$$

or, if ℓ is even,

$$\begin{cases}\mu_{\{1,2\}}=\mu_{\{3,4\}}=\cdots=\mu_{\{\ell-1,\ell\}}=a\\ \mu_{\{2,3\}}=\mu_{\{4,5\}}=\cdots=\mu_{\{\ell,1\}}=b.\end{cases}$$

Let $\mathbf{t}^\delta\in K[E_{G^\flat}]$ be the monomial supported on $G^\flat$, given by

$$\begin{cases}\mathbf{t}^\mu=(t_{12}t_{34}\cdots t_{\ell(\ell+1)})^a(t_{23}t_{45}\cdots t_{(\ell-1)\ell})^b\,\mathbf{t}^\delta, & \text{if } \ell \text{ is odd, or}\\ \mathbf{t}^\mu=(t_{12}t_{34}\cdots t_{(\ell-1)\ell})^a(t_{23}t_{45}\cdots t_{\ell 1})^b\,\mathbf{t}^\delta, & \text{if } \ell \text{ is even.}\end{cases}$$

In view of (9), we deduce that there exists $\mathbf{t}^\beta\in K[E_{G^\flat}]$ such that

$$\beta_{\{k,l\}}\geq\mu_{\{k,l\}}>0$$

and

$$\begin{cases}\mathbf{t}^\mu-(t_{23}t_{45}\cdots t_{(\ell-1)\ell})^{q-2}\,\mathbf{t}^\beta\in I_q(G), & \text{if } \ell \text{ is odd, or}\\ \mathbf{t}^\mu-(t_{23}t_{45}\cdots t_{\ell 1})^{q-2}\,\mathbf{t}^\beta\in I_q(G), & \text{if } \ell \text{ is even.}\end{cases}\tag{11}$$

Note that if $a+b>q-2$ then, using $t_{ij}^{q-1}-t_{kl}^{q-1}\in I_q(G)$, the powers of the variables t_{23}, $t_{45},\dots,t_{(\ell-1)\ell}$ can be reduced to $q-2$. Combining (10) and (11) we obtain

$$\begin{cases}\mathbf{t}^\alpha(t_{23}t_{45}\cdots t_{(\ell-1)\ell})^{q-2}-(t_{23}t_{45}\cdots t_{(\ell-1)\ell})^{q-2}\,\mathbf{t}^\beta\in I_q(G), & \text{if } \ell \text{ is odd, or}\\ \mathbf{t}^\alpha(t_{23}t_{45}\cdots t_{\ell 1})^{q-2}-(t_{23}t_{45}\cdots t_{\ell 1})^{q-2}\,\mathbf{t}^\beta\in I_q(G), & \text{if } \ell \text{ is even.}\end{cases}$$

Since any product of variables is regular in $K[E_G]/I_q(G)$, we get

$$\mathbf{t}^\alpha-\mathbf{t}^\beta\in I_q(G),$$

where, recall $\beta\{k,l\} > 0$. But this binomial, being supported on $G^\flat$, also belongs to $I_q(G^\flat)$. This contradicts the assumptions on $\mathbf{t}^\alpha$. Therefore, by Proposition 2.5,

$$\operatorname{reg} G \geq \deg(\mathbf{t}^\nu) = \operatorname{reg}(G^\flat) + \left\lfloor \tfrac{\ell}{2} \right\rfloor (q-2).$$

To prove the opposite inequality we will use Proposition 2.9. If ℓ is odd (see Figure 2a), consider the decomposition of G given by $\mathcal{P} \cup \{1, \ell+1\}$ and $G^\flat$. Then

$$\operatorname{reg} G \leq \operatorname{reg} G^\flat + \operatorname{reg}(\mathcal{P} \cup \{1, \ell+1\}) = \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell}{2} \right\rfloor (q-2).$$

If ℓ is even (see Figure 2b), we consider the decomposition of G into the subgraph $H = G^\flat \cup \{1, 2\}$ and the cycle $\mathcal{P}$. Using Propositions 2.9 and 2.11, we get

$$\operatorname{reg} G \leq \operatorname{reg} G^\flat + (q-2) + \operatorname{reg} \mathcal{P} = \operatorname{reg} G^\flat + \tfrac{\ell}{2}(q-2). \quad \blacksquare$$

Definition 3.3. Let G be a bipartite graph and $\mathcal{I} \subset G$ an open ear of G. A bipartite ear modification of G along $\mathcal{I}$ is the *simple* graph obtained by either: (i) replacing $\mathcal{I}$ by another open ear $\mathcal{P}$, with the same end-vertices and length of the same parity as $\ell(\mathcal{I})$, or (ii), if $\ell(\mathcal{I})$ is even, by identifying the end-vertices of $\mathcal{I}$ in $G - \mathcal{I}^\circ$. We say that G satisfies the bipartite ear modification hypothesis on $\mathcal{I}$ if, whenever G' is a bipartite ear modification of G along $\mathcal{I}$, we have

$$\operatorname{reg} G' = \operatorname{reg} G + \tfrac{|V_{G'}| - |V_G|}{2}(q-2). \tag{12}$$

Notice that, since G is assumed to be bipartite, if $\ell(\mathcal{I})$ is even, then its end-vertices are not adjacent and in the bipartite ear modification described in (ii), no loop is created. However, in both cases, to obtain a simple graph it may be necessary to remove the multiple edges created.

It is easy to see that an even cycle satisfies the bipartite ear modification assumption on any of its open ears. Given that the regularity of a tree on n vertices is $(n-2)(q-2)$, it is clear that trees do not.

Theorem 3.4. *Let G be a bipartite graph and $\mathcal{I}$ and $\mathcal{P}$ be two open ears of G sharing the same end-vertices. Let $G^\flat$ denote the graph $G - \mathcal{P}^\circ$, if $\ell(\mathcal{P}) > 1$, or $G \backslash E_{\mathcal{P}}$, if $\ell(\mathcal{P}) = 1$. Assume that $G^\flat$ satisfies the bipartite ear modification hypothesis on $\mathcal{I}$. Then*

$$\operatorname{reg} G = \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor (q-2).$$

Proof: Note that, since G is bipartite, the lengths of $\mathcal{I}$ and $\mathcal{P}$ have the same parity. We may assume, without loss of generality, that $\mathcal{P}$ is the path $(1, \ldots, \ell_1 + 1)$, where $\ell_1 = \ell(\mathcal{P})$ and $\mathcal{I}$ is the path $(\ell_1 + 1, \ldots, \ell_2, 1)$, where $\ell_2 = \ell(\mathcal{P}) + \ell(\mathcal{I})$, as illustrated in Figure 3.

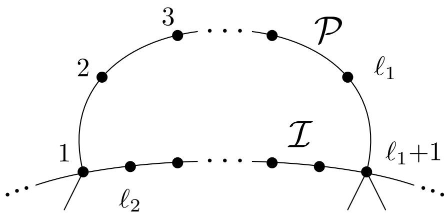

FIGURE 3.

If the vertices 1 and $\ell_1 + 1$ are neighbors in $G^\flat$ then, by Proposition 3.2, the result holds. Assume $\ell(\mathcal{P}) = 1$ and $\ell(\mathcal{I}) > 1$. Then the graph $G - \mathcal{I}^\circ$ is isomorphic to a bipartite ear modification of $G^\flat$ and, accordingly,

$$\operatorname{reg}(G - \mathcal{I}^\circ) = \operatorname{reg} G^\flat - \lfloor \tfrac{\ell(\mathcal{I})}{2} \rfloor (q-2).$$

On the other hand, using again Proposition 3.2 we get

$$\operatorname{reg} G = \operatorname{reg}(G - \mathcal{I}^\circ) + \lfloor \tfrac{\ell(\mathcal{I})}{2} \rfloor (q-2) = \operatorname{reg} G^\flat.$$

Thus, from now on, we may assume that the vertices 1 and $\ell_1 + 1$ are not neighbors in $G^\flat$ and $\ell(\mathcal{I}), \ell(\mathcal{P}) > 1$. We will split the proof into two cases according to the parity of $\ell(\mathcal{I})$.

We start by assuming that $\ell(\mathcal{I})$ is odd. Consider the graph $G^\sharp$ obtained by adding the edge $E = \{1, \ell_1 + 1\}$ to G. Then, as G is a spanning subgraph of $G^\sharp$, we have $\operatorname{reg} G \geq \operatorname{reg} G^\sharp$. Denote the graph $(G^\flat - \mathcal{I}^\circ) \cup E$ by $(G^\flat)'$. (See Figure 4.)
Since $(G^\flat)'$ is a bipartite ear modification of $G^\flat$ along $\mathcal{I}$,

$$\operatorname{reg}(G^\flat)' = \operatorname{reg} G^\flat - \lfloor \tfrac{\ell(\mathcal{I})}{2} \rfloor (q-2).$$

On the other hand, using Proposition 3.2,

$$\operatorname{reg} G^\sharp = \operatorname{reg}(G^\flat)' + \left(\lfloor \tfrac{\ell(\mathcal{P})}{2} \rfloor + \lfloor \tfrac{\ell(\mathcal{I})}{2} \rfloor\right)(q-2)$$

and therefore,

$$\operatorname{reg} G \geq \operatorname{reg} G^\sharp = \operatorname{reg} G^\flat + \lfloor \tfrac{\ell(\mathcal{P})}{2} \rfloor (q-2).$$

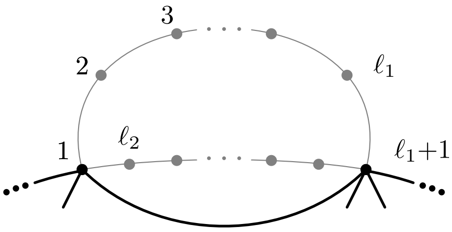

FIGURE 4. The graph $(G^\flat)'$.

Let us now prove the opposite inequality. We will use induction on $\frac{\ell(\mathcal{P})+\ell(\mathcal{I})}{2}$. Consider the following monomials in $K[E_G]$:

$$\begin{aligned} \mathbf{t}^{\delta_1} &= t_{23}t_{45}\cdots t_{(\ell_1-1)\ell_1}, \\ \mathbf{t}^{\epsilon_1} &= t_{12}t_{34}\cdots t_{\ell_1(\ell_1+1)}, \\ \mathbf{t}^{\delta_2} &= t_{(\ell_1+2)(\ell_1+3)}\cdots t_{(\ell_2-1)\ell_2}, \\ \mathbf{t}^{\epsilon_2} &= t_{(\ell_1+1)(\ell_1+2)}\cdots t_{\ell_2 1}. \end{aligned} \tag{13}$$

The monomial $\mathbf{t}^{\delta_1}$ is the monomial given by the multiplication of the variables associated to every other edge of $\mathcal{P}$ starting from the second. The monomial $\mathbf{t}^{\epsilon_1}$ is the monomial given by the multiplication of the other edges of $\mathcal{P}$. The monomials $\mathbf{t}^{\delta_2}$ and $\mathbf{t}^{\epsilon_2}$ are described similarly with respect to $\mathcal{I}$. (See Figure 5.)

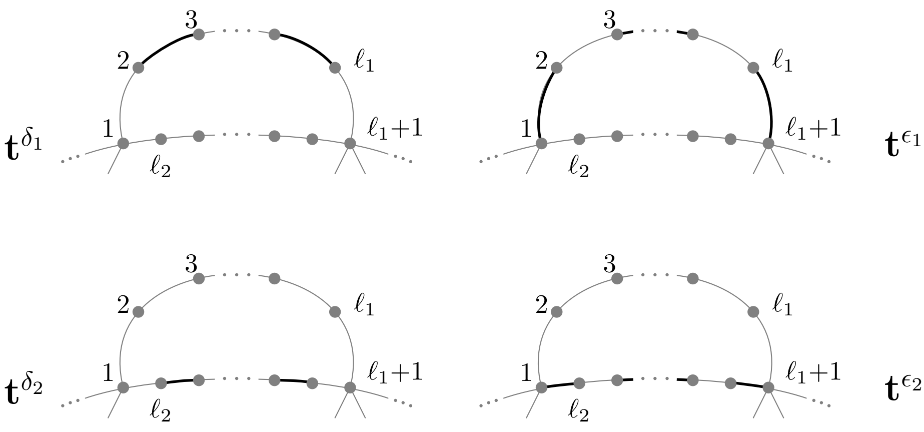

FIGURE 5. Edges in the support of $\mathbf{t}^{\delta_1}$, $\mathbf{t}^{\epsilon_1}$, $\mathbf{t}^{\delta_2}$ and $\mathbf{t}^{\epsilon_2}$.

Notice that

$$\deg(\mathbf{t}^{\delta_1}) = \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor, \;\; \deg(\mathbf{t}^{\epsilon_1}) = \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor + 1,$$

$$\deg(\mathbf{t}^{\delta_2}) = \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor, \;\; \deg(\mathbf{t}^{\epsilon_2}) = \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor + 1.$$

Also, since $\mathbf{t}^{\epsilon_1}\mathbf{t}^{\delta_2} - \mathbf{t}^{\delta_1}\mathbf{t}^{\epsilon_2}$ is a generator of the toric ideal of the even cycle $\mathcal{P} \cup \mathcal{I}$,

$$\mathbf{t}^{\epsilon_1}\mathbf{t}^{\delta_2} - \mathbf{t}^{\delta_1}\mathbf{t}^{\epsilon_2} \in I_q(\mathcal{P} \cup \mathcal{I}) \subset I_q(G). \tag{14}$$

By Proposition 2.5, to show that

$$\operatorname{reg} G \leq \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor (q-2) \tag{15}$$

it suffices to prove that for any monomial $\mathbf{t}^\nu \in K[E_G]$ of degree

$$\operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor (q-2) + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor + \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor$$

there exists $\mathbf{t}^\mu \in K[E_G]$, of the same degree as $\mathbf{t}^\nu$, divisible by $\mathbf{t}^{\delta_1}\mathbf{t}^{\delta_2}$, such that $\mathbf{t}^\nu - \mathbf{t}^\mu$ belongs to $I_q(G)$. Set $\mathbf{t}^\nu = \mathbf{t}^\alpha\mathbf{t}^\beta\mathbf{t}^\gamma$ for some $\alpha, \beta, \gamma \in \mathbb{N}^{E_G}$ satisfying $\mathbf{t}^\alpha \in K[E_\mathcal{P}]$, $\mathbf{t}^\beta \in K[E_\mathcal{I}]$, $\mathbf{t}^\gamma \in K[E_{G^\flat - \mathcal{I}^\circ}]$. Then

$$\deg(\mathbf{t}^\alpha) + \deg(\mathbf{t}^\beta) + \deg(\mathbf{t}^\gamma) = \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor (q-1) + \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor. \tag{16}$$

Suppose that

$$\deg(\mathbf{t}^\alpha) \geq \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor (q-1) \quad \text{and} \quad \deg(\mathbf{t}^\beta) \geq \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor (q-1). \tag{17}$$

Then $\mathbf{t}^\alpha\mathbf{t}^\beta$, which is supported on the cycle $\mathcal{P} \cup \mathcal{I}$, is such that

$$\deg(\mathbf{t}^\alpha\mathbf{t}^\beta) \geq \operatorname{reg}(\mathcal{P} \cup \mathcal{I}) + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor + \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor$$

and then, by Proposition 2.5, applied to $\mathcal{P} \cup \mathcal{I}$, there exists $\mathbf{t}^\mu \in K[E_{\mathcal{P}\cup\mathcal{I}}]$, divisible by $\mathbf{t}^{\delta_1}\mathbf{t}^{\delta_2}$ such that $\mathbf{t}^\alpha\mathbf{t}^\beta \in I_q(\mathcal{P} \cup \mathcal{I}) \subset I_q(G)$. We deduce that

$$\mathbf{t}^\alpha\mathbf{t}^\beta\mathbf{t}^\gamma - \mathbf{t}^\mu\mathbf{t}^\gamma \in I_q(G),$$

as desired. Assume now that (17) does not hold. Now, directly from (16),

$$\deg(\mathbf{t}^\alpha) < \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor (q-1) \iff \deg(\mathbf{t}^\beta\mathbf{t}^\gamma) \geq \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor + 1.$$

On the other hand, since $G - \mathcal{I}^\circ$ is a bipartite ear modification of $G^\flat$ along $\mathcal{I}$, and therefore by our assumptions,

$$\operatorname{reg}(G - \mathcal{I}^\circ) = \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor (q-2) - \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor (q-2),$$

we get from (16):

$$\deg(\mathbf{t}^\beta) < \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor (q-1) \iff \deg(\mathbf{t}^\alpha\mathbf{t}^\gamma) \geq \operatorname{reg}(G - \mathcal{I}^\circ) + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor + 1.$$

Hence, by symmetry, we may assume that

$$\deg(\mathbf{t}^\alpha) < \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor (q-1) \iff \deg(\mathbf{t}^\beta \mathbf{t}^\gamma) \geq \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor + 1. \tag{18}$$

Then, by Proposition 2.5, there exists $\mathbf{t}^\mu \in K[E_{G^\flat}]$, of degree equal to $\deg(\mathbf{t}^\beta \mathbf{t}^\gamma)$, divisible by $\mathbf{t}^{\delta_2}$, such that

$$\mathbf{t}^\beta \mathbf{t}^\gamma - \mathbf{t}^\mu \in I_q(G^\flat) \subset I_q(G),$$

which implies that

$$\mathbf{t}^\alpha \mathbf{t}^\beta \mathbf{t}^\gamma - \mathbf{t}^\alpha \mathbf{t}^\mu \in I_q(G).$$

If $\mathbf{t}^{\delta_1}$ divides $\mathbf{t}^\alpha$ we have finished. Assume $\mathbf{t}^{\delta_1}$ does not divide $\mathbf{t}^\alpha$. If $\mathbf{t}^{\epsilon_1}$ divides $\mathbf{t}^\alpha$, then, since

$$\mathbf{t}^\alpha \mathbf{t}^\mu = (\mathbf{t}^\alpha \mathbf{t}^{-\epsilon_1})(\mathbf{t}^{\epsilon_1} \mathbf{t}^{\delta_2})(\mathbf{t}^{-\delta_2} \mathbf{t}^\mu)$$

and $\mathbf{t}^{\epsilon_1} \mathbf{t}^{\delta_2} - \mathbf{t}^{\delta_1} \mathbf{t}^{\epsilon_2} \in I_q(G)$, we get:

$$\mathbf{t}^\alpha \mathbf{t}^\beta \mathbf{t}^\gamma - (\mathbf{t}^\alpha \mathbf{t}^{-\epsilon_1} \mathbf{t}^{\delta_1})(\mathbf{t}^{\epsilon_2} \mathbf{t}^{-\delta_2} \mathbf{t}^\mu) \in I_q(G), \tag{19}$$

where,

$$\deg(\mathbf{t}^{\epsilon_2} \mathbf{t}^{-\delta_2} \mathbf{t}^\mu) = \deg(\mathbf{t}^\beta \mathbf{t}^\gamma) + 1 \geq \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor + 2.$$

Hence, by Proposition 2.5, there exists $\mathbf{t}^\rho \in K[E_{G^\flat}]$ divisible by $\mathbf{t}^{\delta_2}$ such that

$$\mathbf{t}^{\epsilon_2} \mathbf{t}^{-\delta_2} \mathbf{t}^\mu - \mathbf{t}^\rho \in I_q(G^\flat) \subset I_q(G). \tag{20}$$

From (19) and (20), we deduce that

$$\mathbf{t}^\alpha \mathbf{t}^\beta \mathbf{t}^\gamma - (\mathbf{t}^\alpha \mathbf{t}^{-\epsilon_1} \mathbf{t}^{\delta_1}) \mathbf{t}^\rho \in I_q(G)$$

where $(\mathbf{t}^\alpha \mathbf{t}^{-\epsilon_1} \mathbf{t}^{\delta_1}) \mathbf{t}^\rho$ is divisible by $\mathbf{t}^{\delta_1} \mathbf{t}^{\delta_2}$, as required.

We may assume from now on that $\mathbf{t}^\alpha$ is divisible by neither $\mathbf{t}^{\epsilon_1}$ nor $\mathbf{t}^{\delta_1}$. Since showing that there exists a monomial $\mathbf{t}^\mu \in K[E_G]$ of degree equal to the degree of $\mathbf{t}^\nu = \mathbf{t}^\alpha \mathbf{t}^\beta \mathbf{t}^\gamma$, divisible by $\mathbf{t}^{\delta_1} \mathbf{t}^{\delta_2}$ such that $\mathbf{t}^\nu - \mathbf{t}^\mu \in I_q(G)$ is, by [11, Lemma 2.1], equivalent to showing that the same holds for the monomial obtained by permuting the variables of the support of $\mathbf{t}^{\delta_1}$ and permuting the variables of the support of $\mathbf{t}^{\epsilon_1}$, we may assume that neither t_{12} nor t_{23} divides $\mathbf{t}^\nu$.

Consider the graph H obtained from G by removing the edges $\{1,2\}$ and $\{2,3\}$, and identifying the vertices 1 and 3, as illustrated in Figure 6. Denote the ear obtained from $\mathcal{P}$ after this operation by $\mathcal{Q}$. By induction:

$$\operatorname{reg} H = \operatorname{reg}(H - \mathcal{Q}^\circ) + \left\lfloor \tfrac{\ell(\mathcal{Q})}{2} \right\rfloor (q-2) = \operatorname{reg} G^\flat + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor (q-2) - (q-2).$$

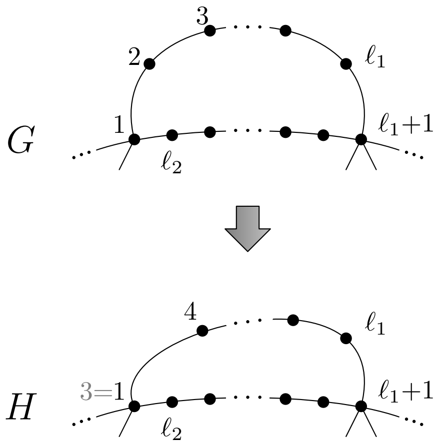

FIGURE 6.

Let $d = \nu_{\{3,4\}}$ and let $\mathbf{t}^{\overline{\nu}} \in K[E_H]$ be given by:

$$\mathbf{t}^{\overline{\nu}} = \mathbf{t}^{\nu} t_{34}^{-d} t_{14}^{d}.$$

Then, since

$$\deg(\mathbf{t}^{\overline{\nu}}) = \deg(\mathbf{t}^{\nu}) = \operatorname{reg} H + \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor + \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor + (q-2)$$

and $\mathbf{t}^{\delta_1} t_{23}^{-1} \mathbf{t}^{\delta_2} t_{1\ell_2}^{q-1} \in K[E_H]$ is such that

$$\deg(\mathbf{t}^{\delta_1} t_{23}^{-1} \mathbf{t}^{\delta_2} t_{1\ell_2}^{q-1}) = \left\lfloor \tfrac{\ell(\mathcal{P})}{2} \right\rfloor + \left\lfloor \tfrac{\ell(\mathcal{I})}{2} \right\rfloor + (q-2),$$

by Proposition 2.5 applied to the graph H, there exists $\mathbf{t}^{\overline{\mu}} \in K[E_H]$ such that $\mathbf{t}^{\delta_1} t_{23}^{-1} \mathbf{t}^{\delta_2} t_{1\ell_2}^{q-1}$ divides $\mathbf{t}^{\overline{\mu}}$ and $\mathbf{t}^{\overline{\nu}} - \mathbf{t}^{\overline{\mu}} \in I_q(H)$. Let $c = \overline{\mu}_{\{1,4\}}$ and let $\mathbf{t}^{\mu} \in K[E_G]$ be given by:

$$\mathbf{t}^{\mu} = \mathbf{t}^{\overline{\mu}} t_{14}^{-c} t_{34}^{c}.$$

By Proposition 2.2, the binomial $\mathbf{t}^{\overline{\nu}} - \mathbf{t}^{\overline{\mu}}$ satisfies a set of congruences modulo $q-1$, one for each vertex of H. In particular, at $1 \in V_H$, we have:

$$d + \sum_{k \in N_G(1)} \nu_{\{1,k\}} \;=\; \sum_{k \in N_H(1)} \overline{\nu}_{\{1,k\}} \;\equiv\; \sum_{k \in N_H(1)} \overline{\mu}_{\{1,k\}} \;=\; c + \sum_{k \in N_G(1)} \mu_{\{1,k\}}\,. \tag{21}$$

Let $a \in \{1, \ldots, q-1\}$ be such that $a \equiv d - c$ and let $b = (q-1) - a$. Then, as $\mathbf{t}^{\delta_1} t_{23}^{-1} \mathbf{t}^{\delta_2} t_{1\ell_2}^{q-1}$ divides $\mathbf{t}^{\overline{\mu}}$ and hence it divides $\mathbf{t}^{\mu}$, the binomial

$$\mathbf{t}^{\nu} - \mathbf{t}^{\mu} t_{1\ell_2}^{-(q-1)} t_{12}^{b} t_{23}^{a} \tag{22}$$

is a homogeneous binomial of the ring $K[E_G]$. Moreover, since $a \geq 1$, we deduce that $\mathbf{t}^{\delta_1}\mathbf{t}^{\delta_2}$ divides the monomial on the right side of (22).

Let us prove that the binomial (22) belongs to $I_q(G)$. It suffices to check that the corresponding congruences at vertices 1, 2 and 3 of G are satisfied, since at any other vertex the corresponding congruence is identical to the one in H. At the vertices 2 and 3, we get, respectively,

$$0 \equiv a + b \quad \text{and} \quad d \equiv c + a \iff d - c \equiv a$$

which hold, by the definitions of b and a. At the vertex 1, we get:

$$\sum_{k \in N_G(1)} \nu_{\{1,k\}} \equiv b - (q-1) + \sum_{k \in N_G(1)} \mu_{\{1,k\}} = c - d + \sum_{k \in N_G(1)} \mu_{\{1,k\}}\,,$$

which holds by (21). Hence (22) belongs to $I_q(G)$. This concludes the proof, by induction, of the inequality (15) in the case of $\ell(\mathcal{I})$ odd.

Let us now consider the case of $\ell(\mathcal{I})$ and $\ell(\mathcal{P})$ even. We start by proving that

$$\operatorname{reg} G \geq \operatorname{reg} G^\flat + \tfrac{\ell(\mathcal{P})}{2}(q-2). \tag{23}$$

Consider the simple graph H obtained from G by identifying the vertices 1 and $\ell_1 + 1$ and denote by H' be the subgraph of H obtained from $G^\flat - \mathcal{I}^\circ$ under the same identification. (See Figure 7.) Since H' is a bipartite ear

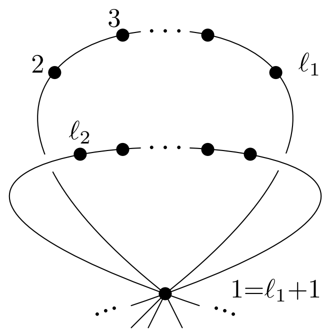

Figure 7.

modification of $G^\flat$,

$$\operatorname{reg} H' = \operatorname{reg} G^\flat - \tfrac{\ell(I)}{2}(q-2).$$

On the other hand, using Propositions 2.7 and 3.2, or, in the case of $\ell(\mathcal{P}) = 2$ or $\ell(\mathcal{I}) = 2$, Proposition 2.11,

$$\operatorname{reg} G \geq \operatorname{reg} H = \operatorname{reg} H' + \tfrac{\ell(\mathcal{P})}{2}(q-2) + \tfrac{\ell(\mathcal{I})}{2}(q-2) = \operatorname{reg} G^\flat + \tfrac{\ell(\mathcal{P})}{2}(q-2),$$

which proves (23). To prove that

$$\operatorname{reg} G \leq \operatorname{reg} G^\flat + \tfrac{\ell(\mathcal{P})}{2}(q-2), \tag{24}$$

by Proposition 2.5, it suffices to show that for every $\mathbf{t}^\nu \in K[E_G]$ with

$$\deg(\mathbf{t}^\nu) = \operatorname{reg} G^\flat + \tfrac{\ell(\mathcal{P})}{2}(q-2) + 1, \tag{25}$$

there exists $\mathbf{t}^\mu$, divisible by t_{12}, such that $\mathbf{t}^\nu - \mathbf{t}^\mu \in I_q(G)$. Consider the following graphs:

$$(G^\flat)^* = G^\flat \cup \{1,2\}\,, \quad \mathcal{C} = \mathcal{P} \cup \mathcal{I} \quad \text{and} \quad G - \mathcal{I}^\circ.$$

(See Figure 8.) By Proposition 2.11, $\operatorname{reg}(G^\flat)^* = \operatorname{reg} G^\flat + (q-2)$. Since $\mathcal{C}$ is

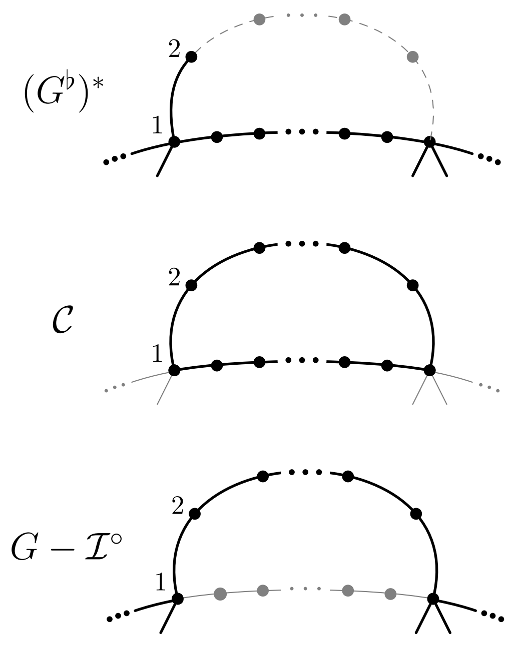

FIGURE 8.

an even cycle,

$$\operatorname{reg} \mathcal{C} = \tfrac{\ell(\mathcal{P})}{2}(q-2) + \tfrac{\ell(\mathcal{I})}{2}(q-2) - (q-2).$$

Finally, since $G^\flat$ satisfies the bipartite ear modification hypothesis,

$$\operatorname{reg}(G-\mathcal{I}^\circ)=\operatorname{reg}G^\flat-\tfrac{\ell(\mathcal{I})}{2}(q-2)+\tfrac{\ell(\mathcal{P})}{2}(q-2).$$

Let us write $\mathbf{t}^\nu=\mathbf{t}^\alpha\mathbf{t}^\beta\mathbf{t}^\gamma$ for some $\alpha,\beta,\gamma\in\mathbb{N}^{E_G}$ satisfying

$$\mathbf{t}^\alpha\in K[E_{\mathcal{P}}],\quad \mathbf{t}^\beta\in K[E_{\mathcal{I}}]\quad\text{and}\quad \mathbf{t}^\gamma\in K[E_{G^\flat-\mathcal{I}^\circ}].$$

Suppose that

$$\begin{cases}\deg(\mathbf{t}^\alpha)+\deg(\mathbf{t}^\beta)\leq\operatorname{reg}\mathcal{C}\\ \deg(\mathbf{t}^\beta)+\deg(\mathbf{t}^\gamma)\leq\operatorname{reg}(G^\flat)^*\\ \deg(\mathbf{t}^\alpha)+\deg(\mathbf{t}^\gamma)\leq\operatorname{reg}(G-\mathcal{I}^\circ)\end{cases}\tag{26}$$

Then

$$\deg(\mathbf{t}^\nu)\leq\tfrac{1}{2}\big[\operatorname{reg}\mathcal{C}+\operatorname{reg}(G^\flat)^*+\operatorname{reg}(G-\mathcal{I}^\circ)\big]=\operatorname{reg}G^\flat+\tfrac{\ell(\mathcal{P})}{2}(q-2),$$

which is in contradiction with (25). Hence the opposite inequality of one of the inequalities in (26) must hold. For instance if it is the opposite of the first inequality, then, by Proposition 2.5, there exists $\mathbf{t}^\mu\in K[E_{\mathcal{C}}]$, divisible by t_{12}, such that

$$\mathbf{t}^\alpha\mathbf{t}^\beta-\mathbf{t}^\mu\in I_q(\mathcal{C})\subset I_q(G)$$

which implies that

$$\mathbf{t}^\nu-\mathbf{t}^\mu\mathbf{t}^\gamma\in I_q(G),$$

as desired. We argue similarly for the other two cases. This proves (24) and concludes the proof of the theorem. ∎

The next two examples show that the assumptions of Theorem 3.4 are strictly necessary.

Example 3.5. Let $G^\flat$ be the graph of Figure 9. This graph decomposes into a cycle of length six and two cycles of length four. By Proposition 2.9, $\operatorname{reg}G^\flat\leq 4(q-2)$. On the other hand, the set $V=\{2,4,6,8\}$ is an independent set for which $G-V$ has a nonempty edge set. Hence, by Proposition 2.6, $\operatorname{reg}G^\flat\geq 4(q-2)$. We conclude that $\operatorname{reg}G^\flat=4(q-2)$. Let G be the graph obtained by adding the edge $\{2,8\}$ to $G^\flat$. Unlike $G^\flat$, the graph G has now spanning cycle (of length 8). By Proposition 2.8, we get $\operatorname{reg}G\leq 3(q-2)$. By a similar argument as above, one can show that indeed $\operatorname{reg}G=3(q-2)$. Thus the conclusion of Theorem 3.4, which in this case would state that $\operatorname{reg}G^\flat=\operatorname{reg}G$, does not hold. This is because the hypothesis that, besides

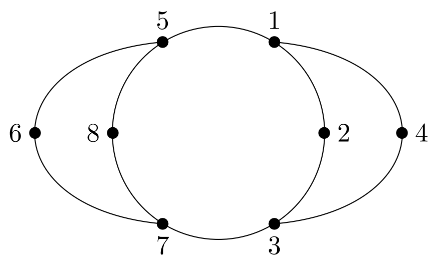

FIGURE 9.

the edge $\{2, 8\}$, there should be another ear in G with end-vertices 2 and 8, is not satisfied.

Example 3.6. Consider the graph, G, illustrated in Figure 10. G is a non-bipartite parallel composition of paths; two of length two and a path of length three. According to [11, Theorem 1.2], $\operatorname{reg} G = 4(q-2)$. Consider $G^\flat \subset G$ the

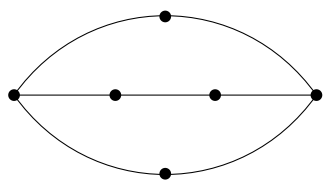

FIGURE 10.

subgraph given by the parallel composition of one of the paths of length two and the path of length three. Then $G^\flat$ is a cycle of length 5 and, accordingly, $\operatorname{reg} G^\flat = 4(q-2)$. If we take $G^\flat$ to be the parallel composition of the two paths of length two. Then $\operatorname{reg} G^\flat = q-2$. In both cases, the conclusion of Theorem 3.4 does not hold.

4. Nested Ear Decompositions

The goal of this section is to give a formula for the Castelnuovo–Mumford regularity of a graph endowed with a special decomposition into paths.

An ear decomposition of a graph consists of a collection of $r > 0$ subgraphs

$$\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r,$$

the edge sets of which form a partition of E_G, such that $\mathcal{P}_0$ is a vertex and, for all $1 \leq i \leq r$, $\mathcal{P}_i$ is a path with end-vertices in $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{i-1}$ while *none* of its inner vertices belong to $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{i-1}$. The paths $\mathcal{P}_1, \ldots, \mathcal{P}_r$

are called ears of the decomposition of G. We note that $\mathcal{P}_i$ is not necessarily an ear of G, according to Definition 3.1, as its inner vertices may become end-vertices of the following ears. An ear decomposition is called *open* if all of paths $\mathcal{P}_2, \ldots, \mathcal{P}_r$ have distinct end-vertices. It is well known that a graph is 2-vertex-connected if and only if it has an open ear decomposition (Whitney's Theorem). More generally, a graph is 2-edge-connected if and only if it has an ear decomposition.

Definition 4.1. Let $\mathcal{P}_0$, $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be an ear decomposition of a graph, G. If a path $\mathcal{P}_i$ has both its end-vertices in $\mathcal{P}_j$ we say that $\mathcal{P}_i$ is nested in $\mathcal{P}_j$ and we define the corresponding *nest interval* to be the subpath of $\mathcal{P}_j$ determined by the end-vertices of $\mathcal{P}_i$, if they are distinct, or, if they coincide, to be that single end-vertex. An ear decomposition of G is *nested* if, for all $1 \leq i \leq r$, the path $\mathcal{P}_i$ is nested in a previous subgraph of the decomposition, $\mathcal{P}_j$, with $j < i$, and, in addition, if two paths $\mathcal{P}_i$ and $\mathcal{P}_l$ are nested in $\mathcal{P}_j$, with $j < i, l$, then either the corresponding nest intervals in $\mathcal{P}_j$ have disjoint edge sets or one edge set is contained in the other.

Nested ear decompositions were introduced by Eppstein in [3]. In the original definition P_0 is allowed to be a path and thus the graphs considered in [3] are not necessarily 2-edge-connected.

The main result of this section is Theorem 4.4, which gives a formula for the Castelnuovo–Mumford regularity of a bipartite graph endowed with a nested ear decomposition. In the proof of this result we will need to show that a graph endowed with a nested ear decomposition satisfies the bipartite ear modification hypothesis along a certain ear. However, an instance of a bipartite ear modification, namely the one involving removing the ear and identifying its end-vertices can modify the ear decomposition structure by introducing *pendant edges* and, thus, producing a graph G' which may well not be 2-edge connected. We remedy this by working on a wider class of graphs, that of graphs endowed with a weaker form of nested ear decompositions.

Definition 4.2. A *weak nested ear decomposition* of a graph is a collection of subgraphs $\mathcal{P}_0, \ldots, \mathcal{P}_r$, with $r > 0$, the edge sets of which form a partition of E_G, such that $\mathcal{P}_0$ is a vertex and, for every $1 \leq i \leq r$, $\mathcal{P}_i$ is a path with either

(i) both end-vertices in some $\mathcal{P}_j$, with $j < i$ and none of its inner vertices in $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{i-1}$, or

(ii) if $\ell(\mathcal{P}_i) = 1$, only one end-vertex in $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{i-1}$.

If $\mathcal{P}_i$ has both its end-vertices in $\mathcal{P}_j$, the nest interval of $\mathcal{P}_i$ in $\mathcal{P}_j$ is defined as before. If $\ell(\mathcal{P}_i) = 1$ and only one end-vertex belongs to $\mathcal{P}_j$ then the nest interval is defined to be this vertex. The nesting condition is the same as the one of Definition 4.1.

If both end-vertices of $\mathcal{P}_i$ belong to a previous $\mathcal{P}_j$, then $\mathcal{P}_i$ will be referred to as an ear of the decomposition, otherwise, if $\ell(\mathcal{P}_i) = 1$ and $\mathcal{P}_i$ has only one end-vertex in a previous $\mathcal{P}_j$, then $\mathcal{P}_i$ will referred to as a *pendant edge* of the decomposition. An ear with coinciding end-vertices will also be referred to as a *pending cycle* of the decomposition. Notice that when $\ell(P_i) = 1$, $\mathcal{P}_i$ can either be an ear (so-called trivial ear) or a pendant edge of the decomposition.

Figure 11 shows a graph that can be endowed with a weak nested ear decomposition. For instance, $\mathcal{P}_0 = 1$, $\mathcal{P}_1 = (1, 2)$, $\mathcal{P}_2 = (1, 5)$, $\mathcal{P}_3 = (1, 3, 4, 5)$,

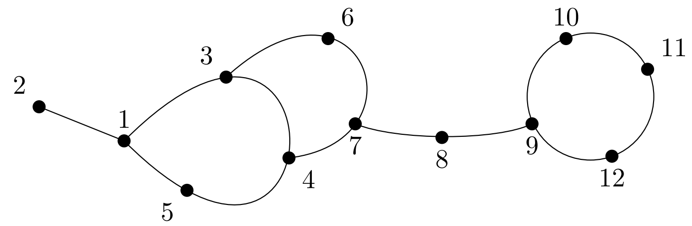

FIGURE 11.

$\mathcal{P}_4 = (3, 6, 7, 4)$, $\mathcal{P}_5 = (7, 8)$, $\mathcal{P}_6 = (8, 9)$, $\mathcal{P}_7 = (9, 10, 11, 12, 9)$. Another weak nested ear decomposition of this graph can be given by $\mathcal{P}_0 = 1$, $\mathcal{P}_1 = (1, 2)$, $\mathcal{P}_2 = (1, 3, 4, 5, 1)$, etc., as in the previous decomposition. We note that the number of even ears and pendant edges of these decompositions is five, the same in both.

It is clear that a nested ear decomposition of a graph is a weak nested ear decomposition. As the example above shows, not all graphs endowed with a weak nested ear decomposition are 2-edge-connected.

We will prove Theorem 4.4 by induction on the number of ears of the decomposition. We will need the following lemma.

Lemma 4.3. *Let $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ be a weak nested ear decomposition of a graph G. Then there exists $i > 0$ such that either $\mathcal{P}_i$ is a pendant edge of G, or a* *pendant cycle of G, or an ear of G with distinct end-vertices such that for any $\mathcal{P}_k$ containing both end-vertices of $\mathcal{P}_i$, the subpath of $\mathcal{P}_k$ induced by them is an ear of G.*

Proof: We argue by induction on $r \geq 1$. If $r = 1$ then it suffices to take $i = 1$. If $r > 1$, consider the graph $G^\flat = \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{r-1}$. By induction, there exist $i > 0$ and $\mathcal{P}_i$ satisfying the conditions in the statement. If $\mathcal{P}_r$ is

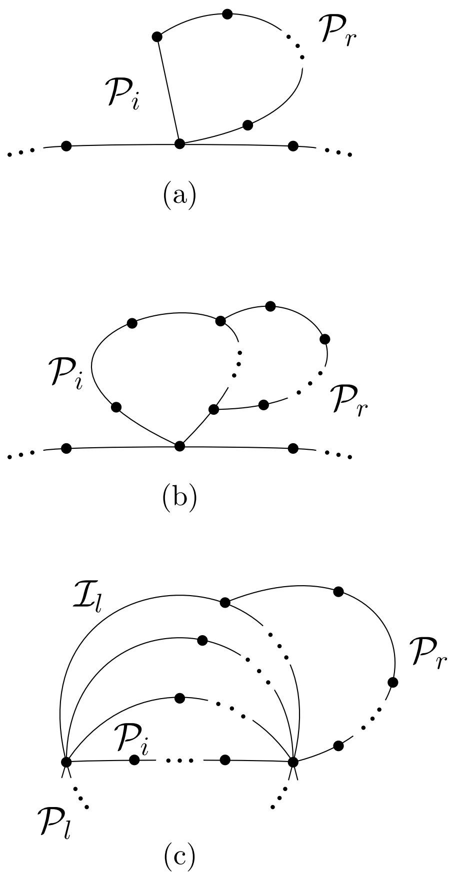

FIGURE 12.

pendant edge we may take $i = r$ for G. The same applies if $\mathcal{P}_r$ is a pending cycle. Assume then that $\mathcal{P}_r$ has distinct end-vertices. If $\mathcal{P}_i$ is a pendant edge of $G^\flat$ and it ceases to be so in G, then the end-vertices of $\mathcal{P}_r$ must coincide with the end-vertices of $\mathcal{P}_i$ and the only $\mathcal{P}_k$ that contain these vertices are then $\mathcal{P}_r$ and $\mathcal{P}_i$ which are both ears of G. (See Figure 12a.) In this case, $\mathcal{P}_r$ satisfies the conditions for G. If $\mathcal{P}_i$ is a pending cycle of $G^\flat$ which ceases to be an ear of G then one of the end-vertices of $\mathcal{P}_r$ must be an inner vertex of $\mathcal{P}_i$. Then, arguing as before, we see that $\mathcal{P}_r$ satisfies the conditions. (See Figure 12b.) Finally, assume that $\mathcal{P}_i$ is an ear of $G^\flat$ with distinct end-vertices. Let $\mathcal{I}_1, \ldots, \mathcal{I}_{r-1}$ be the subpaths induced by the end-vertices of $\mathcal{P}_i$ in the paths $\mathcal{P}_1, \ldots, \mathcal{P}_{r-1}$. For ease of notation consider $\mathcal{I}_j$ equal to the empty set if $\mathcal{P}_j$ does not contain both end-vertices of $\mathcal{P}_i$. If the end-vertices of $\mathcal{P}_r$ do not coincide with any inner vertex of the paths $\mathcal{I}_i, \ldots, \mathcal{I}_{r-1}$ then $\mathcal{P}_i$ satisfies the conditions of the statement for G. Assume that an end-vertex of $\mathcal{P}_r$ is an inner vertex of $\mathcal{I}_l$. Then, as $\mathcal{I}_l$ is an ear of $G^\flat$, $\mathcal{P}_r$ has to be nested in $\mathcal{P}_l$. Since $\mathcal{P}_i$ is also nested in $\mathcal{P}_l$ the nest intervals must be nested. This means that the subpath induced by the end-vertices of $\mathcal{P}_r$ in $\mathcal{P}_l$ must be contained in $\mathcal{I}_l$. (See Figure 12c.) Then $\mathcal{P}_r$ satisfies the conditions of the statement for G as the only paths that contain the end-vertices of $\mathcal{P}_r$ are then $\mathcal{P}_r$ and $\mathcal{P}_l$. ■

Theorem 4.4. *Let $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ be a weak nested ear decomposition of a bipartite graph, G. Let ϵ denote the number of even ears and pendant edges of the decomposition. Then*

$$\operatorname{reg} G = \tfrac{|V_G|+\epsilon-3}{2}(q-2). \tag{27}$$

Proof: We will argue by induction on $r \geq 1$. If $r = 1$ then G is either an even cycle or a single edge. In both cases $\epsilon = 1$ and (27) gives $\operatorname{reg} G = \tfrac{|V_G|-2}{2}(q-2)$, in the case of the even cycle and $\operatorname{reg} G = 0$, in the case of the edge. Both are correct. Assume that (27) holds for any bipartite graph endowed with a weak nested ear decomposition with r paths and consider G a graph endowed with a weak nested ear decomposition $\mathcal{P}_0, \ldots, \mathcal{P}_{r+1}$ with $r+1$ paths. Throughout the remainder of the proof, denote by ϵ the number of even ears and pendant edges of this decomposition. By Lemma 4.3, there exist $i > 0$, such that $\mathcal{P}_i$ is either a pendant edge, or a pendant cycle, or an ear of G with distinct end-vertices such that for any $\mathcal{P}_k$ containing both end-vertices of $\mathcal{P}_i$, the subpath of $\mathcal{P}_k$ induced by them is an ear of G. It is clear that in any of the cases

$$G^\flat = \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{i-1} \cup \mathcal{P}_{i+1} \cup \cdots \cup \mathcal{P}_{r+1} \tag{28}$$

is a bipartite graph endowed with a weak nested ear decomposition. If $\mathcal{P}_i$ is a pendant edge of G, then by Proposition 2.11 and induction,

$$\operatorname{reg} G = \operatorname{reg} G^\flat + (q-2) = \tfrac{|V_G|-1+(\epsilon-1)-3}{2}(q-2) + (q-2) = \tfrac{|V_G|+\epsilon-3}{2}(q-2).$$

If $\mathcal{P}_i$ is pendant cycle of G, then $\ell(P_i)$ is even and, by induction and Proposition 3.2,

$$\operatorname{reg} G = \tfrac{|V_G|-\ell(\mathcal{P}_i)+1+(\epsilon-1)-3}{2}(q-2) + \tfrac{\ell(\mathcal{P}_i)}{2}(q-2) = \tfrac{|V_G|+\epsilon-3}{2}(q-2).$$

Assume that $\mathcal{P}_i$ has distinct end-vertices and that for any $\mathcal{P}_k$ containing both end-vertices of $\mathcal{P}_i$, the subpath of $\mathcal{P}_k$ induced by them is an ear of G. Denote the end-vertices of $\mathcal{P}_i$ by v and w. If v and w are adjacent in G then $\ell(\mathcal{P}_i)$ must be odd. Accordingly, by induction and Proposition 3.2,

$$\operatorname{reg} G = \tfrac{|V_G|-\ell(\mathcal{P}_i)+1+\epsilon-3}{2}(q-2) + \tfrac{\ell(\mathcal{P}_i)-1}{2}(q-2) = \tfrac{|V_G|+\epsilon-3}{2}(q-2).$$

Assume now that v and w are not adjacent in G and let $j > 0$ be the least positive integer such that $\mathcal{P}_i$ is nested in $\mathcal{P}_j$. By the minimality of j one of v or w, say v, belongs to $\mathcal{P}_j^\circ$. Denote the nest interval of $\mathcal{P}_i$ in $\mathcal{P}_j$ by $\mathcal{I}$. (See top of Figure 13.) To be able to use Theorem 3.4, it will now suffice

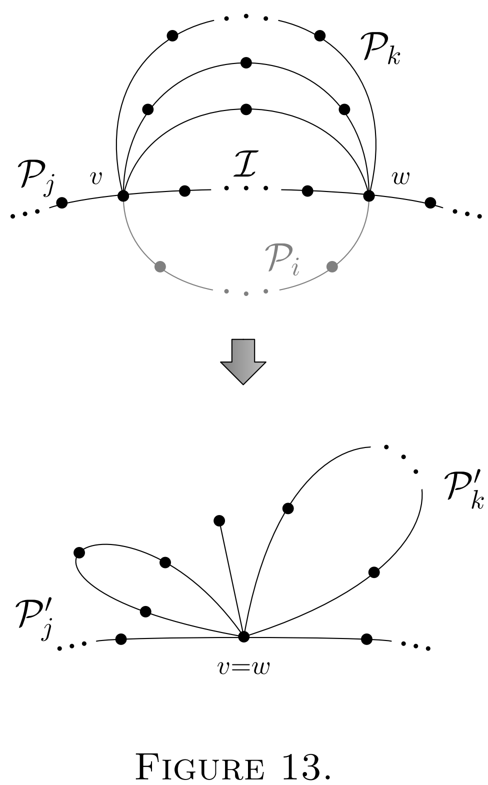

FIGURE 13.

to show that $G^\flat$ satisfies the bipartite ear modification hypothesis on $\mathcal{I}$. If G' is a bipartite ear modification of $G^\flat$ along $\mathcal{I}$, which does not involve the identification of the end-vertices of $\mathcal{I}$, then, as v and w are not adjacent, no multiple edges arise and the weak nested ear decomposition of $G^\flat$ induces a weak nested ear decomposition of G' in which the only change is in the length of $\mathcal{P}_j$, which, nevertheless, remains of the same parity. By induction, we can use (27) on both $G^\flat$ and G'. It follows that

$$\operatorname{reg} G' = \operatorname{reg} G^\flat + \tfrac{|V_{G^\flat}|-|V_{G'}|}{2}(q-2),$$

which is condition (12) of the bipartite ear modification hypothesis.

Suppose now that $\ell(\mathcal{I})$ is even and that G' is obtained by identifying the end-vertices of $\mathcal{I}$ in $G^\flat - \mathcal{I}^\circ$ and removing the multiple edges created. For $k \neq i$, let $\mathcal{P}'_k \subset G'$ denote the graph obtained by identifying v with w in $\mathcal{P}_k - \mathcal{I}^\circ$ and removing the multiple edges created. (See Figure 13.) We note that since $\mathcal{I}$ is an ear of G, for $k \neq j$, the graph $\mathcal{P}'_k$ is obtained by simply identifying v and w in $\mathcal{P}_k$ and removing all multiple edges created. It follows that $\mathcal{P}_k$ is isomorphic to $\mathcal{P}'_k$ if one of v or w does not belong to $\mathcal{P}_k$. If both v and w belong to $\mathcal{P}_k$ then, by the minimality of j, we must have $j < k$. But then none of v or w can be an inner vertex of $\mathcal{P}_k$. Accordingly, they must coincide with the end-vertices of $\mathcal{P}_k$. (See top of Figure 13.) Then $\mathcal{I} \cup \mathcal{P}_k$ is a cycle of G and, since it must be of even length, we deduce that $\ell(\mathcal{P}_k)$ is also even. In this situation $\mathcal{P}'_k$ is either a pending (even) cycle of G', if $\ell(\mathcal{P}_k) > 2$, or a pending edge, if $\ell(\mathcal{P}_k) = 2$. (See the bottom of Figure 13.) As for $\mathcal{P}'_j$, it may be a single edge if $\ell(\mathcal{P}_j) - \ell(\mathcal{I}) = 1$ or if $\ell(\mathcal{P}_j) - \ell(\mathcal{I}) = 2$ and $\mathcal{P}_j$ has coinciding end-vertices.

It is clear that

$$G' = \mathcal{P}_0 \cup \mathcal{P}'_1 \cup \cdots \cup \mathcal{P}'_{i-1} \cup \mathcal{P}'_{i+1} \cup \cdots \cup \mathcal{P}'_{r+1}, \tag{29}$$

as any edge in G' comes from an edge in some path $\mathcal{P}_k$. If (29) does not induce a partition of the edge set of G', then there exist a vertex u and two edges $\{u,v\}$, $\{u,w\}$ belonging to different paths, which become the same edges after the identification of v with w. Consider the least k for which the path $\mathcal{P}_k$ contains both vertices u, v and the least l for which $\mathcal{P}_l$ contains u, w. We claim that $\mathcal{P}_j = \mathcal{P}_k$, and, consequently, that u must belong to $\mathcal{P}_j$.

Assume, to the contrary, that $j \neq k$. Since $v \in \mathcal{P}_j^\circ$ we have $j < k$ and then v is an end-vertex of $\mathcal{P}_k$. If u in an inner vertex of $\mathcal{P}_k$ then we must have $k \leq l$. If u is an end-vertex of $\mathcal{P}_k$ then, by the minimality of k, $\mathcal{P}_k$ has to be a pending edge and we get the same conclusion, $k \leq l$. Assume that $k = l$. Then $j < k = l$ implies that $\mathcal{P}_k = \mathcal{P}_l$ is a path with end-vertices

v and w, containing u as an interior point. However if $\{u, v\}$ and $\{u, w\}$ belong to different paths then the degree of u is not two, which contradicts the assumption on v and w, stating that these vertices induce on any path of the weak decomposition a subpath the inner vertices of which have degree two. Hence we must have $k < l$. Then $j < k < l$ implies that $\mathcal{P}_l$ is the edge $\{u, w\}$. Since both its end-vertices belong to earlier paths, this contradicts the minimality of l.

We have proved that $\mathcal{P}_j = \mathcal{P}_k$ and, in particular, that $u \in \mathcal{P}_j$. Resetting notation, Let now $\mathcal{P}_k$ and $\mathcal{P}_l$ be *any* two (distinct) paths containing the edges $\{u, v\}$ and $\{u, w\}$, respectively. We claim that either $\mathcal{P}_k$ is a non-pending odd path of the weak ear decomposition of $G^\flat$ and $\mathcal{P}'_k$ is an edge or $\mathcal{P}_l$ is a non-pending odd path of the weak ear decomposition of $G^\flat$ and $\mathcal{P}'_l$ is an edge.

To see this, we start by noting that, since $v \in \mathcal{P}_j^\circ$, we have $j \leq k$. If $j < k$, then as u and v belong to $\mathcal{P}_j$ they must be end-vertices of $\mathcal{P}_k$. In this case, $\mathcal{P}_k = \{u, v\}$, which is a non-pending odd path and $\mathcal{P}'_k$ an edge. If $j = k$ then there are two sub-cases. Either $j = k < l$ and then $\mathcal{P}_l = \{u, w\}$, which is a non-pending odd ear with $\mathcal{P}'_l$ an edge (see Figure 14a), or we have $l < j = k$ and then u and w must be the end-vertices of $\mathcal{P}_j = \mathcal{P}_k$. This implies that $\ell(P_j) = \ell(\mathcal{I}) + 1$, which is a odd integer, and that $\mathcal{P}'_j$ is an edge. (See Figure 14b.)

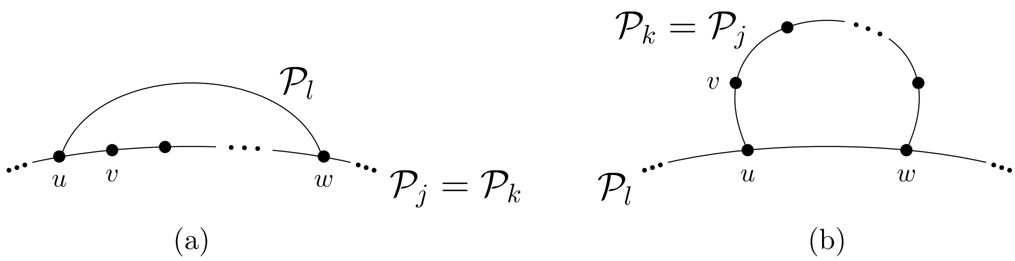

FIGURE 14.

We conclude that, for (29) to induce a partition of the edge set of G' it suffices to remove appropriately from (29) the paths that consist of a single repeated edge coming from paths as described above. We note that, in this situation, the number of even ears and pending edges, after removing all repeated edges in (29), coincides with the number of even ears and pending edges of the weak nested decomposition of $G^\flat$ given by (28).

Let us now prove that, after the exclusion from (29) of the repeated edges, we obtain a weak nested ear decomposition of G'. Since $\mathcal{I}$ is an ear of G, it is clear that the end-vertices of $\mathcal{P}'_k$ belong to $\mathcal{P}'_l$, for some $l < k$. An inner vertex of $\mathcal{P}'_k$ always comes from an inner vertex of $\mathcal{P}_k$. If, following the bipartite ear modification, such vertex is identified with a vertex of a previous $\mathcal{P}'_l$, then the two vertices in question must v and w. We deduce that $\mathcal{P}_k = \mathcal{P}_j$. If w is also an inner vertex of $\mathcal{P}_j$ then $v = w$ in $\mathcal{P}'_j$ cannot belong to any earlier $\mathcal{P}'_l$. If w is an end-vertex of $\mathcal{P}_j$ then $v = w$ becomes an end-vertex of $\mathcal{P}'_j$. As for the nesting condition, let $\mathcal{P}'_k$ and $\mathcal{P}'_l$ be two paths nested in $\mathcal{P}'_s$, with $s < k, l$. We may assume that none of $\mathcal{P}'_k$ or $\mathcal{P}'_l$ is a pending edge or cycle for otherwise there will be nothing to show. Let $v_1 \neq v_2$ be the end-vertices of $\mathcal{P}'_k$ and $w_1 \neq w_2$ those of $\mathcal{P}'_l$. If the nest intervals of $\mathcal{P}'_k$ and $\mathcal{P}'_l$ in $\mathcal{P}'_s$ have inner vertices in common and are not nested, then, in the order of vertices of $\mathcal{P}'_s$, we must have, without loss of generality,

$$(\dots, v_1, \dots, w_1, \dots, v_2, \dots, w_2, \dots).$$

This is impossible before the identification of v with w, since we are starting from a weak nested ear decomposition of $G^\flat$. Hence both v and w belong to $\mathcal{P}_s$ and one of w_1 or v_2 must be the vertex obtained by identifying v with w. Assume, without loss of generality, that this vertex is w_1. Then v_1, v_2 come from $\mathcal{P}_s$ unchanged and one of them is an inner vertex of the subpath induced by v and w in $\mathcal{P}_s$, but this is impossible since, by assumption, v and w induce on $\mathcal{P}_s$ a subpath whose inner vertices have degree two in G.

Having established that, after removing all redundant edges, (29) induces a weak nested ear decomposition of G' we can now use induction to compute its regularity. Accordingly,

$$\operatorname{reg} G' = \tfrac{|V_{G'}|+(\epsilon-1)-3}{2}(q-2) = \operatorname{reg} G^\flat + \tfrac{|V_{G'}|-|V_{G^\flat}|}{2}(q-2),$$

and therefore, $G^\flat$ satisfies the bipartite ear modification assumption along $\mathcal{I}$. This finishes the proof of the theorem. ■

The number of even length paths in an ear decomposition of a graph is not necessarily constant, even if we restrict to bipartite graphs. Take, for example, the graph obtained from the graph in Figure 9 by adding the edge $\{2, 8\}$. As a first ear decomposition, consider the one obtained by starting from vertex 1 and adding consecutively the paths $(1, 2, 3, 7, 8, 5, 1)$, $(1, 4, 3)$, $(5, 6, 7)$ and $(2, 8)$. This decomposition has three even length ears. Alternatively, consider the ear decomposition starting from vertex 1, using first the

Hamiltonian cycle: $(1, 2, 8, 5, 6, 7, 3, 4, 1)$, followed by $(1, 5)$, $(7, 8)$ and $(2, 3)$. This decomposition has only one even length ear.

The following purely combinatorial result follows easily from Theorem 4.4.

Corollary 4.5. *The number of even ears and pendant edges of any weak nested ear decomposition of a bipartite graph remains constant.*

We will finish by giving another application of Theorem 4.4. As mentioned in Section 2, it follows from Proposition 2.6 that

$$\operatorname{reg} G \geq (\alpha(G) - 1)(q - 2),$$

where $\alpha(G)$ denotes the independence number of G. This bound is not sharp if G is not bipartite or 2-connected, but equality does hold if G is an even cycle, or a complete bipartite graph, or a bipartite parallel composition of paths (see [11]), among many other examples. This could suggest that for a bipartite 2-connected graph the Castelnuovo–Mumford regularity of a graph is closely related with $\alpha(G)$. In the following example we want to show that this is not the case.

Example 4.6. Fix an even positive integer k. Consider the graph G, in Figure 15, below, obtained from a cycle of length $3k$, by attaching k ears of length two at the pairs of vertices $3i - 2$ and $3i$, for each $i = 1, \ldots, k$. This graph is endowed with a nested ear decomposition with $k + 1$ even ears.

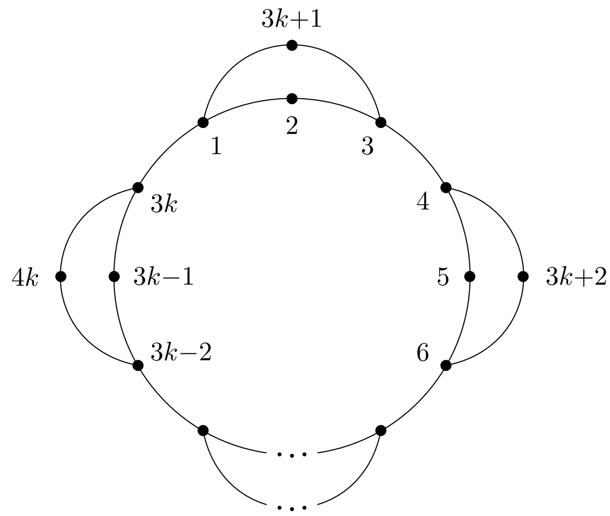

FIGURE 15.

According to Theorem 4.4,

$$\operatorname{reg} G = \tfrac{4k+k+1-3}{2}(q-2) = (\tfrac{5k}{2}-1)(q-2).$$

On the other hand, as any set of independent vertices must have at most two elements in the k cycles of length 4 created by the addition of the k ears and the vertex sets of these cycles cover V_G, we deduce that $\alpha(G)$ is $2k$. In conclusion, this example shows that, indeed,

$$\operatorname{reg} G - (\alpha(G)-1)(q-2) = \tfrac{k}{2}(q-2)$$

can be arbitrarily large.

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