Pré-Publicações do Departamento de Matemática Universidade de Coimbra Preprint Number 18–29

TOPOLOGICAL GROUPS HAVE REPRESENTABLE ACTIONS

FRANCESCA CAGLIARI AND MARIA MANUEL CLEMENTINO

ABSTRACT: This paper shows that the group of auto-homeomorphisms of a topological group can be endowed with a topology so that the resulting topological group plays, for topological groups, the role of the group of automorphisms of a group: it represents the internal actions on the given topological group.

KEYWORDS: topological group, representability of actions, split extension classifier, group of homeomorphisms, Stone-Čech compactification. MATH. SUBJECT CLASSIFICATION (2010): 57S05, 54H11, 22A05, 18B30.

1. Introduction

It is well known that the category **Grp** of groups is *action representative*, with internal actions represented by the group Aut(X) of automorphisms of X. This means that the functor

$$\operatorname{Act}(-, X) : \operatorname{\mathbf{Grp}} \to \operatorname{\mathbf{Set}},$$

assigning to each group Y the set of internal actions of Y on X, is represented by the group $\operatorname{Aut}(X)$. As observed in [2], representability of this functor is equivalent to the existence of *split extension classifier* for the group X, meaning that, for each split extension with kernel X

$$0 \longrightarrow X \longrightarrow A \xrightarrow[p]{s} B \longrightarrow 0$$

there exists a unique homomorphism $\varphi: B \to \operatorname{Aut}(X)$ making the following diagram commute

$$\begin{array}{ccc} X & \stackrel{k}{\longrightarrow} A & \stackrel{s}{\longleftarrow} B \\ \| & \varphi_1 & & & \downarrow \varphi \\ X & \longrightarrow \operatorname{Hol}(X) & \stackrel{s}{\longleftarrow} \operatorname{Aut}(X), \end{array}$$

Received August 31, 2018.

Research partially supported by Centro de Matemática da Universidade de Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

where Hol(X) is the semidirect product of X and Aut(X), with respect to the evaluation action (i.e. the classical holomorph of the group X).

In [1] it was investigated whether the category **TopGrp** of topological groups has the same property. As shown there, this ends up on investigating whether, for a given topological group X, the set $\underline{Aut}(X)$ of autohomeomorphisms of X, as a subspace of the pseudotopological space $X^X \times X^X$, is topological. When X is quasi-locally compact, so that X is exponentiable in **Top**, or, equivalently, the pseudotopological space Y^X is topological for every topological space Y, $\underline{Aut}(X)$ is surely topological, since $X^X \times X^X$ is.

In this paper we show that, in fact, $\underline{\operatorname{Aut}}(X)$ is topological for every topological group X, concluding that the category **TopGrp** has representable actions. In order to do that, in Section 2 we revisit the results used in [1], in Section 3 we reduce the problem to the study of the pseudotopology on the subspace $\operatorname{Iso}(X)$ of X^X consisting of homeomorphisms of a Tychonoff space X, and finally in Section 4 we prove our key result:

Theorem. If X is a Tychonoff space, then Iso(X) is a subspace of $Iso(\beta X^{\beta X})$. In particular, it is a topological space.

2. $\underline{\operatorname{Aut}}(X)$ as a subspace of $X^X \times X^X$

We start by recalling a key result published (without proof) in [3]:

Theorem. If \mathbf{C} is a finitely complete Cartesian closed category, then the category of internal groups in \mathbf{C} is action representative.

The proof of this Theorem presented in [1] shows how to build the internal group that represents the functor Act(-, X), out of the exponential X^X . Here we just detail the construction described in [1] for internal groups in the category **Top** of topological spaces. Since **Top** is not Cartesian closed, one embeds **Top** in the (complete and) Cartesian closed category **PsTop** of pseudotopological spaces, so that for every pseudotopological space X there is an adjunction:

$$\mathbf{PsTop} \underbrace{\overset{()\times X}{\longleftarrow}}_{()^X} \mathbf{PsTop}$$

The category **Top** is in fact a bireflective subcategory of **PsTop**, meaning that the unit of the adjunction is pointwise both a mono and an epimorphism. (For more information on **PsTop** see [4].)

Given a topological group X, one first considers the pseudotopological space

$$X^X = \{ f : X \to X \mid f \text{ is continuous} \},\$$

which is in fact an internal monoid with respect to the composition, then its subspace – and submonoid –

 $\operatorname{Hom}(X, X) = \{f : X \to X \mid f \text{ is a continuous homomorphism}\},\$

and then one obtains $\underline{Aut}(X)$ as the pullback

$$\underbrace{\operatorname{Aut}(X)}_{\downarrow} \xrightarrow{1}_{\downarrow < 1_X, 1_X >} \operatorname{Hom}(X, X) \times \operatorname{Hom}(X, X) \xrightarrow{\mu \times \mu^{\operatorname{op}}} \operatorname{Hom}(X, X) \times \operatorname{Hom}(X, X)$$

(where μ is the composition and μ^{op} the reverse composition); that is, we identify the internal group

 $\underline{\operatorname{Aut}}(X) = \{f : X \to X \mid f \text{ is an auto-homeomorphism}\}\$

with the subspace $\{(f, f^{-1}) | f : X \to X \text{ is an auto-homeomorphism}\}$ of $X^X \times X^X$. We point out that, when X is compact and Hausdorff, the topology in <u>Aut(X)</u> is not the compact-open topology but the product of the compact-open topology in each factor.

In order to show that the pseudotopology of $\underline{\operatorname{Aut}}(X)$ is in fact a topology for every topological group X, in the next sections we will show that the subspace

 $Iso(X) = \{ f \in X^X \mid f \text{ is a homeomorphism} \}$

of X^X is a topological space. Then we may conclude that $\underline{\operatorname{Aut}}(X)$, as a subspace of $\operatorname{Iso}(X) \times \operatorname{Iso}(X)$, is also a topological space.

3. $\underline{\operatorname{Aut}}(X)$ in PsTop

Throughout this section X is a topological space.

Given a directed set Λ , we denote by Λ_{∞} the set $\Lambda \cup \{\infty\}$ equipped with the topology

 $\{A \subseteq \Lambda_{\infty} \mid \infty \notin A \text{ or there exists } \lambda \in \Lambda \text{ such that } \uparrow \lambda \subseteq A\}.$

Lemma. Given a net $(f_{\lambda})_{\lambda \in \Lambda}$ and f in X^X , (f_{λ}) converges to f if, and only if, the map $F : \Lambda_{\infty} \times X \to X$, defined by $F(\lambda, x) = f_{\lambda}(x)$ and $F(\infty, x) = f(x)$, is continuous. *Proof*: Continuity of the map F is equivalent to continuity of the map $\tilde{F} : \Lambda_{\infty} \to X^X$, with $\tilde{F}(\lambda) = f_{\lambda}$ and $\tilde{F}(\infty) = f$, and this is clearly equivalent to the convergence of (f_{λ}) to f. (Note that in general \tilde{F} is a morphism in **PsTop**, not in **Top**.)

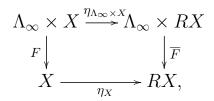
From now on we denote by $\eta = (\eta_X : X \to RX)_{X \in \mathbf{Top}}$ the unit of the adjunction

$$\operatorname{Top}_0 \xrightarrow{R} \operatorname{Top}$$

that is η_X is the T0-reflection of X. Note that T0-reflections are both initial and final maps (see [5]).

Theorem. If Iso(RX) is a topological space, then so is Iso(X).

Proof: Since **Top** is bireflective in **PsTop**, it is closed under initial continuous maps. We will show now that the map $\varphi : \operatorname{Iso}(X) \to \operatorname{Iso}(RX)$, assigning to each element f of $\operatorname{Iso}(X)$ its T0-reflection Rf, is both continuous and initial. Let $(f_{\lambda})_{\lambda \in \Lambda}$ and f belong to $\operatorname{Iso}(X)$. Observing that the T0-reflection of $\Lambda_{\infty} \times X$ is $\Lambda_{\infty} \times RX$, we have the following commutative diagram



where $\overline{F}(\lambda, y) = Rf_{\lambda}(y)$ and $\overline{F}(\infty, y) = Rf(y)$. Then φ is continuous and initial if, and only if, continuity of F is equivalent to continuity of \overline{F} . If (f_{λ}) converges to f in Iso(X), so that F is continuous, then $\overline{F} \cdot \eta_{\Lambda_{\infty} \times X}$ is continuous, from which it follows that \overline{F} is continuous, due to finality of η . Conversely, if (Rf_{λ}) converges to Rf in Iso(RX), that is, \overline{F} is continuous, then $\eta_X \cdot F$ is continuous, and initiality of η_X gives the continuity of F.

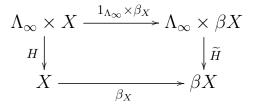
4. Iso(X) is topological

Throughout this section X is a Tychonoff space, so that we consider its embedding $\beta_X : X \to \beta X$ into its Stone-Čech compactification.

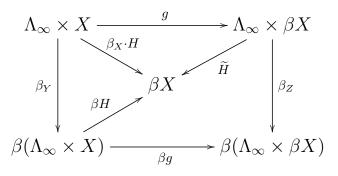
Theorem. If X is a Tychonoff space, then Iso(X) is a subspace of $Iso(\beta X)$. In particular, it is a topological space. *Proof*: First we point out that, since βX is compact and Hausdorff, $\beta X^{\beta X}$ is a topological space, endowed with the compact-open topology. Therefore its subspace Iso (βX) is also topological.

Consider the map $\widetilde{\beta}_X : \operatorname{Iso}(X) \to \operatorname{Iso}(\beta X)$ which assigns to each homeomorphism $h: X \to X$ the homeomorphism $\beta(h)$. Then $\widetilde{\beta}_X$ is injective since β_X is. To conclude that it is an embedding we need to show that, for $(h_\lambda)_{\lambda \in \Lambda}$ and h in $\operatorname{Iso}(X)$, (h_λ) converges to h if, and only if, $(\widetilde{\beta}_X(h_\lambda))$ converges to $\widetilde{\beta}_X(h)$. As we have already observed, (h_λ) converges to h if, and only if, the corresponding map $H : \Lambda_\infty \times X \to X$ is continuous, while $(\widetilde{\beta}_X(h_\lambda))$ converges to $\widetilde{\beta}_X(h)$ exactly when the corresponding map $\widetilde{H} : \Lambda_\infty \times \beta X \to \beta X$ is continuous.

Consider the following commutative diagram



If \widetilde{H} is continuous, then $\beta_X \cdot H$ is continuous, and so is H since β_X is an embedding. The converse implication is the non-trivial one. In order to prove it, assume that H is continuous and consider the following commutative diagram, where $g := 1_X \times \beta_X$, $Y = \Lambda_\infty \times X$ and $Z = \Lambda_\infty \times \beta X$:

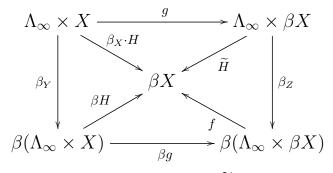


The map βg is dense, since both β_Z and g are. As a continuous map between compact Hausdorff spaces, it is thus surjective, and, moreover, a quotient. Below we will show that:

$$\forall \mathfrak{y}, \mathfrak{y}' \in \beta(\Lambda_{\infty} \times X) \ \beta g(\mathfrak{y}) = \beta g(\mathfrak{y}') \implies \beta H(\mathfrak{y}) = \beta H(\mathfrak{y}'). \quad (\diamondsuit)$$

From (\Diamond) it follows that βH factors through βg , via a map $f : \beta(\Lambda_{\infty} \times \beta X) \to \beta X$. Since βg is a quotient and $f \cdot \beta g = \beta H$ is continuous, f is in fact

continuous.



Now, from $(f \cdot \beta_Z) \cdot g = \beta H \cdot \beta_Y = \beta_X \cdot H = \widetilde{H} \cdot g$ and since, by construction, \widetilde{H} is the unique map such that $\widetilde{H} \cdot g = \beta_X \cdot H$, it follows that $\widetilde{H} = f \cdot \beta_Z$, and then it is continuous as claimed.

Therefore to finish our proof we only need to show (\diamond) .

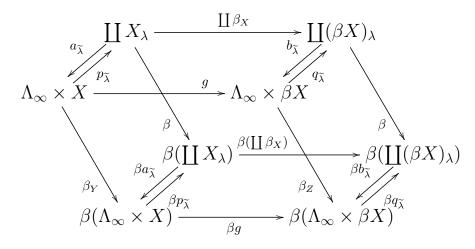
First we prove it for $\mathfrak{y}, \mathfrak{y}' \in \beta(\coprod_{\widetilde{\lambda}} X_{\lambda}) \subseteq \beta(\Lambda_{\infty} \times X)$, where $\coprod_{\widetilde{\lambda}} X_{\lambda}$ is the coproduct in **Top** of $(X_{\lambda})_{\lambda \leq \widetilde{\lambda}}$, with $\widetilde{\lambda} \in \Lambda$ and $X_{\lambda} = X$ for every $\lambda \leq \widetilde{\lambda}$. Note that the maps

$$\coprod_{\widetilde{\lambda}} X_{\lambda} \xrightarrow[p_{\widetilde{\lambda}}]{a_{\widetilde{\lambda}}} \Lambda_{\infty} \times X,$$

with $a_{\tilde{\lambda}}(x) = (\lambda, x)$ for $x \in X_{\lambda}$, and $p_{\tilde{\lambda}}(\mu, x) = x \in X_{\mu}$ if $\mu \leq \tilde{\lambda}$ and $p_{\tilde{\lambda}}(\mu, x) = x \in X_{\tilde{\lambda}}$ otherwise, are continuous, and $p_{\tilde{\lambda}} \cdot a_{\tilde{\lambda}} = 1$. Analogously we define

$$\coprod_{\widetilde{\lambda}} (\beta X)_{\lambda} \xrightarrow[q_{\widetilde{\lambda}}]{b_{\widetilde{\lambda}}} \Lambda_{\infty} \times \beta X.$$

Observing that the following diagram commutes and $\beta(\coprod \beta_X)$ is an isomorphism because the functor β is a left adjoint

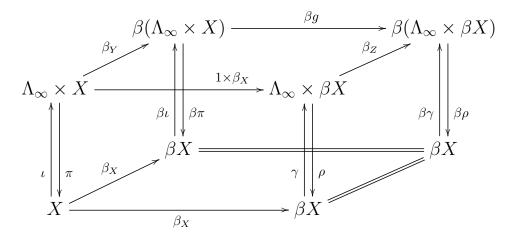


we conclude that $\beta g \cdot \beta a_{\tilde{\lambda}} = \beta b_{\tilde{\lambda}} \cdot \beta(\prod \beta_X)$ is an embedding. This leads to the conclusion that βg is injective when restricted to the image of the closure of $\prod_{\tilde{\lambda}} X_{\lambda}$: $\beta(\overline{\prod_{\tilde{\lambda}} X_{\lambda}}) \subseteq \overline{\beta(\prod_{\tilde{\lambda}} X_{\lambda})} = \beta(\prod_{\tilde{\lambda}} X_{\lambda})$. Now, for each $\tilde{\mu} \geq \tilde{\lambda}$, we have a section $\iota_{\tilde{\lambda},\tilde{\mu}}$, with retraction $\pi_{\tilde{\lambda},\tilde{\mu}}$,

$$\coprod_{\tilde{\lambda}} X_{\lambda} \xrightarrow{\iota_{\tilde{\lambda},\tilde{\mu}}} \coprod_{\tilde{\mu}} X_{\lambda}$$

where $\iota_{\tilde{\lambda},\tilde{\mu}}(\lambda,x) = (\lambda,x)$, and $\pi_{\tilde{\lambda},\tilde{\mu}}(\lambda,x) = (\lambda,x)$ if $\lambda \leq \tilde{\lambda}$ and $\pi_{\tilde{\lambda},\tilde{\mu}}(\lambda,x) = (\tilde{\lambda},x)$ elsewhere. Therefore, since Λ is directed, this allows us to conclude that βg is injective when restricted to $\bigcup_{\tilde{\lambda}} \beta(\coprod_{\tilde{\lambda}} X_{\lambda}) \hookrightarrow \beta(\Lambda_{\infty} \times X)$.

For the remaining proof it is useful to consider the diagram



where π and ρ are projections, $\iota(x) = (\infty, x)$, and $\gamma(\mathfrak{x}) = (\infty, \mathfrak{x})$. (For simplicity we will also denote by (∞, \mathfrak{x}) the image of $\mathfrak{x} \in \beta X$ under $\beta \gamma$, i.e. we will identify (∞, \mathfrak{x}) with $\beta_Z(\infty, \mathfrak{x})$.)

Let $\mathfrak{y} \in \widetilde{Y} := \beta(\Lambda_{\infty} \times X) \setminus \bigcup_{\lambda \in \Lambda} \beta(\coprod_{\lambda} X_{\lambda})$. Then \mathfrak{y} must be the limit point of a net $(\mathfrak{y}_{\mu} = \beta_{Y}(\lambda_{\mu}, x_{\mu}))_{\mu \in M}$, in the image of $\Lambda \times X$ via β_{Y} , cofinal with Λ , since $\Lambda \times X \to \Lambda_{\infty} \times X \to \beta(\Lambda_{\infty} \times X)$ is dense. The net $(\beta g(\mathfrak{y}_{\mu}))_{\mu}$ converges to $\beta g(\mathfrak{y})$, and its image under $\beta \rho$ converges to $\mathfrak{x} := \beta \rho(\beta g(\mathfrak{y})) = \beta \pi(\mathfrak{y})$. Any neighbourhood of $\beta \gamma(\mathfrak{x}) = (\infty, \mathfrak{x})$ contains a queue of $(\beta g(\mathfrak{y}_{\mu}))_{\mu}$, since this net is cofinal with Λ . We may choose a neighbourhood U of $\beta \gamma(\mathfrak{x})$ so that $U \cap (\Lambda_{\infty} \times \beta X) = V \times A$, with $V = \{\lambda; \lambda \geq \overline{\lambda}\} \cup \{\infty\}$, and A an open subset of βX containing \mathfrak{x} . Hence in A there is a queue of $(\beta \rho(\beta g(\mathfrak{y}_{\mu})))_{\mu}$ and then in U there is a queue of $(\beta g(\mathfrak{y}_{\mu}))_{\mu}$, which implies that $\beta g(\mathfrak{y}) = (\infty, \mathfrak{x})$. We may then conclude that any $\mathfrak{y}' \in Y$ with $\beta g(\mathfrak{y}) = \beta g(\mathfrak{y}')$ belongs necessarily to \widetilde{Y} .

To complete the proof we consider the continuous map $K = \langle 1, H \rangle : \Lambda_{\infty} \times X \to \Lambda_{\infty} \times X$ defined by $K(\lambda, x) = (\lambda, h_{\lambda}(x))$ and $K(\infty, x) = (\infty, h(x))$, so that $H = \pi \cdot K$. Note that in the diagram below βg is the (unique) continuous extension of the top composite, since the vertical maps are dense

(here we consider (co)restrictions of K and β_X although we use the same notations).

Our next goal is to show that

$$\beta g(\beta K(\mathfrak{y})) = \beta \gamma(\beta h(\beta \pi(\mathfrak{y}))). \qquad (\nabla)$$

By definition of K it is clear that $\beta K(\mathfrak{y}) \in \widetilde{Y}$. Therefore $\beta g(\beta K(\mathfrak{y})) = (\infty, \beta \pi(\beta K(\mathfrak{y})))$. To show that $(\infty, \beta \pi(\beta K(\mathfrak{y}))) = (\infty, \beta h(\beta \pi(\mathfrak{y})))$ – that is, (∇) – we use the following diagram

Indeed, applying $(\beta h)^{-1} \cdot \beta \rho$ to the latter, one gets $\beta \pi(\mathfrak{y})$; but the image of $(\beta h)^{-1}(\beta \rho(\beta \pi(\beta K(\mathfrak{y}))))$ must be also $\beta \pi(\mathfrak{y})$, since $\beta \pi(\mathfrak{y}) = \beta \rho(\beta g(\mathfrak{y})) = (\beta h)^{-1}(\beta \rho(\beta g(\beta K(\mathfrak{y}))))$, and so (∇) holds.

Now we are able to conclude (\Diamond) for $\mathfrak{y}, \mathfrak{y}' \in \widetilde{Y}$. From $\beta g(\mathfrak{y}) = \beta g(\mathfrak{y}')$ it follows that $\beta \pi(\mathfrak{y}) = \beta \pi(\mathfrak{y}')$, and then, by equality $(\nabla), \beta g(\beta K(\mathfrak{y})) = \beta g(\beta K(\mathfrak{y}'))$, with $\beta K(\mathfrak{y})$ and $\beta K(\mathfrak{y}')$ in \widetilde{Y} . Therefore also $\beta \pi(\beta K(\mathfrak{y})) = \beta \pi(\beta K(\mathfrak{y}'))$, which means exactly that $\beta H(\mathfrak{y}) = \beta H(\mathfrak{y}')$, and this ends the proof.

References

- F. Borceux, M.M. Clementino, A. Montoli, On the representability of actions for topological algebras. In: *Categorical Methods in Algebra and Topology*, Textos de Matemática, DMUC, Vol. 46 (2014) 41–66.
- [2] F. Borceux, G. Janelidze, G.M. Kelly, On the representability of actions in a semi-abelian category. *Theory Appl. Categ.* 14 (2005), 244–286.
- [3] F. Borceux, G. Janelidze, G.M. Kelly, Internal object actions. Comment. Math. Univ. Carolin. 46 (2005), no.2, 235–255.
- [4] H. Herrlich, E. Lowen-Colebunders, F. Schwarz, Improving Top: PrTop and PsTop. In: Category theory at work (Bremen, 1990), Res. Exp. Math. 18 (1991), 21–34, Heldermann, Berlin, 1991.
- [5] O. Wyler, Injective spaces and essential extensions in TOP. General Topology and Appl. 7 (1977), no. 3, 247–249.

FRANCESCA CAGLIARI

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI BOLOGNA, PIAZZA PORTA SAN DO-NATO, 5, 40127 BOLOGNA, ITALIA

E-mail address: francesca.cagliari@unibo.it

MARIA MANUEL CLEMENTINO

CMUC, DEPARTMENT OF MATHEMATICS, UNIV. COIMBRA, 3001-501 COIMBRA, PORTUGAL *E-mail address*: mmc@mat.uc.pt