Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–29

# TOPOLOGICAL GROUPS HAVE REPRESENTABLE ACTIONS

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Abstract: This paper shows that the group of auto-homeomorphisms of a topological group can be endowed with a topology so that the resulting topological group plays, for topological groups, the role of the group of automorphisms of a group: it represents the internal actions on the given topological group.

Keywords: topological group, representability of actions, split extension classifier, group of homeomorphisms, Stone-Čech compactification.
Math. Subject Classification (2010): 57S05, 54H11, 22A05, 18B30.

## 1. Introduction

It is well known that the category **Grp** of groups is *action representative*, with internal actions represented by the group $\mathrm{Aut}(X)$ of automorphisms of $X$. This means that the functor

$$\mathrm{Act}(-,X) : \mathbf{Grp} \to \mathbf{Set},$$

assigning to each group $Y$ the set of internal actions of $Y$ on $X$, is represented by the group $\mathrm{Aut}(X)$. As observed in [2], representability of this functor is equivalent to the existence of *split extension classifier* for the group $X$, meaning that, for each split extension with kernel $X$

$$0 \longrightarrow X \longrightarrow A \underset{p}{\overset{s}{\rightleftarrows}} B \longrightarrow 0$$

there exists a unique homomorphism $\varphi : B \to \mathrm{Aut}(X)$ making the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift right, "p"'] \arrow[d, "\varphi_1"'] & B \arrow[l, shift right, "s"'] \arrow[d, "\varphi"] \\
X \arrow[r] & \mathrm{Hol}(X) \arrow[r, shift right] & \mathrm{Aut}(X), \arrow[l, shift right]
\end{tikzcd}$$

Received August 31, 2018.
Research partially supported by Centro de Matemática da Universidade de Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

where $\mathrm{Hol}(X)$ is the semidirect product of $X$ and $\mathrm{Aut}(X)$, with respect to the evaluation action (i.e. the classical holomorph of the group $X$).

In [1] it was investigated whether the category **TopGrp** of topological groups has the same property. As shown there, this ends up on investigating whether, for a given topological group $X$, the set $\underline{\mathrm{Aut}}(X)$ of autohomeomorphisms of $X$, as a subspace of the pseudotopological space $X^X \times X^X$, is topological. When $X$ is quasi-locally compact, so that $X$ is exponentiable in **Top**, or, equivalently, the pseudotopological space $Y^X$ is topological for every topological space $Y$, $\underline{\mathrm{Aut}}(X)$ is surely topological, since $X^X \times X^X$ is.

In this paper we show that, in fact, $\underline{\mathrm{Aut}}(X)$ is topological for every topological group $X$, concluding that the category **TopGrp** has representable actions. In order to do that, in Section 2 we revisit the results used in [1], in Section 3 we reduce the problem to the study of the pseudotopology on the subspace $\mathrm{Iso}(X)$ of $X^X$ consisting of homeomorphisms of a Tychonoff space $X$, and finally in Section 4 we prove our key result:

**Theorem.** *If $X$ is a Tychonoff space, then $\mathrm{Iso}(X)$ is a subspace of $\mathrm{Iso}(\beta X^{\beta X})$. In particular, it is a topological space.*

## 2. $\underline{\mathrm{Aut}}(X)$ as a subspace of $X^X \times X^X$

We start by recalling a key result published (without proof) in [3]:

**Theorem.** *If $\mathbf{C}$ is a finitely complete Cartesian closed category, then the category of internal groups in $\mathbf{C}$ is action representative.*

The proof of this Theorem presented in [1] shows how to build the internal group that represents the functor $\mathrm{Act}(-, X)$, out of the exponential $X^X$. Here we just detail the construction described in [1] for internal groups in the category **Top** of topological spaces. Since **Top** is not Cartesian closed, one embeds **Top** in the (complete and) Cartesian closed category **PsTop** of pseudotopological spaces, so that for every pseudotopological space $X$ there is an adjunction:

$$\begin{tikzcd} \mathbf{PsTop} \arrow[r, shift right, "(\ )^X"'] \arrow[r, phantom, "\perp"] & \mathbf{PsTop} \arrow[l, shift right, "(\ )\times X"'] \end{tikzcd}$$

The category **Top** is in fact a bireflective subcategory of **PsTop**, meaning that the unit of the adjunction is pointwise both a mono and an epimorphism. (For more information on **PsTop** see [4].)

Given a topological group $X$, one first considers the pseudotopological space
$$X^X = \{f : X \to X \,|\, f \text{ is continuous}\},$$
which is in fact an internal monoid with respect to the composition, then its subspace – and submonoid –
$$\mathrm{Hom}(X,X) = \{f : X \to X \,|\, f \text{ is a continuous homomorphism}\},$$
and then one obtains $\underline{\mathrm{Aut}}(X)$ as the pullback

$$\begin{tikzcd}
\underline{\mathrm{Aut}}(X) \arrow[rr] \arrow[d] & & 1 \arrow[d, "<1_X{,}1_X>"] \\
\mathrm{Hom}(X,X)\times\mathrm{Hom}(X,X) \arrow[rr, "\mu\times\mu^{\mathrm{op}}"'] & & \mathrm{Hom}(X,X)\times\mathrm{Hom}(X,X)
\end{tikzcd}$$

(where $\mu$ is the composition and $\mu^{\mathrm{op}}$ the reverse composition); that is, we identify the internal group
$$\underline{\mathrm{Aut}}(X) = \{f : X \to X \,|\, f \text{ is an auto-homeomorphism}\}$$
with the subspace $\{(f, f^{-1}) \,|\, f : X \to X \text{ is an auto-homeomorphism}\}$ of $X^X \times X^X$. We point out that, when $X$ is compact and Hausdorff, the topology in $\underline{\mathrm{Aut}}(X)$ is not the compact-open topology but the product of the compact-open topology in each factor.

In order to show that the pseudotopology of $\underline{\mathrm{Aut}}(X)$ is in fact a topology for every topological group $X$, in the next sections we will show that the subspace
$$\mathrm{Iso}(X) = \{f \in X^X \,|\, f \text{ is a homeomorphism}\}$$
of $X^X$ is a topological space. Then we may conclude that $\underline{\mathrm{Aut}}(X)$, as a subspace of $\mathrm{Iso}(X) \times \mathrm{Iso}(X)$, is also a topological space.

## 3. $\underline{\mathrm{Aut}}(X)$ in PsTop

Throughout this section $X$ is a topological space.

Given a directed set $\Lambda$, we denote by $\Lambda_\infty$ the set $\Lambda \,\dot\cup\, \{\infty\}$ equipped with the topology
$$\{A \subseteq \Lambda_\infty \,|\, \infty \notin A \text{ or there exists } \lambda \in \Lambda \text{ such that } \uparrow\lambda \subseteq A\}.$$

**Lemma.** *Given a net $(f_\lambda)_{\lambda\in\Lambda}$ and $f$ in $X^X$, $(f_\lambda)$ converges to $f$ if, and only if, the map $F : \Lambda_\infty \times X \to X$, defined by $F(\lambda, x) = f_\lambda(x)$ and $F(\infty, x) = f(x)$, is continuous.*

*Proof*: Continuity of the map $F$ is equivalent to continuity of the map $\tilde{F} : \Lambda_\infty \to X^X$, with $\tilde{F}(\lambda) = f_\lambda$ and $\tilde{F}(\infty) = f$, and this is clearly equivalent to the convergence of $(f_\lambda)$ to $f$. (Note that in general $\tilde{F}$ is a morphism in **PsTop**, not in **Top**.) ■

From now on we denote by $\eta = (\eta_X : X \to RX)_{X \in \mathbf{Top}}$ the unit of the adjunction

$$\begin{tikzcd} \mathbf{Top}_0 \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, shift right, "R"'] \end{tikzcd}$$

that is $\eta_X$ is the $T0$-reflection of $X$. Note that $T0$-reflections are both initial and final maps (see [5]).

**Theorem.** *If* $\mathrm{Iso}(RX)$ *is a topological space, then so is* $\mathrm{Iso}(X)$*.*

*Proof*: Since **Top** is bireflective in **PsTop**, it is closed under initial continuous maps. We will show now that the map $\varphi : \mathrm{Iso}(X) \to \mathrm{Iso}(RX)$, assigning to each element $f$ of $\mathrm{Iso}(X)$ its $T0$-reflection $Rf$, is both continuous and initial. Let $(f_\lambda)_{\lambda\in\Lambda}$ and $f$ belong to $\mathrm{Iso}(X)$. Observing that the $T0$-reflection of $\Lambda_\infty \times X$ is $\Lambda_\infty \times RX$, we have the following commutative diagram

$$\begin{tikzcd} \Lambda_\infty \times X \arrow[r, "\eta_{\Lambda_\infty \times X}"] \arrow[d, "F"'] & \Lambda_\infty \times RX \arrow[d, "\overline{F}"] \\ X \arrow[r, "\eta_X"'] & RX, \end{tikzcd}$$

where $\overline{F}(\lambda, y) = Rf_\lambda(y)$ and $\overline{F}(\infty, y) = Rf(y)$. Then $\varphi$ is continuous and initial if, and only if, continuity of $F$ is equivalent to continuity of $\overline{F}$. If $(f_\lambda)$ converges to $f$ in $\mathrm{Iso}(X)$, so that $F$ is continuous, then $\overline{F} \cdot \eta_{\Lambda_\infty \times X}$ is continuous, from which it follows that $\overline{F}$ is continuous, due to finality of $\eta$. Conversely, if $(Rf_\lambda)$ converges to $Rf$ in $\mathrm{Iso}(RX)$, that is, $\overline{F}$ is continuous, then $\eta_X \cdot F$ is continuous, and initiality of $\eta_X$ gives the continuity of $F$. ■

## 4. $\mathrm{Iso}(X)$ is topological

Throughout this section $X$ is a Tychonoff space, so that we consider its embedding $\beta_X : X \to \beta X$ into its Stone-Čech compactification.

**Theorem.** *If* $X$ *is a Tychonoff space, then* $\mathrm{Iso}(X)$ *is a subspace of* $\mathrm{Iso}(\beta X)$*. In particular, it is a topological space.*

*Proof*: First we point out that, since $\beta X$ is compact and Hausdorff, $\beta X^{\beta X}$ is a topological space, endowed with the compact-open topology. Therefore its subspace $\mathrm{Iso}(\beta X)$ is also topological.

Consider the map $\widetilde{\beta}_X : \mathrm{Iso}(X) \to \mathrm{Iso}(\beta X)$ which assigns to each homeomorphism $h : X \to X$ the homeomorphism $\beta(h)$. Then $\widetilde{\beta}_X$ is injective since $\beta_X$ is. To conclude that it is an embedding we need to show that, for $(h_\lambda)_{\lambda\in\Lambda}$ and $h$ in $\mathrm{Iso}(X)$, $(h_\lambda)$ converges to $h$ if, and only if, $(\widetilde{\beta}_X(h_\lambda))$ converges to $\widetilde{\beta}_X(h)$. As we have already observed, $(h_\lambda)$ converges to $h$ if, and only if, the corresponding map $H : \Lambda_\infty \times X \to X$ is continuous, while $(\widetilde{\beta}_X(h_\lambda))$ converges to $\widetilde{\beta}_X(h)$ exactly when the corresponding map $\widetilde{H} : \Lambda_\infty \times \beta X \to \beta X$ is continuous.

Consider the following commutative diagram

$$\begin{tikzcd}
\Lambda_\infty \times X \arrow[r, "1_{\Lambda_\infty}\times\beta_X"] \arrow[d, "H"'] & \Lambda_\infty \times \beta X \arrow[d, "\widetilde{H}"] \\
X \arrow[r, "\beta_X"'] & \beta X
\end{tikzcd}$$

If $\widetilde{H}$ is continuous, then $\beta_X \cdot H$ is continuous, and so is $H$ since $\beta_X$ is an embedding. The converse implication is the non-trivial one. In order to prove it, assume that $H$ is continuous and consider the following commutative diagram, where $g := 1_X \times \beta_X$, $Y = \Lambda_\infty \times X$ and $Z = \Lambda_\infty \times \beta X$:

$$\begin{tikzcd}
\Lambda_\infty \times X \arrow[rr, "g"] \arrow[dr, "\beta_X\cdot H"] \arrow[dd, "\beta_Y"'] & & \Lambda_\infty \times \beta X \arrow[dl, "\widetilde{H}"'] \arrow[dd, "\beta_Z"] \\
& \beta X & \\
\beta(\Lambda_\infty \times X) \arrow[rr, "\beta g"'] \arrow[ur, "\beta H"] & & \beta(\Lambda_\infty \times \beta X)
\end{tikzcd}$$

The map $\beta g$ is dense, since both $\beta_Z$ and $g$ are. As a continuous map between compact Hausdorff spaces, it is thus surjective, and, moreover, a quotient. Below we will show that:

$$\forall \mathfrak{y}, \mathfrak{y}' \in \beta(\Lambda_\infty \times X) \;\; \beta g(\mathfrak{y}) = \beta g(\mathfrak{y}') \implies \beta H(\mathfrak{y}) = \beta H(\mathfrak{y}'). \qquad (\Diamond)$$

From $(\Diamond)$ it follows that $\beta H$ factors through $\beta g$, via a map $f : \beta(\Lambda_\infty \times \beta X) \to \beta X$. Since $\beta g$ is a quotient and $f \cdot \beta g = \beta H$ is continuous, $f$ is in fact continuous.

$$\begin{tikzcd}
\Lambda_\infty \times X \arrow[rr, "g"] \arrow[dr, "\beta_X\cdot H"] \arrow[dd, "\beta_Y"'] & & \Lambda_\infty \times \beta X \arrow[dl, "\widetilde{H}"] \arrow[dd, "\beta_Z"] \\
& \beta X & \\
\beta(\Lambda_\infty \times X) \arrow[ur, "\beta H"] \arrow[rr, "\beta g"'] & & \beta(\Lambda_\infty \times \beta X) \arrow[ul, "f"']
\end{tikzcd}$$

Now, from $(f\cdot\beta_Z)\cdot g=\beta H\cdot\beta_Y=\beta_X\cdot H=\widetilde{H}\cdot g$ and since, by construction, $\widetilde{H}$ is the unique map such that $\widetilde{H}\cdot g=\beta_X\cdot H$, it follows that $\widetilde{H}=f\cdot\beta_Z$, and then it is continuous as claimed.

Therefore to finish our proof we only need to show ($\Diamond$).

*First we prove it for* $\mathfrak{y},\mathfrak{y}'\in\beta(\coprod_{\widetilde{\lambda}}X_\lambda)\subseteq\beta(\Lambda_\infty\times X)$, where $\coprod_{\widetilde{\lambda}}X_\lambda$ is the coproduct in **Top** of $(X_\lambda)_{\lambda\leq\widetilde{\lambda}}$, with $\widetilde{\lambda}\in\Lambda$ and $X_\lambda=X$ for every $\lambda\leq\widetilde{\lambda}$. Note that the maps

$$\textstyle\coprod_{\widetilde{\lambda}}X_\lambda \underset{p_{\widetilde{\lambda}}}{\overset{a_{\widetilde{\lambda}}}{\rightleftarrows}} \Lambda_\infty\times X,$$

with $a_{\widetilde{\lambda}}(x)=(\lambda,x)$ for $x\in X_\lambda$, and $p_{\widetilde{\lambda}}(\mu,x)=x\in X_\mu$ if $\mu\leq\widetilde{\lambda}$ and $p_{\widetilde{\lambda}}(\mu,x)=x\in X_{\widetilde{\lambda}}$ otherwise, are continuous, and $p_{\widetilde{\lambda}}\cdot a_{\widetilde{\lambda}}=1$. Analogously we define

$$\textstyle\coprod_{\widetilde{\lambda}}(\beta X)_\lambda \underset{q_{\widetilde{\lambda}}}{\overset{b_{\widetilde{\lambda}}}{\rightleftarrows}} \Lambda_\infty\times \beta X.$$

Observing that the following diagram commutes and $\beta(\coprod\beta_X)$ is an isomorphism because the functor $\beta$ is a left adjoint

$$\begin{tikzcd}
& \coprod X_\lambda \arrow[rr, "\coprod\beta_X"] \arrow[dl, shift right, "a_{\widetilde{\lambda}}"'] \arrow[dd, "\beta"] & & \coprod(\beta X)_\lambda \arrow[dl, shift right, "b_{\widetilde{\lambda}}"'] \arrow[dd, "\beta"] \\
\Lambda_\infty\times X \arrow[ur, shift right, "p_{\widetilde{\lambda}}"'] \arrow[rr, "g"] \arrow[dd, "\beta_Y"'] & & \Lambda_\infty\times\beta X \arrow[ur, shift right, "q_{\widetilde{\lambda}}"'] \arrow[dd, "\beta_Z"] & \\
& \beta(\coprod X_\lambda) \arrow[rr, "\beta(\coprod\beta_X)"] \arrow[dl, shift right, "\beta a_{\widetilde{\lambda}}"'] & & \beta(\coprod(\beta X)_\lambda) \arrow[dl, shift right, "\beta b_{\widetilde{\lambda}}"'] \\
\beta(\Lambda_\infty\times X) \arrow[ur, shift right, "\beta p_{\widetilde{\lambda}}"'] \arrow[rr, "\beta g"'] & & \beta(\Lambda_\infty\times\beta X) \arrow[ur, shift right, "\beta q_{\widetilde{\lambda}}"'] &
\end{tikzcd}$$

we conclude that $\beta g\cdot\beta a_{\tilde\lambda}=\beta b_{\tilde\lambda}\cdot\beta(\coprod\beta_X)$ is an embedding. This leads to the conclusion that $\beta g$ is injective when restricted to the image of the closure of $\coprod_{\tilde\lambda}X_\lambda$: $\beta(\overline{\coprod_{\tilde\lambda}X_\lambda})\subseteq\overline{\beta(\coprod_{\tilde\lambda}X_\lambda)}=\beta(\coprod_{\tilde\lambda}X_\lambda)$. Now, for each $\tilde\mu\geq\tilde\lambda$, we have a section $\iota_{\tilde\lambda,\tilde\mu}$, with retraction $\pi_{\tilde\lambda,\tilde\mu}$,

$$\begin{tikzcd} \coprod_{\tilde\lambda}X_\lambda \arrow[r, shift left, "\iota_{\tilde\lambda,\tilde\mu}"] & \coprod_{\tilde\mu}X_\lambda \arrow[l, shift left, "\pi_{\tilde\lambda,\tilde\mu}"] \end{tikzcd}$$

where $\iota_{\tilde\lambda,\tilde\mu}(\lambda,x)=(\lambda,x)$, and $\pi_{\tilde\lambda,\tilde\mu}(\lambda,x)=(\lambda,x)$ if $\lambda\leq\tilde\lambda$ and $\pi_{\tilde\lambda,\tilde\mu}(\lambda,x)=(\tilde\lambda,x)$ elsewhere. Therefore, since $\Lambda$ is directed, this allows us to conclude that *$\beta g$ is injective when restricted to* $\bigcup\limits_{\tilde\lambda\in\Lambda}\beta(\coprod_{\tilde\lambda}X_\lambda)\hookrightarrow\beta(\Lambda_\infty\times X)$.

For the remaining proof it is useful to consider the diagram

$$\begin{tikzcd}
 & & \beta(\Lambda_\infty\times X) \arrow[rrrr, "\beta g"] \arrow[dd, shift left, "\beta\pi"] & & & & \beta(\Lambda_\infty\times\beta X) \arrow[dd, shift left, "\beta\rho"] \\
\Lambda_\infty\times X \arrow[urr, "\beta_Y"] \arrow[rrrr, "1\times\beta_X"] \arrow[dd, shift left, "\pi"] & & & & \Lambda_\infty\times\beta X \arrow[urr, "\beta_Z"] \arrow[dd, shift left, "\rho"] & & \\
 & & \beta X \arrow[uu, shift left, "\beta\iota"] \arrow[rrrr, equal] & & & & \beta X \arrow[uu, shift left, "\beta\gamma"] \\
X \arrow[uu, shift left, "\iota"] \arrow[urr, "\beta_X"] \arrow[rrrr, "\beta_X"'] & & & & \beta X \arrow[uu, shift left, "\gamma"] \arrow[urr, equal] & &
\end{tikzcd}$$

where $\pi$ and $\rho$ are projections, $\iota(x)=(\infty,x)$, and $\gamma(\mathfrak{x})=(\infty,\mathfrak{x})$. (For simplicity we will also denote by $(\infty,\mathfrak{x})$ the image of $\mathfrak{x}\in\beta X$ under $\beta\gamma$, i.e. we will identify $(\infty,\mathfrak{x})$ with $\beta_Z(\infty,\mathfrak{x})$.)

Let $\mathfrak{y}\in\widetilde{Y}:=\beta(\Lambda_\infty\times X)\setminus\bigcup_{\tilde\lambda\in\Lambda}\beta(\coprod_{\tilde\lambda}X_\lambda)$. Then $\mathfrak{y}$ must be the limit point of a net $(\mathfrak{y}_\mu=\beta_Y(\lambda_\mu,x_\mu))_{\mu\in M}$, in the image of $\Lambda\times X$ via $\beta_Y$, cofinal with $\Lambda$, since $\Lambda\times X\to\Lambda_\infty\times X\to\beta(\Lambda_\infty\times X)$ is dense. The net $(\beta g(\mathfrak{y}_\mu))_\mu$ converges to $\beta g(\mathfrak{y})$, and its image under $\beta\rho$ converges to $\mathfrak{x}:=\beta\rho(\beta g(\mathfrak{y}))=\beta\pi(\mathfrak{y})$. Any neighbourhood of $\beta\gamma(\mathfrak{x})=(\infty,\mathfrak{x})$ contains a queue of $(\beta g(\mathfrak{y}_\mu))_\mu$, since this net is cofinal with $\Lambda$. We may choose a neighbourhood $U$ of $\beta\gamma(\mathfrak{x})$ so that $U\cap(\Lambda_\infty\times\beta X)=V\times A$, with $V=\{\lambda\,;\,\lambda\geq\overline{\lambda}\}\cup\{\infty\}$, and $A$ an open subset of $\beta X$ containing $\mathfrak{x}$. Hence in $A$ there is a queue of $(\beta\rho(\beta g(\mathfrak{y}_\mu)))_\mu$ and then in $U$ there is a queue of $(\beta g(\mathfrak{y}_\mu))_\mu$, which implies that $\beta g(\mathfrak{y})=(\infty,\mathfrak{x})$. We may then conclude that any $\mathfrak{y}' \in Y$ with $\beta g(\mathfrak{y}) = \beta g(\mathfrak{y}')$ belongs necessarily to $\widetilde{Y}$.

To complete the proof we consider the continuous map $K = \langle 1, H\rangle : \Lambda_\infty \times X \to \Lambda_\infty \times X$ defined by $K(\lambda, x) = (\lambda, h_\lambda(x))$ and $K(\infty, x) = (\infty, h(x))$, so that $H = \pi \cdot K$. Note that in the diagram below $\beta g$ is the (unique) continuous extension of the top composite, since the vertical maps are dense

$$\begin{tikzcd}
\Lambda \times X \arrow[r, "K"] \arrow[d] & \Lambda \times X \arrow[r, "1\times\beta_X"] & \Lambda \times \beta_X(X) \arrow[r, "(1\times\beta_X)^{-1}"] & \Lambda \times X \arrow[r, "K^{-1}"] & \Lambda \times X \arrow[r, "1\times\beta_X"] & \Lambda \times \beta_X(X) \arrow[d] \\
\beta(\Lambda_\infty \times X) \arrow[rrrrr, "\beta g"'] & & & & & \beta(\Lambda_\infty \times \beta X)
\end{tikzcd}$$

(here we consider (co)restrictions of $K$ and $\beta_X$ although we use the same notations).

Our next goal is to show that

$$\beta g(\beta K(\mathfrak{y})) = \beta\gamma(\beta h(\beta\pi(\mathfrak{y}))). \tag{$\nabla$}$$

By definition of $K$ it is clear that $\beta K(\mathfrak{y}) \in \widetilde{Y}$. Therefore $\beta g(\beta K(\mathfrak{y})) = (\infty, \beta\pi(\beta K(\mathfrak{y})))$. To show that $(\infty, \beta\pi(\beta K(\mathfrak{y})) = (\infty, \beta h(\beta\pi(\mathfrak{y})))$ – that is, $(\nabla)$ – we use the following diagram

$$\begin{tikzcd}
\Lambda \times X \arrow[r, "K"] \arrow[d] \arrow[drrrrr, bend left, "\beta g"] & \Lambda \times X \arrow[r, "1\times\beta_X"] \arrow[d] & \Lambda \times \beta_X(X) \arrow[r, "(1\times\beta_X)^{-1}"] \arrow[d] & \Lambda \times X \arrow[r, "K^{-1}"] & \Lambda \times X \arrow[r, "1\times\beta_X"] & \Lambda \times \beta_X(X) \\
\beta(\Lambda_\infty \times X) \arrow[r, "\beta K"'] & \beta(\Lambda_\infty \times X) \arrow[r, "\beta g"'] & \beta(\Lambda_\infty \times \beta X) \arrow[d, "\beta\rho"] & & & \beta(\Lambda_\infty \times \beta X) \arrow[d, "\beta\rho"] \\
\beta X \arrow[r, "\beta h"'] \arrow[u, "\beta\iota"] & \beta X \arrow[r, equal] \arrow[u, "\beta\iota"] & \beta X \arrow[r, equal] & \beta X \arrow[r, "(\beta h)^{-1}"'] & \beta X \arrow[r, equal] & \beta X
\end{tikzcd}$$

Indeed, applying $(\beta h)^{-1} \cdot \beta\rho$ to the latter, one gets $\beta\pi(\mathfrak{y})$; but the image of $(\beta h)^{-1}(\beta\rho(\beta\pi(\beta K(\mathfrak{y}))))$ must be also $\beta\pi(\mathfrak{y})$, since $\beta\pi(\mathfrak{y}) = \beta\rho(\beta g(\mathfrak{y})) = (\beta h)^{-1}(\beta\rho(\beta g(\beta K(\mathfrak{y}))))$, and so $(\nabla)$ holds.

Now we are able to conclude $(\Diamond)$ for $\mathfrak{y}, \mathfrak{y}' \in \widetilde{Y}$. From $\beta g(\mathfrak{y}) = \beta g(\mathfrak{y}')$ it follows that $\beta\pi(\mathfrak{y}) = \beta\pi(\mathfrak{y}')$, and then, by equality $(\nabla)$, $\beta g(\beta K(\mathfrak{y})) = \beta g(\beta K(\mathfrak{y}'))$, with $\beta K(\mathfrak{y})$ and $\beta K(\mathfrak{y}')$ in $\widetilde{Y}$. Therefore also $\beta\pi(\beta K(\mathfrak{y})) = \beta\pi(\beta K(\mathfrak{y}'))$, which means exactly that $\beta H(\mathfrak{y}) = \beta H(\mathfrak{y}')$, and this ends the proof. ∎

# References

[1] F. Borceux, M.M. Clementino, A. Montoli, On the representability of actions for topological algebras. In: *Categorical Methods in Algebra and Topology*, Textos de Matemática, DMUC, Vol. 46 (2014) 41–66.

[2] F. Borceux, G. Janelidze, G.M. Kelly, On the representability of actions in a semi-abelian category. *Theory Appl. Categ.* 14 (2005), 244–286.

[3] F. Borceux, G. Janelidze, G.M. Kelly, Internal object actions. *Comment. Math. Univ. Carolin.* 46 (2005), no.2, 235–255.

[4] H. Herrlich, E. Lowen-Colebunders, F. Schwarz, Improving Top: PrTop and PsTop. In: *Category theory at work* (Bremen, 1990), Res. Exp. Math. 18 (1991), 21–34, Heldermann, Berlin, 1991.

[5] O. Wyler, Injective spaces and essential extensions in TOP. *General Topology and Appl.* 7 (1977), no. 3, 247–249.

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