Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–32

ANOTHER NOTE ON EFFECTIVE DESCENT MORPHISMS OF TOPOLOGICAL SPACES AND RELATIONAL ALGEBRAS

MARIA MANUEL CLEMENTINO AND GEORGE JANELIDZE

Dedicated to Aleš Pultr on the occasion of his eightieth birthday

Abstract: We make three independent observations on characterizing effective descent morphisms in the category of topological spaces. The first of them proposes a new modification of known characterizations of effective descent morphisms of general spaces, while the other two are devoted to locally finite and Hausdorff spaces, respectively. The Hausdorff case is considered, as far as we could, at the more general level of relational algebras in the sense of M. Barr.

Keywords: monad, effective descent morphism, relational algebra, ultrafilter monad, locally finite space, Hausdorff space, Alexandrov space.
Math. Subject Classification (2000): 18C20, 18A40, 18A20, 54C10, 18B30.

Introduction

As the Reader might conclude from its title, this note is closely related to our previous note [5]. In fact it consists of three independent additional observations on characterizing effective descent morphisms in the category **Top** of topological spaces, presented in three sections, respectively, as follows:

Just as in [5] and in several other papers we refer to, saying that $f : X \to Y$ is an effective descent morphism in **Top** we simply mean that the pullback functor

$$f^* : (\mathbf{Top} \downarrow Y) \to (\mathbf{Top} \downarrow X)$$

is monadic. In Section 1 we recall two known characterizations of such maps, due to J. Reiterman and W. Tholen [10] and to M. M. Clementino and D. Hofmann [2], and add a modified version of the first of them. Since each characterization is quite sophisticated, such an addition seems to be useful.

Section 2 is devoted to locally finite spaces, and its main purpose is to show that a morphism there is an effective descent morphism if and only if it is

Received September 12, 2018.
The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020, and by South African NRF.

an effective descent morphism in **Top**. A counter-example shows that this result does not extend to Alexandrov spaces.

Section 3 is devoted to the 'opposite extreme', namely to Hausdorff spaces, and again, a morphism there is an effective descent morphism if and only if it is an effective descent morphism in **Top**. As far as we could, we use there the more general context of relational algebras in the sense of M. Barr [1]; such an algebra is a Hausdorff algebra if its structure relation is a partial map. A remark at the end recalls one of the open questions mentioned in [5] and explains that, as follows from our results, it has affirmative answers in both locally finite and Hausdorff cases.

1. Reiterman-Tholen and Clementino-Hofmann characterizations of effective descent morphisms of general topological spaces: a reformulation

For a continuous map $f : X \to Y$ of topological spaces, we have:

Theorem 1.1 ([10])**.** *A surjective continuous map $f : X \to Y$ is an effective descent morphism in* **Top** *if and only if, for every family of ultrafilters $\mathfrak{y}_i$ on Y converging to $y_i \in Y$, $i \in I$, such that the y_i's converge to $y \in Y$ with respect to an ultrafilter $\mathfrak{u}$ on I, there is an ultrafilter $\mathfrak{x}$ on X converging to a point $x \in f^{-1}(y)$ such that $\bigcup_{i\in U} A_i \in \mathfrak{x}$ for all $U \in \mathfrak{u}$, where A_i is the set of adherence points of the filterbase $f^{-1}\mathfrak{y}_i$ which belong to $f^{-1}(y_i)$.*

Theorem 1.2 ([2])**.** *A continuous map $f : X \to Y$ between topological spaces is of effective descent if and only if* $\mathrm{Ult}(\mathrm{Ult}(f))$ *is surjective.*

Denoting the ultrafilter monad on **Top** by $T = (T, \eta, \mu)$, we are going to reformulate these theorems as one theorem (Theorem 1.4 below) expressed in a language that uses *only* T and the convergence relations $R \subseteq T(X) \times X$ and $S \subseteq T(Y) \times Y$ on X and Y, respectively. Following [2], the map $R \to S$ induced by f will be denoted by f_1, while the map $T(R) \times_{T(X)} R \to T(S) \times_{T(Y)} S$ induced by f_1 will be denoted by f_2. That is, $f_1 = \mathrm{Ult}(f)$ and $f_2 = \mathrm{Ult}(\mathrm{Ult}(f))$ as defined in [2].

Lemma 1.3. (*'Folklore'*) *Let*

$$A \xleftarrow{\;u\;} C \xrightarrow{\;v\;} B$$

be a span of sets with $\mathfrak{a} \in T(A)$ and $\mathfrak{b} \in T(B)$. The following conditions are equivalent:

(i) *there exists* $\mathfrak{c} \in T(C)$ *with* $T(u)(\mathfrak{c}) = \mathfrak{a}$ *and* $T(v)(\mathfrak{c}) = \mathfrak{b}$*;*
(ii) $D \in \mathfrak{a} \Rightarrow v(u^{-1}(D)) \in \mathfrak{b}$.

Proof: (i)⇒(ii): $D \in \mathfrak{a} \Rightarrow u^{-1}(D) \in \mathfrak{c} \Rightarrow v^{-1}(v(u^{-1}(D))) \in \mathfrak{c} \Rightarrow v(u^{-1}(D)) \in T(v)(\mathfrak{c}) = \mathfrak{b}$.

(ii)⇒(i): The equalities $T(u)(\mathfrak{c}) = \mathfrak{a}$ and $T(v)(\mathfrak{c}) = \mathfrak{b}$ hold if and only if, for every $D \in \mathfrak{a}$ and every $E \in \mathfrak{b}$, the sets $u^{-1}(D)$ and $v^{-1}(E)$ belong to $\mathfrak{c}$. To prove the existence of such $\mathfrak{c}$ is to prove that $u^{-1}(D) \cap v^{-1}(E)$ is always non-empty. And it is indeed non-empty since so is $v(u^{-1}(D)) \cap E$ being the intersection of two elements of the filter $\mathfrak{b}$. ■

Theorem 1.4. *Let* $X = (X, R)$ *and* $Y = (Y, S)$ *be topological spaces. The following conditions on a continuous map* $f : X \to Y$ *are equivalent:*

(i) f *is an effective descent morphism in* **Top***;*
(ii) *for every* $(\mathfrak{s}, (\mathfrak{y}, y)) \in T(S) \times_{T(Y)} S$*, there exists* $(\mathfrak{x}, x) \in R$ *with* $f(x) = y$ *and* $\rho_2(f_1^{-1}(U))$ *in* $\mathfrak{x}$ *for each* $U \in \mathfrak{s}$*, where* ρ_2 *is the second projection map* $R \to X$*;*
(iii) *for every* $(\mathfrak{s}, (\mathfrak{y}, y)) \in T(S) \times_{T(Y)} S$*, there exists* $(\mathfrak{x}, x) \in R$ *with* $f(x) = y$ *and* $f_1(\rho_2^{-1}(V))$ *in* $\mathfrak{x}$ *for each* $V \in \mathfrak{x}$*, where* ρ_2 *is as above;*
(iv) f_2 *is surjective, that is, for every* $(\mathfrak{s}, (\mathfrak{y}, y)) \in T(S) \times_{T(Y)} S$*, there exists* $(\mathfrak{r}, (\mathfrak{x}, x)) \in T(R) \times_{T(X)} R$ *with* $f(x) = y$ *and* $T(f_1)(\mathfrak{r}) = \mathfrak{s}$ *(which also implies* $T(f)(\mathfrak{x}) = \mathfrak{y}$*).*

Moreover, it can be assumed that for a given $(\mathfrak{s}, (\mathfrak{y}, y)) \in T(S) \times_{T(Y)} S$*, the pair* $(\mathfrak{x}, x)$ *involved in conditions* (ii)*,* (iii)*, and* (iv) *is the same.*

Proof: (i)⇔(iv) is a trivial copy of Theorem 1.2. (ii)⇔(iv) follows from Lemma 1.3 applied to the span

$$S \xleftarrow{f_1} R \xrightarrow{\rho_2} X,$$

while (iii)⇔(iv) follows from Lemma 1.3 applied to the opposite span. ■

Remark 1.5. A careful comparison with Section 5 in [2] could explain that our reformulations and proof of Theorem 1.4 are hidden there, and, on the other hand, they cover the proof of Theorem 5.2 of [2]. Not going into the full story, let us only point out the following:

(a) Our Lemma 1.3, which indeed seems to be a known 'folklore observation' is actually useful for several purposes. For example, it easily implies the Beck–Chevalley property of T, using a much simpler

Beck–Chevalley property of the power set functor, namely the fact that, for a pullback diagram

$$\begin{array}{ccc} C & \xrightarrow{v} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & U \end{array}$$

and any subset D of A, we have $v(u^{-1}(D)) = \beta^{-1}(\alpha(D))$.

(b) The definition of A_i in Theorem 1.1 is equivalent to

$$A_i = \{a \in X \,|\, (\exists_{\mathfrak{a} \in T(X)} (T(f)(\mathfrak{a}) = \mathfrak{y}_i \;\&\; (\mathfrak{a}, a) \in R)) \;\&\; f(a) = y_i\}, \qquad (1.\mathrm{i})$$

and, for $U \in \mathfrak{u}$, we can calculate subsequently:

$$\phi(U) = \{(\mathfrak{y}_i, y_i) \,|\, i \in U\},$$

$$f_1^{-1}(\phi(U)) = \{(\mathfrak{a}, a) \in R \,|\, \exists_{i \in U} (T(f)(\mathfrak{a}), f(a)) = (\mathfrak{y}_i, y_i)\},$$

$$\rho_2(f_1^{-1}(\phi(U))) = \{a \in X \,|\, \exists_{i \in U} \exists_{\mathfrak{a} \in T(X)} ((T(f)(\mathfrak{a}), f(a)) = (\mathfrak{y}_i, y_i) \,\&\, (\mathfrak{a}, a) \in R)\},$$

which, together with (1.i), gives

$$\bigcup_{i \in U} A_i = \rho_2(f_1^{-1}(\phi(U))),$$

and easily shows the equivalence of conditions (i) and (ii) in Theorem 1.4, independently of Theorem 1.2.

2. Locally finite descent

For a topological space X, let $R = \mathrm{Ult}(X)$ and $T(R) \times_{T(X)} R = \mathrm{Ult}(\mathrm{Ult}(X))$ be as in Section 1.

Theorem 2.1. *The following conditions on a space $X = (X, R)$ are equivalent:*

(i) *X is locally finite, that is every point in X has a finite neighbourhood;*
(ii) *X is an Alexandrov space (which means that its set of open subsets is closed under intersections, or, equivalently, its topology is determined by a preorder), in which all minimal open subsets are finite;*
(iii) *if $(\mathfrak{x}, x)$ belongs to R, then the ultrafilter $\mathfrak{x}$ is principal;*
(iv) *if $(\mathfrak{r}, (\mathfrak{x}, x))$ belongs to $T(R) \times_{T(X)} R$, then the ultrafilters $\mathfrak{x}$ and $\mathfrak{r}$ are principal.*

Proof: (i)⇔(ii) is well known.

(i)⇒(iii): Given $(\mathfrak{x}, x) \in R$, let X' be a finite neighbourhood of x; then $X' \in \mathfrak{x}$ and, since X' is finite, this implies that $\mathfrak{x}$ is principal.

(iii)⇒(i): Suppose x is a point in X and $\mathfrak{u}$ the set of all subsets of X of the form $U \cap V$, where U is a neighbourhood of x and V has a finite complement. If $\mathfrak{x} \in T(X)$ contains $\mathfrak{u}$ as a subset, then $\mathfrak{x}$ is a non-principal ultrafilter with $(\mathfrak{x}, x)$ in R, which is impossible by (iii). Since $\mathfrak{u}$ is closed under finite intersections, this means that $U \cap V = \emptyset$ for some neighbourhood U of x and some V with a finite complement, making U finite.

(iii)⇒(iv): Assuming (iii), if $(\mathfrak{r}, (\mathfrak{x}, x))$ belongs to $T(R) \times_{T(X)} R$, then $\mathfrak{x}$ is principal and $T(\rho_2)(\mathfrak{r}) = \mathfrak{x}$. Suppose $\mathfrak{x}$ is generated by $\{x'\}$, that is, $\mathfrak{x} = \{U \subseteq X \mid x' \in U\}$. We observe:

(a) $\{x'\} \in \mathfrak{x}$, and so $\rho_2^{-1}(\{x'\})$ belongs to $\mathfrak{r}$.
(b) We already know that (iii) implies (i), and so there exists a finite neighbourhood A of x'.
(c) $\rho_2^{-1}(\{x'\})$ consists of all elements of R of the form $(\mathfrak{a}, x')$, and, since every ultrafilter converging to x' contains all neighbourhoods of x', each such $\mathfrak{a}$ contains A.
(d) An ultrafilter containing a finite set must be a principal ultrafilter generated by one of its one-element subsets.
(e) As follows from (c) and (d), the set $\rho_2^{-1}(\{x'\})$ is finite.
(f) As follows from (a) and (e), $\mathfrak{r}$ is a principal ultrafilter.

(iv)⇒(iii): For $(\mathfrak{x}, x) \in R$, choose $\mathfrak{r}$ with $T(\rho_2)(\mathfrak{r}) = \mathfrak{x}$, which is possible since ρ_2 is surjective making $T(\rho_2)$ as well. Then $(\mathfrak{r}, (\mathfrak{x}, x))$ belongs to $T(R) \times_{T(X)} R$, and we can apply (iv). ■

Let us write

$$\leqslant_R = \{(x_1, x_0) \in X \times X \mid x_0 \in \overline{\{x_1\}}\},$$

$$\leqslant_R^{(2)} = \{(x_2, x_1, x_0) \mid x_2 \leqslant x_1 \leqslant x_0\} \approx \leqslant_R \times_X \leqslant_R .$$

Consider the map

$$\leqslant_R^{(2)} \to T(R) \times_{T(X)} R \tag{2.i}$$

defined by $(x_2, x_1, x_0) \mapsto (\mathfrak{r}, (\mathfrak{x}, x_0))$, where $\mathfrak{x} = \dot{x}_1$ is the ultrafilter on X generated by $\{x_1\}$, and $\mathfrak{r}$ is the ultrafilter on R generated by $\{(\dot{x}_2, x_1)\}$. It is natural in X, and, from Theorem 2.1, we easily obtain:

Corollary 2.2. *Is X is locally finite, then the map* (2.i) *is bijective.*

This gives us a simple characterization of effective descent morphisms of locally finite spaces:

Theorem 2.3. *Let $X = (X, R)$ and $Y = (Y, S)$ be locally finite topological spaces. The following conditions on a continuous map $f : X \to Y$ are equivalent:*

(i) *f is an effective descent morphism in* **Top***;*
(ii) *f is an effective descent morphism in the category of Alexandrov spaces;*
(iii) *f is an effective descent morphism in the category of locally finite spaces;*
(iv) *the map $\leqslant_R^{(2)} \to \leqslant_S^{(2)}$ induced by f is surjective.*

Proof: (i)⇔(iv) follows from Theorem 1.4(i)⇔(iv) (i.e., from Theorem 1.2) and Corollary 2.2.

(ii)⇔(iv) is nothing but the characterization of effective descent morphisms of preordered sets (Proposition 3.4 in [8]).

(ii)⇒(iii): As follows from a well-known observation in descent theory, e.g. recalled as Corollary 2.7.2 in [9], we only need to prove that, for a pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{2.ii}$$

in the category of Alexandrov spaces, B is a locally finite space whenever so are A, X, and Y. Replacing again Alexandrov spaces with preorders and continuous maps with preorder-preserving maps, we have to prove that, for every $b \in B$, the set $b\downarrow = \{b' \in B \mid b' \leq b\}$ is finite, whenever all elements of A, of X, and of Y have such properties. Suppose $b\downarrow$ is infinite, and observe:

- Since $\beta(b)\downarrow$ is a finite subset of Y, there exists an infinite subset B' of $b\downarrow$, on which β is constant; say, $\beta(B') = \{y\}$.
- Since β is preorder-preserving, we have $y \leq \beta(b)$.
- Since f satisfies (ii), it also satisfies (iv). Therefore there exist $x' \leq x$ in X with $f(x) = \beta(b)$ and $f(x') = y$.
- For every $b' \in B'$, we have $(x', b') \leq (x, b)$ in $A = X \times_Y B$. Since B' is infinite, this gives an infinite subset of $(x, b)\downarrow$ in A, which is a contradiction.

(iii)⇒(iv): The proof of Proposition 3.4(b)⇒(c) of [8] can be used since A there, which plays the role of our B, is finite. ■

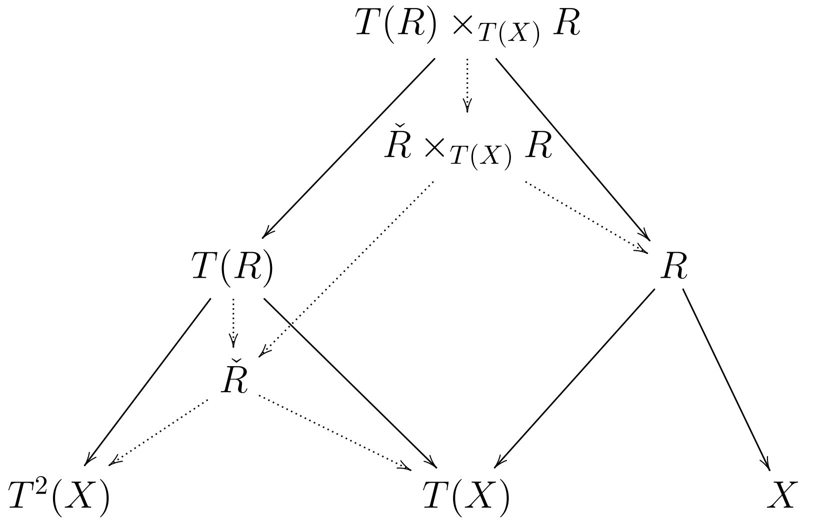
Remark 2.4. (a) Having a simple description of effective descent morphisms in the category of locally finite spaces might seem surprising since this category does not admit some coequalizers.

(b) Assuming X and Y in Theorem 2.3 to be not just Alexandrov spaces, but locally finite spaces, is essential. Indeed, take: Y to be the set of integers with the topology determined by the usual order; X to be the coproduct of all sets of all three-element subspaces of Y; $f : X \to Y$ to be induced by the inclusion maps. Then f satisfies condition (iv) of Theorem 2.3 (which implies that it satisfies condition (ii) there, by Proposition 3.4 of [8]), but it is not even a descent map (=pullback stable regular epimorphism) in **Top**, which can be easily shown using either the convergence approach, or the Day–Kelly characterization [7] of pullback stable regular epimorphisms in **Top**.

3. Hausdorff descent

We begin this section with a context considered in [5], where $T = (T, \eta, \mu)$ is an arbitrary non-trivial monad on **Sets**, and consider relational T-algebras in the sense of M. Barr [1]. As mentioned in [5], they can also be seen as 'T-preorders', and they are special cases of reflexive and transitive lax algebras in the sense of [3] and of more special reflexive and transitive (T, V)-algebras, also called (T, V)-categories, in the sense of [6]; when T is the ultrafilter monad they are the same as topological spaces as considered in Sections 1 and 2.

Given a relational T-algebra $X = (X, R)$, let us repeat diagram (2.2) of [5]:



(3.i)

in which (also repeating from [5]):

- the solid arrows represent R as a span $T(X) \to X$ and $T(R)$ as a span $T^2(R) \to T(X)$, and then represent the composite of these spans as a span $T^2(X) \to X$;
- $\check{R}$ is the relation $T^2(X) \to T(X)$ associated with the span $T(R) : T^2(X) \to T(X)$, that is, $\check{R}$ is simply the image of $T(R)$ in $T^2(X) \times T(X)$;
- the dotted arrows are the canonical maps defined accordingly.

Then, given a morphism $f : (X, R) \to (Y, S)$ of relational T-algebras, consider the commutative diagram

$$\begin{array}{ccc} T(R) \times_{T(X)} R & \xrightarrow{f_2} & T(S) \times_{T(Y)} S \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \check{R} \times_{T(X)} R & \xrightarrow[\check{f}_2]{} & \check{S} \times_{T(Y)} S, \end{array} \qquad (3.\mathrm{ii})$$

in which:

- f_2 and $\check{f}_2$ are induced by f, and they are the same as the maps (2.5) and (2.6), respectively, in [2]; note also that, when T is the ultrafilter monad, f_2 is the same as $\mathrm{Ult}(\mathrm{Ult}(f))$ used in Section 1.
- π_X is the same as the top vertical dotted arrow in (3.i) and π_Y is the similar canonical map associated with $Y = (Y, S)$.

Recall that, following the special case of the ultrafilter monad, a relational T-algebra $X = (X, R)$ is said to be a *Hausdorff T-algebra* if R is a partial map, that is, if X satisfies the implication

$$((\mathfrak{x}, x), (\mathfrak{x}, x') \in R) \Rightarrow x = x'$$

(of course the algebraic viewpoint would rather suggest to say "partial" instead of "Hausdorff").

Theorem 3.1. *Let $f : X \to Y$ be a morphism of relational T-algebras. Then:*

(a) *if Y is a Hausdorff T-algebra, then f_2 is surjective if and only if so is $\check{f}_2$;*
(b) *if the functor T has the Beck–Chevalley property, X and Y are Hausdorff T-algebras, and f is an effective descent morphism in the category of relational T-algebras, then f is an effective descent morphism in the category of Hausdorff T-algebras.*

Proof: (a) is obvious: just note that, when Y is a Hausdorff T-algebra, the map π_Y involved in diagram (3.ii) is bijective.

(b): Referring again Corollary 2.7.2 in [9] (similarly to our proof of Theorem 2.3(ii)⇒(iii)), we only need to prove that, for a pullback diagram (2.ii) in the category of relational T-algebras with f being an effective descent morphism, B is a Hausdorff T-algebra whenever so are A, X, and Y. Writing here $A = (A, R_A)$, etc., suppose $(\mathfrak{b}, b) \in R_B$ and $(\mathfrak{b}, b') \in R_B$, and observe:

- Since $(\mathfrak{b}, b) \in R_B$ and $(\mathfrak{b}, b') \in R_B$, we have $(T(\beta)(\mathfrak{b}), \beta(b)) \in R_Y$ and $(T(\beta)(\mathfrak{b}), \beta(b')) \in R_Y$, which gives $\beta(b) = \beta(b')$, since Y is a Hausdorff T-algebra.
- Since f is an effective descent morphism, the map $R_X \to R_Y$ induced by f is surjective, as follows e.g. from Theorem 2.4 of [4] (in fact this follows from various results proved by the same authors before).
- Therefore there exists $(\mathfrak{x}, x) \in R_X$ with $T(f)(\mathfrak{x}) = T(\beta)(\mathfrak{b})$ and $f(x) = \beta(b) = \beta(b')$.
- Since the functor T has the Beck–Chevalley property, there exists $\mathfrak{a} \in A$ with $T(\alpha)(\mathfrak{a}) = \mathfrak{x}$ and $T(g)(\mathfrak{a}) = \mathfrak{b}$.
- Then, since (2.ii) is a pullback diagram, $(\mathfrak{a}, (x, b)) \in R_A$ and $(\mathfrak{a}, (x, b')) \in R_A$.
- Since A is Hausdorff, this gives $(x, b) = (x, b')$, and so $b = b'$.

That is, B is a Hausdorff T-algebra. ■

Theorem 3.2. *For a continuous map $f : X \to Y$ of Hausdorff spaces, the following conditions are equivalent:*

(i) *f is an effective descent morphism in* **Top***;*
(ii) *f is an effective descent morphism in the category of Hausdorff spaces;*
(iii) *f_2 is surjective;*
(iv) *$\check{f}_2$ is surjective.*

Proof: (i)⇔(iii) is a special case of Theorem 1.2 (or of Theorem 1.4(i)⇔(iv), which is the same). (iii)⇔(iv) is a special case of Theorem 3.1(a). (i)⇒(ii) follows from Theorem 3.1(b). To prove (ii)⇒(i) we can simply copy the arguments of subsection 4.2 of [10], having in mind that the space $E \times_B A$ constructed there is obviously a Hausdorff space provided so is E. ■

Remark 3.3. As follows from Corollary 2.2, when X is a locally finite space, the canonical maps $T(R) \to T^2(X)$ and $T(R) \to T(X)$ in diagram (3.i) are jointly monic. Therefore, when $f : X \to Y$ is a continuous map of locally finite spaces, the vertical arrows in diagram (3.ii) are bijections. It follows that the equivalence of conditions (iii) and (iv) in Theorem 3.2 also holds for locally finite spaces. Does it hold for general topological spaces? This problem, mentioned as one of the open problems in [5], is still open.

References

[1] M. Barr, Relational algebras, in: Reports of the Midwest Category Seminar, IV, Lecture Notes in Mathematics 137, Springer, Berlin, 1970, pp. 39–55

[2] M. M. Clementino and D. Hofmann, Triquotient maps via ultrafilter convergence, Proceedings of the American Mathematical Society 130, 2002, 3423–3431

[3] M. M. Clementino and D. Hofmann, Topological features of lax algebras, Applied Categorical Structures 11, 2003, 267–286

[4] M. M. Clementino and D. Hofmann, Descent morphisms and a Van Kampen Theorem in categories of lax algebras, Topology and its Applications 159, 9, 2012, 2310–2319

[5] M. M. Clementino and G. Janelidze, A note on effective descent morphisms of topological spaces and relational algebras, Topology and its Applications 158, 17, 2011, 2431–2436

[6] M. M. Clementino, and W. Tholen, Metric, topology and multicategory – a common approach, Journal of Pure and Applied Algebra 179, 2003, 13–47

[7] B. Day and G. M. Kelly, On topological quotient maps preserved by pullbacks, Proceedings of the Cambridge Philosophical Society 67, 1970, 553–558

[8] G. Janelidze and M. Sobral, Finite preorders and topological descent I, Journal of Pure and Applied Algebra 175(1-3), 2002, 187–205

[9] G. Janelidze and W. Tholen, Facets of Descent I, Applied Categorical Structures 2, 1994, 245–281

[10] J. Reiterman and W. Tholen, Effective descent maps of topological spaces, Topology and its Applications 57, 1994, 53–69

Maria Manuel Clementino
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: mmc@mat.uc.pt

George Janelidze
Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, Cape Town, South Africa
E-mail address: George.Janelidze@uct.ac.za