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STRICT MONADIC TOPOLOGY I: FIRST SEPARATION AXIOMS AND REFLECTIONS

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Dedicated to Aleš Pultr on the occasion of his eightieth birthday

ABSTRACT: Given a monad T on the category of sets, we consider reflections of $\mathrm{Alg}(T)$ into its full subcategories formed by algebras satisfying natural counterparts of topological separation axioms T_0, T_1, T_2, T_{ts}, and T_{ths}; here *ts* stands for *totally separated* and *ths* for what we call *totally homomorphically separated*, which coincides with *ts* in the (compact Hausdorff) topological case. We ask whether these reflections satisfy simple conditions useful in categorical Galois theory, and give some partial answers in easy cases.

KEYWORDS: monadic topology, monad, separation axiom, Galois structure.
MATH. SUBJECT CLASSIFICATION (2010): 18C99, 18A40, 18C15, 08B99.

1. Introduction

The category **CHSpaces** of compact Hausdorff spaces defined via the ultrafilter convergence can be identified with the Eilenberg–Moore category $\mathrm{Alg}(T)$ of T-algebras, where T is the ultrafilter monad on the category **Sets** of sets – which can therefore be also called the compact Hausdorff monad. This result, due E. G. Manes [25], was mentioned many times in literature, including the Introduction of [13], written by R. Lowen and W. Tholen. It is also interesting that the monadicity of **CHSpaces** over **Sets** can be deduced directly from R. Paré's version of the Beck monadicity theorem, not using the notion of convergence [26] (see also Section VI.9 of [24]).

As we know from M. Barr [1], replacing algebras with what Barr called *relational algebras* and what were later called *lax algebras*, one characterizes not just compact Hausdorff, but all topological spaces. Replacing then the ultrafilter monad with a general monad leads to developing what W. Gähler called "monadic topology" (see [12], which also refers to his several other

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papers about it). Next, "Monoidal topology" [13] goes much further and has a much more general context giving a number of new important examples. It was initiated in [10] and in [11], with many subsequent papers of M. M. Clementino, D. Hofmann, W. Tholen and others (see also [23], [7], and [9] for a 'prehistory').

In this paper we go back to strict (not lax) monadicy, and consider it as a suggestion to treat algebraic categories as topological ones. Specifically, we copy some of the very first separation axioms of general topology, namely T_0, T_1, T_2, and T_{ts} (where *ts* stands for *totally separated*), and introduce one more, which we call T_{ths} (*totally homomorphically separated*), and try to characterize them in several purely algebraic cases. These axioms produce reflective subcategories

$$\mathrm{Alg}_0(T) \supseteq \mathrm{Alg}_1(T) \supseteq \mathrm{Alg}_2(T) \supseteq \mathrm{Alg}_{ts}(T) \supseteq \mathrm{Alg}_{ths}(T)$$

of $\mathrm{Alg}(T)$, for any monad T on **Sets**, and we ask, how good the resulting reflections are from the viewpoint of categorical Galois theory. We give only simple answers to simple questions, and yet we think these answers show that further development of our strict monadic topology is going to be useful, especially in finding new interesting examples of Galois theories.

Remark 1.1. (a) *The term "monadic topology" also exists in mathematical logic (see e.g. page 505 in* [2]*), but it is not related to what we are doing.*

(b) *Our* T_2 *is not related to Hausdorff-ness in monoidal topology.*

Throughout this paper T denotes a monad on **Sets**, and F the corresponding free functor $\mathbf{Sets} \to \mathrm{Alg}(T)$; T-algebras will be usually simply called algebras.

2. Introducing separation axioms

The following theorem is a combination of several well-known facts (and its last assertion trivially follows from definitions):

Theorem 2.1. *The following conditions on a full replete subcategory* $\mathcal{X}$ *of* $\mathrm{Alg}(T)$ *are equivalent:*

(a) $\mathcal{X}$ *is a regular-epi-reflective subcategory of* $\mathrm{Alg}(T)$*;*
(b) $\mathcal{X}$ *is closed under products and subalgebras in* $\mathrm{Alg}(T)$*.*

Furthermore, under these conditions, $\mathcal{X}$ is a Birkhoff subcategory of $\mathrm{Alg}(T)$ *in the sense of* [20] *if and only if it is a Birkhoff subcategory in the sense of* [25]*, and if and only if it closed under quotient algebras in* $\mathrm{Alg}(T)$*.*

Next we introduce:

Definition 2.2. An algebra A is said to be:

(a) a T_0-algebra, if for every two distinct elements a_1 and a_2 in A, there exists a subalgebra S of A with $(a_1 \in S \wedge a_2 \notin S) \vee (a_1 \notin S \wedge a_2 \in S)$;

(b) a T_1-algebra, if for every two distinct elements a_1 and a_2 in A, there exists a subalgebra S of A with $a_1 \in S \wedge a_2 \notin S$, or, equivalently, if for every element $a \in A$, the set $\{a\}$ is a subalgebra of A;

(c) a T_2-algebra, if for every two distinct elements a_1 and a_2 in A, there exist subalgebras S_1 and S_2 of A with $a_1 \in S_1, a_2 \notin S_1, a_1 \notin S_2, a_2 \in S_2$, and $S_1 \cup S_2 = A$.

(d) a totally separated algebra, if for every two distinct elements a_1 and a_2 in A, there exist subalgebras S_1 and S_2 of A with $a_1 \in S_1, a_2 \notin S_1, a_1 \notin S_2, a_2 \in S_2$, $S_1 \cup S_2 = A$ and $S_1 \cap S_2 = \emptyset$.

(e) a totally homomorphically separated algebra, if for every two distinct elements a_1 and a_2 in A, there exist subalgebras S_1 and S_2 of A with $a_1 \in S_1, a_2 \notin S_1, a_1 \notin S_2, a_2 \in S_2$, $S_1 \cup S_2 = A$, $S_1 \cap S_2 = \emptyset$, and $(S_1 \times S_1) \cup (S_2 \times S_2)$ being a congruence on A.

The full subcategories of $\mathrm{Alg}(T)$ with objects all T_i-algebras $(i = 0, 1, 2)$, all totally separated algebras, and all totally homomorphically separated algebras will be denoted by $\mathrm{Alg}_i(T)$, $\mathrm{Alg}_{ts}(T)$, and $\mathrm{Alg}_{ths}(T)$, respectively.

And it is easy to prove (and, moreover, 2.3(a) is trivial):

Theorem 2.3. (a) $\mathrm{Alg}(T) \supseteq \mathrm{Alg}_0(T) \supseteq \mathrm{Alg}_1(T) \supseteq \mathrm{Alg}_2(T) \supseteq \mathrm{Alg}_{ts}(T) \supseteq \mathrm{Alg}_{ths}(T)$.

(b) *The subcategories* $\mathrm{Alg}_i(T)$ $(i = 0, 1, 2)$*,* $\mathrm{Alg}_{ts}(T)$*, and* $\mathrm{Alg}_{ths}(T)$*, of* $\mathrm{Alg}(T)$*, satisfy the condition* 2.1(b) *(and so also* 2.1(a)*).*

(c) $\mathrm{Alg}_1(T)$ *is a Birkhoff subcategory of* $\mathrm{Alg}(T)$*.*

(d) $\mathrm{Alg}_{ths}(T)$ *consists of all those algebras that can be presented as subdirect products of* 2*-element algebras.*

Remark 2.4. *It is also easy to see that if A is a* T_0*-algebra, then the image of the (unique) T-algebra homomorphism $F(\emptyset) \to A$ has at most one element.*

Some trivial situations are described in the following three examples:

Example 2.5. *If $T(\emptyset) \neq \emptyset$, the following conditions on a T-algebra are equivalent:*

(a) *A is a T_1-algebra;*
(b) *A is a T_2-algebra;*
(c) *A is totally separated;*
(d) *A is totally homomorphically separated;*
(e) *A has at most one element.*

Example 2.6. *We have* $\mathrm{Alg}_0(T) = \mathrm{Alg}_1(T) = \mathrm{Alg}_2(T)$ *when T is:*

(a) *one of the two trivial monads, that is, $T(X)$ has at most one element for each set X;*
(b) *the identity monad;*
(c) *the compact Hausdorff space monad (=the ultrafilter monad).*

Furthermore, in the cases (a) and (b) we also have $\mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T)$*, while in the case (c)* $\mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T)$ *is the category of Stone spaces.*

Example 2.7. *We have* $\mathrm{Alg}_0(T) = \mathrm{Alg}_1(T) = \mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T) = \mathbf{Sets}$ *(with the obvious meaning of the last equality) when T is:*

(a) *the identity monad;*
(b) *more generally, the G-set monad, where G is a group.*

3. Universal algebras

When $\mathrm{Alg}(T)$ is a variety of universal algebras, which is the case if and only if T is a finitary monad, it is convenient to use the terminology and notation of universal algebra, and in particular to mention:

Proposition 3.1. *For every finitary monad T:*

(a) $\mathrm{Alg}_0(T)$ *is the quasi-variety of T-algebras determined by the quasi-identities*

$$(t(x, ..., x) = y \wedge u(y, ..., y) = x) \Rightarrow x = y,$$

required for each pair (t, u) of T-terms.

(b) $\mathrm{Alg}_1(T)$ *is the variety of T-algebras determined by the identity*

$$t(x, ..., x) = x,$$

required for each T-term t, or, equivalently, for each basic operation t in any presentation of $\mathrm{Alg}(T)$ *as a variety of universal algebras.*

What seem to be the first natural examples are:

Example 3.2. *Let M be a monoid and T the M-set monad, making* $\mathrm{Alg}(T)$ *to be the category of M-sets. Then:*

(a) $\mathrm{Alg}_0(T)$ *is the quasi-variety of M-sets determined by the quasi-identities*

$$(mx = y \wedge ny = x) \Rightarrow x = y,$$

required for each pair (m, n) of element of M.

(b) $\mathrm{Alg}_1(T) = \mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T) =$ **Sets** *(which we already mentioned in the case of M being a group).*

Example 3.3. *Let T be the semigroup monad, making* $\mathrm{Alg}(T)$ *to be the category of semigroups. Then:*

(a) $\mathrm{Alg}_0(T)$ *is the quasi-variety of semigroups determined by the quasi-identities*

$$(x^m = y \wedge y^n = x) \Rightarrow x = y,$$

required for each pair (m, n) of natural numbers.

(b) $\mathrm{Alg}_1(T) = \mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T)$ *is the category* **Bands** *of idempotent semigroups (=bands). For, having in mind Proposition 3.1(b) (and Theorem 2.3(a)), we only need to show that, for every idempotent semigroup A and distinct elements a_1 and a_2 in A, there exist subalgebras S_1 and S_2 of A with $a_1 \in S_1, a_2 \notin S_1, a_1 \notin S_2, a_2 \in S_2, S_1 \cup S_2 = A$ and $S_1 \cap S_2 = \emptyset$. Since $a_1 \neq a_2$, we can assume, without loss of generality (replacing, if necessary, the multiplication of A with the opposite one), that $a_1 \neq a_1a_2$. And assuming that we take $S_1 = \{a \in A \,|\, a_1 = a_1a\}$ and take S_2 to be the complement of S_1, that is, $S_2 = \{a \in A | a_1 \neq a_1a\}$. To check that S_1 and S_2 are subsemigroups of A is easy:*

- *for $a, a' \in S_1$, we have $a_1aa' = a_1a' = a_1$, and so $aa' \in S_1$;*
- *for $a, a' \in S_2$ with $aa' \notin S_2$, we have $a_1 = a_1aa' = a_1aa'a' = a_1a'$, which is a contradiction.*

Note that when A is commutative, S_1 is nothing but the up-closure of $\{a_1\}$ with respect to order on A defined by $x \leq y \Leftrightarrow x = xy$.

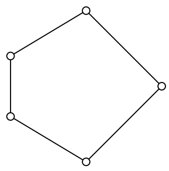
(c) $\mathrm{Alg}_{ths}(T)$ *is the category* **SLat** *of commutative idempotent semigroups (=semilattices), which follows from Theorem 2.3(d) and* $\mathrm{Alg}_{ths}(T) \subseteq \mathrm{Alg}_1(T) =$ **Bands**.

Example 3.4. *Let T be the unbounded lattice monad, making* $\mathrm{Alg}(T)$ *to be the category of not-necessarily bounded lattices. We don't have a good characterization of* $\mathrm{Alg}_2(T)$ *and* $\mathrm{Alg}_{ts}(T)$*, but:*

(a) $\mathrm{Alg}(T) = \mathrm{Alg}_0(T) = \mathrm{Alg}_1(T)$.

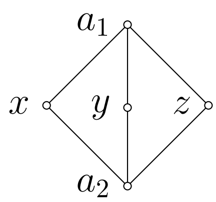
(b) $\mathrm{Alg}_{ths}(T)$ *is the category* **DLat** *of (not-necessarily bounded) distributive lattices, as follows from Theorem 2.3(d).*

(c) *It is easy to check directly that the lattice*



is (not totally homomorphically separated but) totally separated, showing that $\mathrm{Alg}_{ts}(T) \neq \mathrm{Alg}_{ths}(T)$.

(d) *However,* $\mathrm{Alg}_2(T) \cap \mathbf{MLat} = \mathrm{Alg}_{ts}(T) \cap \mathbf{MLat} = \mathrm{Alg}_{ths}(T) = \mathbf{DLat}$*, where* **MLat** *is the category of (not-necessarily bounded) modular lattices. To prove this, consider a modular non-distributive lattice A. As shown in* [4]*, A must have a sublattice L of the form*



and since A is a T_2-lattice, A must have sublattices S_1 and S_2 with $a_1 \in S_1$, $a_2 \notin S_1$, $a_1 \notin S_2$, $a_2 \in S_2$, and $S_1 \cup S_2 = A$. Since $S_1 \cup S_2 = A$, two out of the three elements x, y, and z belong to S_i, where i is either 1 *or* 2*. But if so, then both a_1 and a_2 belong to the same S_i, which a contradiction. Of course this argument in fact proves more, namely that no T_2-lattice has a sublattice isomorphic to L above.*

Example 3.5. *Let T be the pregroup monad making* $\mathrm{Alg}(T)$ *to be the category of pairs $A = (A, p)$, where p is ternary operation on A satisfying the identities*

$$p(x, y, y) = x = p(y, y, x),\ p(x, y, p(z, t, u)) = p(p(x, y, z), t, u).$$

Each such algebra A becomes a group once one chooses an element c in A and defines the group operations by

$$xy = p(x, c, y),\ 1 = c,\ x^{-1} = p(c, x, c);$$

this group will be denoted by A_c. Recall also that, for every $c, d \in A$, the map

$$p(-, c, d) : A_c \to A_d$$

is a group isomorphism; furthermore, we have

$$p(-, c, c) = 1_{A_c},\ p(-, d, e)p(-, c, d) = p(-, c, e),\ p(-, c, d)^{-1} = p(-, d, c)$$

and, on the other hand, A can be recovered from any A_c via

$$p(x, y, z) = xy^{-1}z.$$

Note also that subgroups of A_c are the same as subalgebras of A containing c. We have:

(a) $\mathrm{Alg}(T) = \mathrm{Alg}_0(T) = \mathrm{Alg}_1(T)$ *(which is obviously also true under the much weaker assumption that p satisfies the identity $p(x, x, x) = x$).*

(b) $\mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T)$ *and it consists of all T-algebras A satisfying the identity*

$$p(x, y, x) = y,$$

or, equivalently, of all T-algebras A for which (any, hence every) A_c is a vector space over the two-element field $\mathbb{F}_2$. The fact that, when A_c is such a vector space, A is totally homomorphically separated, follows from Theorem 2.3(d). To prove that, for every T_2-pregroup A, any A_c is a vector space over $\mathbb{F}_2$, take any $d \in A$, different from c, and subalgebras S_1 and S_2 of A with $c \in S_1$, $d \notin S_1$, $c \notin S_2$, $d \in S_2$, and $S_1 \cup S_2 = A$. In the language of A_c, we have $c = 1$ and $S_2 = dG$ where $G = d^{-1}S_2$ is a subgroup of A_c. We observe:

- *since d is not in S_1, there is no $s \in S_1$ with ds in S_1;*
- *since $S_1 \cup dG = S_1 \cup S_2 = A$, it follows that for every $s \in S_1$ there exists $g \in G$ with $ds = dg$;*
- *that is, S_1 is a subgroup of G;*
- *since $S_1 \cup dG = A$ and $S_1 \subseteq G$, we have $G \cup dG = A$;*
- *since $1 = c$ does not belong to $dG = S_2$, d does not belong to G;*
- *as follows from the last two observations, G has index 2.*

Now, since d was an arbitrarily chosen element of A, we can conclude that the intersection of all subgroups in A_c of index 2 is trivial, which implies that A_c satisfies the identity $x^2 = 1$ (indeed, it is a standard argument that if G has index 2, then G is a normal subgroup and $x^2G = (xG)(xG) = G$ in the quotient group, which implies $x^2 \in G$).

4. Monadic Galois structures

We will use very special kinds of Galois structures in the sense of [15] (see also [19] and references therein), defined as follows:

Definition 4.1. A monadic Galois structure (over **Sets**) consists of a monad T on **Sets** (for which we will use the terminology and notation introduced above) and a full (replete) subcategory $\mathcal{X}$ of Alg(T), closed under subobjects and products, together with a reflection $I = (I, \eta)$ of $\mathrm{Alg}(T)$ into $\mathcal{X}$; in particular, for each algebra A, $\eta_A : A \to I(A)$ is a surjective homomorphism satisfying the suitable universal property. Such a structure is said to be:

(a) a Birkhoff Galois structure, if $\mathcal{X}$ is a Birkhoff subcategory of $\mathrm{Alg}(T)$;
(b) regular-epi-admissible, if for every algebra A and every surjective homomorphism $X \to I(A)$ in $\mathcal{X}$, the canonical morphisms $I(A \times_{I(A)} X) \to X$ is an isomorphism;
(c) admissible, if the same condition holds for every morphism $X \to I(A)$ in $\mathcal{X}$.
(d) trivial, if either $\mathcal{X} = \mathrm{Alg}(T)$, or every algebra in $\mathcal{X}$ has at most one element.

Recall that admissibility is the same as semi-left-exactness in the sense of [6]: see [5] and [22]; see also [8] for the context where *semisimple = attainable = admissible = semi-left-exact = fibered.*

Problem 4.2. *Given a non-trivial monad T on* **Sets**, *consider the reflections*

$$\begin{tikzcd}
\mathrm{Alg}(T) \arrow[rr] \arrow[drrrr] \arrow[ddrrrr] \arrow[ddd] \arrow[dddrr] & & \mathrm{Alg}_0(T) \arrow[drr, dotted] \arrow[ddrr, dotted] \arrow[ddd, dotted] \arrow[dddll, dotted] & & \\
& & & & \mathrm{Alg}_1(T) \arrow[d, dotted] \arrow[ddllll, dotted] \arrow[ddll, dotted] \\
& & & & \mathrm{Alg}_2(T) \arrow[dllll, dotted] \arrow[dll, dotted] \\
\mathrm{Alg}_{ths}(T) & & \mathrm{Alg}_{ts}(T) \arrow[ll, dotted] & &
\end{tikzcd}$$

As follows from Theorem 2.3(b), each of the solid arrows determines a monadic Galois structure, while a dotted one does if and only if its domain is *monadic over Sets. When do these Galois structures satisfy any of the conditions of 4.1(a)-(d)?*

Examples 2.6 and 3.2-3.5 give the following five theorems, respectively (which in fact are also just examples):

Theorem 4.3. *Let T is the compact Hausdorff space monad (=the ultrafilter monad). The Galois structure determined by the reflection of* $\mathrm{Alg}(T)$ *into* $\mathrm{Alg}_{ts}(T)$ = **Stone Spaces** *is an admissible Galois structure that is not a Birkhoff Galois structure.*

Proof: This reflection, mentioned in Example 2.6, is a familiar one, and its admissibility is proved in [5]; the fact the corresponding Galois structure is not a Birkhoff one is obvious. ∎

Theorem 4.4. *Let M be a monoid and T the M-set monad. Then:*

(a) *When $M = \mathbb{N}$ is the additive monoid of natural numbers, the Galois structure determined by the reflection* $\mathrm{Alg}(T) \to \mathrm{Alg}_0(T)$ *is neither regular-epi-admissible nor a Birkhoff Galois structure.*
(b) *The Galois structure determined by the reflection of* $\mathrm{Alg}(T)$ *into* $\mathrm{Alg}_1(T) = \mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T)$ = **Sets** *is an admissible Birkhoff Galois structure.*

Proof: (a): The second assertion is obvious (see Example 3.2(a)). To prove the first one, consider the pullback diagram

$$\begin{tikzcd}
\{a,b\} \times \mathbb{N} \arrow[r] \arrow[d] & \mathbb{N} \arrow[d] \\
\{a,b\} \arrow[r] & 1
\end{tikzcd}$$

in $\mathrm{Alg}(T)$, in which:

- $\{a,b\}$ has the $\mathbb{N}$-set structure, for which each even number determines the identity map of $\{a,b\}$, while each odd number determines the non-identity permutation of $\{a,b\}$;
- $\mathbb{N}$ acts on itself via its addition;
- $1 = I(\{a,b\})$ is the trivial (=one-element) $\mathbb{N}$-set, where I is the reflection $\mathrm{Alg}(T) \to \mathrm{Alg}_0(T)$.

Here both $\{a,b\} \times \mathbb{N}$ and $\mathbb{N}$ belong to $\mathrm{Alg}_0(T)$, and the canonical morphism $I(\{a,b\} \times_{I(\{a,b\})} \mathbb{N}) \to \mathbb{N}$, which is nothing but the projection $\{a,b\} \times \mathbb{N} \to \mathbb{N}$, is not an isomorphism.

(b): The 'Birkhoff part' is obvious, while the admissibility in well known: e.g. it a special case of the situation considered in Example 1.5 in [17], and in fact the story goes back at least to [3] (see also [16]). ■

Note that when M is a group, the reflection $\mathrm{Alg}(T) \to \mathrm{Alg}_0(T)$ determines an admissible Birkhoff Galois structure, simply because in that case $\mathrm{Alg}_0(T) = \mathrm{Alg}_1(T)$, as mentioned in Example 2.7.

Theorem 4.5. (a) *Let T be either the semigroup monad or the commutative semigroup monad. Then the Galois structure determined by the reflection* $\mathrm{Alg}(T) \to \mathrm{Alg}_0(T)$ *is neither regular-epi-admissible nor a Birkhoff Galois structure.*

(b) *Let T be the semigroup monad. Then the Galois structures determined by the reflections of* $\mathrm{Alg}(T)$ *into* $\mathrm{Alg}_1(T) = \mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) =$ **Bands** *and into* $\mathrm{Alg}_{ths}(T)$ *are Birkhoff Galois structures; the first of them is not admissible, while the second one is.*

(c) *Let T is the commutative semigroup monad. Then the Galois structures determined by the reflection of* $\mathrm{Alg}(T)$ *into* $\mathrm{Alg}_1(T) = \mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) =$ **SLat** *is an admissible Birkhoff Galois structure.*

Proof: We only need to check the admissibility:

(a): Consider the pullback diagram

$$\begin{tikzcd} (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{N} \setminus \{0\}) \arrow[r] \arrow[d] & \mathbb{N} \setminus \{0\} \arrow[d] \\ \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

$\mathrm{Alg}(T)$, in which:

- $\mathbb{Z}/3\mathbb{Z}$ is the three-element cyclic group, considered as an additive semigroup;
- $\mathbb{N} \setminus \{0\}$ is the additive semigroup of natural numbers with 0 removed;
- $0 = I(\mathbb{Z}/3\mathbb{Z})$ is the trivial (semi)group, where I is the reflection $\mathrm{Alg}(T) \to \mathrm{Alg}_0(T)$.

Here again (cf. the proof of Theorem 4.4(a)) both $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{N} \setminus \{0\})$ and $\mathbb{N} \setminus \{0\}$ belong to $\mathrm{Alg}_0(T)$, and the canonical morphism

$$I((\mathbb{Z}/3\mathbb{Z}) \times_{I(\mathbb{Z}/3\mathbb{Z})} (\mathbb{N} \setminus \{0\})) \to \mathbb{N} \setminus \{0\},$$

which is nothing but the projection $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{N} \setminus \{0\}) \to \mathbb{N} \setminus \{0\}$, is not an isomorphism.

(b): As follows from the explanation in Example 7 (page 862) of [21], based on the results of [27], there exists a pullback diagram of the form

$$\begin{tikzcd} A \times_{I(A)} 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ A \arrow[r, "\eta_A"'] & I(A) \end{tikzcd}$$

in $\mathrm{Alg}(T)$, in which:

- $I = (I, \eta)$ is the reflection of $Alg(T)$ into $\mathrm{Alg}_1(T) = \mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathbf{Bands}$ the category of idempotent semigroups (bands);
- 1 denotes the trivial (semi)group;
- the (unique) morphism $I(A \times_{I(A)} 1) \to 1$ is not an isomorphism.

This shows that the first reflection in (b) is not admissible; the admissibility of the second one follows from Theorem 3 of [21] (see also [28] and references therein).

(c) follows from (b) since when T is the commutative semigroup monad we have $\mathrm{Alg}_1(T) = \mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T) = \mathbf{SLat}$. ■

Theorem 4.6. (a) *Let T be the lattice monad. Then the Galois structures determined by the reflection of* $\mathrm{Alg}(T)$ *into* $\mathrm{Alg}_{ths}(T) = \mathbf{DLat}$ *is a Birkhoff Galois structure.*

(b) *Let T be the modular lattice monad. Then the Galois structures determined by the reflection of* $\mathrm{Alg}(T)$ *into*

$$\mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T) = \mathbf{DLat}$$

is a Birkhoff Galois structure.

Proof: This is trivial once the equalities mentioned in the theorem are established, which was done in Example 3.4. ■

Theorem 4.7. *Let T be the pregroup monad. Then the Galois structures determined by the reflection of* $\mathrm{Alg}(T)$ *into* $\mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T)$ *is a regular-epi-admissible Birkhoff Galois structure that is not admissible.*

Proof: The fact that it is indeed a Birkhoff Galois structure follows from the description of $\mathrm{Alg}_2(T) = \mathrm{Alg}_{ts}(T) = \mathrm{Alg}_{ths}(T)$ in Example 3.5. The regular-epi-admissibility follows from Theorem 3.4 in [20], and, having in mind the simple relationship between pregroups and groups, the non-admissibility follows from Theorem 3.1 in [14]. ■

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