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AN EPIPERIMETRIC INEQUALITY APPROACH TO THE PARABOLIC SIGNORINI PROBLEM

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ABSTRACT: In this note, we use an epiperimetric inequality approach to study the regularity of the free boundary for the parabolic Signorini problem. We show that if the "vanishing order" of a solution at a free boundary point is close to 3/2 or an even integer, then the solution is asymptotically homogeneous. Furthermore, one can derive a convergence rate estimate towards the asymptotic homogeneous solution. As a consequence, we obtain the regularity of the regular free boundary as well as the singular set.

KEYWORDS: parabolic Signorini problem, epiperimetric inequality, free boundary regularity, singular set.
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CONTENTS

1. Introduction

In this note, we develop a new approach to study the asymptotics of solutions to the parabolic Signorini problem at the free boundary points. More precisely, let u be a solution to

$$\begin{gathered}\partial_t u - \Delta u = f \text{ in } S_2^+\\ u \geq 0,\ \partial_n u \leq 0,\ u\partial_n u = 0 \text{ on } S_2'.\end{gathered} \tag{1}$$

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Here for $R>0$ and space dimension $n\geq 2$

$$S_R^+ := \{(x,t) : x\in\mathbb{R}^n_+,\ t\in[-R,0]\},\quad \mathbb{R}^n_+ := \mathbb{R}_n\cap\{x_n>0\}$$
$$S_R' := \{(x,t) : x\in\mathbb{R}^n\cap\{x_n=0\},\ t\in[-R,0]\}$$

and $f\in L^\infty(S_2^+)$ is a given inhomogeneity. Throughout the paper we will assume that

(A) u is normalized such that

$$\int_{S_2^+} u(x,t)^2 G(x,t)dxdt = 1,$$

where

$$G(x,t) := \begin{cases} (-4\pi t)^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}}, & t<0,\\ 0, & t\geq 0\end{cases}$$

denotes the backward heat kernel in $\mathbb{R}^n\times\mathbb{R}$.

(B) u satisfies the Sobolev regularity in the Gaussian space: there exists C depending on n and $\|f\|_{L^\infty(S_2^+)}$ such that

$$\int_{S_1^+}(-t)^2\left(|D^2u(x,t)|^2+|\partial_t u(x,t)|^2\right)G(x,t)dxdt$$
$$+\sup_{t\in[-1,0]}\int_{\mathbb{R}^n_+}(|\nabla u(x,t)|^2+|u(x,t)|^2)G(x,t)dx\leq C.$$

(C) u satisfies the interior Hölder estimate: there exists $\alpha\in(0,1)$ such that for any $U\Subset\mathbb{R}^n_+\cup(\mathbb{R}^{n-1}\times\{0\})$,

$$\|\nabla u\|_{C^{\alpha,\alpha/2}(U\times[-1,0])}\leq C$$

for some $C>0$ depending on n, α, $\|f\|_{L^\infty(S_2^+)}$ and U. Here $C^{\alpha,\alpha/2}$ is the parabolic Hölder class.

Solutions to (1) can come from solving a variational inequality for the initial value problem in the class of functions with mild growth at infinity and satisfy $u\in L^2([-2,0];W^{1,2}_{loc}(\mathbb{R}^n_+))$, $\partial_t u\in L^2([-2,0];L^2_{loc}(\mathbb{R}^n_+))$, or they can come from solutions to the Signorini problem in a bounded domain (one applies suitable cut-offs to extend them into full space solutions). In both cases, the Sobolev estimate in (B) and the interior Hölder estimate in (C) hold true, cf. [1, 8]. The normalization assumption (A) is put simply to make the constants in (B) and (C) independent of u. Under our regularity assumption, the Signorini boundary condition in (1) holds in the classical sense.

We denote the contact set

$$\Lambda_u := \{(x,t) \in S_2' : u(x,t) = 0\}$$

and the free boundary

$$\Gamma_u := \partial_{S_2'} \Lambda_u.$$

The behavior of a solution around a free boundary point depends very much on how fast it vanishes towards it. In this note we will show that if the "vanishing order" of a solution at a free boundary point is close to $3/2$, which is the expected lowest vanishing order, or $2m$, $m \in \mathbb{N}_+$, which are the eigenvalues of the Ornstein-Uhlenbeck operator $-\frac{1}{2}\Delta - x \cdot \nabla$ in $\mathbb{R}^n_+$ with the vanishing Neumann boundary condition, then the solution is asymptotically a homogeneous solution, and furthermore one can derive a convergence rate estimate towards it. More precisely, assume $(0,0) \in \Gamma_u$, and for $r \in (0,1)$, let

$$H_u(r) := \frac{1}{r^2} \int_{S^+_{r^2}} u(x,t)^2 G(x,t) dx dt$$

be the weighted L^2 space-time average at $(0,0)$. The first theorem is about the asymptotics of the solution around the free boundary point where the vanishing order is around $3/2$:

Theorem 1. *Let u be a solution to* (1) *and satisfy the assumptions (A)-(C). Assume that* $(0,0) \in \Gamma_u$. *Then there exists a constant* $\gamma_0 \in (0,1)$ *depending on n and* $\|f\|_{L^\infty}$ *such that if*

$$c_u r^{3+\gamma_0} \le H_u(r) \le C_u r^{3-\gamma_0}, \text{ for all } r \in (0, r_u) \tag{2}$$

for some $r_u, c_u, C_u > 0$, *then there exists a unique function* $u_0(x) = c_0 \operatorname{Re}(x' \cdot e_0 + i|x_n|)^{3/2}$ *with* $c_0 > 0$ *and* $e_0 \in \mathbb{S}^{n-1} \cap \{x_n = 0\}$ *such that*

$$\int_{\mathbb{R}^n_+} |u(x,t) - u_0(x)|^2 G(x,t) dx \le C(\sqrt{-t})^{3+2\gamma_0}$$

for all $t \in [-1, 0)$. *Here* $C > 0$ *depends on n and* $\|f\|_{L^\infty(S_2^+)}$.

Theorem 1 still holds true, if we assume instead of the lower bound in (2) that the solution at $t = -1$ is sufficiently close to the asymptotic profile, cf. Remark 4.1. We remark that (2) is satisfied if $(0,0)$ is a free boundary point with frequency $\kappa = 3/2$, where the frequency is defined as the limit of the Almgren-Poon frequency function at $(0,0)$, cf. [8] for the precise definition. However, our assumption (2) is much weaker, since it does not rely on the existence of the limit of the frequency function or the optimal spacial regularity. On the other hand, if $\alpha \geq \frac{1-\gamma_0}{2}$ in the assumption (C), then we can instead derive the optimal interior regularity $\nabla u \in C^{1/2,1/4}$ from Theorem 1.

The next theorem is about the logarithmic convergence of the solution towards the asymptotic homogeneous solution around the free boundary point with the vanishing order close to even integers. For that, we let $\mathcal{E}_{2m}$ be the finite dimensional space spanned by the $2m$-order Hermite polynomials in $\mathbb{R}^n$ which are even about $\{x_n = 0\}$. Note that $\mathcal{E}_{2m}$ is the $2m$-eigenspace of the Ornstein-Uhlenbeck operator $-\frac{1}{2}\Delta + x\cdot\nabla$ on $\mathbb{R}^n_+$ with the vanishing Neumann boundary condition on $\{x_n = 0\}$. Let $\mathcal{E}^+_{2m} = \{p \in \mathcal{E}_{2m} : p \geq 0 \text{ on } \{x_n = 0\}\}$. We remark that given $\bar{p} \in \mathcal{E}^+_{2m}$, the function $p(x,t) = (\sqrt{-t})^{2m}\bar{p}(\frac{x}{2\sqrt{-t}})$, $t \in (-\infty, 0)$, is a $2m$-parabolic homogeneous solution to (1).

Theorem 2. *Let u be a solution to* (1) *and satisfy the assumptions (A)-(C) with $f = 0$. Let $m \in \mathbb{N}_+$ be an arbitrary positive integer. Assume that $(0,0) \in \Gamma_u$. Then there exist small constants $\gamma_m, \delta_0 \in (0,1)$ depending only on m, n, such that if*

$$H_u(r) \leq C_u r^{2\kappa - \gamma_m}, \quad \kappa = 2m,$$

for each $r \in (0, r_u)$ for some $r_u, C_u > 0$, and at $t = -1$

$$\begin{aligned} &\mathrm{dist}_{L^2_{\tilde\mu}(\mathbb{R}^n_+)}(u(\cdot,-1), \mathcal{E}^+_{2m})^2 \leq \delta_0 \|u(\cdot,-1)\|^2_{L^2_{\tilde\mu}(\mathbb{R}^n_+)}, \\ &d\tilde\mu(x) := G(x,-1)dx = c_n e^{-\frac{|x|^2}{4}} dx, \end{aligned}$$

then there exists a unique nonzero $p_0(x,t) = (\sqrt{-t})^{2m}\bar{p}_0(\frac{x}{2\sqrt{-t}})$ with $\bar{p}_0 \in \mathcal{E}^+_{2m}$ such that

$$\int_{\mathbb{R}^n_+} |u(x,t) - p_0(x,t)|^2 G(x,t)dx \leq C_{n,m}(\sqrt{-t})^{4m}(-\ln t)^{-\frac{2\gamma}{1-\gamma}}, \quad \gamma = \frac{1}{n+1}$$

for all $t \in [-1, 0)$.

It is possible to generalize Theorem 2 to a nonzero inhomogeneity f, where f satisfies an additional vanishing property at $(0,0)$: $|f(x,t)| \leq M(|x| + \sqrt{-t})^{2(m-1+\epsilon_0)}$ for some $\epsilon_0 > 0$ and $M > 0$, cf. Section 5. We state and prove the theorem for $f = 0$ to avoid the technicalities caused by the inhomogeneity such that the proof looks neater. The assumptions of Theorem (2) are satisfied at the free boundary points with the frequency $2m$, cf. [8], but here we do not rely on the existence of the limit of the frequency function. Note that in Theorem 2 we obtain a polynomial decay in $\ln(-t)$ instead of an exponential decay as in Theorem 1. A polynomial decay rate of this kind towards the asymptotic solutions was obtained originally in the elliptic case [5, 6, 7]. Moreover, in the classical obstacle problem it was shown, that there is in general no exponential decay rate at singular points, cf. [10].

The proofs for Theorem 1 and Theorem 2 are based on establishing a decay rate for the Weiss energy W_κ, $\kappa = 3/2$ and $\kappa = 2m$, in the self-similar conformal coordinates. In the elliptic problems, such change of coordinates corresponds to $(r, \theta) \mapsto (t, \theta) = (-\ln r, \theta)$ from $(0, 1] \times \mathbb{S}^{n-1}$ to $[0, \infty) \times \mathbb{S}^{n-1}$, which transforms the original problem around the free boundary point at 0 to an evolutionary problem on $\mathbb{S}^{n-1}$. The long time asymptotics of the solution then corresponds to (back in the original coordinates) the blow-up limits at the origin. In the parabolic setting, we can (formally) formulate our problem in the self-similar conformal coordinates as a gradient flow of the Weiss energy over an admissible set of functions under the convex constraint $u \geq 0$ on $\{x_n = 0\}$. The relation between the Weiss energy and the evolution of certain quantities thus becomes more transparent, cf. Lemma 2.4.

The main step in the proof are discrete decay estimates for the Weiss energy W_κ, $\kappa = 3/2$ and $\kappa = 2m$, which can be viewed as parabolic epiperimetric inequalities. Epiperimetric inequalities were introduced by G. Weiss [14] to the classical obstacle problem and they continue to be a subject of intense research interest in the elliptic setting [5, 6, 7, 11, 12]. We treat the free boundary points with frequency $3/2$ and $2m$, because these are the cases where one can classify all the stationary solutions, and furthermore, we strongly make use of the special structures of these stationary solutions.

We briefly comment about our results and the related literature. Theorem 1 allows to provide a simpler proof for the openness of the regular free boundary and its space-time regularity (cf. Section 4), which was shown in [2, 8, 13] by using a comparison principle and boundary Harnack approach. In particular, the method does not require the optimal spacial regularity or continuity of the time derivative of the solutions. Theorem 2 generalizes the results about the singular set in [8] in the sense that we obtain a logarithmic modulus of continuity, which implies a frequency gap around the

$2m$-frequency free boundary points (cf. Section 4). The logarithmic epiperimetric inequality was recently established for the elliptic obstacle and thin obstacle problems [5, 6, 7], which inspired our paper. It is also possible to generalize our approach to the Signorini problem for the degenerate parabolic operators considered in [2, 3, 4].

The rest of the paper is structured as follows: in Section 2 we introduce the conformal change of coordinates, reformulate our problem in the new coordinates and introduce the corresponding Weiss energy; In Section 3 we prove discrete decay estimates for the Weiss energy W_κ with $\kappa = 3/2$ and $\kappa = 2m$. For simplicity we assume that the inhomogeneity f vanishes. The idea of the proof remains the same for nonzero inhomogeneities, as one can see from Section 5. In Section 4 we show the consequences of the decay estimates, for example, we prove the $C^{1,\alpha}$ regularity of the regular free boundary, frequency gap around $2m$-frequency and the structure of the singular set. In the last section we will show how to modify the proof in Section 3 to the inhomogeneous setting.

2. Conformal self-similar coordinates and Weiss energy

In the sequel, it will prove convenient to work in conformal self-similar coordinates. This will simplify many of the computations, which will be carried out for the Weiss energy.

Thus, we consider the following change of variables, which should be viewed as the analogue of conformal polar coordinates:

Lemma 2.1. *Let $u : S_2^+ \to \mathbb{R}$ be a solution to* (1). *We consider the change of coordinates*

$$\begin{aligned} \mathcal{T} : \mathbb{R}^n \times [-1, 0) &\to \mathbb{R}^n \times [0, \infty) \\ (x, t) &\mapsto (y, \tau) = \left(\frac{x}{2\sqrt{-t}}, -\ln(-t) \right). \end{aligned} \tag{3}$$

For $\kappa > 0$ we denote

$$\tilde{u}_\kappa(y, \tau) := \frac{u(x, t)}{(\sqrt{-t})^\kappa}\Big|_{(x,t)=\mathcal{T}^{-1}(y,\tau)} = e^{\tau\kappa/2} u(2e^{-\tau/2} y, -e^{-\tau}).$$

Then, $\tilde{u}_\kappa$ is a solution to

$$\partial_\tau \tilde{u} + \frac{y}{2} \cdot \nabla \tilde{u} - \frac{1}{4} \Delta \tilde{u} - \frac{\kappa}{2} \tilde{u} = 0 \ \text{in} \ \mathbb{R}^n_+ \times [0, \infty) \tag{4}$$

with the Signorini condition

$$\tilde{u} \geq 0,\ \partial_n \tilde{u}(y) \leq 0,\ \tilde{u}\partial_n \tilde{u} = 0 \text{ on } \{y_n = 0\}. \tag{5}$$

Proof: The proof follows from a direct computation. ∎

Remark 2.2. *For later use, we remark that the above change of coordinates is reversible. In particular, if $\tilde{u}_\kappa$ is a stationary solution to* (4) *and* (5)*, then the function*

$$u(x,t) := (\sqrt{-t})^\kappa \tilde{u}_\kappa \left(\frac{x}{2\sqrt{-t}}\right),\ t < 0$$

is a parabolically κ-homogeneous *solution to* (1)*, i.e. $u(x,t)$ solves the equation* (1) *and satisfies*

$$u(\lambda x, \lambda^2 t) = \lambda^\kappa u(x,t) \text{ for all } \lambda > 0.$$

Remark 2.3. *Let $u : S_1^+ \to \mathbb{R}$ satisfy the Sobolev regularity (A) and (B). Let $\tilde{u}_\kappa$ be obtained from u as in Lemma 2.1. Then it holds*

$$\|\tilde{u}_\kappa(y,0)\|_{L^2_\mu(\mathbb{R}^n_+)} = \pi^{n/2}\|u(x,-1)\|_{L^2_{\tilde{\mu}}(\mathbb{R}^n_+)}, \quad d\mu(y) := e^{-|y|^2}dy,\ d\tilde{\mu}(x) := e^{-\frac{|x|^2}{4}}dx.$$

Moreover, for any $0 \leq \tau_1 < \tau_2 < \infty$, there exists a constant C depending on $\tau_1, \tau_2, \kappa, n, \|f\|_{L^\infty}$ such that

$$\int_{\tau_1}^{\tau_2} \|D^2\tilde{u}_\kappa\|^2_{L^2_\mu} + \|\partial_\tau \tilde{u}_\kappa\|^2_{L^2_\mu} d\tau \leq C,$$
$$\sup_{\tau\in(0,\tau_2]} \|\nabla \tilde{u}_\kappa(\cdot,\tau)\|_{L^2_\mu} + \|\tilde{u}_\kappa(\cdot,\tau)\|_{L^2_\mu} \leq C.$$

Here and in the sequel $\|\cdot\|_{L^2_\mu} := \|\cdot\|_{L^2_\mu(\mathbb{R}^n_+)}$. Thus

$$\tilde{u}_\kappa \in L^\infty_{loc}([0,\infty); W^{1,2}_\mu) \cap L^2_{loc}([0,+\infty); W^{2,2}_\mu),\ \partial_\tau \tilde{u}_\kappa \in L^2_{loc}([0,\infty); L^2_\mu). \tag{6}$$

Now we define the Weiss energy associated to a solution $\tilde{u} := \tilde{u}_\kappa$ to (4)–(5) and derive relevant quantities of the Weiss energy.

Lemma 2.4. *Let $\tilde{u} = \tilde{u}_\kappa$ be a solution to* (4)–(5) *and satisfy* (6)*. We define the Weiss energy*

$$W_\kappa(\tilde{u}(\tau)) := \int_{\mathbb{R}^n_+} \frac{1}{4}|\nabla \tilde{u}(y,\tau)|^2 - \frac{\kappa}{2}\tilde{u}^2(y,\tau)d\mu. \tag{7}$$

Then for $0 \le \tau_1 \le \tau_2 < \infty$

$$W_\kappa(\tilde{u}(\tau_2)) - W_\kappa(\tilde{u}(\tau_1)) \le -2 \int_{\tau_1}^{\tau_2} (\partial_\tau \tilde{u}(y,\tau))^2 d\mu d\tau. \tag{8}$$

Further,

$$W_\kappa(\tilde{u}(\tau)) = -\frac{1}{2}\partial_\tau \|\tilde{u}(\tau)\|^2_{L^2_\mu}, \ \ a.e.\ \tau \in [0,\infty). \tag{9}$$

Proof: On a formal level the estimate (8) can be deduced by differentiating the functional $W_\kappa(\tilde{u}(\tau))$. However, in order to give meaning to the arising boundary contributions, it is necessary to work in a regularized framework which is achieved by penalization. More precisely, we consider the following penalized version of (5), (4): Let $\beta_\epsilon : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying the following properties

$$\begin{aligned} &\beta_\epsilon(s) = 0 \text{ for } s \ge 0,\\ &\beta_\epsilon(s) = \epsilon + \frac{s}{\epsilon} \text{ for all } s \le -2\epsilon^2,\\ &\beta'_\epsilon(s) \ge 0 \text{ for all } s \in \mathbb{R}. \end{aligned}$$

We approximate $\tilde{u}$ by solutions to the penalization problem

$$\begin{aligned} \partial_\tau \tilde{u}^\epsilon + \frac{y}{2}\cdot\nabla\tilde{u}^\epsilon - \frac{1}{4}\Delta\tilde{u}^\epsilon - \frac{\kappa}{2}\tilde{u}^\epsilon &= 0 \text{ on } \mathbb{R}^n_+ \times (0,\tau_2),\\ \partial_n \tilde{u}^\epsilon &= \beta_\epsilon(\tilde{u}^\epsilon) \text{ on } \{y_n = 0\} \times (0,\tau_2),\\ \tilde{u}^\epsilon(y,0) &= \tilde{u}^\epsilon_0(y) \text{ at } \mathbb{R}^n_+ \times \{0\}. \end{aligned}$$

Here $\tilde{u}^\epsilon_0$ is a smooth compact supported function such that $\|\tilde{u}^\epsilon_0 - \tilde{u}(\cdot,0)\|_{L^2_\mu} \to 0$ as $\epsilon \to 0$. There exists a unique solution $\tilde{u}^\epsilon$ with a polynomial growth as $|y| \to \infty$. The function $\tilde{u}^\epsilon$ is smooth and satisfies the uniform bound in the Gaussian space (cf. Chap. 3 in [8]): there exists $C = C(\kappa, \tau_2, \|\tilde{u}^\epsilon_0\|_{L^2_\mu}, n)$ such that

$$\|D^2\tilde{u}^\epsilon\|_{L^2_\tau L^2_\mu} + \|\partial_\tau \tilde{u}^\epsilon\|_{L^2_\tau L^2_\mu} + \|\tilde{u}^\epsilon\|_{L^\infty_\tau L^2_\mu} + \|\nabla\tilde{u}^\epsilon\|_{L^\infty_\tau L^2_\mu} \le C.$$

In particular, using the equation for $\tilde{u}^\epsilon$, it is then possible to compute as follows:

$$\frac{d}{d\tau} W_\kappa(\tilde{u}^\epsilon(\tau)) = -2 \int_{\mathbb{R}^n_+} (\partial_\tau \tilde{u}^\epsilon)^2 d\mu - \frac{1}{2} \int_{\{y_n=0\}} \beta_\epsilon(\tilde{u}^\epsilon)(\partial_\tau \tilde{u}^\epsilon) d\mu.$$

We test this identity with $\varphi \in C_c^\infty((0,\infty))$ which yields

$$\int_0^\infty \varphi(\tau)\frac{d}{d\tau}W_\kappa(\tilde{u}^\epsilon(\tau))d\tau = -2\int_0^\infty \varphi(\tau)\int_{\mathbb{R}^n_+}(\partial_\tau \tilde{u}^\epsilon)^2 d\mu d\tau$$
$$-\frac{1}{2}\int_0^\infty \varphi(\tau)\int_{\{y_n=0\}}\beta_\epsilon(\tilde{u}^\epsilon)(\partial_\tau \tilde{u}^\epsilon)d\mu d\tau.$$

The a priori bounds for $\tilde{u}^\epsilon$ leads to space time $W^{1,2}_\mu$ uniform estimates. Hence it is possible to use lower semi-continuity to pass to the weak limit in the bulk integral. Using that $\beta_\epsilon(\tilde{u}^\epsilon)\partial_\tau\tilde{u}^\epsilon = \partial_\tau \mathcal{B}_\epsilon(\tilde{u}^\epsilon)$, where $\mathcal{B}_\epsilon$ is the primitive function of β_ϵ with $\mathcal{B}_\epsilon(0)=0$, the boundary integral is treated as in Lemma 5.1 (3°) in [8] and can be shown to vanish in the limit. Hence for $\varphi \geq 0$ we infer

$$-\int_0^\infty \varphi'(\tau)W_\kappa(\tilde{u}(\tau))d\tau \leq -2\int_0^\infty \varphi(\tau)\int_{\mathbb{R}^n_+}(\partial_\tau\tilde{u})^2 d\mu d\tau,$$

which yields the desired result. Approximating the characteristic function $\chi_{[\tau_1,\tau_2]}(t)$ by smooth positive functions then yields the claim on the sign of the difference of the Weiss functionals.

Finally, the identity (9) is a consequence of the equation (5) in conjunction with the the Signorini condition $\tilde{u}\partial_n\tilde{u} = 0$ on $\{y_n = 0\}$. ■

As direct consequences of Lemma 2.4, we infer the following properties:

Corollary 2.5. *Let $\tilde{u} : \mathbb{R}^n_+ \times [0,\infty) \to \mathbb{R}$ be a solution to* (5), (4) *which satisfies* (6). *Then, the function*

$$\tau \mapsto \|\tilde{u}(\tau)\|^2_{L^2_\mu}$$

is convex. Moreover, $\tilde{u}$ is a stationary solution if and only if $W_\kappa(\tilde{u}(\tau)) \equiv 0$.

In the sequel, we will in particular exploit the second observation frequently. When $\kappa = 3/2$ and $\kappa = 2m$, $m \in \mathbb{N}_+$, by Liouville type theorems, we can characterize stationary solutions.

Proposition 2.6. *Let $\tilde{u} \in W^{2,2}_\mu$ be a solution to*

$$\begin{aligned} -\frac{1}{2}\Delta\tilde{u} + y\cdot\nabla\tilde{u} - \kappa\tilde{u} = 0 \ \textit{in}\ \mathbb{R}^n_+, \\ \tilde{u} \geq 0,\ \partial_n\tilde{u} \leq 0,\ \tilde{u}\partial_n\tilde{u} = 0 \ \textit{on}\ \mathbb{R}^{n-1}\times\{0\}. \end{aligned} \tag{10}$$

We extend $\tilde{u}$ evenly about $\{y_n = 0\}$. If $\kappa = 3/2$, then

$$\tilde{u} \in \mathcal{E}_{3/2} := \{c\,\mathrm{Re}(y'\cdot e + i|y_n|)^{3/2},\ c\geq 0,\ |e| = 1,\ e\cdot e_n = 0.\}$$

If $\kappa = 2m$ for $m \in \mathbb{N}_+$, then

$$\begin{aligned} \tilde{u} \in \mathcal{E}^+_{2m} := \{p : p = \sum_{\alpha:|\alpha|=2m} \lambda_\alpha H_{\alpha_1}(y_1)\cdots H_{\alpha_n}(y_n),\ \lambda_\alpha \in \mathbb{R}, \\ p(y', y_n) = p(y', -y_n),\ p(y', 0) \geq 0\}. \end{aligned}$$

Here $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, H_{α_i} is the 1*d-Hermite polynomial of order α_i, $i \in \{1, \cdots, n\}$.*

Proof: *(i) Case $\kappa = 3/2$.* We will prove that $\tilde{u}$ is two dimensional. Given any tangential direction e with $|e| = 1$ and $e\cdot e_n = 0$, $v := \partial_e\tilde{u}$ solves the Dirichlet eigenvalue problem for $L_0 := -\frac{1}{2}\Delta + y\cdot\nabla$ on $W^{1,2}_0(\mathbb{R}^n \setminus \Lambda_{\tilde{u}}; d\mu) \subset L^2_\mu(\mathbb{R}^n)$:

$$L_0 v = \frac{1}{2}v \text{ in } \mathbb{R}^n \setminus \Lambda_{\tilde{u}}, \quad v = 0 \text{ on } \Lambda_{\tilde{u}},$$

where $\Lambda_{\tilde{u}} := \{(y', 0) : \tilde{u}(y', 0) = 0\}$. We claim that v does not change the sign in $\mathbb{R}^n$. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ denote the Dirichlet eigenvalue. Assume that v changes the sign, then necessarily $\lambda_2 \leq \frac{1}{2}$. By the min-max theorem,

$$\begin{aligned} \lambda_2 = \inf\left\{\sup\left\{\frac{1}{2}\int |\nabla v|^2 d\mu : v \in M,\ \|v\|_{L^2_\mu} = 1\right\} : \right. \\ \left. M \subset W^{1,2}_0(\mathbb{R}^n \setminus \Lambda_{\tilde{u}}; d\mu) \text{ subspace and } dim(M) = 2\right\}. \end{aligned}$$

Since $v(y) = y_n \in W^{1,2}_0(\mathbb{R}^n \setminus \Lambda_{\tilde{u}}; d\mu)$, the Rayleigh quotient of which is equal to 1, then necessarily $\lambda_2 \geq 1$. This is a contradiction. Therefore we conclude that $v = \partial_e\tilde{u}$ is nonpositive or nonnegative in the whole space $\mathbb{R}^n$. Since this holds for any tangential direction, it follows that $\tilde{u}$ is of the form $\tilde{u}(y) = \tilde{u}(y'\cdot e, y_n)$ for some tangential direction e. In other words, $\tilde{u}$ is two dimensional. Direct computation shows that the function $\mathrm{Re}(y'\cdot e + i|y_n|)^{1/2}$ is an eigenfunction. The uniqueness of the principal eigenfunction implies that actually $\partial_e\tilde{u} = c\,\mathrm{Re}(y'\cdot e + i|y_n|)^{1/2}$ for some $c \in \mathbb{R}$. Thus $\tilde{u} \in \mathcal{E}_{3/2}$.

(ii) Case $\kappa = 2m$. This follows directly from Lemma 12.4 of [8] and Remark 2.2. ∎

At the end of this section we compare the Weiss energy in the original coordinates and the conformal coordinates. Firstly, if $\tilde{u}_\kappa$ is associated with a solution $u : S_2^+ \to \mathbb{R}$ to the parabolic Signorini problem (1) as in Lemma 2.1, then the Weiss energy of $\tilde{u}_\kappa$ in (7) can be rewritten in terms of u as

$$W_\kappa(u(t)) = \frac{1}{(-t)^{\kappa-1}} \int\limits_{\mathbb{R}^n_+} |\nabla u(x,t)|^2 G(x,t)dx - \frac{\kappa}{2(-t)^\kappa} \int\limits_{\mathbb{R}^n_+} u(x,t)^2 G(x,t)dx. \tag{11}$$

Next, for $\lambda > 0$, let

$$u_\lambda(x,t) := \frac{u(\lambda x, \lambda^2 t)}{\lambda^\kappa},$$

be the (parabolic) κ-homogeneous scaling, and let

$$\tilde{u}_{\kappa,\lambda}(y,\tau) := \frac{u_\lambda(x,t)}{(\sqrt{-t})^\kappa}\Big|_{(x,t)=\mathcal{T}^{-1}(y,\tau)}$$

as in Lemma 2.1. Then

$$\tilde{u}_{\kappa,\lambda}(y,\tau) = \tilde{u}_\kappa(y, \tau - 2\ln\lambda), \tag{12}$$

i.e. the homogeneous κ scaling for $u(x,t)$ corresponds to the time shift for $\tilde{u}_\kappa(y,\tau)$ by $-2\ln\lambda$. The Weiss energy in the original coordinates is well-behaved with respect to the parabolic rescaling, i.e. $W_\kappa(u_\lambda(t)) = W_\kappa(u(\lambda^2 t))$. In the conformal coordinates this leads to

$$W_\kappa(\tilde{u}_\kappa(\tau - 2\ln\lambda)) = W_\kappa(\tilde{u}_{\kappa,\lambda}(\tau)). \tag{13}$$

3. Parabolic epiperimetric inequality

We describe a dynamical system approach for deriving the decay of the Weiss energy $W_\kappa(\tilde{u}(\tau))$ along solutions to (4)–(5) with $\kappa = 3/2$ or $\kappa = 2m$, $m \in \mathbb{N}_+$.

3.1. The case $\kappa = \frac{3}{2}$. In this case, stationary solutions to (4)–(5) are in $\mathcal{E}_{3/2}$ by Proposition 2.6. We will project our solution $\tilde{u}(\tau) := \tilde{u}_{3/2}(\tau)$ to $\mathcal{E}_{3/2}$ for each $\tau > 0$, i.e. let

$$\lambda(\tau) h_{e(\tau)}(y) := \lambda(\tau) h(e(\tau) \cdot y', y_n),$$

where $h(x_1, x_2) := c_n \operatorname{Re}(x_1 + i|x_2|)^{3/2}$, $c_n > 0$ and $\|h\|_{L^2_\mu(\mathbb{R}^n_+)} = 1$, be such that

$$\left\|\tilde{u}(\tau) - \lambda(\tau)h_{e(\tau)}\right\|_{L^2_\mu} = \operatorname{dist}_{L^2_\mu}(\tilde{u}(\tau), \mathcal{E}_{3/2}) = \inf_{\substack{\lambda \geq 0,\\ e \in \mathbb{S}^{n-1} \cap \{y_n = 0\}}} \|\tilde{u}(\tau, \cdot) - \lambda h_e\|_{L^2_\mu},$$

and study the evolution of $\operatorname{dist}_{L^2_\mu}(\tilde{u}(\tau, \cdot), \mathcal{E}_{3/2})$.

Due to the non-convexity of $\mathcal{E}_{3/2}$ the projection λh_e of $\tilde{u}(\tau, \cdot)$ onto $\mathcal{E}_{3/2}$ is not necessarily unique, hence the regularity of the parameters λ, e in dependence of τ is in question. In particular this implies that we have to take care in our dynamical systems argument and can not directly work with the evolution equations for the parameters $\lambda(\tau)$ and $e(\tau)$. Instead, we rely on robust (energy type) identities for the Weiss energy.

To this end, we split $\tilde{u}$ into its leading order profile and an error:

$$\tilde{u}(y, \tau) = \lambda(\tau)h_{e(\tau)}(y) + \tilde{v}(y, \tau).$$

Here $\lambda(\tau)h_{e(\tau)}(y)$ is chosen such as to minimize the L^2_μ distance of $\tilde{u}$ to the set $\mathcal{E}_{3/2}$. We stress again, that this decomposition is a priori not necessarily unique. From the minimality of $\|\tilde{u}(y, \tau) - \lambda(\tau)h_{e(\tau)}(y)\|_{L^2_\mu}$ we infer the following orthogonality conditions

$$\lambda(\tau) \int_{\mathbb{R}^n_+} h_{e(\tau)}(y)\tilde{v}(y, \tau) d\mu = 0, \tag{14}$$

$$\lambda(\tau) \int_{\mathbb{R}^n_+} \operatorname{Re}(y \cdot \tilde{e}(\tau) + i|y_n|)^{1/2}(y \cdot \tilde{e}(\tau))\tilde{v}(y, \tau) d\mu = 0,$$
$$\forall \tilde{e} \in \mathbb{S}^{n-1}, \tilde{e} \perp \operatorname{span}\{e_n, e(\tau)\}. \tag{15}$$

Here we have used that $c \operatorname{Re}(x_1 + i|x_2|)^{1/2} = \partial_{x_1} h(x_1, x_2)$. We rewrite the Weiss energy $W(\tilde{u}) := W_{3/2}(\tilde{u})$ in terms of $\tilde{v}$ as

$$W(\tilde{u}) = W(\tilde{v}) - \frac{\lambda}{2} \int_{\{y_n = 0\}} \tilde{u} \partial_n h_e d\mu. \tag{16}$$

To see this, we observe that since $W(h_e)=0$,

$$\begin{aligned} W(\tilde{u}) &= W(\tilde{v}) + \frac{2\lambda}{4}\int\limits_{\mathbb{R}^n_+} \nabla\tilde{v}\cdot\nabla h_e d\mu \\ &= W(\tilde{v}) - \frac{\lambda}{2}\int_{\{y_n=0\}} \tilde{v}\partial_n h_e d\mu - \frac{\lambda}{2}\int\limits_{\mathbb{R}^n_+} \tilde{v}\operatorname{div}(e^{-|y|^2}\nabla h_e)dy. \end{aligned}$$

Using that $\operatorname{div}(e^{-|y|^2}\nabla h_e) = e^{-|y|^2}(\Delta h_e - 2y\cdot\nabla h_e) = -3h_e e^{-|y|^2}$, the orthogonality condition (14) and that $\tilde{u}\partial_n h_e = \tilde{v}\partial_n h_e$ on $\{y_n=0\}$, we obtain (16).

As our main auxiliary result, we deduce the following contraction argument, which is of the flavour of an epiperimetric inequality.

Proposition 3.1. *Let $\tilde{u}:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ be a solution of the parabolic thin obstacle problem* (4)–(5) *and satisfy* (6)*. Then there exists a constant $c_0\in(0,1)$ depending only on n, such that*

$$W(\tilde{u}(\tau+1)) \leq (1-c_0)W(\tilde{u}(\tau)) \text{ for any } \tau\in(0,\infty).$$

Proof: We argue by contradiction and use the contradiction assumption in combination with (16) to derive enough compactness.

(i). Assume that the statement were not true. Then there exists a sequence $\{c_j\}$ with $c_j\in(0,1/2)$, $c_j\to 0$, solutions $\tilde{u}_j$ and times τ_j such that

$$W(\tilde{u}_j(\tau_j+1)) > (1-c_j)W(\tilde{u}_j(\tau_j)). \tag{17}$$

The contradiction assumption (17) implies that

$$W(\tilde{u}_j(\tau_j+1)) - W(\tilde{u}_j(\tau_j)) > \frac{-c_j}{1-c_j}W(\tilde{u}_j(\tau_j+1)).$$

Using (8) and the monotone decreasing property of $\tau\mapsto W(\tilde{u}(\tau))$ we infer

$$\int\limits_{\tau_j}^{\tau_j+1} \|\partial_\tau\tilde{u}_j\|^2_{L^2_\mu}d\tau < \frac{c_j}{2(1-c_j)}W(\tilde{u}_j(\tau_j+1)) \leq c_j\int\limits_{\tau_j}^{\tau_j+1} W(\tilde{u}_j(\tau))d\tau. \tag{18}$$

(ii). We show that

$$\int_{\tau_j}^{\tau_j+1} W(\tilde{v}_j(\tau))d\tau \leq 2c_j\int_{\tau_j}^{\tau_j+1} \|\tilde{v}_j(\tau)\|^2_{L^2_\mu}d\tau. \tag{19}$$

First we observe that for any solution $\tilde{u}$ and any time interval $I = [\tau_a, \tau_b]$,

$$\int\limits_I W(\tilde{u}(\tau))d\tau \overset{(9)}{=} -\int\limits_I \frac{1}{2}\partial_\tau \|\tilde{u}\|^2_{L^2_\mu} d\tau = -\int\limits_{\mathbb{R}^n_+ \times I} \lambda \partial_\tau \tilde{u} h_e d\mu d\tau - \int\limits_{\mathbb{R}^n_+ \times I} \partial_\tau \tilde{u}\tilde{v} d\mu d\tau$$

Using that $\partial_\tau \tilde{u} = \mathcal{L}\tilde{u}$ in $\mathbb{R}^n_+ \times (0,\infty)$, where $\mathcal{L} := \frac{1}{4}\Delta - \frac{y}{2}\cdot\nabla + \frac{3}{4}$, $\mathcal{L}h_e = 0$ in $\mathbb{R}^n_+$ and an integration by parts we have

$$\begin{aligned}\int\limits_I W(\tilde{u}(\tau))d\tau &= -\int\limits_{\mathbb{R}^n_+ \times I} \partial_\tau \tilde{u}\tilde{v} d\mu d\tau - \int\limits_{\mathbb{R}^n_+ \times I} \mathcal{L}\tilde{u}(\lambda h_e) d\mu d\tau \\ &= -\int\limits_{\mathbb{R}^n_+ \times I} \partial_\tau \tilde{u}\tilde{v} d\mu d\tau + \frac{\lambda}{4}\int_{\{y_n=0\}} \left(\partial_n \tilde{u} h_e - \tilde{u}\partial_n h_e\right) d\mu d\tau.\end{aligned} \tag{20}$$

We apply (20) to $\tilde{u}_j$ with $I_j := [\tau_j, \tau_j + 1]$. For the first integral we use Hölder and (18) to get

$$\left| \int_{\mathbb{R}^n_+ \times I_j} \partial_\tau \tilde{u}_j \tilde{v}_j d\mu d\tau \right| \leq \left(c_j \int_{I_j} W(\tilde{u}_j(\tau)) d\tau \right)^{1/2} \left(\int_{I_j} \|\tilde{v}_j\|^2_{L^2_\mu} d\tau \right)^{1/2}. \tag{21}$$

Combining (20) and (21) and using Young's inequality, we obtain

$$\frac{3}{4}\int_{I_j} W(\tilde{u}_j(\tau))d\tau \leq c_j \int_{I_j} \|\tilde{v}_j\|^2_{L^2_\mu} d\tau + \int\limits_{I_j}\int\limits_{\{y_n=0\}} \frac{\lambda_j}{4}\left(\partial_n \tilde{u}_j h_{e_j} - \tilde{u}_j \partial_n h_{e_j}\right) d\mu d\tau.$$

Recalling the relation between $W(\tilde{u})$ and $W(\tilde{v})$ in (16), we infer

$$\frac{3}{4}\int_{I_j} W(\tilde{v}_j(\tau))d\tau \leq c_j \int_{I_j} \|\tilde{v}_j\|^2_{L^2_\mu} d\tau + \int\limits_{I_j}\int\limits_{\{y_n=0\}} \left(\frac{\lambda_j}{4}\partial_n \tilde{u}_j h_{e_j} + \frac{\lambda_j}{8}\tilde{u}_j \partial_n h_{e_j}\right) d\mu d\tau. \tag{22}$$

Since, by the Signorini conditions, the second integral on the right hand side is less or equal to zero, we obtain the upper bound in (19). We remark that by rearrangement (22) also entails that

$$\begin{aligned}&-\frac{\lambda_j}{8}\int_{I_j}\int\limits_{\{y_n=0\}} \left(\partial_n \tilde{u}_j h_{e_j} + \tilde{u}_j \partial_n h_{e_j}\right) d\mu d\tau \\ &\leq c_j \int_{I_j} \|\tilde{v}_j\|^2_{L^2_\mu} d\tau - \frac{3}{4}\int_{I_j} W(\tilde{v}_j(\tau))d\tau \leq \left(c_j + \frac{3}{4}\right)\int_{I_j} \|\tilde{v}_j\|^2_{L^2_\mu} d\tau.\end{aligned} \tag{23}$$

In the next steps *(iii)-(iv)* we will use a compactness argument to arrive at a contradiction. The main idea is that one can find sequences $\hat{\tau}_j \in I_j$ and $\hat{v}_j(y) := \tilde{v}_j(\hat{\tau}_j, y)/\|\tilde{v}_j(\hat{\tau}_j)\|_{L^2_\mu}$, such that $\hat{v}_j$ converges to a nonzero blow-up profile in $\mathcal{E}_{3/2}$. This leads to a contradiction. The bounds on the Weiss energy for $\tilde{v}$ in step *(ii)* are used to derive the desired compactness properties.

(iii). We seek to prove that up to a subsequence

$$\hat{u}_j(y,\tau) := \frac{\tilde{u}_j(y, \tau_j + \tau)}{\|\tilde{u}_j\|_{L^2(I_j;L^2_\mu)}}, \ \tau \in [0,1] \tag{24}$$

converges weakly in $L^2([0,1]; W^{1,2}_\mu)$, strongly in $C([0,1]; L^2_\mu)$ and locally in $C^1_x C^0_t$ up to $\{y_n = 0\}$ to some stationary solution $\hat{u}_0$ to (4)–(5). Note that $\|\hat{u}_j\|_{L^2([0,1];L^2_\mu)} = 1$ by our normalization, thus the strong L^2-convergence implies that $\|\hat{u}_0\|_{L^2_\mu} = 1$. By Proposition 2.6 necessarily $\tilde{u}_0 = c_n h(y' \cdot e_0, y_n) \in \mathcal{E}_{3/2}$ for some tangential direction e_0.

First we note that by (17) and the monotone decreasing property of $\tau \mapsto W(\tilde{u}(\tau))$, we have $W(\tilde{u}_j(\tau)) \le 2 \int_{I_j} W(\tilde{u}_j(\tau)) d\tau$ for all $\tau \in I_j$. Then by (16), (19) and (23), for some absolute constant $C > 0$,

$$W(\tilde{u}_j(\tau)) \le C \int_{I_j} \|\tilde{v}_j(\tau)\|^2_{L^2_\mu} d\tau \le C \int_{I_j} \|\tilde{u}_j(\tau)\|^2_{L^2_\mu} d\tau, \quad \tau \in I_j. \tag{25}$$

Recalling the definition of the Weiss energy in (7) and the normalization (24), the above inequality can be rewritten as

$$\sup_{\tau \in [0,1]} \frac{1}{4} \|\nabla \hat{u}_j(\tau)\|^2_{L^2_\mu} \le C + \frac{3}{4}.$$

Thus $\hat{u}_j \in L^\infty([0,1]; W^{1,2}_\mu)$. Next, (25) together with (18) leads to

$$\int_{I_j} \|\partial_\tau \tilde{u}_j\|^2_{L^2_\mu(\mathbb{R}^n_+)} d\tau \le C c_j \int_{I_j} \|\tilde{v}_j\|^2_{L^2_\mu} d\tau \le C c_j \int_{I_j} \|\tilde{u}_j\|^2_{L^2_\mu} d\tau, \tag{26}$$

which gives $\partial_\tau \hat{u}_j \in L^2([0,1]; L^2_\mu)$ with $\|\partial_\tau \hat{u}_j\|_{L^2([0,1];L^2_\mu)} \le C c_j$. Since the embedding $W^{1,2}_\mu \hookrightarrow L^2_\mu$ is compact, by Aubin-Lions lemma, up to a subsequence $\hat{u}_j \to \hat{u}_0$ strongly in $C([0,1]; L^2_\mu)$ for some function $\hat{u}_0$. Note that $\hat{u}_j$ solves the variational inequality

$$\int_0^1 \int_{\mathbb{R}^n_+} \left[\partial_\tau \hat{u}_j (v - \hat{u}_j) + \frac{1}{4} \nabla \hat{u}_j \cdot \nabla (v - \hat{u}_j) - \frac{3}{4} \hat{u}_j (v - \hat{u}_j) \right] d\mu d\tau \ge 0,$$

for any $v \in L^2([0,1]; W^{1,2}_\mu)$, $v \geq 0$ on $\{y_n = 0\}$, $v(0,\cdot) = \hat{u}_j(0,\cdot)$ and $v(\tau,\cdot) - \hat{u}_j(\tau,\cdot)$ has compact support in $\mathbb{R}^n_+$ for any $\tau \in [0,1]$. The interior regularity estimate for the solution to the variational inequality [1] entails that (after taking another subsequence) $\nabla \hat{u}_j \to \nabla \hat{u}_0$ locally in $C^{\alpha,\alpha/2}$. In the end using (26) and passing to the limit in the variational inequality of $\hat{u}_j$ we conclude that $\hat{u}_0$ is a stationary solution.

(iv). Let $\tilde{u}_j$ be the sequence from step (iii) such that $\hat{u}_j \to \hat{u}_0 \in \mathcal{E}_{3/2}$. Let $\hat{\tau}_j \in I_j$ be such that $\|\tilde{v}_j(\hat{\tau}_j)\|^2_{L^2_\mu} = \|\tilde{u}(\hat{\tau}_j) - \lambda_j(\hat{\tau}_j) h_{e(\hat{\tau}_j)}\|^2_{L^2_\mu} = \int_{I_j} \|\tilde{v}_j(\tau)\|^2_{L^2_\mu} d\tau$. Consider

$$\hat{w}_j(y,\tau) := \frac{\tilde{u}_j(y, \tau_j + \tau) - \lambda_j(\hat{\tau}_j) h_{e_j(\hat{\tau}_j)}(y)}{\|\tilde{v}_j(\hat{\tau}_j)\|_{L^2_\mu}}, \ (y,\tau) \in \mathbb{R}^n_+ \times [0,1].$$

We will prove that up to a subsequence $\hat{w}_j$ converge in $C([0,1]; L^2_\mu)$ to a nonzero function $\hat{w}_0 \in \mathcal{E}_{3/2}$. This leads to a contradiction, since at each τ such that $\tau_j + \tau = \hat{\tau}_j$ we have projected out $\mathcal{E}_{3/2}$ from $\tilde{u}_j(\hat{\tau}_j)$.

Invoking (25) and that

$$\begin{aligned} W(\tilde{u}_j(\tau) - \lambda_j(\hat{\tau}_j) h_{e_j(\hat{\tau}_j)}) &= W(\tilde{u}_j(\tau)) + \frac{\lambda_j(\hat{\tau}_j)}{2} \int_{\{y_n=0\}} \tilde{u}_j(\tau) \partial_n h_{e_j(\hat{\tau}_j)} d\mu \\ &\leq W(\tilde{u}_j(\tau)) \end{aligned}$$

we have $W(\hat{w}_j(\tau)) \leq Cc_j$ for any $\tau \in [0,1]$. As in step (iii) this implies that $\hat{w}_j$ is uniformly bounded in $L^\infty([0,1]; W^{1,2}_\mu)$. By (26),

$$\int_{I_j} \|\partial_\tau \hat{w}_j\|^2_{L^2_\mu} d\tau \leq Cc_j. \tag{27}$$

Fundamental theorem of calculus together with (27) gives that $\|\hat{w}_j(\tau)\|_{L^2_\mu}$ stays uniformly away from zero, i.e.

$$\left| \|\hat{w}_j(\tau)\|^2_{L^2_\mu} - \|\hat{w}_j(\hat{\tau}_j)\|^2_{L^2_\mu} \right| = \left| \|\hat{w}_j(\tau)\|^2_{L^2_\mu} - 1 \right| \leq Cc_j, \ \tau \in [0,1].$$

Thus by Aubin-Lions lemma, up to a subsequence $\hat{w}_j$ converges strongly in $C^0([0,1]; L^2_\mu)$ to a nonzero function $\hat{w}_0$. Since $\partial_\tau \hat{w}_j - \mathcal{L}\hat{w}_j = 0$ in $\mathbb{R}^n_+$, by the interior estimates $\hat{w}_j$ converges locally smoothly in $\mathbb{R}^n_+$. This together with (27) implies that $\hat{w}_0$ is stationary and it solves $\mathcal{L}\hat{w}_0 = 0$ in $\mathbb{R}^n_+$.

We claim that the limiting function $\hat{w}_0$ satisfies

$$\hat{w}_0 = 0 \text{ on } \Lambda_0 := \{y_n = 0, \ y' \cdot e_0 \leq 0\}, \ \partial_n \hat{w}_0 = 0 \text{ on } \Omega_0 := \{y_n = 0\} \setminus \Lambda_0,$$

where e_0 is the tangential direction from step (iii).

This is a consequence of the complementary boundary conditions satisfied by $\hat{u}_j$ and the uniform convergence. Indeed, given $U \Subset \Omega_0$, using $c_n h_{e_0} > c > 0$ in U and the uniform convergence of $\hat{u}_j$ to $c_n h_{e_0}$ in $\overline{U} \times [0,1]$ we have, for sufficiently large j depending on U, $\hat{u}_j > 0$ in $U \times [0,1]$. By the complementary condition in terms of $\tilde{u}_j$ we have $\partial_n \hat{u}_j = 0$ in $U \times [0,1]$. Next since $e_j(\hat{\tau}_j)$ converges to e_0, which follows from the convergence of $\hat{u}_j$ to $c_n h_{e_0}$ in $C^0([0,1]; L^2_\mu)$, one has $\partial_n h_{e_j(\hat{\tau})} = 0$ in U for j sufficiently large. Thus $\partial_n \hat{w}_j = 0$ in $U \times [0,1]$ for sufficiently large j. Therefore, after extending $\hat{w}_j$ evenly about $\{y_n = 0\}$, $\hat{w}_j$ solves $\partial_t \hat{w}_j - \mathcal{L}\hat{w}_j = 0$ in $\tilde{U} \times [0,1]$, where $\tilde{U}$ is an open neighborhood of U in $\mathbb{R}^n$ with $\tilde{U} \cap \{y_n = 0\} = U$. By the interior estimates $\hat{w}_j \to \hat{w}_0$ in $C^1(\tilde{U} \times [0,1])$. This implies that in the limit $\partial_n \hat{w}_0 = 0$ on U. Since U is arbitrary we have $\partial_n \hat{w}_0 = 0$ on Ω_0. Using the fact that $c_n \partial_n h_{e_0} < -c < 0$ in $U \Subset \operatorname{int}(\Lambda_0)$ and arguing similarly we can conclude that $\hat{w}_0 = 0$ on $\operatorname{int}(\Lambda_0)$.

Now we have shown that $\hat{w}_0 \in W^{1,2}_\mu$ solves an eigenvalue problem

$$(-\frac{1}{2}\Delta + y \cdot \nabla)\hat{w}_0 = \frac{3}{2}\hat{w}_0 \text{ in } \mathbb{R}^n_+$$

with the Dirichlet-Neumann boundary data $\hat{w}_0 = 0$ on Λ_0 and $\partial_n \hat{w}_0 = 0$ on Ω_0. By the characterization of the eigenfunctions for the second Dirichlet-Neumann eigenvalue and after a rotation of coordinate (such that $e_0 = e_{n-1}$),

$$\hat{w}_0(y) = \lambda_n h(y_{n-1}, y_n) + \sum_{i=1}^{n-2} \lambda_i x_i h_{1/2}(y_{n-1}, y_n), \quad \lambda_i \in \mathbb{R}.$$

Here $h_{1/2}(y_{n-1}, y_n) := c_n \operatorname{Re}(y_{n-1} + i|y_n|)^{1/2}$. The orthogonality condition (29) implies that $\lambda_i = 0$ for $i = 1, \cdots, n-2$. Thus $\lambda_n > 0$ and we have shown that $\hat{w}_0$ is a nonzero function in $\mathcal{E}_{3/2}$. ∎

A very similar but simpler argument as for Proposition 3.1 gives the decay estimate of the Weiss energy if it becomes negative starting from some time τ_0. After a shift in time, we may assume $\tau_0 = 0$.

Proposition 3.2. *Let $\tilde{u} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a solution to (4)–(5) with $\kappa = 3/2$ and satisfy (6). Assume that $W(\tilde{u}(\tau)) \le 0$ for all $\tau > 0$. Then there exists a constant $c_0 \in (0,1)$ depending only on n, such that*

$$W(\tilde{u}(\tau+1)) \le (1 + c_0) W(\tilde{u}(\tau)) \text{ for any } \tau \in (0,\infty).$$

As an immediate consequence of Proposition 3.1 and Proposition 3.2, one obtains the exponential decay rate of the Weiss energy:

Corollary 3.3. *Let $\tilde{u}$ be a solution to* (4)–(5) *with $\kappa = 3/2$ and satisfy* (6). *Then there exists $\gamma_0 \in (0,1)$ depending only on n such that*

(i) *If $W(\tilde{u}(\tau)) \geq 0$ for all $\tau \in (0,\infty)$, then $W(\tilde{u}(\tau)) \leq e^{-\gamma_0\tau} W(\tilde{u}(0))$. Moreover, the limit $\lim_{\tau\to\infty} \tilde{u}(\tau) =: \tilde{u}(\infty) = \lambda(\infty) h_{e(\infty)} \in \mathcal{E}_{3/2}$ exists, and it satisfies*

$$\|\tilde{u}(\tau) - \tilde{u}(\infty)\|^2_{L^2_\mu} \leq C_n W(\tilde{u}(0)) e^{-\gamma_0\tau},$$
$$\|\tilde{v}(\tau)\|^2_{L^2_\mu} + |\lambda(\tau)^2 - \lambda(\infty)^2| \leq C_n W(\tilde{u}(0)) e^{-\gamma_0\tau}$$

for all $\tau \in (0,\infty)$.

(ii) *If $W(\tilde{u}(0)) < 0$, then $W(\tilde{u}(\tau)) \leq e^{\gamma_0\tau} W(\tilde{u}(0))$. Moreover,*

$$\|\tilde{u}(\tau)\|^2_{L^2_\mu} \geq -\frac{2W(\tilde{u}(0))}{\gamma_0}(e^{\gamma_0\tau} - 1) + \|\tilde{u}(0)\|^2_{L^2_\mu}.$$

Proof: We only provide the proof for the case of nonnegative Weiss energy. The proof for (ii) is the same.

Assuming Proposition 3.1 and arguing inductively we then obtain

$$W(\tilde{u}(\tau + k)) \leq (1 - c_0)^k W(\tilde{u}(\tau))$$

for any $\tau \in (0,\infty)$ and $k \in \mathbb{N}_+$. This together with the monotonicity property of $\tau \mapsto W(\tilde{u}(\tau))$ implies that there exists $\gamma_0 \in (0,1)$ depending only on c_0 such that

$$W(\tilde{u}(\tau)) \leq e^{-\gamma_0\tau} W(\tilde{u}(0)).$$

For any $0 < \tau_1 < \tau_2 \leq \tau_1 + 1 < \infty$, by (8) and Hölder's inequality,

$$\begin{aligned}
\|\tilde{u}(\tau_1) - \tilde{u}(\tau_2)\|_{L^2_\mu} &\leq \int_{\tau_1}^{\tau_2} \|\partial_\tau \tilde{u}\|_{L^2_\mu} d\tau \\
&\leq \left(W(\tilde{u}(\tau_1)) - W(\tilde{u}(\tau_2))\right)^{1/2} (\tau_2 - \tau_1)^{1/2} \\
&\leq W(\tilde{u}(0))^{1/2} e^{-\gamma_0\tau_1/2}.
\end{aligned}$$

Here in the second last inequality we have used that $W(\tilde{u}(\tau_2)) \geq 0$. An iterative argument then yields for any $0 < \tau_1 < \tau_2 < \infty$,

$$\|\tilde{u}(\tau_1) - \tilde{u}(\tau_2)\|_{L^2_\mu} \leq \frac{W(\tilde{u}(0))^{1/2}}{1 - e^{-\gamma_0/2}} e^{-\gamma_0\tau_1/2}.$$

Thus $\lim_{\tau\to\infty}\tilde{u}(\tau) =: \tilde{u}(\infty) \in L^2_\mu$ exists and the convergence rate is exponential. To show the exponential convergence of $\|\tilde{v}(\tau)\|_{L^2_\mu} = \mathrm{dist}_{L^2_\mu}(\tilde{u}(\tau), \mathcal{E}_{3/2})$ and $\lambda(\tau) = \|\mathrm{proj}_{L^2_\mu}(\tilde{u}(\tau), \mathcal{E}_{3/2})\|_{L^2_\mu}$ we only need to observe

$$\|\tilde{v}(\tau)\|^2_{L^2_\mu} \leq \|\tilde{u}(\tau)\|^2_{L^2_\mu} - \|\tilde{u}(\infty)\|^2_{L^2_\mu},$$
$$|\lambda(\tau)^2 - \lambda(\infty)^2| \leq \|\tilde{v}(\tau)\|^2_{L^2_\mu} + \|\tilde{u}(\tau)\|^2_{L^2_\mu} - \|\tilde{u}(\infty)\|^2_{L^2_\mu}$$

and using the exponential convergence of $\tilde{u}(\tau)$ to $\tilde{u}(\infty)$. ■

Remark 3.4. *At this stage it is possible that $\tilde{u}(\infty)$ is zero. However, if at the initial time*

$$W(\tilde{u}(0)) \leq \delta_n\|\tilde{u}(0)\|^2_{L^2_\mu}, \quad \mathrm{dist}_{L^2_\mu}(\tilde{u}(0), \mathcal{E}_{3/2})^2 \leq \delta_n\|\tilde{u}(0)\|^2_{L^2_\mu} \tag{28}$$

for some small $\delta_n > 0$, then in the limit $\lambda(\infty) > 0$. To see this, we note that the bound on the Weiss energy together with the exponential convergence of $\lambda(\tau)^2$ from Corollary 3.3 (i) implies

$$|\lambda(0)^2 - \lambda(\tau)^2| \leq C_n\delta_n\|\tilde{u}(0)\|^2_{L^2_\mu}, \text{ for all } \tau > 0.$$

From the bound on the distance we have $\lambda(0)^2 \geq (1-\delta_n)\|\tilde{u}(0)\|^2_{L^2_\mu}$. Combining together leads to $\lambda(\tau)^2 \geq (1 - C_n\delta_n)\|\tilde{u}(0)\|^2_{L^2_\mu}$ for any $\tau > 0$. Thus $\lambda(\infty) > \frac{1}{2}\|\tilde{u}(0)\|_{L^2_\mu} > 0$ if δ_n is chosen sufficiently small.

We also note that (28) is satisfied by requiring that the solution stays close to $\mathcal{E}_{3/2}$ in $W^{1,2}_\mu$ norm at $\tau = 0$, i.e.

$$\mathrm{dist}_{W^{1,2}_\mu}\left(\frac{\tilde{u}(\cdot,0)}{\|\tilde{u}(\cdot,0)\|_{L^2_\mu}}, \mathcal{E}_{3/2}\right) \leq \delta_n.$$

3.2. The case $\kappa = 2m$. Let $\tilde{u} := \tilde{u}_{2m}$, $m \in \mathbb{N}_+$, be a solution to (4)–(5) which satisfies (6). In this section we derive the decay of the associated Weiss energy

$$W_{2m}(\tilde{u}(\tau)) = \frac{1}{4}\int_{\mathbb{R}^n_+} |\nabla\tilde{u}(\tau)|^2 d\mu - m\int_{\mathbb{R}^n_+} |\tilde{u}(\tau)|^2 d\mu$$

Recall that stationary solutions are in the space $\mathcal{E}^+_{2m}$ by Proposition 2.6, where $\mathcal{E}^+_{2m}$ is a subset of zero eigenspace of the Ornstein-Uhlenbeck operator $\mathcal{L}_{2m} := \frac{1}{4}\Delta - \frac{y}{2}\cdot\nabla + m$.

To fix the notation, let $\{p_\alpha = c_\alpha H_{\alpha_1}(y_1)\cdots H_{\alpha_n}(y_n)\}_{\alpha\in\mathbb{N}^n}$ be the set of Hermite polynomials in $\mathbb{R}^n$, where H_k for $k\in\mathbb{N}$ is the $1d$-Hermite polynomial of order k, i.e. it solves the eigenfunction equation $U''-2xU=-2kU$ in $\mathbb{R}$. Here c_α is chosen such that $\|p_\alpha\|_{L^2_\mu}=1$. Then $\{p_\alpha\}$ are eigenfunctions of $\mathcal{L}_{2m}$:

$$\mathcal{L}_{2m}p_\alpha = \left(m-\frac{|\alpha|}{2}\right)p_\alpha,$$

and they form an orthonormal basis for L^2_μ. Let

$$\mathcal{E}_{2m}=\{\sum_{\alpha:|\alpha|=2m}\lambda_\alpha p_\alpha,\ p_\alpha(y',y_n)=p_\alpha(y',-y_n),\ \lambda_\alpha\in\mathbb{R}\}$$

be the subspace generated by $2m$-Hermite polynomials with symmetry. Given a solution $\tilde{u}(\tau)$ we consider the L^2_μ projection of $\tilde{u}(\tau)$ onto $\mathcal{E}_{2m}$

$$\tilde{u}(y,\tau)=\sum_{|\alpha|=2m}\lambda_\alpha(\tau)p_\alpha(y)+\tilde{v}(y,\tau),\quad \lambda_\alpha(\tau)=\int_{\mathbb{R}^n_+}\tilde{u}(\tau,y)p_\alpha(y)d\mu$$

and study the evolution of $\mathrm{dist}_{L^2_\mu}(\tilde{u}(\tau),\mathcal{E}_{2m})=\|\tilde{v}(\tau)\|_{L^2_\mu}$ and the parameters $\lambda_\alpha(\tau)$. Note that since $\partial_\tau\tilde{u}\in L^2_{loc}(\mathbb{R}_+;L^2_\mu)$, we have that $\dot{\lambda}_\alpha\in L^2_{loc}(\mathbb{R}_+)$.

Due to the minimality, $\tilde{v}$ satisfies the orthogonality condition

$$\int_{\mathbb{R}^n}\tilde{v}p_\alpha d\mu=0,\ \text{for any } p_\alpha\in\mathcal{E}_{2m}. \tag{29}$$

This together with the equation of $\tilde{u}$ yields the following evolution equations:

$$\frac{1}{2}\partial_\tau\|\tilde{v}(\tau)\|^2_{L^2_\mu}=-W_{2m}(\tilde{v}(\tau))+\sum_{|\alpha|=2m}\frac{\lambda_\alpha(\tau)}{4}\int_{\{y_n=0\}}p_\alpha\partial_n\tilde{v}(\tau)d\mu, \tag{30}$$

$$\dot{\lambda}_\alpha(\tau)=-\frac{1}{4}\int_{\{y_n=0\}}p_\alpha\partial_n\tilde{v}(\tau)d\mu,\ \text{for each } \alpha \text{ s.t. } |\alpha|=2m. \tag{31}$$

To see these, we note that from the equation of $\tilde{u}$ and that $\mathcal{L}_{2m}p_\alpha=0$, $\tilde{v}$ satisfies

$$\partial_\tau\tilde{v}=\mathcal{L}_{2m}\tilde{v}-\sum_{|\alpha|=2m}\dot{\lambda}_\alpha p_\alpha \text{ in } \mathbb{R}^n_+\times(0,\infty)$$

with the Signorini condition

$$\sum_{|\alpha|=2m} \lambda_\alpha p_\alpha \partial_n \tilde{v} + \tilde{v}\partial_n \tilde{v} = 0 \text{ on } \{y_n = 0\}.$$

Multiplying $\tilde{v}$ on both sides of the equation, using the Signorini condition and the orthogonality (29) we obtain (30). Multiplying p_α on both sides of the equation, using the orthogonality condition (which gives $\int_{\mathbb{R}^n} \partial_\tau \tilde{v} p_\alpha d\mu = 0$ for a.e. τ) we get the evolution equation for λ_α in (31).

Now we write the Weiss energy $W_{2m}(\tilde{u})$ in terms of $\tilde{v}$. First, using $W_{2m}(p_\alpha) = 0$ for $|\alpha| = 2m$ we have

$$W_{2m}(\tilde{u}) = W_{2m}(\tilde{v}) + \frac{1}{4}\sum_\alpha \lambda_\alpha \int_{\mathbb{R}^n_+} \nabla \tilde{v} \cdot \nabla p_\alpha d\mu.$$

An integration by parts, $\partial_n p_\alpha = 0$ on $\{y_n = 0\}$ and (29) yield that the last term is zero. Thus,

$$W_{2m}(\tilde{u}) = W_{2m}(\tilde{v}). \tag{32}$$

Next by (7) and orthogonality (29), for any $0 < \tau_a < \tau_b < \infty$,

$$W_{2m}(\tilde{u}(\tau_b)) - W_{2m}(\tilde{u}(\tau_a)) \le -2\int_{\tau_a}^{\tau_b} (\|\partial_\tau \tilde{v}\|^2_{L^2_\mu} + \sum_{|\alpha|=2m} \dot{\lambda}^2_\alpha) d\tau. \tag{33}$$

If we decompose further $\tilde{v}(\tau) = \sum_{|\alpha|<2m, |\alpha|>2m} \lambda_\alpha(\tau) p_\alpha$, then

$$W_{2m}(\tilde{u}(\tau_b)) - W_{2m}(\tilde{u}(\tau_a)) \le -2\int_{\tau_a}^{\tau_b} \sum_\alpha \dot{\lambda}^2_\alpha d\tau. \tag{34}$$

In the sequel, we will frequently use the following auxiliary function. Let h_{2k} denote the $(m-k)$-eigenfunction of $\mathcal{L}_{2m}$, which has the expression

$$h_{2k}(y) = C^{-1}_{k,n}\left(\sum_{j=1}^{n-1} 2^{2k}\operatorname{Re}(y_j + iy_n)^{2k} + k!\sum_{\ell=0}^{k} \frac{(-1)^\ell}{(k-\ell)!(2\ell)!}(2y_n)^{2\ell}\right), \tag{35}$$

where $C_{k,n} = c_n 2^{2k} 2^k k!$, $c_n > 0$, is a normalization factor such that $\|h_{2k}\|_{L^2_\mu} = 1$. Note that

$$h_{2k}(y', 0) = C^{-1}_{k,n}(2^{2k}|y'|^2 + 1).$$

In the sequel we will denote

$$\lambda_{2k} := \int \tilde{u} h_{2k} d\mu.$$

Since $\partial_n \tilde{v} = \partial_n \tilde{u} \leq 0$ and $h_{2k} \geq 0$ on $\{y_n = 0\}$, we see from (31) that

$$\dot{\lambda}_{2k} = -\frac{1}{4} \int_{\{y_n=0\}} h_{2k} \partial_n \tilde{v} d\mu \geq 0. \tag{36}$$

The first proposition concerns the evolution of the Weiss energy $W_{2m}(\tilde{u}(\tau))$ if it is negative.

Proposition 3.5. *Let $\tilde{u}$ be a solution to the Signorini problem* (4)–(5) *with $\kappa = 2m$ and satisfy* (6)*. Assume that $W_{2m}(\tilde{u}(0)) \leq 0$. Then there exists a constant $c_0 \in (0,1)$ depending on m, n such that*

$$W_{2m}(\tilde{u}(\tau+1)) \leq (1+c_0) W_{2m}(\tilde{u}(\tau)), \quad \textit{for any } \tau \in (0,\infty).$$

Proof: Assume it were not true, then there exists a sequence of solutions $\tilde{u}_j$, $\tau_j \in (0,\infty)$ and $\epsilon_j \to 0$ such that

$$W_{2m}(\tilde{u}_j(\tau_j+1)) \geq (1+\epsilon_j) W_{2m}(\tilde{u}_j(\tau_j)).$$

For the rest of the proof we drop the dependence on j for simplicity. We decompose $\tilde{u}(\tau,\cdot)$ into $\tilde{u}(\tau,\cdot) = p_{<2m}(\tau,\cdot) + p(\tau,\cdot) + \tilde{w}(\tau,\cdot)$, where $p \in \mathcal{E}_{2m}$, and $p_{<2m}(\tau,y) = \sum_{|\alpha|<2m} \lambda_\alpha(\tau) p_\alpha(y)$ is the projection of $\tilde{u}$ to the subspace $\mathcal{E}_{<2m}$ generated by k-Hermite polynomials $k < 2m$ with symmetry, i.e.

$$\mathcal{E}_{<2m} := \{ \sum_{\alpha:|\alpha|<2m} c_\alpha p_\alpha,\ p_\alpha(y', y_n) = p_\alpha(y', -y_n) \}.$$

Note that

$$W_{2m}(p_{<2m}) \leq 0, \quad W_{2m}(\tilde{w}) \geq 0.$$

Thus the contradiction assumption implies that

$$\epsilon_j W_{2m}(\tilde{w}(\tau_j)) - [W_{2m}(\tilde{u}(\tau_{j+1})) - W_{2m}(\tilde{u}(\tau_j))] \leq -\epsilon_j W_{2m}(p_{<2m}(\tau_j)).$$

This together with (34) and the monotone decreasing property of $\tau \mapsto W_{2m}(\tilde{u}(\tau))$ implies that

$$2 \int_{I_j} \sum_\alpha \dot{\lambda}_\alpha(\tau)^2 d\tau \leq -\epsilon_j \int_{I_j} W_{2m}(p_{<2m}(\tau)) d\tau, \quad I_j := [\tau_j, \tau_{j+1}]. \tag{37}$$

Multiplying the equation of $\tilde{u}$ by $p_{<2m}(\tau)$ and an integration by parts in space yield

$$\int_{I_j} W_{2m}(p_{<2m}(\tau)) = -\frac{1}{2}\int_{I_j} \partial_\tau \|p_{<2m}(\tau)\|^2_{L^2_\mu} - \frac{1}{4}\int_{I_j}\int_{\{y_n=0\}} p_{<2m}(\tau)\partial_n\tilde{u}(\tau)d\mu.$$

By (37) the first integral can be estimated from below as

$$\begin{aligned}
&-\frac{1}{2}\int_{I_j} \partial_\tau \|p_{<2m}(\tau)\|^2_{L^2_\mu} d\tau = -\int_{I_j} \sum_{|\alpha|<2m} \lambda_\alpha \dot{\lambda}_\alpha d\tau \\
&\geq -\epsilon_j^{1/2}\left(\int_{I_j} \|p_{<2m}(\tau)\|_{L^2_\mu} d\tau\right)^{1/2}\left(\int_{I_j} -W_{2m}(p_{<2m}(\tau))d\tau\right)^{1/2}.
\end{aligned}$$

To estimate the boundary integral we observe that for each $\tau > 0$

$$\sup_{y'\in\mathbb{R}^{n-1}} \frac{|p_{<2m}(y',0,\tau)|}{h_{2m}(y')} \leq c_{n,m}\|p_{<2m}(y,\tau)\|_{L^2_\mu}.$$

Using $\partial_n\tilde{u}(y',0) \leq 0$ and recalling the expression of $\dot{\lambda}_{2m}$ in (36), we can estimate the boundary term from below by

$$\begin{aligned}
-\frac{1}{4}\int_{I_j}\int_{\{y_n=0\}} p_{<2m}\partial_n\tilde{u}d\mu d\tau &\geq \frac{c_{m,n}}{4}\int_{I_j} \|p_{<2m}(\tau)\|_{L^2_\mu}\int_{\{y_n=0\}} h_{2m}\partial_n\tilde{u}d\mu d\tau \\
&= -c_{m,n}\int_{I_j} \|p_{<2m}(\tau)\|_{L^2_\mu}\dot{\lambda}_{2m}(\tau)d\tau.
\end{aligned}$$

Invoking (33) again we thus have

$$\begin{aligned}
&-\frac{1}{4}\int_{I_j}\int_{\{y_n=0\}} p_{<2m}\partial_n\tilde{u}d\mu d\tau \\
&\geq -c_{m,n}\epsilon_j^{1/2}\left(\int_{I_j} \|p_{<2m}(\tau)\|_{L^2_\mu} d\tau\right)^{1/2}\left(-\int_{I_j} W_{2m}(p_{<2m}(\tau))d\tau\right)^{1/2}.
\end{aligned}$$

Combining together we obtain

$$\int_{I_j} W_{2m}(p_{<2m}(\tau))d\tau \geq -C\epsilon_j\int_{I_j} \|p_{<2m}(\tau)\|^2_{L^2_\mu} d\tau \tag{38}$$

for some $C = C(m,n) > 0$. This however leads to a contradiction, since for any $\tau > 0$

$$W_{2m}(p_{<2m}(\tau)) = \sum_{|\alpha|<2m, |\alpha|\in\mathbb{N}} -\left(m - \frac{|\alpha|}{2}\right) \|p_\alpha(\tau)\|^2_{L^2_\mu} \le -\|p_{<2m}(\tau)\|^2_{L^2_\mu}.$$

∎

In the next proposition we derive a discrete algebraic decay of the Weiss energy under the assumption that $W_{2m}(\tilde{u}(\tau)) \ge 0$ for all τ.

Proposition 3.6. *Let $\tilde{u}$ be a solution to (4)–(5) with $\kappa = 2m$ and satisfy (6). Assume that $W_{2m}(\tilde{u}(\tau)) \ge 0$ for all $\tau \ge 0$ and that*

$$\operatorname{dist}_{L^2_\mu}(\tilde{u}(0), \mathcal{E}^+_{2m}) \le \delta_0,$$

for some $\delta_0 = \delta_0(m,n) > 0$ small. Then there exists a $c_0 \in (0,1)$ depending only on n and m such that

$$W_{2m}(\tilde{u}(\tau+1)) \le \left(1 - c_0 W_{2m}(\tilde{u}(\tau+1))^{1-\gamma}\right) W_{2m}(\tilde{u}(\tau)), \quad \gamma := \frac{1}{n+1}$$

for all $\tau > 0$.

Proof: (i) Assume that the statement were wrong, then there exists a sequence of solutions $\tilde{u}_j$ with $\operatorname{dist}_{L^2_\mu}(\tilde{u}(0), \mathcal{E}^+_{2m}) \le \delta_0$, a sequence of positive constants $\epsilon_j \to 0$ and $\tau_j > 0$, such that

$$W_{2m}(\tilde{u}_j(\tau_j+1)) \ge \left(1 - \epsilon_j W_{2m}(\tilde{u}_j(\tau_j+1))^{1-\gamma}\right) W_{2m}(\tilde{u}_j(\tau_j)).$$

Thus after rearranging the terms and using (33) as well as that $\tau \mapsto W_{2m}(\tilde{u}_j(\tau))$ is monotone decreasing, we have

$$\begin{aligned} 2\int_{I_j} (\|\partial_\tau \tilde{v}_j\|^2_{L^2_\mu} + \sum_{|\alpha|=2m} \dot{\lambda}^2_\alpha) d\tau &\le W_{2m}(\tilde{u}_j(\tau_j)) - W_{2m}(\tilde{u}_j(\tau_j+1)) \\ &\le \epsilon_j \left(\int_{I_j} W_{2m}(\tilde{u}(\tau)) d\tau\right)^{2-\gamma}, \quad I_j := [\tau_j, \tau_{j+1}]. \end{aligned} \tag{39}$$

(ii). For simplicity we denote $p_j(\tau,\cdot) := \sum_{|\alpha|=2m} \lambda^j_\alpha(\tau) p_\alpha(\cdot) \in \mathcal{E}_{2m}$, which is the projection of $\tilde{u}_j(\tau)$ onto $\mathcal{E}_{2m}$. In the light of (30), to show the decay estimate of the Weiss energy we mainly need to estimate the boundary integral

$$-\int_{\{y_n=0\}} p_j(\tau)\partial_n \tilde{v}_j(\tau)d\mu.$$

Let $(p_j)_- := \max\{-p_j, 0\}$. We aim to show that there exists $C = C(m,n)$ such that

$$\begin{aligned}
&-\int\limits_{I_j}\int\limits_{\{y_n=0\}} (p_j)_-(\tau)\partial_n\tilde{v}_j(\tau)d\mu d\tau \\
&\leq C\epsilon_j^{1/2}\left(\int_{I_j}\|\tilde{v}_j(\tau)\|_{W^{1,2}_\mu}d\tau\right)^{\frac{1}{2(n+1)}}\left(\int_{I_j}W_{2m}(\tilde{v}_j(\tau))d\tau\right)^{\frac{2-\gamma}{2}} \\
&+ C\delta_0^{1/2}\int_{I_j}\|\tilde{v}_j(\tau)\|^2_{W^{1,2}_\mu}\, d\tau.
\end{aligned} \tag{40}$$

Due to the non-compactness of $\mathbb{R}^n_+$, the proof for (40) is more involved than that of [5]. We will divide the proof into three parts (a)-(c). Since the estimate is trivial if $(p_j)_- = 0$, in the sequel we assume that $(p_j)_-$ is not identically zero on $\{y_n = 0\}$. We start with the observation that $\|p_j(\tau)_-\|_{L^2_\mu(\{y_n=0\})} \leq c_{m,n}\delta_0$ for all j and τ. To see this, let $\hat{p}_j \in \mathcal{E}^+_{2m}$, $\|\hat{p}_j\|_{L^2_\mu} = 1$, be such that $\operatorname{dist}_{L^2_\mu}(\tilde{u}_j(0), \mathcal{E}^+_{2m}) = \|\tilde{u}_j(0) - \hat{\lambda}_j(0)\hat{p}_j\|_{L^2_\mu}$ for some $\hat{\lambda}_j(0) \geq 0$. We further let $\hat{\lambda}_j(\tau) := \int_{\mathbb{R}^n_+}\tilde{u}_j(\tau)\hat{p}_j d\mu$. Similar as in (36), $\tau \mapsto \hat{\lambda}_j(\tau)$ is monotone increasing. Thus,

$$\begin{aligned}
&\|p_j(\tau)_-\|^2_{L^2_\mu(\{y_n=0\})} \leq \|p_j(\tau) - \hat{\lambda}_j(\tau)\hat{p}_j\|^2_{L^2_\mu(\{y_n=0\})} \leq c_{m,n}\|p_j(\tau) - \hat{\lambda}_j(\tau)\hat{p}_j\|^2_{L^2_\mu} \\
&\leq c_{m,n}\|\tilde{u}_j(\tau) - \hat{\lambda}_j(\tau)\hat{p}_j\|^2_{L^2_\mu} \leq c_{m,n}\|\tilde{u}_j(0) - \hat{\lambda}_j(0)\hat{p}_j\|^2_{L^2_\mu} \leq c_{m,n}\delta_0^2.
\end{aligned} \tag{41}$$

Here the first inequality is due to $\hat{\lambda}_j\hat{p}_j \geq 0$ on $\{y_n = 0\}$, in the last two inequalities we used the orthogonality and the monotone decreasing property for $\tau \mapsto \|\tilde{u}_j(\tau)\|_{L^2_\mu}$, which follows from (9) and the nonnegativity assumption of the Weiss energy, and that $\tau \mapsto \hat{\lambda}_j(\tau)$ is monotone increasing.

(a.) We show that the integral in $B'_{R_0} \subset \{y_n = 0\}$ with $R_0^2 = R_0(j,\tau)^2 := -\ln\|(p_j)_-(\tau)\|_{L^2_\mu(\{y_n=0\})}$ satisfies the following estimate for a.e. τ:

$$-\int_{B'_{R_0}}(p_j)_-(\tau)\partial_n\tilde{v}_j(\tau)d\mu \leq C\|(p_j)_-(\tau)\|^{\frac{1}{n+1}}_{L^2_\mu(\{y_n=0\})}\dot{\lambda}_{2m}(\tau) \tag{42}$$

where $\dot{\lambda}_{2m}$ is defined in (36). Indeed,

$$\begin{aligned}
-\int_{B'_{R_0}} (p_j)_- \partial_n \tilde{v}_j d\mu &= -\int_{B'_{R_0}} \frac{(p_j)_-}{h_{2m}} h_{2m} \partial_n \tilde{v}_j d\mu \\
&\leq \sup_{B'_{R_0}} \frac{(p_j)_-}{h_{2m}} \int_{B'_{R_0}} -h_{2m} \partial_n \tilde{v}_j d\mu \\
&\leq 4 \sup_{B'_{R_0}} \frac{(p_j)_-}{h_{2m}} \dot{\lambda}_{2m}.
\end{aligned}$$

Thus it suffices to estimate $\sup_{B'_{R_0}} \frac{(p_j)_-}{h_{2m}}$. Assume that for some $y_0 \in \overline{B'_{R_0}}$

$$M := \max_{\overline{B'_{R_0}}} \frac{(p_j)_-}{h_{2m}} = \frac{(p_j)_-(y_0, 0)}{h_{2m}(y_0, 0)} > 0.$$

Since $\|p_j(\tau)\|^2_{L^2_\mu} = \sum_{|\alpha|=2m} \lambda^2_{\alpha,j}(\tau) \leq c_{m,n} \delta_0^2$ by (41), there exists $C = C(m, n) > 0$ such that

$$M \leq C\|p_j(\tau)\|_{L^2_\mu} \leq C, \quad L := [(p_j)_-/h_{2m}]_{\dot{C}^{0,1}(\{y_n=0\})} \leq C.$$

Let $r_0 := M/(c_n L) > 0$. In $B'_{r_0}(y_0) \cap B'_{R_0}$ we have $(p_j)_-/h_{2m} \geq M/2$. Thus

$$\int_{B'_{r_0}(y_0) \cap B'_{R_0}} \left| \frac{(p_j)_-}{h_{2m}} \right|^2 dy' \geq \frac{M^2}{4} |B'_{r_0}(y_0) \cap B'_{R_0}| \geq \tilde{c}_n \frac{M^{n+1}}{L^{n-1}}.$$

Therefore, there exists a constant $C > 0$ depending only on m, n such that

$$\begin{aligned}
M &\leq C \left(\int_{B'_{r_0}(y_0) \cap B'_{R_0}} \left| \frac{(p_j)_-}{h_{2m}} \right|^2 dy' \right)^{\frac{1}{n+1}} \\
&\leq C e^{\frac{R_0^2}{n+1}} \left(\int_{\{y_n=0\}} \left| \frac{(p_j)_-}{h_{2m}} \right|^2 e^{-|y'|^2} dy' \right)^{\frac{1}{n+1}}.
\end{aligned}$$

Since h_{2m} is uniformly bounded away from zero and recalling our choice of R_0, we thus have

$$M \leq C \|(p_j)_-\|_{L^2_\mu}^{-\frac{1}{n+1}} \|(p_j)_-\|_{L^2_\mu}^{\frac{2}{n+1}} \leq C \|(p_j)_-\|_{L^2_\mu}^{\frac{1}{n+1}}.$$

Thus the proof for (42) is complete.

(b). Outside B'_{R_1}, where $R_1^2 = -3\ln\|(p_j)_-\|_{L^2_\mu(\{y_n=0\})}$, the boundary integral can be estimated as

$$-\int_{\mathbb{R}^{n-1}\setminus B'_{R_1}} (p_j)_-\partial_n\tilde{v}_j d\mu \leq C_{m,n}(R_1)^{4m-1}e^{-R_1^2}.$$

To see this, we note that $|p_j(y',0)| \leq C|y'|^{2m}$ and

$$|\partial_n\tilde{v}_j(y',0)| = |\partial_n\tilde{u}_j(y',0)| \leq C(1+|y'|^{2m-1})$$

for some $C = C(m,n)$, which follows from the spacial $C^{1,\alpha}$ interior estimate for the Signorini problem [1]. Using $(R_1)^{4m-1}e^{-R_1^2} \leq c_m e^{-5R_1^2/6}$ and recalling the definition of R_1^2 we have

$$-\int_{\mathbb{R}^{n-1}\setminus B'_{R_1}} (p_j)_-\partial_n\tilde{v}_j d\mu \leq C\|(p_j)_-\|^{5/2}_{L^2_\mu(\{y_n=0\})}.$$

(c). We estimate the integral over the annulus

$$A = A(j,\tau) := \{y' \in \mathbb{R}^{n-1}\times\{0\} : -\ln\|(p_j)_-\|_{L^2_\mu(\{y_n=0\})} \leq |y'|^2 \leq -3\ln\|(p_j)_-\|_{L^2_\mu(\{y_n=0\})}\}.$$

For that we take $k \in \mathbb{N}_+$ and $k \in (-\ln\|(p_j)_-\|_{L^2_\mu(\{y_n=0\})}, -3\ln\|(p_j)_-\|_{L^2_\mu(\{y_n=0\})})$ and write

$$-\int_A (p_j)_-\partial_n\tilde{v}_j d\mu = -\int_A \frac{(p_j)_-}{h_{2k}} h_{2k}\partial_n\tilde{v}_j d\mu \leq 4\sup_A \frac{(p_j)_-}{h_{2k}}\ \dot{\lambda}_{2k}.$$

Recalling the definition of h_{2k} in (35) we have that for $y' \in A$

$$\frac{(p_j)_-}{h_{2k}} \leq \frac{|y|^{2m}C_{k,n}}{1+2^{2k}|y'|^{2k}} \leq c_n\frac{|y'|^{2m}2^{2k}2^k k!}{1+2^{2k}|y'|^{2k}} \leq c_n\frac{2^k k!}{|y'|^{2(k-m)}}.$$

By the Stirling's approximation formula $\frac{k!}{k^{k+1/2}} \sim e^{-k}$, the above right hand side can be further bounded from above by

$$Ce^{\frac{1}{3}\ln\|(p_j)_-\|_{L^2_\mu(\{y_n=0\})}} = C\|(p_j)_-\|^{1/3}_{L^2_\mu(\{y_n=0\})} \text{ for } y' \in A.$$

Thus we have

$$-\int_A (p_j)_-\partial_n\tilde{v}_j d\mu \leq C\|(p_j)_-\|^{1/3}_{L^2_\mu(\{y_n=0\})}\ \dot{\lambda}_{2k}.$$

Combining *(a)-(c)* we obtain that for a.e. $\tau \in (0,\infty)$

$$
\begin{aligned}
-\int_{\{y_n=0\}} (p_j)_-(\tau)\partial_n \tilde{v}_j(\tau) d\mu \leq C\|(p_j)_-(\tau)\|^{1/(n+1)}_{L^2_\mu(\{y_n=0\})} \left(\dot{\lambda}_{2m} + \dot{\lambda}_{2k}\right) \\
+ C\|(p_j)_-(\tau)\|^{5/2}_{L^2_\mu(\{y_n=0\})}.
\end{aligned}
$$

Here C is a constant depending on n and m, and we have used that $n \geq 2$ (which implies that $1/(n+1) \leq 1/3$). We first integrate the above inequality in τ, then apply (39) to estimate $\dot{\lambda}_{2m}$ and $\dot{\lambda}_{2k}$ (cf.(34)) and apply (41) to estimate the last integral. Then

$$
\begin{aligned}
-\int_{I_j} (p_j)_- \partial_n \tilde{v}_j d\mu d\tau \leq C\epsilon_j^{1/2} \left(\int_{I_j} \|(p_j)_-\|_{L^2_\mu(\{y_n=0\})} d\tau\right)^{\frac{1}{n+1}} \left(\int_{I_j} W_{2m}(\tilde{v}_j(\tau)) d\tau\right)^{\frac{2-\gamma}{2}} \\
+ C\delta_0^{1/2} \int_{I_j} \|(p_j)_-\|^2_{L^2_\mu(\{y_n=0\})} d\tau.
\end{aligned}
$$

Using $\tilde{u}_j = p_j + \tilde{v}_j \geq 0$ on $\{y_n = 0\}$ we have

$$
\|(p_j)_-\|^2_{L^2_\mu(\{y_n=0\})} \leq \|\tilde{v}_j\|^2_{L^2_\mu(\{y_n=0\})} \leq C_n(\|\nabla \tilde{v}_j\|^2_{L^2_\mu} + \|\tilde{v}_j\|^2_{L^2_\mu}),
$$

where the second inequality follows from the trace lemma. Combining the above two inequalities we obtain (40).

(iii). Now we estimate the Weiss energy for $\tilde{v}_j$ from above. By (30) and $\partial_n \tilde{v}_j = \partial_n \tilde{u}_j \leq 0$ on $\{y_n = 0\}$,

$$
\int_{I_j} W_{2m}(\tilde{v}_j(\tau)) d\tau \leq -\int_{I_j} \int \tilde{v}_j \partial_\tau \tilde{v}_j d\mu - \frac{1}{4} \int_{I_j} \int_{\{y_n=0\}} (p_j)_- \partial_n \tilde{v}_j d\mu d\tau.
$$

By (39) and (40) the right side can be estimated by $W_{2m}(\tilde{v}_j)$ and $\|\tilde{v}_j\|_{W^{1,2}_\mu}$ as

$$\begin{aligned}
\int_{I_j} W_{2m}(\tilde{v}_j(\tau))d\tau &\le \int_{I_j} \|\tilde{v}_j\|_{L^2_\mu}\|\partial_\tau \tilde{v}_j\|_{L^2_\mu} d\tau \\
&+ C\epsilon_j^{1/2}\left(\int_{I_j}\|\tilde{v}_j\|^2_{W^{1,2}_\mu}d\tau\right)^{\frac{1}{2(n+1)}}\left(\int_{I_j} W_{2m}(\tilde{v}_j)d\tau\right)^{\frac{2-\gamma}{2}} + C\delta_0^{1/2}\int_{I_j}\|\tilde{v}_j\|^2_{W^{1,2}_\mu}\,d\tau \\
&\le C\epsilon_j^{1/2}\left(\int_{I_j}\|\tilde{v}_j\|^2_{L^2_\mu}d\tau\right)^{\frac{1}{2}}\left(\int_{I_j} W_{2m}(\tilde{v}_j)d\tau\right)^{\frac{2-\gamma}{2}} \\
&+ C\epsilon_j^{1/2}\left(\int_{I_j}\|\tilde{v}_j\|^2_{W^{1,2}_\mu}d\tau\right)^{\frac{1}{2(n+1)}}\left(\int_{I_j} W_{2m}(\tilde{v}_j)d\tau\right)^{\frac{2-\gamma}{2}} + C\delta_0^{1/2}\int_{I_j}\|\tilde{v}_j\|^2_{W^{1,2}_\mu}\,d\tau
\end{aligned}$$

for some $C = C_{m,n} > 0$. By Young's inequality, for $\gamma = \frac{1}{n+1}$ and large enough j such that $C\epsilon_j \le 1/2$ one obtains

$$\int_{I_j} W_{2m}(\tilde{v}_j(\tau))d\tau \le C\epsilon_j^{1/2}\int_{I_j}\|\tilde{v}_j\|^2_{W^{1,2}_\mu}d\tau + C\delta_0^{1/2}\int_{I_j}\|\tilde{v}_j\|^2_{W^{1,2}_\mu}d\tau.$$

Recalling the definition of the Weiss energy and rearranging the terms we have

$$\left(\frac{1}{4} - C\epsilon_j^{1/2} - C\delta_0^{1/2}\right)\int_{I_j}\|\nabla\tilde{v}_j\|^2_{L^2_\mu}d\tau \le \left(m + C\epsilon_j^{1/2} + C\delta_0^{1/2}\right)\int_{I_j}\|\tilde{v}_j\|^2_{L^2_\mu}d\tau.$$

Thus if $\delta_0 = \delta_0(m,n)$ is a priori chosen small, then

$$\int_{I_j}\|\nabla\tilde{v}_j\|^2_{L^2_\mu}d\tau \le 8m\int_{I_j}\|\tilde{v}_j\|^2_{L^2_\mu}d\tau, \tag{43}$$

and thus by (39)

$$\int_{I_j}\|\partial_\tau\tilde{v}_j\|^2_{L^2_\mu}d\tau \le C\epsilon_j\int_{I_j}\|\tilde{v}_j\|^2_{L^2_\mu}d\tau. \tag{44}$$

(iv). Consider $\tilde{w}_j(y,\tau) := \frac{\tilde{v}_j(y,\tau_j+\tau)}{\|\tilde{v}_j\|_{L^2(I_j;L^2_\mu)}}$, $(y,\tau) \in \mathbb{R}^n_+ \times [0,1]$, which satisfies

$$\partial_\tau\tilde{w}_j = \mathcal{L}_{2m}\tilde{w}_j \text{ in } \mathbb{R}^n_+ \times (0,1], \quad \partial_n\tilde{w}_j \le 0 \text{ on } \{y_n = 0\}.$$

With (39), (43) and (44) at hand and arguing as in (iii) of Proposition 3.1 we have that $\tilde{w}_j \in L^\infty([0,1]; W^{1,2}_\mu)$ and $\partial_\tau\tilde{w}_j \in L^2([0,1]; L^2_\mu)$. Up to a subsequence, $\tilde{w}_j$ converges weakly in $L^2([0,1]; W^{1,2}_\mu)$ and strongly in $C([0,1]; L^2_\mu)$

to a nonzero function $\tilde{w}_0$. The convergence is locally C^∞ in $\mathbb{R}^n_+ \times (0,1)$ by using the interior estimate of the equation. By (44) and the equation for $\tilde{w}_j$, $\tilde{w}_0$ solves the stationary equation $\mathcal{L}_{2m}\tilde{w}_0 = 0$ in $\mathbb{R}^n_+$ and after an even reflection about $\{y_n = 0\}$ satisfies $\mathcal{L}_{2m}\tilde{w}_0 \leq 0$ in $\mathbb{R}^n$. By Proposition 2.6 (or Lemma 12.4 in [8] in the conformal coordinates), we conclude that $\tilde{w}_0 \in \mathcal{E}_{2m}$. This is a contradiction. ∎

Similar as for the case $\kappa = 3/2$, Proposition 3.5 and Proposition 3.6 imply a decay estimate for the Weiss energy.

Corollary 3.7. *Let $\tilde{u}$ be a solution to* (4)–(5) *with $\kappa = 2m$ and satisfy* (6). *Then*

(i) *Weiss energy goes to $-\infty$ exponentially fast if at the initial time it is negative: there exists $\gamma_m \in (0,1)$ such that if $W_{2m}(\tilde{u}(0)) < 0$, then*
$$W_{2m}(\tilde{u}(\tau)) \leq e^{\gamma_m \tau} W_{2m}(\tilde{u}(0)). \tag{45}$$
Moreover, in this case we have
$$\|\tilde{u}(\tau)\|^2_{L^2_\mu} \geq -\frac{2W_{2m}(\tilde{u}(0))}{\gamma_m}\left(e^{\gamma_m\tau} - 1\right) + \|\tilde{u}(0)\|^2_{L^2_\mu}.$$

(ii) *If $W_{2m}(\tilde{u}(\tau)) \geq 0$ for all $\tau \in (0,\infty)$ and $\mathrm{dist}_{L^2_\mu}(\tilde{u}(0), \mathcal{E}^+_{2m}) \leq \delta_0$ for some $\delta_0 = \delta_0(m,n) > 0$, then the Weiss energy has the algebraic decay: there exists $c_0 \in (0,1)$ depending on n, m such that for $\gamma = \frac{1}{n+1}$*
$$W_{2m}(\tilde{u}(\tau)) \leq \left(c_0(1-\gamma)\tau + W_{2m}(\tilde{u}(0))^{\gamma-1}\right)^{-\frac{1}{1-\gamma}}, \quad \tau \in (0,\infty).$$
Moreover, there exists a unique $\tilde{u}(\infty) := \lim_{\tau\to\infty}\tilde{u}(\tau) \in \mathcal{E}^+_{2m}$ and $C > 0$ depending on $W_{2m}(\tilde{u}(0))$, $\|\tilde{u}(0)\|_{L^2_\mu}$, m and n, such that
$$\|\tilde{u}(\tau) - \tilde{u}(\infty)\|_{L^2_\mu} \leq C\tau^{-\frac{\gamma}{1-\gamma}}.$$

Proof: The proof for (i) is the same as for Corollary 3.3. We will only provide the proof for (ii).

We first show the decay estimate for the Weiss energy. By the same arguments as above we can get a slightly more general decay estimate for the Weiss energy: under the same assumptions of Proposition 3.6, there exists a $c_0 \in (0,1)$ only depending on m, n such that
$$W_{2m}(\tilde{u}(\tau+h)) \leq \left(1 - c_0 h W_{2m}(\tilde{u}(\tau+h))^{1-\gamma}\right) W_{2m}(\tilde{u}(\tau)), \quad \gamma = 1/(n+1)$$

for all $\tau \in (0, \infty)$ and $h \in (0, 1]$. This implies that for any $0 < \tau_1 < \tau_2 < \infty$,

$$W_{2m}(\tilde{u}_{2m}(\tau_2)) - W_{2m}(\tilde{u}_{2m}(\tau_1)) = \int_{\tau_1}^{\tau_2} \frac{d}{d\tau} W_{2m}(\tilde{u}(\tau)) \leq -c_0 \int_{\tau_1}^{\tau_2} W_{2m}(\tilde{u}(\tau))^{2-\gamma}. \tag{46}$$

Solving this differential inequality we obtain the algebraic decay estimate for the Weiss energy:

$$W_{2m}(\tilde{u}(\tau)) \leq (A_0\tau + A_1)^{-\frac{1}{1-\gamma}}, \ A_0 := c_0(1-\gamma), \ A_1 := W_{2m}(\tilde{u}(0))^{\gamma-1}$$

Next, for $0 < \tau_1 < \tau_2 < \infty$ with $\tau_1 \in [2^i, 2^{i+1})$ and $\tau_2 \in [2^j, 2^{j+1}]$, using (8), Hölder's inequality, monotone decreasing property of the Weiss energy and (46),

$$\|\tilde{u}(\tau_1) - \tilde{u}(\tau_2)\|_{L^2_\mu} \leq \sum_{k=i}^{j} \int_{2^k}^{2^{k+1}} \|\partial_\tau \tilde{u}\|_{L^2_\mu} d\tau \leq \sum_{k=i}^{j} \left(\int_{2^k}^{2^{k+1}} \|\partial_\tau \tilde{u}\|^2_{L^2_\mu} d\tau \right)^{1/2} (2^k)^{1/2}$$

$$\leq \sum_{k=i}^{j} \left(W_{2m}(\tilde{u}(2^k)) - W_{2m}(\tilde{u}(2^{k+1})) \right)^{1/2} (2^k)^{1/2}$$

$$\leq \sum_{k=i}^{j} (A_0 2^k + A_1)^{-\frac{1}{2(1-\gamma)}} 2^{k/2} \leq C 2^{-\frac{\gamma}{1-\gamma} i} \leq C \tau_1^{-\frac{\gamma}{1-\gamma}}.$$

Therefore, $\tau \mapsto \tilde{u}(\tau)$ forms a Cauchy sequence in L^2_μ. Thus the limit $\lim_{\tau\to\infty} \tilde{u}(\tau)$ exists. Let $\tilde{u}(\infty)$ denote the limit. By Proposition 2.6, $\tilde{u}(\infty) \in \mathcal{E}^+_{2m}$. ∎

We remark that at this stage we do not know whether or not the limit $\tilde{u}(\infty)$ is zero.

4. Consequences of the epiperimetric inequality

4.1. The case $\kappa = 3/2$: Uniqueness of blow-ups and regularity of the regular free boundary. In this section we apply the decay estimates for the Weiss energy in Section 3 to our original Signorini problem to derive the regularity of the free boundary.

Proof for Theorem 1: We will prove the following:

$$\left(\int_{\mathbb{R}^n_+} |u_\lambda(x,t)-u_0(x)|^2 G(x,t)dx\right)^{1/2} \leq C(\sqrt{-t})^{\kappa+\gamma_0}\lambda^{\gamma_0}, \tag{47}$$

where for any $\lambda \in (0,1]$, $u_\lambda(x,t) := \lambda^{-3/2}u(\lambda x, \lambda^2 t)$. Theorem 1 follows by taking $\lambda = 1$.

Let $\tilde{u} := \tilde{u}_{3/2}$ be the 3/2-normalized solution in the conformal coordinates as in Lemma 2.1. Firstly we want to show that there exists $\gamma_0 \in (0,1)$ universal, such that the upper bound in (2) yields $W_{3/2}(\tilde{u}(\tau)) \geq 0$ for all τ. Indeed, assume that $W_{3/2}(\tilde{u}(\tau_0)) < 0$ for some $\tau_0 > 0$. Then by Corollary 3.3, there exists a universal $\gamma_0 \in (0,1)$ and $C_0 > 0$, such that $\|\tilde{u}(\tau)\|^2_{L^2_\mu} \geq C_0 e^{\gamma_0(\tau-\tau_0)}$ for $\tau > \tau_0$. Transforming back to the original coordinate we have $\int_{\mathbb{R}^n_+} u^2(x,t)G(x,t)dx \geq C(\sqrt{-t})^{3-2\gamma_0}$ for some $C > 0$ depending on τ_0 and for $|t|$ sufficiently small. This is however a contradiction to the upper bound in (2).

With nonnegative Weiss energy at hand we apply (i) in Corollary 3.3 to conclude that there exists a unique $\tilde{u}(\infty) =: u_0 \in \mathcal{E}_{3/2}$ such that

$$\begin{aligned}\int_{\mathbb{R}^n_+} (\sqrt{-t})^3 |u_\lambda(x,t)-u_0(x)|^2 G(x,t)dx &= c_n \int_{\mathbb{R}^n_+} |\tilde{u}(y,\tau-2\ln\lambda)-u_0(y)|^2 d\mu \\ &\leq Ce^{-\gamma_0(\tau-2\ln\lambda)} = C(\sqrt{-t})^{2\gamma_0}\lambda^{2\gamma_0}.\end{aligned}$$

Here in the first equation we have used (12) and that u_0 is 3/2-homogeneous.

In the end we will show that the lower bound in (2) implies that $u_0 \neq 0$. Indeed, if u_0 vanishes identically, then it holds

$$\int_{\mathbb{R}^n_+} |u(x,t)|^2 G(x,t)dx \leq C(\sqrt{-t})^{3+2\gamma_0}, \quad t \in [-1,0].$$

The above estimate yields that $H_u(r) \leq Cr^{3+2\gamma_0}$. This is a contradiction to the lower bound in (2). ■

Remark 4.1. *Rewriting* (28) *in Remark 3.4 into the original variable, we see that instead of the lower bound assumption in* (2)*, we can assume the solution is close to* $\mathcal{E}_{3/2}$ *at* $t = -1$ *to guarantee the non-triviality of the* 3/2*-blowup* *limit. More precisely, assume that at* $t=-1$

$$W_{3/2}(u(-1)) \leq \delta_0 \int_{\mathbb{R}^n_+} |u(x,-1)|^2 d\tilde{\mu}(x), \quad d\tilde{\mu}(x) := G(x,-1)dx, \tag{48}$$

where $W_{3/2}(u_0)$ *is the Weiss energy in the original variable as in* (11)*, and*

$$\inf_{p\in\mathcal{E}_{3/2}} \int_{\mathbb{R}^n_+} |u(x,-1)-p(x)|^2 d\tilde{\mu}(x) \leq \delta_0 \int_{\mathbb{R}^n_+} |u(x,-1)|^2 d\tilde{\mu}(x). \tag{49}$$

Then if δ_0 *is sufficiently small depending on* n *and* $\|f\|_{L^\infty}$*, there is a unique* $u_0 = c_0 \operatorname{Re}(x'\cdot e_0 + i|x_n|)^{3/2} \in \mathcal{E}_{3/2}$ *with* $c_0 \geq c_n > 0$ *such that* (47) *holds true. We note that conditions* (48)*–*(49) *are satisfied if*

$$\operatorname{dist}_{W^{1,2}_{\tilde{\mu}}}\left(\frac{u(\cdot,-1)}{\|u(\cdot,-1)\|_{L^2_\mu}}, \mathcal{E}_{3/2}\right) \leq \delta_0.$$

We also note that under the assumptions of Theorem 1, (48)*–*(49) *are satisfied for* u_λ *for sufficiently small* $\lambda>0$ *depending on* u_0*.*

An advantage of the conditions (48)–(49) is that they are stable under the translation. More precisely, by the Hölder continuity of u, (48)–(49) hold with constant $2\delta_n$ for $u(x-x_0,t-t_0)$, where (x_0,t_0) varies in a small neighborhood of $(0,0)$. We let $\Gamma_{3/2}(u)$ denote the set of the free boundary points at which the 3/2-homogeneous scaling $u_{(x_0,t_0),\lambda}$ has a unique *nonzero* blow-up limit in $\mathcal{E}_{3/2}$ as $\lambda\to 0$. Then the above discussion leads to the openness of $\Gamma_{3/2}(u)$:

Proposition 4.2. *Let* $u : S_2^+ \to \mathbb{R}$ *be a solution to* (1) *which satisfies assumptions (A)-(C). Let* $H_u^{(x_0,t_0)}(r) := \frac{1}{r^2}\int_{S^+_{r^2}} u(x-x_0,t-t_0)^2 G(x,t)dxdt$*,* $r>0$*. Assume that for each* $(x_0,t_0)\in\Gamma_u$*,*

$$H_u^{(x_0,t_0)}(r) \leq C r^{3-\gamma_0} \tag{50}$$

for r *sufficiently small depending on* u *and* (x_0,t_0)*. Assume that* $(\bar{x}_0,\bar{t}_0)\in\Gamma_{3/2}(u)\cap S_1'$*. Then there exists a small* $r>0$ *depending on* $(\bar{x}_0,\bar{t}_0)$ *such that* $\Gamma_u \cap (B_r(\bar{x}_0)\times(\bar{t}_0-r^2,\bar{t}_0+r^2)) \subset \Gamma_{3/2}(u)$*.*

Given $(x_0,t_0)\in\Gamma_{3/2}(u)$, we let

$$u_{(x_0,t_0)}(x) = c_{(x_0,t_0)} \operatorname{Re}(e_{(x_0,t_0)}\cdot x + i|x_n|)^{3/2}, \quad c_{(x_0,t_0)} > 0$$

denote the blow-up limit. Next we prove the continuity of the maps $\Gamma_{3/2}(u)\ni(x_0,t_0)\mapsto c_{(x_0,t_0)}$ and $\Gamma_{3/2}(u)\ni(x_0,t_0)\mapsto e_{(x_0,t_0)}$.

Proposition 4.3. *Let $u : S_1^+ \to \mathbb{R}$ be a solution to* (1)*. Assume that Let* $(x_0, t_0), (y_0, s_0) \in \Gamma_{3/2}(u) \cap Q_1$*, where* $Q_1 := B_1(0) \times (-1, 0)$*. Then, the maps*

$$\begin{aligned}\Gamma_{3/2}(u) \ni (x_0, t_0) &\mapsto c_{(x_0,t_0)} \in \mathbb{R}_+,\\ \Gamma_{3/2}(u) \ni (x_0, t_0) &\mapsto e_{(x_0,t_0)} \in \mathbb{S}^{n-1} \cap \{y_n = 0\}\end{aligned}$$

are parabolically θ-Hölder continuous for some $\theta \in (0, 1)$.

Proof: We note that by rotation invariance $c_{(x_0,t_0)} = c_n \|u_{(x_0,t_0)}\|_{L^2_{\tilde{\mu}}}$ for $c_n > 0$. Hence, for $(x_0, t_0), (y_0, s_0) \in \Gamma_{3/2}(u) \cap Q_1$ and $\lambda > 0$

$$\begin{aligned}&|c_{(x_0,t_0)} - c_{(y_0,s_0)}| \leq c_n \left| \|u_{(x_0,t_0)}\|_{L^2_{\tilde{\mu}}} - \|u_{(y_0,s_0)}\|_{L^2_{\tilde{\mu}}} \right|\\ &\leq c_n \Big(\|u_{(x_0,t_0)} - u_{(x_0,t_0),\lambda}(\cdot, -1)\|_{L^2_{\tilde{\mu}}} + \|u_{(y_0,s_0)} - u_{(y_0,s_0),\lambda}(\cdot, -1)\|_{L^2_{\tilde{\mu}}}\\ &\quad + \left| \|u_{(x_0,t_0),\lambda}(\cdot, -1)\|_{L^2_{\tilde{\mu}}} - \|u_{(y_0,s_0),\lambda}(\cdot, -1)\|_{L^2_{\tilde{\mu}}} \right| \Big)\\ &\leq C\lambda^{\gamma_0} + Cd((x_0, t_0), (y_0, s_0))^{\alpha} \lambda^{-3/2}.\end{aligned}$$

Here $d((x_0, t_0), (y_0, s_0)) = |x_0 - y_0| + |t_0 - s_0|^{1/2}$ is the parabolic distance, and to estimate the three terms coming from the triangle inequality, we have used (47) to bound the first two integrals and the interior Hölder $C^{\alpha,\alpha/2}$ estimate of the solution to bound the third integral. Balancing the above two bounds we get

$$|c_{(x_0,t_0)} - c_{(y_0,s_0)}| \leq Cd((x_0, t_0), (y_0, s_0))^{\theta}, \quad \theta = \frac{\gamma_0 \alpha}{\gamma_0 + 3/2} \in (0, 1).$$

Next we note that

$$\left\| u_{(x_0,t_0)}/c_{(x_0,t_0)} - u_{(y_0,s_0)}/c_{(y_0,s_0)} \right\|_{L^2_{\tilde{\mu}}} \geq C_n |e_{(x_0,t_0)} - e_{(y_0,s_0)}|.$$

Using similar estimate as above and combining it with the estimate for $c_{(x_0,t_0)}$ then yields the claimed Hölder continuity of $e_{(x_0,t_0)}$. ∎

With the previous results at hand, we can prove the regularity of the regular free boundary.

Proposition 4.4. *Let $u : S_1^+ \to \mathbb{R}$ be a solution to* (1)*. Assume that* $(x_0, t_0) \in \Gamma_{3/2}(u)$*. Then, there exists a radius $r \in (0, 1)$ depending on (x_0, t_0) such that $\Gamma_u \cap Q_r(x_0, t_0)$, where $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$, can be represented as a graph (after a suitable choice of coordinates)*

$$\Gamma_{3/2}(u) \cap Q_r(x_0, t_0) := \{(x', 0, t) : x_{n-1} = g(x'', t)\}.$$

Moreover, there exists $\theta \in (0,1)$ *such that* $\nabla'' g \in C^{\theta,\theta/2}$.

Proof: From Proposition 4.2 we know that there exists $r > 0$ depending on (x_0, t_0) such that $\Gamma_u \cap Q_r(x_0,t_0)$ consist of $\Gamma_{3/2}(u)$ free boundary points. Let $c_{(x_0,t_0),\lambda} > 0$ and $e_{(x_0,t_0),\lambda}$ be the direction such that $c_{(x_0,t_0),\lambda}\operatorname{Re}(x' \cdot e_{(x_0,t_0),\lambda} + i|x_n|)^{3/2} \in \mathcal{E}_{3/2}$ realizes the $L^2_{\tilde{\mu}}$ distance between $u_\lambda(\cdot,-1)$ and $\mathcal{E}_{3/2}$. Then from the proof for Proposition 4.3 we have that for each λ sufficiently small,

$$|e_{(x_0,t_0),\lambda} - e_{(y_0,s_0),\lambda}| \le Cd((x_0,t_0),(y_0,s_0))^\theta.$$

for any $(x_0,t_0),(y_0,s_0) \in \Gamma_u \cap Q_r(x_0,t_0)$, and $C > 0$ independent of λ. Thus we find a parameter family of hypersurfaces Γ_u^λ, the normals of which are spacial and equal to $e_{(x_0,t_0),\lambda}$ at each $(x_0,t_0) \in \Gamma_u^\lambda$, and they are uniformly C^θ regular with respect to the parabolic distance. Passing to the limit as $\lambda \to 0$ we thus obtain that the limiting hypersurface, which is the free boundary $\Gamma_u \cap Q_r(x_0,t_0)$, is a C^θ hypersurface. Thus up to a rotation of the spacial cooridnates, it can be represented as the graph $x_{n-1} = g(x'',t)$ for some function g, where $\nabla'' g \in C^{\theta,\theta/2}$. ■

4.2. The case $\kappa = 2m$.

4.2.1. *Uniqueness and nondegeneracy.* We first prove Theorem 2 by using decay estimate of the Weiss energy in Corollary 3.7.

Proof for Theorem 2: *1.Uniqueness.* We will show that there exists a unique parabolically $2m$-homogeneous polynomial solution $p_0(x,t)$ such that for all $\lambda \in (0,1]$ and $t \in [-1,0)$

$$\left(\int_{\mathbb{R}^n_+} |u_\lambda(x,t) - p_0(x,t)|^2 G(x,t)dx\right)^{1/2} \le C(\sqrt{-t})^{2m}\left(-\ln(-t) - 2\ln\lambda\right)^{-\frac{\gamma}{1-\gamma}}.$$

Here $\gamma := 1/(n+1)$ and $u_\lambda(x,t) := \frac{u(\lambda x, \lambda^2 t)}{\lambda^\kappa}$, $\lambda > 0$.

Let $\tilde{u} = \tilde{u}_{2m}$ be the $2m$ conformal normalized solution as in Lemma 2.1. First, the upper bound on $H_u(r)$ implies that $W_{2m}(\tilde{u}(\tau)) \ge 0$ for all $\tau \in (0,\infty)$. Indeed, if $W_{2m}(\tilde{u}(\tau_0)) < 0$ for some $\tau_0 > 0$, then by Corollary 3.7, there exist $\gamma_m \in (0,1)$ and $C_0 > 0$ such that $\|\tilde{u}(\tau)\|^2_{L^2_\mu} \ge C_0 e^{\gamma_m(\tau-\tau_0)}$ for each $\tau > \tau_0$. Back in the original coordinates we have $\int_{\mathbb{R}^n_+} u^2(x,t)G(x,t)dx \ge C(\sqrt{-t})^{4m-2\gamma_m}$ for some $C > 0$ depending on τ_0 and for each $t \in (-e^{-\tau_0}, 0)$. This is however a contradiction to our assumption on H_u when $|t|$ is sufficiently small.

Thus by Corollary 3.7 (ii), there is a unique $p(y) \in \mathcal{E}_{2m}^{+}$ such that

$$\|\tilde{u}(\tau) - p\|_{L^2_\mu} \le C\tau^{-\frac{\gamma}{1-\gamma}}, \quad \gamma = \frac{1}{n+1}$$

for some $C > 0$ depending on m, n. Let $p_0(x,t) := (\sqrt{-t})^{\kappa} p(\frac{x}{2\sqrt{-t}})$. Writing the above inequality by the original variables and by (12), we obtain the desired estimate.

2.Nondegeneracy. In this step we show that if for some $\lambda \in (0,1]$, the smallness assumption

$$\mathrm{dist}_{L^2_{\tilde{\mu}}}(u_\lambda(\cdot,-1), \mathcal{E}_{2m}^{+})^2 \le \delta_0 \|u_\lambda(\cdot,-1)\|^2_{L^2_{\tilde{\mu}}}$$

is satisfied, then the unique blow-up limit obtained in *step 1* is not zero.

Indeed, our smallness assumption implies that the L^2_μ projection of $\tilde{u}(\tau_0)$, $\tau_0 = -2\lambda$, to $\mathcal{E}_{2m}^{+}$ is not zero, i.e. there exists $\lambda_0 \bar{p}_0(y) \in \mathcal{E}_{2m}^{+}$ with $\lambda_0 > 0$ and $\|\bar{p}_0\|_{L^2_\mu} = 1$ which realizes the distance between $\tilde{u}(\tau_0)$ and $\mathcal{E}_{2m}^{+}$. We now consider the orthogonal decomposition: $\tilde{u}(\tau) = \lambda_0(\tau)\bar{p}_0 + \tilde{w}(\tau)$ for each $\tau \ge \tau_0$. By the orthogonality $\lambda_0(\tau) = \int_{\mathbb{R}^n_+} \tilde{u}(y,\tau)\bar{p}_0(y) d\mu$. Using the equation of $\tilde{u}$ and that $\mathcal{L}_{2m}\bar{p}_0 = 0$ in $\mathbb{R}^n$, we have $\dot{\lambda}_0(\tau) = -\frac{1}{4}\int_{\{y_n=0\}} \bar{p}_0 \partial_n \tilde{u}(\tau) d\mu$ for a.e. $\tau > \tau_0$. Since $\bar{p}_0 \ge 0$ and $\partial_n \tilde{u} \le 0$ on $\{y_n = 0\}$, we have that $\dot{\lambda}_0(\tau) \ge 0$. Thus $\lambda_0(\tau) \ge \lambda_0 > 0$ for all $\tau > \tau_0$. This implies that the limit of $\tilde{u}(\tau)$ as $\tau \to \infty$ is nontrivial. Thus the blow-up limit p_0 obtained in *step 1* is nontrivial. ∎

4.2.2. *Frequency gap and structure of the singular set.* By the Almgren-Poon's monotonicity formula for solutions to the parabolic Signorini problem (with $f = 0$), each free boundary point $(x_0, t_0) \in \Gamma_u$ can be associated with a frequency $\kappa_{(x_0,t_0)} \in \{3/2\} \cup [2, \infty)$. Furthermore, blow-ups at (x_0, t_0) are parabolic $\kappa_{(x_0,t_0)}$-homogeneous solutions, cf. [8]. Another consequence of the epiperimetric inequality is the gap of the frequency around $2m$, $m \in \mathbb{N}_+$.

Proposition 4.5 (Frequency gap). *Let $u : S_2^+ \to \mathbb{R}$ be a solution to the parabolic Signorini problem* (1) *with $f = 0$. Assume that u satisfies assumptions (A)-(C). Then there exist positive constants c_- and c_+ depending on m, n such that*

$$\{(x,t) \in \Gamma_u : \kappa_{(x,t)} \in (2m - c_-, 2m + c_+)\} = \emptyset.$$

Proof: *1.* Assume that $(0,0) \in \Gamma_u$ is a free boundary point with the frequency $2m + \epsilon$ for some $\epsilon > 0$. Let $u(x,t)$ be a nontrivial $2m + \epsilon$ parabolic homogeneous blow-up limit at $(0,0)$. In the conformal coordinate it is of the form $e^{-(m+\epsilon/2)\tau}v(y)$ for some nontrivial function v, which solves the stationary Signorini problem

$$\mathcal{L}_{2m}v + \frac{\epsilon}{2}v = 0 \text{ in } \mathbb{R}^n_+,$$
$$v \geq 0,\ \partial_n v \leq 0,\ v\partial_n v = 0 \text{ on } \{y_n = 0\}.$$

We consider the $2m$-normalized solution $\tilde{u}_{2m}(y,\tau) = e^{m\tau}u(x(y,\tau), t(y,\tau)) = e^{-\epsilon\tau/2}v(y)$ as in Lemma 2.1. By Proposition 3.6,

$$W_{2m}(\tilde{u}_{2m}(\tau+1)) \leq \left(1 - c_0 W_{2m}(\tilde{u}_{2m}(\tau+1))^{1-\gamma}\right) W_{2m}(\tilde{u}_{2m}(\tau)), \quad \gamma = 1/(n+1).$$

Note that $W_{2m}(\tilde{u}(\tau)) = e^{-\epsilon\tau}W_{2m}(v)$ and $W_{2m}(v) = \epsilon\|v\|^2_{L^2_\mu}/2$. Thus the above inequality can be rewritten as (after dividing by $W_{2m}(v)$)

$$e^{-\epsilon(\tau+1)} \leq \left(1 - c_0 e^{-\epsilon(\tau+1)}(\epsilon\|v\|^2_{L^2_\mu}/2)^{1-\gamma}\right) e^{-\epsilon\tau}.$$

Evaluating at $\tau = 0$ we have

$$e^{-\epsilon} \leq 1 - \tilde{c}e^{-\epsilon}\epsilon^{1-\gamma}, \quad \tilde{c} := \frac{c_0\|v\|^{2(1-\gamma)}_{L^2_\mu}}{2^{1-\gamma}}.$$

Necessarily $\epsilon \geq \epsilon_0$, thus $c_+ \geq \epsilon_0$, for some $\epsilon_0 > 0$ depending on $\tilde{c}$.

2. Assume that $(0,0)$ is a free boundary point with frequency $2m - \epsilon$, $\epsilon > 0$. Then $\epsilon > \gamma_m$, where $\gamma_m \in (0,1)$ is the constant from Corollary 3.7 (i). Indeed, similarly as in step 1 we consider a nontrivial blow-up limit at $(0,0)$. After $2m$-normalization and in the conformal coordinates this leads to $\tilde{u}_{2m}(y,\tau) = e^{\epsilon\tau/2}v(y)$, where v solves

$$\mathcal{L}_{2m}v - \frac{\epsilon}{2}v = 0 \text{ in } \mathbb{R}^n_+,$$
$$v \geq 0,\ \partial_n v \leq 0,\ v\partial_n v = 0 \text{ on } \{y_n = 0\}.$$

Thus one has

$$W_{2m}(\tilde{u}_{2m}(\tau)) = -\frac{\epsilon}{2}e^{\epsilon\tau}\|v\|^2_{L^2_\mu} < 0, \quad \tau \in (0,\infty).$$

By (45), $W_{2m}(\tilde{u}_{2m}(\tau)) \leq e^{\gamma_m\tau}(-\frac{\epsilon}{2}\|v\|_{L^2_\mu})$, which implies that $\epsilon \geq \gamma_m > 0$. ■

The decay estimate in Theorem 2 gives finer regularity results for the singular set $\Sigma(u)$. We briefly recall the definition and the main properties of the singular set and refer to [8] for detailed statements. Let $\Sigma(u)$ denote the set of free boundary points (x_0,t_0) such that the parabolic density of the contact set Λ_u at (x_0,t_0) is zero, i.e.

$$\lim_{r\to 0_+}\frac{\mathcal{H}^n(\Lambda_u\cap Q_r'(x_0,t_0))}{\mathcal{H}^n(Q_r')}=0.$$

By Proposition 12.2 in [8], $(x_0,t_0)\in\Sigma(u)$ if and only if $\kappa_{(x_0,t_0)}=2m$, $m\in\mathbb{N}_+$, if and only if blow-ups are $2m$-parabolic homogeneous polynomial solutions. Thus around the singular point our Theorem 2 is applicable, which yields a unique nonzero blow-up limit at each singular point. Moreover, let $\Sigma_{2m}(u):=\{(x,t)\in\Sigma(u):\kappa_{(x,t)}=2m\}$, we can show that the blow-up limit $p_{(x_0,t_0)}$, $(x_0,t_0)\in\Sigma_{2m}(u)$, varies continuously on $\Sigma_{2m}(u)$ with a modulus of continuity $\omega(s)=-\ln s$.

Proposition 4.6. *Let $(x_0,t_0),(y_0,s_0)\in\Sigma_{2m}(u)\cap Q_{1/2}$. Let $p_{(x_0,t_0)}$ and $p_{(y_0,s_0)}$ be the unique blow-up limits. Then there exists a positive constant C depending on n,m,α such that*

$$\|p_{(x_0,t_0)}-p_{(y_0,s_0)}\|_{L^\infty(Q_1)}\le C(-\ln d((x_0,t_0),(y_0,s_0)))^{-\frac{\gamma}{1-\gamma}},\quad \gamma=\frac{1}{n+1},$$

where $d((x_0,t_0),(y_0,s_0)):=|x_0-y_0|+|t_0-s_0|^{1/2}$ is the parabolic distance.

Proof: Consider the rescaled family $u_{(x_0,t_0),\lambda}(x,t):=\frac{u(x_0+\lambda x,t_0+\lambda^2 t)}{\lambda^{2m}}$, $\lambda>0$. By Theorem 2, there exists a unique nonzero $2m$-parabolic homogeneous polynomial $p_{(x_0,t_0)}$ such that

$$\|u_{(x_0,t_0),\lambda}(x,-1)-p_{(x_0,t_0)}(x,-1)\|_{L^2_{\tilde{\mu}}}\le C_{m,n}(-2\ln\lambda)^{-\frac{\gamma}{1-\gamma}},$$

for all $\lambda\in(0,1/2)$. Similarly, for $(y_0,s_0)\in\Sigma_{2m}(u)\cap Q_{1/2}$,

$$\|u_{(y_0,s_0),\lambda}(x,-1)-p_{(y_0,s_0)}(x,-1)\|_{L^2_{\tilde{\mu}}}\le C_{m,n}(-2\ln\lambda)^{-\frac{\gamma}{1-\gamma}}.$$

Therefore, by the triangle inequality

$$\begin{aligned}
&\|p_{(x_0,t_0),\lambda}(x,-1)-p_{(y_0,s_0),\lambda}(x,-1)\|_{L^2_{\tilde{\mu}}}\\
&\le\|u_{(x_0,t_0),\lambda}(x,-1)-p_{(x_0,t_0),\lambda}(x,-1)\|_{L^2_{\tilde{\mu}}}+\|u_{(y_0,s_0),\lambda}(x,-1)-p_{(y_0,s_0)}(x,-1)\|_{L^2_{\tilde{\mu}}}\\
&\quad+\|u_{(x_0,t_0),\lambda}-u_{(y_0,s_0),\lambda}\|_{L^2_{\tilde{\mu}}}\\
&\le C_{m,n}(-2\ln\lambda)^{-\frac{\gamma}{1-\gamma}}+C_{m,n,\alpha}d((x_0,t_0),(y_0,s_0))^\alpha\lambda^{-2m}.
\end{aligned}$$

Here in the last line we have used the interior $C^{\alpha,\alpha/2}$ estimate for the solution to the Signorini problem. Balancing the two terms by taking $\lambda=$

$d((x_0,t_0),(y_0,s_0))^{\frac{\alpha}{2m+2/n}}$ we obtain

$$\|p_{(x_0,t_0),\lambda}(x,-1) - p_{(y_0,s_0),\lambda}(x,-1)\|_{L^2_{\tilde{\mu}}} \le C\left(-\ln d((x_0,t_0),(y_0,s_0))\right)^{-\frac{\gamma}{1-\gamma}}.$$

Note that $p_{(x_0,t_0)}(x,-1) = \bar{p}_{(x_0,t_0)}(2y)$, where $\bar{p}_{(x_0,t_0)}(y) \in \mathcal{E}_{2m}$ is an eigenfunction for $\mathcal{L}_{2m}$. Thus by writing $\bar{p}_{(x_0,t_0)} = \sum_{|\alpha|=2m} \lambda_\alpha^{(x_0,t_0)} p_\alpha$ and $\bar{p}_{(y_0,s_0)} = \sum_{|\alpha|=2m} \lambda_\alpha^{(y_0,s_0)} p_\alpha$, where $\{p_\alpha\}$ is an orthonormal basis for $\mathcal{E}_{2m}$, we obtain from the bound on the $L^2_{\tilde{\mu}}$ norm that

$$\sum_\alpha |\lambda_\alpha^{(x_0,t_0)} - \lambda_\alpha^{(y_0,s_0)}|^2 \le C\left(-\ln d((x_0,t_0),(y_0,s_0))\right)^{-\frac{2\gamma}{1-\gamma}}.$$

Note that $p(x,t) = (\sqrt{-t})^{2m})p(x/\sqrt{-t},-1)$. Thus from the bound on the differences of the corresponding coefficients of the polynomials $p_{(x_0,t_0)}(x,t)$ and $p_{(y_0,s_0)}(x,t)$, we obtain

$$\|p_{(x_0,t_0)}(x,t) - p_{(y_0,s_0)}(x,t)\|_{L^\infty(Q_1)} \le C\left(-\ln d((x_0,t_0),(y_0,s_0))\right)^{-\frac{\gamma}{1-\gamma}},$$

for any $(x_0,t_0),(y_0,s_0) \in \Sigma_{2m}(u) \cap Q_{1/2}$. ■

Proposition 4.6 together with the parabolic Whitney extension theorem in [8] would yield the regularity of the singular set. We remark that with the precise logarithmic modulus of continuity at hand, it is plausible to conclude a finer regularity property for the singular set by generalizing the Whitney extension theorem considered in [9] to the parabolic setting.

5. Perturbation

In this section we show how to modify our proof in Section 3 to the nonzero inhomogeneity setting. We consider global solutions which satisfies (A)-(C) to

$$\partial_t u - \Delta u = f \text{ in } S_2^+$$
$$u \ge 0,\ \partial_n u \le 0,\ u\partial_n u = 0 \text{ on } S_2',$$

where $f = f(x,t) \in L^\infty(S_2^+)$. By chapter 4 of [8], the study of local solutions with nonzero obstacles can be reduced to the study of global solutions to the above inhomogeneous equations by subtracting the obstacle and applying suitable cut-offs.

Assume $(0,0) \in \Gamma_u$ is a free boundary point of frequency κ. Under a similar conformal change of variables around $(0,0)$ as before (cf. Lemma 2.1) we have that $\tilde{u}_\kappa$ solves

$$\begin{aligned} &\partial_\tau \tilde{u}_\kappa + \frac{y}{2}\cdot\nabla\tilde{u}_\kappa - \frac{1}{4}\Delta\tilde{u}_\kappa - \frac{\kappa}{2}\tilde{u}_\kappa = e^{\tau(\kappa/2-1)}\tilde{f} \text{ in } \mathbb{R}^n_+\times[0,\infty),\\ &\tilde{u}_\kappa \geq 0,\ \partial_n\tilde{u}_\kappa\leq 0,\ \tilde{u}_\kappa\partial_n\tilde{u}_\kappa = 0 \text{ on } \{y_n=0\}, \end{aligned} \tag{51}$$

where $\tilde{f}(y,\tau) := f(2e^{-\tau/2}y, -e^{-\tau})$. Note that up to a dimensional constant

$$M_{\tilde{f}} := \|\tilde{f}\|_{L^\infty([0,\infty);L^2_\mu)} = \sup_{t\in[-1,0)}\left(\int_{\mathbb{R}^n_+}|f(x,t)|^2G(x,t)dx\right)^{1/2} \leq \|f\|_{L^\infty(S_1^+)}.$$

We first consider the case $\kappa = 3/2$ and denote $\tilde{u} := \tilde{u}_{3/2}$. Let $W(\tilde{u}(\tau)) := W_{3/2}(\tilde{u}(\tau))$ be the Weiss energy (defined as in (7)) associated with the solution $\tilde{u}$. A direct computation as in Lemma 2.4 gives that $\tau\mapsto W(\tilde{u}(\tau))$ satisfies the almost monotone decreasing property: for any $0<\tau_a<\tau_b<\infty$

$$\begin{aligned} W(\tilde{u}(\tau_b)) - W(\tilde{u}(\tau_a)) &\leq -2\int\limits_{\mathbb{R}^n_+\times[\tau_a,\tau_b]}|\partial_\tau\tilde{u}|^2 d\mu d\tau + 2\int\limits_{\mathbb{R}^n_+\times[\tau_a,\tau_b]} e^{-\tau/4}\partial_\tau\tilde{u}\tilde{f}d\mu d\tau\\ &\leq -\int\limits_{\mathbb{R}^n_+\times[\tau_a,\tau_b]}|\partial_\tau\tilde{u}|^2 d\mu d\tau + \int\limits_{\mathbb{R}^n_+\times[\tau_a,\tau_b]} e^{-\tau/2}\tilde{f}^2 d\mu d\tau. \end{aligned} \tag{52}$$

Relying on this almost monotonicity, we seek to prove the following contraction result:

Proposition 5.1. *Let $\tilde{u}$ be a solution to* (51) *with $\kappa = 3/2$ and $\tilde{u}$ satisfies* (6)*. Then there exists a universal constant $c_0\in(0,1/4)$ such that*

$$W(\tilde{u}(\tau+1)) \leq (1-c_0)W(\tilde{u}(\tau)) + 2e^{-\tau/2}M^2_{\tilde{f}} \textit{ for any } \tau>0.$$

The argument for this contraction is similar as before. Instead of providing the full details, we only outline the proof and point out the differences.

Proof of Proposition 5.1: *(i).* Assume not, then there exists $c_j\in(0,1/4)$, $c_j\to 0$, solutions $\tilde{u}_j$ to (51) with inhomogeneity $\tilde{f}_j$ and times τ_j such that

$$W(\tilde{u}_j(\tau_j+1)) \geq (1-c_j)W(\tilde{u}_j(\tau_j)) + 2e^{-\tau_j/2}M^2_{\tilde{f}_j}.$$

In the sequel for notational simplicity we write $\tilde{u}$ and $\tilde{f}$ instead of $\tilde{u}_j$ and $\tilde{f}_j$. Then, using (52), one has

$$\begin{aligned} c_j W(\tilde{u}(\tau_j+1)) &\geq (1-c_j)\left[W(\tilde{u}(\tau_j)) - W(\tilde{u}(\tau_j+1))\right] + 2e^{-\tau_j/2}M_{\tilde{f}}^2 \\ &\geq (1-c_j)\int_{\tau_j}^{\tau_j+1}\left(\|\partial_\tau \tilde{u}\|_{L^2_\mu}^2 - e^{-\tau/2}\|\tilde{f}\|_{L^2_\mu}^2\right) d\tau + 2e^{-\tau_j/2}M_{\tilde{f}}^2 \\ &\geq (1-c_j)\int_{\tau_j}^{\tau_j+1} \|\partial_\tau \tilde{u}\|_{L^2_\mu}^2 d\tau + e^{-\tau_j/2}M_{\tilde{f}}^2. \end{aligned}$$

By the almost monotonicity (52), for any $\tau \in I_j := [\tau_j, \tau_j+1]$,

$$c_j W(\tilde{u}(\tau)) \geq c_j W(\tilde{u}(\tau_j+1)) - c_j e^{-\tau_j/2}M_{\tilde{f}}^2 \geq (1-c_j)\left[\int_{I_j}\|\partial_\tau \tilde{u}\|_{L^2_\mu}^2 d\tau + e^{-\tau_j/2}M_{\tilde{f}}^2\right].$$

Therefore, we obtain

$$\int_{I_j}\|\partial_\tau \tilde{u}\|_{L^2_\mu}^2 d\tau + e^{-\tau_j/2}M_{\tilde{f}}^2 \leq 2c_j\int_{I_j} W(\tilde{u}(\tau))d\tau. \tag{53}$$

(ii). We seek to estimate the Weiss energy for the error term $\tilde{v}_j$. We remark that the relation (16) still holds for the inhomogeneous problem. First, we can write

$$\begin{aligned} &\int_{I_j} W(\tilde{u}(\tau))d\tau = \int_{\mathbb{R}^n_+\times I_j} \mathcal{L}\tilde{u}\tilde{u} d\mu d\tau \\ &= \int_{\mathbb{R}^n_+\times I_j} (-\partial_\tau\tilde{u} + e^{-\tau/4}\tilde{f})\tilde{v} d\mu d\tau + \int_{I_j}\frac{\lambda_j}{2}\int_{\{y_n=0\}}\left(\partial_n\tilde{u}h_{e_j} - \tilde{u}\partial_n h_{e_j}\right) d\mu d\tau \\ &\leq \int_{I_j}\left(\|\partial_\tau\tilde{u}\|_{L^2_\mu} + e^{-\tau/4}\|\tilde{f}\|_{L^2_\mu}\right)\|\tilde{v}\|_{L^2_\mu} d\tau + \int_{I_j}\frac{\lambda_j}{2}\int_{\{y_n=0\}}\left(\partial_n\tilde{u}h_{e_j} - \tilde{u}\partial_n h_{e_j}\right) d\mu d\tau. \end{aligned}$$

Here as in the zero homogeneity case, $\mathcal{L} := \frac{1}{4}\Delta - \frac{y}{2}\cdot\nabla + \frac{3}{4}$. Applying (53) to the first term on the RHS we obtain

$$\int_{I_j} W(\tilde{u}(\tau))d\tau \le \left(2c_j \int_{I_j} W(\tilde{u}(\tau)d\tau\right)^{1/2} \left(\int_{I_j} \|\tilde{v}\|^2_{L^2_\mu} d\tau\right)^{1/2} + \int\limits_{I_j} \frac{\lambda_j}{2} \int\limits_{\{y_n=0\}} \left(\partial_n \tilde{u} h_{e_j} - \tilde{u}\partial_n h_{e_j}\right) d\mu d\tau.$$

By Cauchy-Schwartz and applying (16) to replace $W(\tilde{u})$ by $W(\tilde{v})$ we get,

$$\int_{I_j} W(\tilde{v}(\tau))d\tau \le 20c_j \int_{I_j} \|\tilde{v}\|^2_{L^2_\mu} d\tau + 4\int_{I_j}\int_{\{y_n=0\}} \left(\frac{\lambda_j}{4}\partial_n \tilde{u} h_{e_j} + \frac{\lambda_j}{8}\tilde{u}\partial_n h_{e_j}\right) d\mu d\tau.$$

Noting that the term involving integral on $\{y_n = 0\}$ is non-positive by the Signorini boundary condition, we thus obtain

$$\int_{I_j} W(\tilde{v}(\tau))d\tau \le Cc_j \int_{I_j} \|\tilde{v}\|^2_{L^2_\mu} d\tau. \tag{54}$$

Using $W(\tilde{v}(\tau)) \ge -\frac{3}{4}\|\tilde{v}(\tau)\|^2_{L^2_\mu}$ we further obtain

$$-\frac{\lambda_j}{8}\int_{I_j}\int_{\{y_n=0\}} \left(\partial_n \tilde{u} h_{e_j} + \tilde{u}\partial_n h_{e_j}\right) d\mu d\tau \le C\int_{I_j} \|\tilde{v}\|^2_{L^2_\mu} d\tau. \tag{55}$$

Here in (54) and (55), the constant $C > 0$ is an absolute constant.

With (53), (54) and (55) at hand, we argue as step (iii) and (iv) in Proposition 3.1 and reach a contradiction. ■

Similar as for the zero inhomogeneity case, if the Weiss energy is negative starting from some time τ_0, one can show a stronger decay estimate:

Proposition 5.2. *Let $\tilde{u}$ be a solution to* (51) *with $\kappa = 3/2$ and $\tilde{u}$ satisfies* (6)*. Assume that $W(\tilde{u}(\tau)) \le -4e^{-\tau/2}M^2_{\tilde{f}}$ for all $\tau > 0$. Then there exists a universal constant $c_0 \in (0, 1/4)$ such that*

$$W(\tilde{u}(\tau+1)) \le (1+c_0)W(\tilde{u}(\tau)) + 2e^{-\tau/2}M^2_{\tilde{f}} \text{ for any } \tau > 0.$$

Now we outline how to obtain the decay rate of the Weiss energy and the convergence rate of $\tilde{u}(\tau)$ from Proposition 5.1 and Proposition 5.2. In the perturbation case, we consider the modified energy

$$\widetilde{W}(\tilde{u}(\tau)) := W(\tilde{u}(\tau)) + 4e^{-\frac{\tau}{2}}M^2_{\tilde{f}}.$$

By (52), $\tau \mapsto \widetilde{W}(\tilde{u}(\tau))$ is monotone decreasing. Moreover, from Proposition 5.1 and Proposition 5.2, there exists a $c_0 \in (0, 1/8)$ (c_0 might be different from that in Proposition 5.1 but remains universal) such that

$$\widetilde{W}(\tilde{u}(\tau+1)) \leq (1-c_0)\widetilde{W}(\tilde{u}(\tau)),$$

and if $\widetilde{W}(\tilde{u}(\tau)) < 0$ for all $\tau > 0$,

$$\widetilde{W}(\tilde{u}(\tau+1)) \leq (1+c_0)\widetilde{W}(\tilde{u}(\tau)).$$

This implies that there is some constant $\gamma_0 \in (0,1)$ depending on n such that

$$\widetilde{W}(\tilde{u}(\tau)) \leq e^{-\gamma_0 \tau}\widetilde{W}(\tilde{u}(0)),$$

and

$$\widetilde{W}(\tilde{u}(\tau)) \leq e^{\gamma_0 \tau}\widetilde{W}(\tilde{u}(0)) \text{ if } \widetilde{W}(\tilde{u}(0)) < 0.$$

From this and arguing similarly as in Corollary 3.3 we conclude that if $\widetilde{W}(\tilde{u}(\tau)) \geq 0$ for all $\tau > 0$, then for all $0 < \tau_1 < \tau_2 < \infty$,

$$\|\tilde{u}(\tau_2) - \tilde{u}(\tau_1)\|^2_{L^2_\mu} \leq C_n \left(W(\tilde{u}(0)) + 4M^2_{\tilde{f}}\right) e^{-\gamma_0 \tau_1};$$

and if $\widetilde{W}(\tilde{u}(\tau_0)) < 0$ for some $\tau_0 > 0$, then

$$\|\tilde{u}(\tau)\|^2_{L^2_\mu} \geq -C_n \widetilde{W}(\tilde{u}(\tau_0)) e^{\gamma_0(\tau-\tau_0)}, \quad \tau \in [\tau_0, \infty).$$

For the case $\kappa = 2m$, $m \in \mathbb{N}_+$ in (51) we assume further that the inhomogeneity in the conformal coordinates satisfies: for some $M > 0$ and $\epsilon_0 \in (0,1)$,

$$\|\tilde{f}(\tau)\|_{L^2_\mu} \leq M e^{-\tau(m-1+\epsilon_0)}, \text{ for all } \tau > 0. \tag{56}$$

Note that (56) is satisfied if in the original coordinates $f(x,t)$ has the vanishing property at $(0,0)$ that $|f(x,t)| \leq M(|x| + \sqrt{-t})^{2(m-1+\epsilon_0)}$. Under such assumption, the inhomogeneity only contributes as a higher order term in our estimates. In particular, similar as (52) in the $\kappa = 3/2$ case, we have the almost monotone decreasing property for the Weiss energy: for $0 < \tau_a < \tau_b < \infty$,

$$W(\tilde{u}(\tau_b)) - W(\tilde{u}(\tau_a)) \leq -\int\limits_{\mathbb{R}^n_+ \times [\tau_a, \tau_b]} |\partial_\tau \tilde{u}|^2 d\mu d\tau + M^2 \int_{\tau_a}^{\tau_b} e^{-2\tau\epsilon_0} d\tau.$$

Thus with slight modification as in the case $\kappa = 3/2$, one can generalize Proposition 3.5 and Proposition 3.6 to the nonzero inhomogeneity case, and we do not repeat the proof here.

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