DISCRETE SEMI-CLASSICAL ORTHOGONAL POLYNOMIALS OF CLASS ONE ON QUADRATIC LATTICES

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ABSTRACT: We study orthogonal polynomials on quadratic lattices with respect to a Stieltjes function, $S$, that satisfies a difference equation $A\mathcal{D}S = CMS + D$, where $A$ is a polynomial of degree less or equal than 3 and $C$ is a polynomial of degree greater or equal than 1 and less or equal than 2. We show systems of difference equations for the orthogonal polynomials that arise from the so-called compatibility conditions. Some closed formulae for the recurrence relation coefficients are obtained.

KEYWORDS: Discrete orthogonal polynomials; quadratic lattice; divided-difference operator; semi-classical class.

MATH. SUBJECT CLASSIFICATION (2010): 33C45, 33C47, 42C05.

1. Introduction

Discrete semi-classical orthogonal polynomials have been widely studied in the literature of special functions [12, 19, 20]. They are defined through a difference equation with polynomial coefficients for the corresponding Stieltjes function,

$$ A\mathcal{D}S = CMS + D. $$

Here, $\mathcal{D}$ is some divided-difference operator and $\mathcal{M}$ is a companion difference operator related to $\mathcal{D}$. The divided-difference calculus is classified in terms of hierarchies of operators and related lattices (see, for instance, [22, Sec. 2.3]). In this paper we shall consider the divided-difference operator $\mathcal{D}$ given by

$$ \mathcal{D}f(x(s)) = \frac{f(x(s+1/2)) - f(x(s-1/2))}{x(s+1/2) - x(s-1/2)}, $$

with the so-called quadratic lattice, $x(s) = c_2 s^2 + c_1 s + c_0$ [16, Sec. 2] (see Section 2 for details). In the literature, these lattices are part of the lattices usually referred to as non-uniform. The calculus on non-uniform lattices

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generalizes the calculus on lattices of lower complexity, such as the linear and $q$–uniform lattices.

There are many papers on semi-classical orthogonal polynomials on quadratic lattices. We refer the interested reader to [7, 8, 11, 22] and their list of references. Standard research topics include the study of structure relations, that is, difference equations involving the polynomials, and systematic classifications or characterizations, given pairs of $(A, C)$ in (1).

In the present paper our goal is twofold. First, to gather some recent results on semi-classical orthogonal polynomials on quadratic lattices, namely, difference equations involving the polynomials and related functions, compatibility relations, and new matrix identities. Essentially, such equations generalize well-known differential systems from [14] (see Section 3). Then, with the help of these results, to describe the sequences of orthogonal polynomials within the class one, that is, under the restrictions $\deg(A) \leq 3, 1 \leq \deg(C) \leq 2$ in (1) (see Section 4). The main results are difference equations for the recurrence relation coefficients of the orthogonal polynomials. For the case $\deg(A) \leq 2, \deg(C) = 1$, we recover closed form formulae for the classical orthogonal polynomials.

Let us emphasize that, for some lattices of lower complexity, the description of class one has been carried out. For instance, [2] gives the classification and integral representation of semi-classical linear functionals of class one when $D$ is the derivative operator; in [17], the authors established the system satisfied by the recurrence relation coefficients of symmetric semi-classical orthogonal polynomials of class one when $D$ is the Hahn’s difference operator. We also note [5], an extensive study on semi-classical orthogonal polynomials of class one when $D$ is the forward difference operator.

The remainder of the paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we show difference equations for semi-classical orthogonal polynomials on quadratic lattices, together with the consequent compatibility relations and matrix identities. In Section 4 we deduce difference equations for the recurrence relation coefficients of the semi-classical orthogonal polynomials. Section 5 is devoted to examples: we show applications on the Dual Hahn polynomials as well as on some of their modifications.
2. Divided-difference calculus on quadratic lattices and orthogonal polynomials

Quadratic lattices are commonly defined through a parametric representation \( x = x(s), s \in \mathbb{Z} \),
\[
x(s) = \hat{c}_2 s^2 + \hat{c}_1 s + \hat{c}_0,
\]
for appropriate constants \( \hat{c}_j \)'s \([19, 20]\). The corresponding divided-difference operator, defined on the space of arbitrary functions, is given by \([1, 18, 19]\)
\[
\mathbb{D}f(x(s)) = \frac{f(x(s + 1/2)) - f(x(s - 1/2))}{x(s + 1/2) - x(s - 1/2)}.
\]
Alternatively, \( \mathbb{D} \) can be defined in terms of two functions, say \( y_+, y_- \), as \([15, 22]\)
\[
(\mathbb{D}f)(x) = \frac{f(y_+(x)) - f(y_-(x))}{y_+(x) - y_-(x)},
\]
where \( y_- \) and \( y_+ \) are the two \( y \)-roots of a quadratic equation
\[
\hat{a}y^2 + 2\hat{b}xy + \hat{c}x^2 + 2\hat{d}y + 2\hat{e}x + \hat{f} = 0, \quad \hat{a} \hat{c} \neq 0, \quad \hat{b}^2 = \hat{a} \hat{c}.
\]
As \( y_-, y_+ \) are the \( y \)-roots of \((4)\), we have
\[
y_-(x) = p(x) - \sqrt{r(x)}, \quad y_+(x) = p(x) + \sqrt{r(x)},
\]
with \( p, r \) polynomials of degree one (in \( x \)) given by
\[
p(x) = -\frac{\hat{b}x + \hat{d}}{\hat{a}}, \quad r(x) = \frac{2(\hat{b}\hat{d} - \hat{a}\hat{e})}{\hat{a}^2}x + \frac{\hat{d}^2 - \hat{a}\hat{f}}{\hat{a}^2}.
\]

The polynomials \( p, r \) defined in \((6)\) will play an important role in the sequel. In the account of \((5)\) and \( y_-(x) = x(s - 1/2), \ y_+(x) = x(s + 1/2) \), we have
\[
x(s + 1/2) + x(s - 1/2) = 2p(x(s)), \quad (x(s + 1/2) - x(s - 1/2))^2 = 4r(x(s)).
\]

We take \( \Delta_y = y_+ - y_- \). From \((5)\), there follows
\[
\Delta_y = 2\sqrt{r}.
\]
Define the operators \( \mathbb{E}^+ \) and \( \mathbb{E}^- \) (see [15]), acting on arbitrary functions \( f \), as
\[
\mathbb{E}^\pm f(x) = f(y_\pm(x)).
\]
With this notation, \((3)\) is given by
\[
(\mathbb{D}f)(x) = \frac{\mathbb{E}^+ f - \mathbb{E}^- f}{\mathbb{E}^+ x - \mathbb{E}^- x}.
\]
The companion operator of $D$ is defined as (see [15])

$$
(Mf)(x) = \frac{E^+ f(x) + E^- f(x)}{2}.
$$

(9)

Note that $D$ has the following property: if $f(x)$ is a polynomial of degree $n$ in $x$, then $Df(x)$ is a polynomial of degree $n - 1$ in $x$. $Mf$ is a polynomial whenever $f$ is a polynomial. Furthermore, if $\text{deg}(f) = n$, then $\text{deg}(Mf) = n$.

We emphasize that, throughout the paper, we will deal with polynomials of the variable $x$, not displaying the parametrization (2).

Let us introduce some notations within the functional approach. We take a linear functional, $L : \mathbb{C}[x] \rightarrow \mathbb{C}$, defined by its moments $(u_n)_{n \geq 0}$,

$$
L[x^n] = u_n, \quad n = 0, 1, \ldots,
$$

under the condition

$$
\det [u_{i+j}]_{i,j=0}^n \neq 0, \quad n \geq 0.
$$

(10)

We shall consider systems of orthogonal polynomials, $\{P_n\}_{n \geq 0}$, with respect to $L$, that is,

$$
L[P_n P_m] = h_n \delta_{n,m}, \quad n, m = 0, 1, \ldots,
$$

where $h_n \neq 0$ and $\delta_{n,m}$ is the Kronecker’s delta. It is well known that (10) is a necessary and sufficient condition for the existence of a sequence of orthogonal polynomials with respect to $L$ [21]. Furthermore, if $\det [u_{i+j}]_{i,j=0}^n > 0, \quad n \geq 0$, then there exists a positive measure $\mu$ such that

$$
L[P] = \int_{\text{supp} \mu} P(x) d\mu(x), \quad \forall P \in \mathbb{C}[x],
$$

(11)

thus the family $\{P_n\}_{n \geq 0}$ is said to be orthogonal with respect to $\mu$.

Closely related to $L$ is the moment generating function, the (formal) Stieltjes function, defined by

$$
S(x) = \sum_{n=0}^{+\infty} u_n x^{-n-1}.
$$

(12)

Throughout this paper the orthogonal polynomials $P_n$ are taken to be monic, $P_n(x) = x^n + \text{lower degree terms, } n \geq 0$, and the sequence $\{P_n\}_{n \geq 0}$ will be denoted by SMOP.

Monic orthogonal polynomials satisfy a three-term recurrence relation [21]

$$
P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots,
$$

(13)
with $P_{-1}(x) = 0$, $P_0(x) = 1$, and $\gamma_n \neq 0$, $n \geq 1$, $\gamma_0 = 1$. The parameters $\beta_n, \gamma_n$ are the so-called recurrence relation coefficients.

Another relevant sequence, related to $\{P_n\}_{n \geq 0}$, is the sequence of associated polynomials of the first kind, denoted by $\{P^{(1)}_n\}_{n \geq 0}$, defined through the three term recurrence relation

$$P^{(1)}_n(x) = (x - \beta_n)P^{(1)}_{n-1}(x) - \gamma_n P^{(1)}_{n-2}(x), \quad n = 1, 2, \ldots ,$$  

with $P^{(1)}_{-1}(x) = 0$, $P^{(1)}_0(x) = 1$.

The sequence of functions of the second kind, $\{q_n\}_{n \geq 0}$, is defined by

$$q_n(x) = S(x)P_n(x) - P^{(1)}_{n-1}(x), \quad n \geq 0,$$

subject to the initial conditions $q_{-1}(x) = 1$, $q_0(x) = S(x)$. It satisfies a three term recurrence relation,

$$q_{n+1}(x) = (x - \beta_n)q_n(x) - \gamma_n q_{n-1}(x), \quad n = 0, 1, 2, \ldots .$$

### 3. Semi-classical orthogonal polynomials on quadratic lattices

Semi-classical orthogonal polynomials on quadratic lattices may be defined through:

(i) a Pearson equation for the linear functional [9, 10],

$$\mathbb{D}(\phi L) = \mathbb{M}(\psi L), \quad \phi \neq 0, \deg(\psi) \geq 1;$$  

(ii) a difference equation for the Stieltjes function [15, 22],

$$A\mathbb{D}S = CMS + D,$$

with $A, C, D$ irreducible polynomials (in $x$);

(iii) a Pearson equation for the weight [3, 22],

$$A\mathbb{D}w = C\mathbb{M}w.$$

The polynomials in (17)–(19) are related via [9, 10]

$$A = \mathbb{M}\phi - r(x)\mathbb{D}\psi - U_1\mathbb{M}\psi, \quad C = -\mathbb{D}\phi + \mathbb{M}\psi + U_1\mathbb{D}\psi,$$

with $U_1 = \tilde{c}_2/2$, being $\tilde{c}_2$ defined by (2) (cf. [9, eq. (16)]), thus, in the account of (7), $U_1 = 2p_0$. $D$ is a polynomial depending on $A, C$.

The polynomials $A, C, D$ in (18) satisfy, in the account of (3), (9), and (12),

$$\deg(A) \leq m + 2, \quad \deg(C) \leq m + 1, \quad \deg(D) \leq m,$$
where \( m \) is some nonnegative integer. When \( m = 0 \) we get the so-called classical polynomials [9, 18].

The class of a linear functional \( L \) on quadratic lattices was defined in [10], as the non-negative integer given by

\[
cl(L) = \min_{(f, g) \in \mathcal{X}} \{\max(\deg(f) - 2, \deg(g) - 1)\},
\]

\[\mathcal{X} = \{ (f, g) \in C[x]^2 : \deg(g) \geq 1 \text{ and } D(fL) = M(gL) \}.\]

In what follows we show some fundamental identities for semi-classical orthogonal polynomials on quadratic lattices.

3.1. The system of difference equations for the polynomials. Let \( S \) be a Stieltjes function satisfying the difference equation (18), \( A^\Delta S = C^M S + D \). Following the same lines as in [22] or [4] (where we take \( B \equiv 0 \) in Theorem 1), we have, for all \( n \geq 1 \),

\[
\begin{align*}
A^\Delta P_n &= (l_{n-1} + \Delta_y \pi_{n-1}) E^- P_n - C/2 E^+ P_n + \Theta_{n-1} E^- P_{n-1}, \\
A^\Delta P_{n-1}^{(1)} &= (l_{n-1} + \Delta_y \pi_{n-1}) E^- P_{n-1}^{(1)} + C/2 E^+ P_{n-1}^{(1)} + D E^+ P_n + \Theta_{n-1} E^- P_{n-2}^{(1)},
\end{align*}
\]

and, for all \( n \geq 0 \),

\[
A^\Delta q_n = (l_{n-1} + \Delta_y \pi_{n-1}) E^- q_n + C/2 E^+ q_n + \Theta_{n-1} E^- q_{n-1}.
\]

The above difference equations (22) are equivalent to

\[
\begin{align*}
A^\Delta P_n &= (l_{n-1} - \Delta_y \pi_{n-1}) E^+ P_n - C/2 E^- P_n + \Theta_{n-1} E^+ P_{n-1}, \\
A^\Delta P_{n-1}^{(1)} &= (l_{n-1} - \Delta_y \pi_{n-1}) E^+ P_{n-1}^{(1)} + C/2 E^- P_{n-1}^{(1)} + D E^- P_n + \Theta_{n-1} E^+ P_{n-2}^{(1)},
\end{align*}
\]

and (23) is equivalent to

\[
A^\Delta q_n = (l_{n-1} - \Delta_y \pi_{n-1}) E^+ q_n + C/2 E^- q_n + \Theta_{n-1} E^+ q_{n-1}.
\]

Remark. Furthermore, the polynomials \( l_n, \Theta_n, \pi_n \) are subject to the following bounds:

\[
\begin{align*}
\deg(\Theta_n) &\leq \max\{\deg(A) - 2, \deg(C) - 1\}, \\
\deg(l_n) &\leq \max\{\deg(A) - 1, \deg(C)\}, \\
\deg(\pi_n) &\leq \deg(C) - 1.
\end{align*}
\]
3.2. Compatibility conditions. Define the matrices

\[ \mathcal{P}_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}, \quad n \geq 0. \tag{28} \]

In the account of (13) and (14), \( \mathcal{P}_n \) satisfies the difference equation

\[ \mathcal{P}_n = \mathcal{A}_n \mathcal{P}_{n-1}, \quad \mathcal{A}_n = \begin{bmatrix} x - \beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1, \tag{29} \]

with initial condition \( \mathcal{P}_0 = \begin{bmatrix} x - \beta_0 & 1 \\ 1 & 0 \end{bmatrix} \).

The previous systems (22) and (24) can be put in the matrix form as [7]

\[ A \mathcal{D} \mathcal{P}_n = \mathcal{B}_n^- \mathcal{E}^- \mathcal{P}_n - (\mathcal{E}^+ \mathcal{P}_n) \mathcal{C}, \tag{30} \]

\[ A \mathcal{D} \mathcal{P}_n = \mathcal{B}_n^+ \mathcal{E}^+ \mathcal{P}_n - (\mathcal{E}^- \mathcal{P}_n) \mathcal{C}, \tag{31} \]

with the matrices \( \mathcal{B}_n^\pm \) and \( \mathcal{C} \) given by

\[ \mathcal{B}^\pm_n = \begin{bmatrix} l_n \mp \Delta_l \pi_n & \Theta_n \\ l_{n-1} \mp \Delta_l \pi_{n-1} + \Theta_{n-1} \gamma_n \mathcal{E}^\pm(x - \beta_n) \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C/2 & -D \\ 0 & -C/2 \end{bmatrix}. \]

From the compatibility of (29) and (30)–(31) we get the equations for the transfer matrices \( \mathcal{A}_n \), for all \( n \geq 1 \) [7]:

\[ A \mathcal{D} \mathcal{A}_n = \mathcal{B}_n^- \mathcal{E}^- \mathcal{A}_n - (\mathcal{E}^+ \mathcal{A}_n) \mathcal{B}^-_{n-1}, \tag{32} \]

\[ A \mathcal{D} \mathcal{A}_n = \mathcal{B}_n^+ \mathcal{E}^+ \mathcal{A}_n - (\mathcal{E}^- \mathcal{A}_n) \mathcal{B}^+_{n-1}. \tag{33} \]

The compatibility conditions (32)–(33) yield the following relations for the polynomials \( \pi_n, l_n, \Theta_n \), for all \( n \geq 0 \) [7, 15]:

\[ \pi_{n+1} = -\frac{1}{2} \sum_{k=0}^{n+1} \Theta_{k-1} \frac{\Theta_n}{\gamma_k}, \tag{34} \]

\[ l_{n+1} + l_n + \mathcal{M}(x - \beta_{n+1}) \frac{\Theta_n}{\gamma_{n+1}} = 0, \tag{35} \]

\[ -A + \mathcal{M}(x - \beta_{n+1})(l_{n+1} - l_n) - \frac{\Delta^2}{2}(\pi_{n+1} + \pi_n) + \Theta_{n+1} \gamma_n \pi_n \Theta_{n+1} = \frac{\gamma_{n+1}}{\gamma_n} \Theta_{n-1}. \tag{36} \]
The following initial conditions hold:
\[\pi_{-1} = 0, \quad \pi_0 = -D/2, \quad (37)\]
\[\Theta_{-1} = D, \quad \Theta_0 = A - \frac{\Delta_y^2}{4}D - (l_0 - C/2)\mathcal{M}(x - \beta_0), \quad (38)\]
\[l_{-1} = C/2, \quad l_0 = -\mathcal{M}(x - \beta_0)D - C/2. \quad (39)\]

3.3. Further matrix identities. The following results extend the differential systems from the continuous orthogonality given in [14] to the discrete orthogonality on systems of nonuniform lattices (see [3, Th. 1] and also [22, Sec. 4]). We stress equation (43) below, the analogue of the so-called Magnus' summation formula [14].

**Theorem 1.** Let \( S \) be a Stieltjes function related to a weight \( w \), satisfying \( A\mathcal{D}S = C\mathcal{M}S + D \), and let \( \{Y_n\}_{n \geq 0} \) be the corresponding sequence given by
\[\{Y_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}\}_{n \geq 0}. \] The following equation holds:
\[A_{n+1} \mathcal{D}Y_n = (B_n - C/2 I)\mathcal{M}Y_n, \quad n \geq 1, \quad (40)\]
where
\[A_{n+1} = A + \frac{\Delta_y^2}{2} \pi_n, \quad I \text{ is the identity matrix, and } B_n \text{ is given as} \]
\[B_n = \begin{bmatrix} l_n & \Theta_n \\ -\frac{\Theta_{n-1}}{\gamma_n} & l_{n-1} + \frac{\Theta_{n-1}}{\gamma_n} \mathcal{M}(x - \beta_n) \end{bmatrix}. \quad (41)\]

**Corollary 1.** The matrix \( B_n \) satisfies the following identities, for all \( n \geq 1 \):
\[\text{tr } B_n = 0, \quad (42)\]
\[\det B_n = -\Delta_y^2 \pi_n^2 + AD - \frac{C^2}{4} + A \sum_{k=1}^{n} \frac{\Theta_{k-1}}{\gamma_k}. \quad (43)\]

**Remark.** Taking into account \( \Theta_{-1}/\gamma_0 = D \) (see (38)) and (34), an equivalent equation for (43) is
\[\det B_n = -\Delta_y^2 \pi_n^2 - \frac{C^2}{4} - 2A\pi_n. \quad (44)\]
In the account of (42), we shall use \( B_n \) in (44) given as
\[
B_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & -l_n \end{bmatrix}.
\]
Therefore, (44) reads as
\[
-l_n^2(x) + \Theta_n(x)\frac{\Theta_{n-1}(x)}{\gamma_n} = -\Delta_y^2\pi_n^2 - \frac{C^2}{4} - 2A\pi_n.
\] (45)

4. Difference equations for the recurrence relation coefficients

4.1. Difference equations when \( m = 1 \) in (21). Let us take \( m = 1 \) in (21), that is, \( A(x)\overline{D}S(x) = C(x)\overline{M}S(x) + D(x) \) with
\[
\deg(A) \leq 3, \quad \deg(C) \leq 2, \quad \deg(D) \leq 1,
\] (46)
were we consider, by writing
\[
A(x) = a_3x^3 + a_2x^2 + a_1x + a_0, \quad C(x) = c_2x^2 + c_1x + c_0,
\]
the condition
\[
a_3 \neq 0 \quad \text{or} \quad c_2 \neq 0.
\] (47)
The polynomial \( D \) is given in terms of \( A, C \). By collecting the coefficient of \( x^3 \) in (38) as well as the coefficient of \( x^2 \) in (39) we get
\[
d_1 = -(a_3 + c_2p_1)/p_1^2.
\] (48)
By collecting the coefficient of \( x^2 \) in (38) as well as the coefficient of \( x \) in (39) we get, using (48),
\[
d_0 = \frac{a_3(2p_0p_1 - r_1 - 2p_1\beta_0) - p_1(a_2p_1 + c_1p_1^2 + c_2(r_1 + p_1\beta_0 - p_0p_1))}{p_1^4}.
\] (49)
In the account of (26)–(27), we have \( \deg(l_n) = 2, \deg(\Theta_n) = \deg(\pi_n) = 1 \).
Set
\[
l_n(x) = \ell_{n,2}x^2 + \ell_{n,1}x + \ell_{n,0}, \quad \Theta_n(x) = \Theta_{n,1}x + \Theta_{n,0}, \quad \pi_n(x) = \pi_{n,1}x + \pi_{n,0}.
\]
Also, recall (8), thus \( \Delta_y^2(x) = 4r(x) \).
Henceforth we adopt the convention that \( \sum_i^j \cdot = 0 \) whenever \( i > j \) and \( \prod_i^j \cdot = 1 \) whenever \( i > j \).
Lemma 1. Under the previous assumptions and notations, the quantities \( \ell_{n,2}, \ell_{n,1}, \Theta_{n,1}/\gamma_{n+1}, \Theta_{n,0}/\gamma_{n+1}, \pi_{n,1}, \pi_{n,0} \) are given, for all \( n \geq 0 \), by

\[
\ell_{n,2} = n \frac{a_3}{p_1} - p_1 d_1 - \frac{c_2}{2},
\]

\[
\Theta_{n,1}/\gamma_{n+1} = \frac{1}{p_1} \left( -(2n+1) \frac{a_3}{p_1} + 2 p_1 d_1 + c_2 \right),
\]

\[
\pi_{n,1} = -\frac{d_1}{2} + \frac{1}{2p_1} \left( \frac{a_3}{p_1} n - 2 p_1 d_1 - c_2 \right) n,
\]

\[
\ell_{n,1} = L_{n,1} + \frac{a_3}{p_1^2} \sum_{k=1}^{n} \beta_k.
\]

with

\[
L_{n,1} = \left( \frac{a_2 p_1 - a_3 p_0}{p_1^2} \right) n
\]

\[
+ \frac{2r_1}{p_1} \left( -n d_1 + \frac{a_3}{2p_1^2} \left( \frac{(n-1)n(2n-1)}{3} + n^2 \right) - \frac{(2p_1 d_1 + c_2)}{2p_1} n^2 \right)
\]

\[- p_1 d_0 - (p_0 - \beta_0) d_1 - \frac{c_1}{2}.
\]

Furthermore, the following relations hold, for all \( n \geq 1 \):

\[
\frac{\Theta_{n,0}}{\gamma_{n+1}} = S_{n,0} + \frac{1}{p_1} \left( \frac{\Theta_{n,1}}{\gamma_{n+1}} - \frac{a_3}{p_1^2} \right) \beta_{n+1} - \frac{2a_3}{p_1^3} \sum_{k=1}^{n} \beta_k,
\]

\[
S_{n,0} = -\frac{(L_{n+1,1} + L_{n,1} + p_0 \Theta_{n,1}/\gamma_{n+1})}{p_1},
\]

(i) if \( a_3 \neq 0 \), then

\[
\pi_{n,0} = \hat{T}_{n,0} + \frac{\ell_{n,2}}{p_1^2} \sum_{k=1}^{n} \beta_k, \quad \hat{T}_{n,0} = \frac{L_{n,1} \ell_{n,2} - 2r_1 \pi_{n,1}^2 - a_2 \pi_{n,1} - c_1 c_2/4}{a_3}.
\]
(ii) if $a_3 = 0$, then

$$
\pi_{n,0} = T_{n,0} - \frac{(2p_1d_1 + c_2)}{2p_1^2} \sum_{k=2}^{n} \beta_k,
$$

$$
T_{n,0} = -\frac{d_0}{2} - \frac{1}{2} \frac{\Theta_{0,0}}{\gamma_1} + \frac{1}{2p_1} \sum_{k=1}^{n-1} \left( L_{k+1,1} + L_{k,1} + \frac{p_0(2p_1d_1 + c_2)}{p_1} \right).
$$

(56)

The initial conditions hold:

$$
\pi_{0,0} = -\frac{d_0}{2},
$$

$$
\frac{\Theta_{0,0}}{\gamma_1} = -\frac{a_2}{p_1^2} + \frac{r_1}{p_1^2} \left( d_1 + \frac{\Theta_{0,1}}{\gamma_1} \right) + \left( -\ell_{0,2} - p_1 \frac{\Theta_{0,1}}{\gamma_1} + \frac{c_2}{2} \right) \left( \frac{p_0 - \beta_1}{p_1^2} \right)
$$

$$
+ \frac{1}{p_1} \left( -\ell_{0,1} - (p_0 - \beta_1) \frac{\Theta_{0,1}}{\gamma_1} + \frac{c_1}{2} \right) + \frac{d_1}{p_1} \left( 2p_0 - (\beta_0 + \beta_1) + \frac{r_1}{p_1} \right) + d_0.
$$

Here, $p_1, p_0, r_1$ are the coefficients of $p(x), r(x)$, defined in (6).

**Proof**: The coefficient of $x^3$ in (36) yields

$$
-a_3 + p_1(\ell_{n+1,2} - \ell_{n,2}) = 0.
$$

This, combined with the initial condition $\ell_{0,2} = -p_1d_1 - c_2/2$, gives us (50).

The use of (50) in the equation that follows from the coefficient of $x^2$ in (35),

$$
\ell_{n+1,2} + \ell_{n,2} + p_1 \Theta_{n,1}/\gamma_{n+1} = 0,
$$

gives us (51) for all $n \geq 1$. In order to get $\Theta_{0,1}/\gamma_1$ we take $n = 1$ in

$$
A_{n+1}D_P^{(1)} = (l_n + C/2)M_{P_n^{(1)}} + DMP_{n+1} + \Theta_n MP_{n-1}^{(1)}.
$$

Indeed, using (34) and (35) with $n = 0$ in the equation above with $n = 1$ we have

$$
A + 2r \left( -\frac{D}{2} - \frac{1}{2} \frac{\Theta_0}{\gamma_1} \right) = \left( -l_0 - M(x - \beta_1) \frac{\Theta_0}{\gamma_1} + C/2 \right) M(x - \beta_1) + DMP_2 + \Theta_1.
$$

(57)

The coefficient of $x^3$ gives us

$$
\frac{\Theta_{0,1}}{\gamma_1} = \frac{1}{p_1} \left( -\frac{a_3}{p_1} + 2p_1d_1 + c_2 \right),
$$

(58)

thus (51) also holds for $n = 0$. 

Equation (52) follows from the use of (51) in the summation formula (34), and the initial condition \( \pi_{0,1} = -d_1/2 \) (cf. (37)).

Let us now obtain (53). Using (50) in the equation that follows from equating the coefficients of \( x^2 \) in (36) we get
\[
\ell_{n+1,1} = \ell_{n,1} - \frac{(p_0 - \beta_{n+1})a_3}{p_1^2} + \frac{2r_1}{p_1} \lambda_{n,1} + \frac{a_2}{p_1}, \quad \lambda_{n,1} = \pi_{n+1,1} + \pi_{n,1}.
\] (59)

Thus, we obtain (53), where we used the initial conditions \( \ell_{0,1} = -p_0d_0 - (p_0 - \beta_0)d_1 - c_1/2. \)

To get (54) we take the \( x \) coefficient in (35),
\[
\ell_{n+1,1} + \ell_{n,1} + p_1 \frac{\Theta_{n,0}}{\gamma_{n+1}} + (p_0 - \beta_{n+1}) \frac{\Theta_{n,1}}{\gamma_{n+1}} = 0,
\]
and substitute (53) and (51) therein. The initial condition \( \pi_{0,0} \) comes from (37) and \( \Theta_{0,0}/\gamma_1 \) follows from taking the coefficient of \( x^2 \) in (57).

\( \pi_{n,0} \) can be obtained via the summation formula (34). Thus, from (34), if \( a_3 = 0 \), we get
\[
\pi_{n,0} = -\frac{1}{2} \frac{\Theta_{-1,0}}{\gamma_0} - \frac{1}{2} \frac{\Theta_{0,0}}{\gamma_1} - \frac{1}{2} \sum_{k=1}^{n-1} \frac{\Theta_{k,0}}{\gamma_{k+1}}.
\] (60)

and (56) follows. The case \( a_3 \neq 0 \) can be alternatively obtained through the use of the \( x^3 \)-coefficient in (45), thus yielding (55).

**Theorem 2.** Let \( S \) be a Stieltjes function satisfying
\[
A(x)D_S(x) = C(x)M_S(x) + D(x)
\]
with \( \deg(A) \leq 3 \), \( \deg(C) \leq 2 \), \( \deg(D) \leq 1 \) subject to the condition (47). Let \( \{P_n\}_{n \geq 0} \) be the corresponding SMOP, satisfying (13),
\[
P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_nP_{n-1}(x), \quad n = 0, 1, 2, \ldots
\]

Under the notations of the previous lemma, the \( \gamma_n \)'s are defined only in terms of the \( \beta_n \)'s and the polynomials \( A, C \), a well as \( p, r \) from (6), related to the quadratic lattice. There holds the formula
\[
\gamma_{n+2} = \gamma_1 \prod_{j=0}^{n} s_j + \sum_{k=0}^{n} t_k \prod_{j=k+1}^{n} s_j, \quad n \geq 0,
\] (61)
\[
\gamma_1 = \frac{p_1\Theta_{0,1}}{-a_3/p_1 + 2p_1d_1 + c_2},
\] (62)
with
\[ s_n = \frac{\Theta_{n-1}/\gamma_n}{\Theta_{n+1}/\gamma_{n+2}}(x_{n+1}), \quad t_n = \frac{A(x_{n+1}) + 2r(x_{n+1})\pi(x_{n+1})}{(\Theta_{n+1}/\gamma_{n+2})(x_{n+1})}, \quad (63) \]
\[ x_{n+1} = (\beta_{n+1} - p_0)/p_1, \]
and
\[ \Theta_{0,1} = a_1 - r_1d_0 - r_0d_1 + (p_1d_0 + (p_0 - \beta_0)d_1 + c_1)(p_0 - \beta_0) + ((p_0 - \beta_0)d_0 + c_0)p_1. \quad (64) \]

**Proof:** We evaluate (36) at \( x_{n+1} = (\beta_{n+1} - p_0)/p_1 \). As \( M(x - \beta_{n+1})(x_{n+1}) = 0 \), we get
\[ -A(x_{n+1}) - 2r(x_{n+1})\lambda_n(x_{n+1}) + \gamma_{n+2}\Theta_{n+1}/\gamma_{n+2}(x_{n+1}) = \gamma_{n+1}\Theta_{n-1}/\gamma_n(x_{n+1}) \]
where it was used the notation \( \lambda_n(x) = \pi_{n+1}(x) + \pi_n(x) \). Hence, we obtain
\[ \gamma_{n+2} = s_n\gamma_{n+1} + t_n, \quad n \geq 0 \quad (66) \]
with \( s_n, t_n \) given in (63). As the solution of the initial value problem
\[ z_{n+1} = a_n z_n + b_n, \quad z_{n_0} = z_0 \]
is [6]
\[ z_n = z_0 \prod_{j=n_0}^{n-1} a_j + \sum_{k=n_0}^{n-1} b_k \prod_{j=k+1}^{n-1} a_j, \]
then equation (61) is a consequence of (66).

To get \( \gamma_1 \), we take the initial conditions
\[ \ell_{0,1} = -p_1d_0 - (p_0 - \beta_0)d_1 - c_1/2, \quad (67) \]
\[ \ell_{0,0} = -(p_0 - \beta_0)d_0 - c_0/2. \quad (68) \]
\[ \Theta_{0,1} = a_1 - r_1d_0 - r_0d_1 - (\ell_{0,1} - c_1/2)(p_0 - \beta_0) - (\ell_{0,0} - c_0/2)p_1, \quad (69) \]
The use of (67) and (68) in (69) yields (64). Using \( \Theta_{0,1}/\gamma_1 \) given by (cf. (58))
\[ \frac{\Theta_{0,1}}{\gamma_1} = \frac{1}{p_1} \left( -a_3/p_1 + 2p_1d_1 + c_2 \right) \]
combined with (64), we get (62).
Remark. The previous theorem gives us the $\gamma_n$’s in terms of the $\beta_n$’s. In order to obtain a recurrence for the $\beta_n$’s we may start by taking the independent term of (35), which gives us

$$
\ell_{n+1,0} = -\ell_{n,0} - (p_0 - \beta_{n+1}) \frac{\Theta_{n,0}}{\gamma_{n+1}}.
$$

(70)

Using this equality into the equation that results from the coefficient of $x$ in (36) we obtain

$$
\ell_{n,0} = -\frac{1}{2p_1} (a_1 + 2r_0 \lambda_{n,1}) - \frac{(p_0 - \beta_{n+1})}{2p_1} \left( p_1 \frac{\Theta_{n,0}}{\gamma_{n+1}} + \ell_{n,1} - \ell_{n+1,1} \right)
$$

$$
- \frac{r_1}{p_1} (\pi_{n,1} + \pi_{n,0}) + \gamma_{n+2} \frac{\nu_{n+1,1}}{p_1} - \gamma_{n+1} \frac{\nu_{n-1,1}}{p_1},
$$

(71)

where we used the notation $\lambda_{n,1} = \pi_{n,1} + \pi_{n,0}$, $\nu_{n,1} = \Theta_{n,1}/\gamma_{n+1}$. Now, by substituting (61) and (71) into equation (70) we get a first order non-linear recurrence relation for the $\beta_n$’s, say $F(\gamma_1, \beta_1, \ldots, \beta_{n+2}) = G(\gamma_1, \beta_1, \ldots, \beta_{n+1})$. Due to the complexity of such a formulae, we shall not give its explicit form here.

4.1.1. The symmetric case. The symmetric case, that is, $\beta_n = 0$, $n \geq 0$, implies simplifications in (53), (54), (55), (56), and all these quantities now depend only on the lattice as well as on the coefficients $A, C, D$ of the difference equation for the Stieltjes function. In such a case, we have the result that follows.

Corollary 2. Let $A(x) \mathbb{D}S(x) = C(x) \mathbb{M}S(x) + D(x)$ with $\deg(A) \leq 3$, $\deg(C) \leq 2$, $\deg(D) \leq 1$ subject to the condition (47). Under the previous notation, let $\beta_n = 0$, $n \geq 0$. Then, the $\gamma_n$’s are determined through:

$$
\gamma_{n+2} = \gamma_1 \prod_{j=0}^{n} s_j + \sum_{k=0}^{n} t_k \prod_{j=k+1}^{n} s_j, \quad n \geq 0,
$$

(72)

$$
\gamma_1 = \frac{p_1 \Theta_{0,1}}{-a_3/p_1 + 2p_1 d_1 + c_2},
$$

(73)

with

$$
s_n = \frac{(\Theta_{n-1}/\gamma_n)(x_0)}{(\Theta_{n+1}/\gamma_{n+2})(x_0)}, \quad t_n = \frac{A(x_0) + 2r(x_0)(\pi_{n+1} + \pi_n)(x_0)}{(\Theta_{n+1}/\gamma_{n+2})(x_0)}, \quad x_0 = -p_0/p_1,
$$

and

$$
\Theta_{0,1} = a_1 - r_1 d_0 - r_0 d_1 + (p_1 d_0 + p_0 d_1 + c_1)p_0 + (p_0 d_0 + c_0)p_1.
$$
Proof: Take $\beta_n = 0$, $n \geq 0$. Evaluate (36) at $x_0 = -p_0/p_1$. Thus, as $M(x_0) = 0$, we get

$$-A(x_0) - 2r(x_0)(\pi_{n+1} + \pi_n)(x_0) + \gamma_{n+2} \frac{\Theta_{n+1}}{\gamma_{n+2}}(x_0) = \gamma_{n+1} \frac{\Theta_{n-1}}{\gamma_n}(x_0).$$

(74)

Therefore, (72) follows. 

\[\blacksquare\]

4.1.2. Condition (47) with $a_3 = 0$. Let us take the case $
\text{deg}(A) \leq 2$, \text{deg}(C) = 2, \text{deg}(D) = 1.
\]

(75)

In such a case, the quantities given in Lemma 1 are as follow:

$$\ell_{n,2} = \frac{c_2}{2}, \ell_{n,1} = L_{n,1},$$

$$\frac{\Theta_{n,1}}{\gamma_{n+1}} = -\frac{c_2}{p_1}, \quad \frac{\Theta_{n,0}}{\gamma_{n+1}} = S_{n,0} - \frac{c_2}{p_1^2} \beta_{n+1},$$

$$\pi_{n,1} = (n + 1) \frac{c_2}{2p_1}, \quad \pi_{n,0} = T_{n,0} + \frac{c_2}{2p_1^2} \sum_{k=2}^{n} \beta_k,$$

with

$$L_{n,1} = \frac{na_2}{p_1} + \frac{2r_1 n a_2}{p_1 p_1} (1 + \frac{n}{2}) - p_1 d_0 - (p_0 - \beta_0) d_1 - \frac{c_1}{2},$$

$$S_{n,0} = \frac{(L_{n+1,1} + L_{n,1} + p_0 \Theta_{n,1}/\gamma_{n+1})}{p_1},$$

$$T_{n,0} = -\frac{d_0}{2} - \frac{1}{2} \frac{\Theta_{0,0}}{\gamma_1} + \frac{1}{2p_1} \sum_{k=1}^{n-1} \left( L_{k+1,1} + L_{k,1} - \frac{c_2 p_0}{p_1} \right).$$

Furthermore, by taking the $x^2$-coefficient of (44), that is,

$$-\ell_{n,1}^2 - 2\ell_{n,2} \ell_{n,0} + \gamma_{n+1} \frac{\Theta_{n,1} \Theta_{n-1,1}}{\gamma_{n+1} \gamma_{n+1}} = -8r_1 \pi_{n,0} \pi_{n,1}$$

$$-4r_0 \pi_{n,1}^2 - (c_1^2 + 2c_2 c_0)/4 - 2a_2 \pi_{n,0} - 2a_1 \pi_{n,1},$$

(76)

we get the expression for $\ell_{n,0}$,

$$\ell_{n,0} = \tau_n + \frac{c_2}{p_1} \gamma_{n+1} + \left( 4r_1 (n + 1) + 2 a_0 / c_2 \right) \pi_{n,0},$$

(77)

with

$$\tau_n = -\frac{\ell_{n,1}^2}{c_2} + r_0 \frac{c_2}{p_1^2} (n + 1)^2 + \frac{(c_1^2 + 2c_2 c_0)}{4c_2}.$$
Theorem 3. Under the degrees (75) and the previous notations, we have the following difference equations for the recurrence coefficients, for all \( n \geq 1 \):

\[
\gamma_{n+2} + \gamma_{n+1} + \frac{p_1}{c_2} \left( 4r_1(n + 2) + 2\frac{a_0}{c_2} \right) \pi_{n+1,0} + \frac{p_1}{c_2} \left( 4r_1(n + 1) + 2\frac{a_0}{c_2} \right) \pi_{n,0} \\
+ \frac{p_1}{c_2} (p_0 - \beta_{n+1}) \frac{\Theta_{n,0}}{\gamma_{n+1}} + \frac{p_1}{c_2} (\tau_{n+1} + \tau_n) = 0, \tag{78}
\]

\[
\gamma_{n+1} = \frac{2L_{n,1}(\tau_n + (4r_1(n + 1) + 2a_0/c_2) \pi_{n,0}) + G_n}{-\frac{2c_2}{p_1^2} L_{n,1} - \frac{c_2}{p_1} \left( \frac{\Theta_{n-1,0}}{\gamma_n} + \frac{\Theta_{n,0}}{\gamma_{n+1}} \right)}, \tag{79}
\]

with the initial condition \( \gamma_1 \) given by (62), and with \( G_n = -4r_1\pi_{n,0}^2 - 8r_0\pi_{n,0}\pi_{n,1} - c_1c_0/2 - 2a_1\pi_{n,0} - 2a_0\pi_{n,1} \).

Proof: To get (78) we use (77) in the equation obtained from the independent term in (35),

\[
\ell_{n+1,0} + \ell_{n,0} + (p_0 - \beta_{n+1}) \frac{\Theta_{n,0}}{\gamma_{n+1}} = 0.
\]

To get (79) we take the \( x \)-coefficient of (44), that is,

\[
-2\ell_{n,1}\ell_{n,0} + \gamma_{n+1} \left( \frac{\Theta_{n,1}}{\gamma_{n+1}} \frac{\Theta_{n-1,0}}{\gamma_n} + \frac{\Theta_{n,0}}{\gamma_{n+1}} \frac{\Theta_{n-1,1}}{\gamma_n} \right) = G_n,
\]

with \( G_n = -4r_1\pi_{n,0}^2 - 8r_0\pi_{n,0}\pi_{n,1} - c_1c_0/2 - 2a_1\pi_{n,0} - 2a_0\pi_{n,1} \). The use of \( \ell_{n,0} \)
given by (77) into the above equation gives us (79).

4.2. \( m = 0 \) in (21): classical orthogonal polynomials on quadratic lattices from compatibility relations. Let us take \( m = 0 \) in (21), that is, \( A(x)\mathbb{D}S(x) = C(x)\mathbb{M}S(x) + D(x) \) with

\[
\deg(A) \leq 2, \ \deg(C) \leq 1, \ \deg(D) = 0. \tag{80}
\]

were we consider, by writing

\[
A(x) = a_2x^2 + a_1x + a_0, \ C(x) = c_1x + c_0,
\]

the condition \( a_2 \neq 0 \) or \( c_1 \neq 0. \tag{81} \)

In the account of (26)–(27), we have \( \deg(l_n) = 1, \ \deg(\Theta_n) = \deg(\pi_n) = 0. \) Set

\[
l_n(x) = \ell_{n,1}x + \ell_{n,0}, \ \Theta_n(x) = \Theta_{n,0}, \ \pi_n(x) = \pi_{n,0}. \]

We have \( D = d_0 = -(a_2 + c_1p_1)/p_1^2. \)
Lemma 2. Under the previous assumptions and notations, we have, for all $n \geq 0$,

$$\ell_{n,1} = n^2 a_2^2 p_1 - p_1 d_0 - c_1^2,$$  \hspace{1cm} (82)

$$\Theta_{n,0}^{\gamma_{n+1}} = \frac{1}{p_1} \left( -(2n + 1) a_2^2 p_1 + 2 p_1 d_0 + c_1 \right),$$  \hspace{1cm} (83)

$$\pi_{n,0} = \frac{d_0}{2} + \frac{1}{2p_1} \left( a_2^2 n - 2p_1 d_0 - c_1 \right) n,$$  \hspace{1cm} (84)

and

$$\ell_{n,0} = \frac{2r_1 \pi_{n,0}^2 + c_0 c_1/4 + a_1 \pi_{n,0}}{\ell_{n,1}}, \quad n \geq 1, \quad \ell_{0,0} = -(p_0 - \beta_0) d_0 - c_0/2.$$  \hspace{1cm} (85)

Here, $p_1, p_0, r_1$ are the coefficients of $p(x), r(x)$, defined in (6).

Proof: The equations (82)–(84) follow from Lemma 1. The $x$-coefficient of (45) gives us (85).

Theorem 4. Let $A(x)D S(x) = C(x) M S(x) + D(x)$ with $\deg(A) \leq 2$, $\deg(C) \leq 1$, $\deg(D) \leq 0$ subject to the condition (81). Consider the notations of the previous lemma. The following holds:

$$\gamma_{n+1} = \frac{\ell_{n,0}^2 - 4r_0 \pi_{n,0}^2 - c_0^2/4 - 2a_0 \pi_{n,0}}{\Theta_{n,0}^{\gamma_{n+1}} \Theta_{n-1,0}^{\gamma_n}}, \quad n \geq 1,$$  \hspace{1cm} (86)

$$\beta_{n+1} = \frac{\ell_{n+1,0} + \ell_{n,0} + p_0 \Theta_{n,0}^{\gamma_{n+1}}/\Theta_{n,0}^{\gamma_{n+1}}}{\Theta_{n,0}^{\gamma_{n+1}}}, \quad n \geq 0,$$  \hspace{1cm} (87)

and the initial conditions $\beta_0$ and $\gamma_1$ given by

$$\beta_0 = \frac{p_0 d_0 + c_0 + (a_1 - r_1 d_0)/p_1 - a_2 p_0/p_1^2}{d_0 - a_2/p_1^2},$$  \hspace{1cm} (88)

$$\gamma_1 = \frac{p_1 (a_0 - r_0 d_0 + ((p_0 - \beta_0) d_0 + c_0)(p_0 - \beta_0))}{-a_2/p_1 + 2 p_1 d_0 + c_1}.$$  \hspace{1cm} (89)

Proof: The equation (86) follows from the independent coefficient of (45),

$$-\ell_{n,0}^2 + \gamma_{n+1} \Theta_{n,0}^{\gamma_{n+1}} \Theta_{n-1,0}^{\gamma_n} = -4r_0 \pi_{n,0}^2 - c_0^2/4 - 2a_0 \pi_{n,0}.$$
The equation (87) is obtained from the independent term of (35),
\[ \ell_{n+1,0} + \ell_{n,0} + (p_0 - \beta_{n+1}) \frac{\Theta_{n,0}}{\gamma_{n+1}} = 0. \]
To obtain \(\beta_0\) and \(\gamma_1\) we equate coefficients in (38) and (39), thus getting
\[
\ell_{0,1} = -p_1 d_0 - c_1/2, \quad (90) \\
\ell_{0,0} = -(p_0 - \beta_0)d_0 - c_0/2. \quad (91) \\
0 = a_1 - r_1 d_0 - (\ell_{0,1} - c_1/2)(p_0 - \beta_0) - (\ell_{0,0} - c_0/2)p_1, \quad (92) \\
\Theta_{0,0} = a_0 - r_0 d_0 - (\ell_{0,0} - c_0/2)(p_0 - \beta_0). \quad (93)
\]
The use of (90) and (91) in (92) yields \(\beta_0\). From (93) we have, using (91),
\[
\Theta_{0,0} = a_0 - r_0 d_0 + ((p_0 - \beta_0)d_0 + c_0)(p_0 - \beta_0). \quad (94)
\]
From \(\Theta_{0,0}/\gamma_1\) given by
\[ \frac{\Theta_{0,0}}{\gamma_1} = \frac{1}{p_1} (-a_2/p_1 + 2p_1 d_0 + c_1) \]
combined with (94) we get \(\gamma_1\).

5. Examples

5.1. Dual Hahn polynomials. The Dual Hahn polynomials have the hypergeometric representation [13]
\[ P_n(x; \gamma, \delta, N) = 3F_2\left(\begin{array}{c} -n, n + \gamma + \delta + 1, -x \\ \gamma + 1, -N \end{array}; 1 \right). \quad (95) \]
The lattice \(x(s)\) and the polynomials \(p, r\) that follow from (7) are
\[ x(s) = s(s + \gamma + \delta + 1), \quad p(x) = x + \frac{1}{4}, \quad r(x) = x + \frac{(\gamma + \delta + 1)^2}{4}. \quad (96) \]
\(\{P_n\}_{n \geq 0}\) is related to a linear functional \(L\) that satisfies \(\mathbb{D}(\phi L) = \mathbb{M}(\psi L)\), where the polynomials \(\phi, \psi\) are given by [10]
\[ \phi(x) = (-1 + 2N + \delta - \gamma)x + N(1 + \gamma)(1 + \gamma + \delta), \quad \psi(x) = -2x + 2N(1 + \gamma). \quad (97) \]
The Stieltjes function satisfies (18), \(A \mathbb{D}S = C \mathbb{M}S + D\), with \(A, C\) given by (20), thus,
\[ A = \mathbb{M}\phi + 2r(x) - \frac{1}{2}\mathbb{M}\psi, \quad C = -1 - \mathbb{D}\phi + \mathbb{M}\psi. \quad (98) \]
where we used $U_1 = 1/2$. The polynomial $D$ is a constant, $D = -c_1/p_1$. As we have $\deg(A) = \deg(C) = 1$, condition (81) of Section 4.2 holds.

From the formulae in Theorem 4 we recover [13, pp. 209], for all $n \geq 1$,
\[
\beta_n = (n+\gamma+1)(n-N)+n(n-\delta-N-1), \quad \gamma_n = n(n+\gamma)(n-1-N)(n-\delta-N-1),
\]
and $\beta_0 = -N(\gamma+1), \gamma_0 = 1$.

5.2. Modification of Dual Hahn polynomials. We consider the following modification of the Dual Hahn polynomials. We take the linear functional [10, Sec. 2.4]
\[
\tilde{L} = \left( x + \frac{(\gamma + \delta + 1)^2}{4} \right) L,
\]
being $L$ the linear functional related to the Dual Hahn polynomials. $\tilde{L}$ satisfies
\[
\mathbb{D}(\tilde{\phi}\tilde{L}) = \mathbb{M}(\tilde{\psi}\tilde{L}),
\]
where the polynomials $\tilde{\phi}, \tilde{\psi}$ are given by (see [10, Eq. (40)])
\[
\tilde{\phi}(x) = (r(x) + 1)\phi(x) + 2r(x)\psi(x), \quad \tilde{\psi}(x) = (r(x) + 1)\psi(x) + 2\phi(x),
\]
with $\phi, \psi$ given in (97). Note that (96) holds. Recall that we are taking $\alpha = 1$ and $x - c = r(x)$, with our notation $r(x)$ for the polynomial $U_2(x)$, in [10, Eq. (40)].

Denote by $\{\tilde{P}_n\}_{n \geq 0}$ the SMOP related to $\tilde{L}$, and its recurrence relation coefficients by $\tilde{\beta}_n, \tilde{\gamma}_n$. The corresponding Stieltjes function satisfies (18), $\tilde{A}\tilde{D}\tilde{S} = \tilde{C}\tilde{M}\tilde{S} + \tilde{D}$, with $\tilde{A}, \tilde{C}$ given by (20), thus,
\[
\tilde{A} = \mathbb{M}\tilde{\phi} - r\mathbb{D}\tilde{\psi} - \frac{1}{2}\mathbb{M}\tilde{\psi}, \quad \tilde{C} = -\mathbb{D}\tilde{\phi} + \mathbb{M}\tilde{\psi} + \frac{1}{2}\mathbb{D}\tilde{\psi}.
\]
$\tilde{D}$ is a polynomial of degree one, with coefficients given by (48) and (49). As we have $\deg(A) = \deg(C) = 2$, condition (75) of Sub-Section 4.1.2 holds. From Theorem 3, the coefficients $\tilde{\gamma}_n, \tilde{\beta}_n$ are governed through the difference system (78)–(79).

Remark . The modification (99) is related to the Christoffel transformation [23, Sec. 3]. In this case the modified recurrence relation coefficients are known to be given in terms of the non-modified ones [23],
\[
\tilde{\beta}_n = \beta_{n+1} - \frac{P_{n+1}(c)}{P_n(c)} + \frac{P_{n+2}(c)}{P_{n+1}(c)}, \quad \tilde{\gamma}_n = \gamma_n - \frac{P_{n-1}(c)P_{n+1}(c)}{P_n^2(c)}, \quad c = -\frac{(\gamma + \delta + 1)^2}{4}.
\]
Note that here the $P_n$’s at $c$ must be evaluated through (95), whilst our formulae in Theorem 3 give a relation for $\tilde{\beta}_n, \tilde{\gamma}_n$ in terms of the lattice and the polynomials involved in the difference equation for $\tilde{S}$.

References


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