Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–44

DISCRETE SEMI-CLASSICAL ORTHOGONAL POLYNOMIALS OF CLASS ONE ON QUADRATIC LATTICES

G. FILIPUK AND M.N. REBOCHO

Abstract: We study orthogonal polynomials on quadratic lattices with respect to a Stieltjes function, S, that satisfies a difference equation $A\mathbb{D}S = C\mathbb{M}S+D$, where A is a polynomial of degree less or equal than 3 and C is a polynomial of degree greater or equal than 1 and less or equal than 2. We show systems of difference equations for the orthogonal polynomials that arise from the so-called compatibility conditions. Some closed formulae for the recurrence relation coefficients are obtained.

Keywords: Discrete orthogonal polynomials; quadratic lattice; divided-difference operator; semi-classical class.
Math. Subject Classification (2010): 33C45, 33C47, 42C05.

1. Introduction

Discrete semi-classical orthogonal polynomials have been widely studied in the literature of special functions [12, 19, 20]. They are defined through a difference equation with polynomial coefficients for the corresponding Stieltjes function,

$$A\mathbb{D}S = C\mathbb{M}S + D\,. \tag{1}$$

Here, $\mathbb{D}$ is some divided-difference operator and $\mathbb{M}$ is a companion difference operator related to $\mathbb{D}$. The divided-difference calculus is classified in terms of hierarchies of operators and related lattices (see, for instance, [22, Sec. 2,3]). In this paper we shall consider the divided-difference operator $\mathbb{D}$ given by

$$\mathbb{D}f(x(s)) = \frac{f(x(s+1/2)) - f(x(s-1/2))}{x(s+1/2) - x(s-1/2)}\,,$$

with the so-called quadratic lattice, $x(s) = c_2s^2 + c_1s + c_0$ [16, Sec. 2] (see Section 2 for details). In the literature, these lattices are part of the lattices usually referred to as non-uniform. The calculus on non-uniform lattices

Received November 07, 2018.
The work of M.N. Rebocho is partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

generalizes the calculus on lattices of lower complexity, such as the linear and $q-$uniform lattices.

There are many papers on semi-classical orthogonal polynomials on quadratic lattices. We refer the interested reader to [7, 8, 11, 22] and their list of references. Standard research topics include the study of structure relations, that is, difference equations involving the polynomials, and systematic classifications or characterizations, given pairs of (A, C) in (1).

In the present paper our goal is twofold. First, to gather some recent results on semi-classical orthogonal polynomials on quadratic lattices, namely, difference equations involving the polynomials and related functions, compatibility relations, and new matrix identities. Essentially, such equations generalize well-known differential systems from [14] (see Section 3). Then, with the help of these results, to describe the sequences of orthogonal polynomials within the class one, that is, under the restrictions $\deg(A) \leq 3, 1 \leq \deg(C) \leq 2$ in (1) (see Section 4). The main results are difference equations for the recurrence relation coefficients of the orthogonal polynomials. For the case $\deg(A) \leq 2, \deg(C) = 1$, we recover closed form formulae for the classical orthogonal polynomials.

Let us emphasize that, for some lattices of lower complexity, the description of class one has been carried out. For instance, [2] gives the classification and integral representation of semi-classical linear functionals of class one when $\mathbb{D}$ is the derivative operator; in [17], the authors established the system satisfied by the recurrence relation coefficients of symmetric semi-classical orthogonal polynomials of class one when $\mathbb{D}$ is the Hahn's difference operator. We also note [5], an extensive study on semi-classical orthogonal polynomials of class one when $\mathbb{D}$ is the forward difference operator.

The remainder of the paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we show difference equations for semi-classical orthogonal polynomials on quadratic lattices, together with the consequent compatibility relations and matrix identities. In Section 4 we deduce difference equations for the recurrence relation coefficients of the semi-classical orthogonal polynomials. Section 5 is devoted to examples: we show applications on the Dual Hahn polynomials as well as on some of their modifications.

2. Divided-difference calculus on quadratic lattices and orthogonal polynomials

Quadratic lattices are commonly defined through a parametric representation $x = x(s)$, $s \in \mathbf{Z}$,

$$x(s) = \check{c}_2 s^2 + \check{c}_1 s + \check{c}_0\,, \tag{2}$$

for appropriate constants $\check{c}_j$'s [19, 20]. The corresponding divided-difference operator, defined on the space of arbitrary functions, is given by [1, 18, 19]

$$\mathbb{D} f(x(s)) = \frac{f(x(s+1/2)) - f(x(s-1/2))}{x(s+1/2) - x(s-1/2)}\,.$$

Alternatively, $\mathbb{D}$ can be defined in terms of two functions, say y_+, y_-, as [15, 22]

$$(\mathbb{D} f)(x) = \frac{f(y_+(x)) - f(y_-(x))}{y_+(x) - y_-(x)}\,, \tag{3}$$

where y_- and y_+ are the two y-roots of a quadratic equation

$$\hat{a}y^2 + 2\hat{b}xy + \hat{c}x^2 + 2\hat{d}y + 2\hat{e}x + \hat{f} = 0\,, \quad \hat{a}\hat{c} \neq 0\,, \;\; \hat{b}^2 = \hat{a}\hat{c}\,. \tag{4}$$

As y_-, y_+ are the y-roots of (4), we have

$$y_-(x) = p(x) - \sqrt{r(x)}\,, \quad y_+(x) = p(x) + \sqrt{r(x)}\,, \tag{5}$$

with p, r polynomials of degree one (in x) given by

$$p(x) = -\frac{\hat{b}x + \hat{d}}{\hat{a}}\,, \quad r(x) = \frac{2(\hat{b}\hat{d} - \hat{a}\hat{e})}{\hat{a}^2}x + \frac{\hat{d}^2 - \hat{a}\hat{f}}{\hat{a}^2}\,. \tag{6}$$

The polynomials p, r defined in (6) will play an important role in the sequel. In the account of (5) and $y_-(x) = x(s-1/2)$, $y_+(x) = x(s+1/2)$, we have

$$x(s+1/2) + x(s-1/2) = 2p(x(s))\,, \;\; (x(s+1/2) - x(s-1/2))^2 = 4r(x(s))\,. \tag{7}$$

We take $\Delta_y = y_+ - y_-$. From (5), there follows

$$\Delta_y = 2\sqrt{r}\,. \tag{8}$$

Define the operators $\mathbb{E}^+$ and $\mathbb{E}^-$ (see [15]), acting on arbitrary functions f, as

$$\mathbb{E}^{\pm} f(x) = f(y_{\pm}(x))\,.$$

With this notation, (3) is given by

$$(\mathbb{D} f)(x) = \frac{\mathbb{E}^+ f - \mathbb{E}^- f}{\mathbb{E}^+ x - \mathbb{E}^- x}\,.$$

The companion operator of $\mathbb{D}$ is defined as (see [15])

$$(\mathbb{M}f)(x) = \frac{\mathbb{E}^+ f(x) + \mathbb{E}^- f(x)}{2}. \tag{9}$$

Note that $\mathbb{D}$ has the following property: if $f(x)$ is a polynomial of degree n in x, then $\mathbb{D}f(x)$ is a polynomial of degree $n-1$ in x. $\mathbb{M}f$ is a polynomial whenever f is a polynomial. Furthermore, if $\deg(f) = n$, then $\deg(\mathbb{M}f) = n$.

We emphasize that, throughout the paper, we will deal with polynomials of the variable x, not displaying the parametrization (2).

Let us introduce some notations within the functional approach. We take a linear functional, $L : \mathbb{C}[x] \longrightarrow \mathbb{C}$, defined by its moments $(u_n)_{n\geq 0}$,

$$L[x^n] = u_n\,, \quad n = 0, 1, \ldots\,,$$

under the condition

$$\det\left[u_{i+j}\right]_{i,j=0}^{n} \neq 0\,, \;\; n \geq 0\,. \tag{10}$$

We shall consider systems of orthogonal polynomials, $\{P_n\}_{n\geq 0}$, with respect to L, that is,

$$L[P_n P_m] = h_n \delta_{n,m}\,, \quad n, m = 0, 1, \ldots\,,$$

where $h_n \neq 0$ and $\delta_{n,m}$ is the Kronecker's delta. It is well known that (10) is a necessary and sufficient condition for the existence of a sequence of orthogonal polynomials with respect to L [21]. Furthermore, if $\det\left[u_{i+j}\right]_{i,j=0}^{n} > 0\,, \;\; n \geq 0$, then there exists a positive measure μ such that

$$L[P] = \int_{supp\,\mu} P(x) d\mu(x)\,, \quad \forall P \in \mathbb{C}[x]\,, \tag{11}$$

thus the family $\{P_n\}_{n\geq 0}$ is said to be orthogonal with respect to μ.

Closely related to $\mathcal{L}$ is the moment generating function, the (formal) Stieltjes function, defined by

$$S(x) = \sum_{n=0}^{+\infty} u_n x^{-n-1}\,. \tag{12}$$

Throughout this paper the orthogonal polynomials P_n are taken to be monic, $P_n(x) = x^n +$ lower degree terms, $n \geq 0$, and the sequence $\{P_n\}_{n\geq 0}$ will be denoted by SMOP.

Monic orthogonal polynomials satisfy a three-term recurrence relation [21]

$$P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0, 1, 2, \ldots\,, \tag{13}$$

with $P_{-1}(x) = 0$, $P_0(x) = 1$, and $\gamma_n \neq 0$, $n \geq 1$, $\gamma_0 = 1$. The parameters β_n, γ_n are the so-called recurrence relation coefficients.

Another relevant sequence, related to $\{P_n\}_{n\geq 0}$, is the sequence of associated polynomials of the first kind, denoted by $\{P_n^{(1)}\}_{n\geq 0}$, defined through the three term recurrence relation

$$P_n^{(1)}(x) = (x - \beta_n)P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x)\,, \quad n = 1, 2, \ldots\,, \tag{14}$$

with $P_{-1}^{(1)}(x) = 0$, $P_0^{(1)}(x) = 1$.

The sequence of functions of the second kind, $\{q_n\}_{n\geq 0}$, is defined by

$$q_n(x) = S(x)P_n(x) - P_{n-1}^{(1)}(x)\,, \quad n \geq 0\,, \tag{15}$$

subject to the initial conditions $q_{-1}(x) = 1$, $q_0(x) = S(x)$. It satisfies a three term recurrence relation,

$$q_{n+1}(x) = (x - \beta_n)q_n(x) - \gamma_n q_{n-1}(x)\,, \quad n = 0, 1, 2, \ldots\,. \tag{16}$$

3. Semi-classical orthogonal polynomials on quadratic lattices

Semi-classical orthogonal polynomials on quadratic lattices may be defined through:

(i) a Pearson equation for the linear functional [9, 10],

$$\mathbb{D}(\phi L) = \mathbb{M}(\psi L)\,, \quad \phi \neq 0\,,\ \deg(\psi) \geq 1\,; \tag{17}$$

(ii) a difference equation for the Stieltjes function [15, 22],

$$A\mathbb{D}S = C\mathbb{M}S + D\,, \tag{18}$$

with A, C, D irreducible polynomials (in x);

(iii) a Pearson equation for the weight [3, 22],

$$A\mathbb{D}w = C\mathbb{M}w\,. \tag{19}$$

The polynomials in (17)–(19) are related via [9, 10]

$$A = \mathbb{M}\phi - r(x)\mathbb{D}\psi - U_1\mathbb{M}\psi\,, \quad C = -\mathbb{D}\phi + \mathbb{M}\psi + U_1\mathbb{D}\psi\,, \tag{20}$$

with $U_1 = \check{c}_2/2$, being $\check{c}_2$ defined by (2) (cf. [9, eq. (16)]), thus, in the account of (7), $U_1 = 2p_0$. D is a polynomial depending on A, C.

The polynomials A, C, D in (18) satisfy, in the account of (3), (9), and (12),

$$\deg(A) \leq m + 2\,,\ \deg(C) \leq m + 1\,,\ \deg(D) \leq m\,, \tag{21}$$

where m is some nonnegative integer. When $m=0$ we get the so-called classical polynomials [9, 18].

The class of a linear functional L on quadratic lattices was defined in [10], as the non-negative integer given by

$$cl(L)=\min_{(f,g)\in\mathcal{X}}\{\max(\deg(f)-2,\deg(g)-1)\}\,,$$

$$\mathcal{X}=\{(f,g)\in C[x]^2\,:\,\deg(g)\geq 1 \text{ and } \mathbb{D}(fL)=\mathbb{M}(gL)\}\,.$$

In what follows we show some fundamental identities for semi-classical orthogonal polynomials on quadratic lattices.

3.1. The system of difference equations for the polynomials. Let S be a Stieltjes function satisfying the difference equation (18), $A\mathbb{D}S=C\mathbb{M}S+D$. Following the same lines as in [22] or [4] (where we take $B\equiv 0$ in Theorem 1), we have, for all $n\geq 1$,

$$\begin{cases} A\mathbb{D}P_n=(l_{n-1}+\Delta_y\pi_{n-1})\mathbb{E}^-P_n-C/2\,\mathbb{E}^+P_n+\Theta_{n-1}\mathbb{E}^-P_{n-1}\,,\\ A\mathbb{D}P^{(1)}_{n-1}=(l_{n-1}+\Delta_y\pi_{n-1})\mathbb{E}^-P^{(1)}_{n-1}+C/2\,\mathbb{E}^+P^{(1)}_{n-1}+D\mathbb{E}^+P_n+\Theta_{n-1}\mathbb{E}^-P^{(1)}_{n-2}\,,\end{cases} \tag{22}$$

and, for all $n\geq 0$,

$$A\mathbb{D}q_n=(l_{n-1}+\Delta_y\pi_{n-1})\mathbb{E}^-q_n+C/2\mathbb{E}^+q_n+\Theta_{n-1}\,\mathbb{E}^-q_{n-1}\,. \tag{23}$$

The above difference equations (22) are equivalent to

$$\begin{cases} A\mathbb{D}P_n=(l_{n-1}-\Delta_y\pi_{n-1})\mathbb{E}^+P_n-C/2\,\mathbb{E}^-P_n+\Theta_{n-1}\mathbb{E}^+P_{n-1}\,,\\ A\mathbb{D}P^{(1)}_{n-1}=(l_{n-1}-\Delta_y\pi_{n-1})\mathbb{E}^+P^{(1)}_{n-1}+C/2\,\mathbb{E}^-P^{(1)}_{n-1}+D\mathbb{E}^-P_n+\Theta_{n-1}\mathbb{E}^+P^{(1)}_{n-2}\,,\end{cases} \tag{24}$$

and (23) is equivalent to

$$A\mathbb{D}q_n=(l_{n-1}-\Delta_y\pi_{n-1})\,\mathbb{E}^+q_n+C/2\mathbb{E}^-q_n+\Theta_{n-1}\,\mathbb{E}^+q_{n-1}\,. \tag{25}$$

Remark . Furthermore, the polynomials l_n,Θ_n,π_n are subject to the following bounds:

$$\deg(\Theta_n)\leq\max\{\deg(A)-2,\deg(C)-1\}\,, \tag{26}$$

$$\deg(l_n)\leq\max\{\deg(A)-1,\deg(C)\}\,,\quad \deg(\pi_n)\leq\deg(C)-1\,. \tag{27}$$

3.2. Compatibility conditions. Define the matrices

$$\mathcal{P}_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}, \quad n \geq 0 . \tag{28}$$

In the account of (13) and (14), $\mathcal{P}_n$ satisfies the difference equation

$$\mathcal{P}_n = \mathcal{A}_n \mathcal{P}_{n-1} , \quad \mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1 , \tag{29}$$

with initial condition $\mathcal{P}_0 = \begin{bmatrix} x-\beta_0 & 1 \\ 1 & 0 \end{bmatrix}$.

The previous systems (22) and (24) can be put in the matrix form as [7]

$$A\,\mathbb{D}\mathcal{P}_n = \mathcal{B}_n^-\,\mathbb{E}^-\mathcal{P}_n - \left(\mathbb{E}^+\mathcal{P}_n\right)\mathcal{C} , \tag{30}$$

$$A\,\mathbb{D}\mathcal{P}_n = \mathcal{B}_n^+\,\mathbb{E}^+\mathcal{P}_n - \left(\mathbb{E}^-\mathcal{P}_n\right)\mathcal{C} , \tag{31}$$

with the matrices $\mathcal{B}_n^{\pm}$ and $\mathcal{C}$ given by

$$\mathcal{B}_n^{\pm} = \begin{bmatrix} l_n \mp \Delta_y \pi_n & \Theta_n \\ -\frac{\Theta_{n-1}}{\gamma_n} & l_{n-1} \mp \Delta_y \pi_{n-1} + \frac{\Theta_{n-1}}{\gamma_n}\,\mathbb{E}^{\pm}(x-\beta_n) \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C/2 & -D \\ 0 & -C/2 \end{bmatrix}.$$

From the compatibility of (29) and (30)–(31) we get the equations for the transfer matrices $\mathcal{A}_n$, for all $n \geq 1$ [7]:

$$A\,\mathbb{D}\mathcal{A}_n = \mathcal{B}_n^-\,\mathbb{E}^-\mathcal{A}_n - \left(\mathbb{E}^+\mathcal{A}_n\right)\mathcal{B}_{n-1}^- , \tag{32}$$

$$A\,\mathbb{D}\mathcal{A}_n = \mathcal{B}_n^+\,\mathbb{E}^+\mathcal{A}_n - \left(\mathbb{E}^-\mathcal{A}_n\right)\mathcal{B}_{n-1}^+ . \tag{33}$$

The compatibility conditions (32)–(33) yield the following relations for the polynomials π_n, l_n, Θ_n, for all $n \geq 0$ [7, 15]:

$$\pi_{n+1} = -\frac{1}{2}\sum_{k=0}^{n+1}\frac{\Theta_{k-1}}{\gamma_k} , \tag{34}$$

$$l_{n+1} + l_n + \mathbb{M}(x-\beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} = 0 , \tag{35}$$

$$-A + \mathbb{M}(x-\beta_{n+1})(l_{n+1}-l_n) - \frac{\Delta_y^2}{2}(\pi_{n+1}+\pi_n) + \Theta_{n+1} = \frac{\gamma_{n+1}}{\gamma_n}\Theta_{n-1} . \tag{36}$$

The following initial conditions hold:

$$\pi_{-1} = 0,\ \pi_0 = -D/2, \tag{37}$$

$$\Theta_{-1} = D,\ \Theta_0 = A - \frac{\Delta_y^2}{4}D - (l_0 - C/2)\mathbb{M}(x-\beta_0)\,, \tag{38}$$

$$l_{-1} = C/2,\ l_0 = -\mathbb{M}(x-\beta_0)D - C/2\,. \tag{39}$$

3.3. Further matrix identities. The following results extend the differential systems from the continuous orthogonality given in [14] to the discrete orthogonality on systems of nonuniform lattices (see [3, Th. 1] and also [22, Sec. 4]). We stress equation (43) below, the analogue of the so-called Magnus' summation formula [14].

Theorem 1. *Let S be a Stieltjes function related to a weight w, satisfying $A\,\mathbb{D}S = C\,\mathbb{M}S + D$, and let $\{\mathcal{Y}_n\}_{n\geq 0}$ be the corresponding sequence given by $\{\mathcal{Y}_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}\}_{n\geq 0}$. The following equation holds:*

$$A_{n+1}\,\mathbb{D}\mathcal{Y}_n = (\mathcal{B}_n - C/2\,I)\mathbb{M}\mathcal{Y}_n\,, \quad n \geq 1\,, \tag{40}$$

where

$$A_{n+1} = A + \frac{\Delta_y^2}{2}\pi_n\,,$$

I is the identity matrix, and $\mathcal{B}_n$ is given as

$$\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\dfrac{\Theta_{n-1}}{\gamma_n} & l_{n-1} + \dfrac{\Theta_{n-1}}{\gamma_n}\mathbb{M}(x-\beta_n) \end{bmatrix}. \tag{41}$$

Corollary 1. *The matrix $\mathcal{B}_n$ satisfies the following identities, for all $n \geq 1$:*

$$\operatorname{tr}\mathcal{B}_n = 0\,, \tag{42}$$

$$\det\mathcal{B}_n = -\Delta_y^2\pi_n^2 + AD - \frac{C^2}{4} + A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k}\,. \tag{43}$$

Remark . Taking into account $\Theta_{-1}/\gamma_0 = D$ (see (38)) and (34), an equivalent equation for (43) is

$$\det\mathcal{B}_n = -\Delta_y^2\pi_n^2 - \frac{C^2}{4} - 2A\pi_n\,. \tag{44}$$

In the account of (42), we shall use $\mathcal{B}_n$ in (44) given as

$$\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & -l_n \end{bmatrix}.$$

Therefore, (44) reads as

$$-l_n^2(x) + \Theta_n(x)\frac{\Theta_{n-1}(x)}{\gamma_n} = -\Delta_y^2\pi_n^2 - \frac{C^2}{4} - 2A\pi_n\,. \tag{45}$$

4. Difference equations for the recurrence relation coefficients

4.1. Difference equations when $m = 1$ in (21). Let us take $m = 1$ in (21), that is, $A(x)\mathbb{D}S(x) = C(x)\mathbb{M}S(x) + D(x)$ with

$$\deg(A) \leq 3\,,\ \deg(C) \leq 2\,,\ \deg(D) \leq 1\,, \tag{46}$$

were we consider, by writing

$$A(x) = a_3x^3 + a_2x^2 + a_1x + a_0\,,\ C(x) = c_2x^2 + c_1x + c_0\,,$$

the condition

$$a_3 \neq 0 \ \text{ or } \ c_2 \neq 0\,. \tag{47}$$

The polynomial D is given in terms of A, C. By collecting the coefficient of x^3 in (38) as well as the coefficient of x^2 in (39) we get

$$d_1 = -(a_3 + c_2p_1)/p_1^2\,. \tag{48}$$

By collecting the coefficient of x^2 in (38) as well as the coefficient of x in (39) we get, using (48),

$$d_0 = \frac{a_3(2p_0p_1 - r_1 - 2p_1\beta_0) - p_1(a_2p_1 + c_1p_1^2 + c_2(r_1 + p_1\beta_0 - p_0p_1))}{p_1^4}\,. \tag{49}$$

In the account of (26)–(27), we have $\deg(l_n) = 2$, $\deg(\Theta_n) = \deg(\pi_n) = 1$. Set

$$l_n(x) = \ell_{n,2}x^2 + \ell_{n,1}x + \ell_{n,0}\,,\ \Theta_n(x) = \Theta_{n,1}x + \Theta_{n,0}\,,\ \pi_n(x) = \pi_{n,1}x + \pi_{n,0}\,.$$

Also, recall (8), thus $\Delta_y^2(x) = 4r(x)$.

Henceforth we adopt the convention that $\sum_i^j \cdot = 0$ whenever $i > j$ and $\prod_i^j \cdot = 1$ whenever $i > j$.

Lemma 1. *Under the previous assumptions and notations, the quantities* $\ell_{n,2},\ \ell_{n,1}, \Theta_{n,1}/\gamma_{n+1}, \Theta_{n,0}/\gamma_{n+1}, \pi_{n,1}, \pi_{n,0}$ *are given, for all* $n \geq 0$, *by*

$$\begin{aligned}
\ell_{n,2} &= n\frac{a_3}{p_1} - p_1 d_1 - \frac{c_2}{2}\,, && (50)\\
\frac{\Theta_{n,1}}{\gamma_{n+1}} &= \frac{1}{p_1}\left(-(2n+1)\frac{a_3}{p_1} + 2p_1 d_1 + c_2\right)\,, && (51)\\
\pi_{n,1} &= -\frac{d_1}{2} + \frac{1}{2p_1}\left(\frac{a_3}{p_1}n - 2p_1 d_1 - c_2\right)n\,, && (52)\\
\ell_{n,1} &= L_{n,1} + \frac{a_3}{p_1^2}\sum_{k=1}^{n}\beta_k\,. && (53)
\end{aligned}$$

with

$$\begin{aligned}
L_{n,1} = &\left(\frac{a_2 p_1 - a_3 p_0}{p_1^2}\right)n\\
&+ \frac{2r_1}{p_1}\left(-nd_1 + \frac{a_3}{2p_1^2}\left(\frac{(n-1)n(2n-1)}{3} + n^2\right) - \frac{(2p_1 d_1 + c_2)}{2p_1}n^2\right)\\
&\qquad\qquad - p_1 d_0 - (p_0 - \beta_0)d_1 - \frac{c_1}{2}\,.
\end{aligned}$$

Furthermore, the following relations hold, for all $n \geq 1$*:*

$$\frac{\Theta_{n,0}}{\gamma_{n+1}} = S_{n,0} + \frac{1}{p_1}\left(\frac{\Theta_{n,1}}{\gamma_{n+1}} - \frac{a_3}{p_1^2}\right)\beta_{n+1} - \frac{2a_3}{p_1^3}\sum_{k=1}^{n}\beta_k\,,$$
$$S_{n,0} = -\frac{(L_{n+1,1} + L_{n,1} + p_0\Theta_{n,1}/\gamma_{n+1})}{p_1}\,, \quad (54)$$

(i) if $a_3 \neq 0$, *then*

$$\pi_{n,0} = \hat{T}_{n,0} + \frac{\ell_{n,2}}{p_1^2}\sum_{k=1}^{n}\beta_k\,,\quad \hat{T}_{n,0} = \frac{L_{n,1}\ell_{n,2} - 2r_1\pi_{n,1}^2 - a_2\pi_{n,1} - c_1c_2/4}{a_3}\,, \quad (55)$$

(ii) if $a_3 = 0$, then

$$\pi_{n,0} = T_{n,0} - \frac{(2p_1d_1 + c_2)}{2p_1^2}\sum_{k=2}^{n}\beta_k\,,$$
$$T_{n,0} = -\frac{d_0}{2} - \frac{1}{2}\frac{\Theta_{0,0}}{\gamma_1} + \frac{1}{2p_1}\sum_{k=1}^{n-1}\left(L_{k+1,1} + L_{k,1} + \frac{p_0(2p_1d_1 + c_2)}{p_1}\right)\,. \tag{56}$$

The initial conditions hold:

$$\begin{aligned}
\pi_{0,0} &= -\frac{d_0}{2}\,,\\
\frac{\Theta_{0,0}}{\gamma_1} &= -\frac{a_2}{p_1^2} + \frac{r_1}{p_1^2}\left(d_1 + \frac{\Theta_{0,1}}{\gamma_1}\right) + \left(-\ell_{0,2} - p_1\frac{\Theta_{0,1}}{\gamma_1} + \frac{c_2}{2}\right)\left(\frac{p_0 - \beta_1}{p_1^2}\right)\\
&+ \frac{1}{p_1}\left(-\ell_{0,1} - (p_0 - \beta_1)\frac{\Theta_{0,1}}{\gamma_1} + \frac{c_1}{2}\right) + \frac{d_1}{p_1}\left(2p_0 - (\beta_0 + \beta_1) + \frac{r_1}{p_1}\right) + d_0\,.
\end{aligned}$$

Here, p_1, p_0, r_1 are the coefficients of $p(x), r(x)$, defined in (6).

Proof: The coefficient of x^3 in (36) yields

$$-a_3 + p_1(\ell_{n+1,2} - \ell_{n,2}) = 0\,.$$

This, combined with the initial condition $\ell_{0,2} = -p_1d_1 - c_2/2$, gives us (50).

The use of (50) in the equation that follows from the coefficient of x^2 in (35),

$$\ell_{n+1,2} + \ell_{n,2} + p_1\Theta_{n,1}/\gamma_{n+1} = 0\,,$$

gives us (51) for all $n \geq 1$. In order to get $\Theta_{0,1}/\gamma_1$ we take $n = 1$ in

$$A_{n+1}\mathbb{D}P_n^{(1)} = (l_n + C/2)\mathbb{M}P_n^{(1)} + D\mathbb{M}P_{n+1} + \Theta_n\mathbb{M}P_{n-1}^{(1)}\,.$$

Indeed, using (34) and (35) with $n = 0$ in the equation above with $n = 1$ we have

$$A + 2r\left(-\frac{D}{2} - \frac{1}{2}\frac{\Theta_0}{\gamma_1}\right) = \left(-l_0 - \mathbb{M}(x - \beta_1)\frac{\Theta_0}{\gamma_1} + C/2\right)\mathbb{M}(x - \beta_1) + D\mathbb{M}P_2 + \Theta_1\,. \tag{57}$$

The coefficient of x^3 gives us

$$\frac{\Theta_{0,1}}{\gamma_1} = \frac{1}{p_1}\left(-\frac{a_3}{p_1} + 2p_1d_1 + c_2\right)\,, \tag{58}$$

thus (51) also holds for $n = 0$.

Equation (52) follows from the use of (51) in the summation formula (34), and the initial condition $\pi_{0,1} = -d_1/2$ (cf. (37)).

Let us now obtain (53). Using (50) in the equation that follows from equating the coefficients of x^2 in (36) we get

$$\ell_{n+1,1} = \ell_{n,1} - \frac{(p_0 - \beta_{n+1})a_3}{p_1^2} + \frac{2r_1}{p_1}\lambda_{n,1} + \frac{a_2}{p_1}, \quad \lambda_{n,1} = \pi_{n+1,1} + \pi_{n,1}. \tag{59}$$

Thus, we obtain (53), where we used the initial conditions $\ell_{0,1} = -p_1 d_0 - (p_0 - \beta_0)d_1 - c_1/2$.

To get (54) we take the x coefficient in (35),

$$\ell_{n+1,1} + \ell_{n,1} + p_1 \frac{\Theta_{n,0}}{\gamma_{n+1}} + (p_0 - \beta_{n+1})\frac{\Theta_{n,1}}{\gamma_{n+1}} = 0,$$

and substitute (53) and (51) therein. The initial condition $\pi_{0,0}$ comes from (37) and $\Theta_{0,0}/\gamma_1$ follows from taking the coefficient of x^2 in (57).

$\pi_{n,0}$ can be obtained via the summation formula (34). Thus, from (34), if $a_3 = 0$, we get

$$\pi_{n,0} = -\frac{1}{2}\frac{\Theta_{-1,0}}{\gamma_0} - \frac{1}{2}\frac{\Theta_{0,0}}{\gamma_1} - \frac{1}{2}\sum_{k=1}^{n-1}\frac{\Theta_{k,0}}{\gamma_{k+1}}. \tag{60}$$

and (56) follows. The case $a_3 \neq 0$ can be alternatively obtained through the use of the x^3-coefficient in (45), thus yielding (55). ■

Theorem 2. *Let S be a Stieltjes function satisfying*

$$A(x)\mathbb{D}S(x) = C(x)\mathbb{M}S(x) + D(x)$$

with $\deg(A) \leq 3,\ \deg(C) \leq 2,\ \deg(D) \leq 1$ *subject to the condition (47). Let $\{P_n\}_{n\geq 0}$ be the corresponding SMOP, satisfying (13),*

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots.$$

Under the notations of the previous lemma, the γ_n's are defined only in terms of the β_n's and the polynomials A, C, a well as p, r from (6), related to the quadratic lattice. There holds the formula

$$\gamma_{n+2} = \gamma_1 \prod_{j=0}^{n} s_j + \sum_{k=0}^{n} t_k \prod_{j=k+1}^{n} s_j, \quad n \geq 0, \tag{61}$$

$$\gamma_1 = \frac{p_1 \Theta_{0,1}}{-a_3/p_1 + 2p_1 d_1 + c_2}, \tag{62}$$

with

$$s_n = \frac{\Theta_{n-1}/\gamma_n}{\Theta_{n+1}/\gamma_{n+2}}(x_{n+1})\,, \quad t_n = \frac{A(x_{n+1}) + 2r(x_{n+1})(\pi_{n+1}+\pi_n)(x_{n+1})}{(\Theta_{n+1}/\gamma_{n+2})(x_{n+1})}\,, \tag{63}$$
$$x_{n+1} = (\beta_{n+1} - p_0)/p_1\,,$$

and

$$\Theta_{0,1} = a_1 - r_1 d_0 - r_0 d_1 + (p_1 d_0 + (p_0 - \beta_0) d_1 + c_1)(p_0 - \beta_0) + ((p_0 - \beta_0) d_0 + c_0) p_1\,. \tag{64}$$

Proof: We evaluate (36) at $x_{n+1} = (\beta_{n+1} - p_0)/p_1$. As $\mathbb{M}(x - \beta_{n+1})(x_{n+1}) = 0$, we get

$$-A(x_{n+1}) - 2r(x_{n+1})\lambda_n(x_{n+1}) + \gamma_{n+2}\frac{\Theta_{n+1}}{\gamma_{n+2}}(x_{n+1}) = \gamma_{n+1}\frac{\Theta_{n-1}}{\gamma_n}(x_{n+1})\,, \tag{65}$$

where it was used the notation $\lambda_n(x) = \pi_{n+1}(x) + \pi_n(x)$. Hence, we obtain

$$\gamma_{n+2} = s_n \gamma_{n+1} + t_n\,, \quad n \geq 0 \tag{66}$$

with s_n, t_n given in (63). As the solution of the initial value problem

$$z_{n+1} = a_n z_n + b_n\,, \quad z_{n_0} = z_0$$

is [6]

$$z_n = z_0 \prod_{j=n_0}^{n-1} a_j + \sum_{k=n_0}^{n-1} b_k \prod_{j=k+1}^{n-1} a_j\,,$$

then equation (61) is a consequence of (66).

To get γ_1, we take the initial conditions

$$\ell_{0,1} = -p_1 d_0 - (p_0 - \beta_0) d_1 - c_1/2\,, \tag{67}$$
$$\ell_{0,0} = -(p_0 - \beta_0) d_0 - c_0/2\,. \tag{68}$$
$$\Theta_{0,1} = a_1 - r_1 d_0 - r_0 d_1 - (\ell_{0,1} - c_1/2)(p_0 - \beta_0) - (\ell_{0,0} - c_0/2) p_1\,, \tag{69}$$

The use of (67) and (68) in (69) yields (64). Using $\Theta_{0,1}/\gamma_1$ given by (cf. (58))

$$\frac{\Theta_{0,1}}{\gamma_1} = \frac{1}{p_1}\left(-a_3/p_1 + 2p_1 d_1 + c_2\right)$$

combined with (64), we get (62). ∎

Remark . The previous theorem gives us the γ_n's in terms of the β_n's. In order to obtain a recurrence for the β_n's we may start by taking the independent term of (35), which gives us

$$\ell_{n+1,0} = -\ell_{n,0} - (p_0 - \beta_{n+1})\frac{\Theta_{n,0}}{\gamma_{n+1}} \,. \tag{70}$$

Using this equality into the equation that results from the coefficient of x in (36) we obtain

$$\begin{aligned} \ell_{n,0} = -\frac{1}{2p_1}(a_1 + 2r_0\lambda_{n,1}) - \frac{(p_0 - \beta_{n+1})}{2p_1}\left(p_1\frac{\Theta_{n,0}}{\gamma_{n+1}} + \ell_{n,1} - \ell_{n+1,1}\right) \\ - \frac{r_1}{p_1}(\pi_{n+1,0} + \pi_{n,0}) + \gamma_{n+2}\frac{\nu_{n+1,1}}{2p_1} - \gamma_{n+1}\frac{\nu_{n-1,1}}{2p_1} \,, \end{aligned} \tag{71}$$

where we used the notation $\lambda_{n,1} = \pi_{n+1,1} + \pi_{n,1}$, $\nu_{n,1} = \Theta_{n,1}/\gamma_{n+1}$. Now, by substituting (61) and (71) into equation (70) we get a first order non-linear recurrence relation for the β_n's, say $F(\gamma_1, \beta_1, \dots, \beta_{n+2}) = G(\gamma_1, \beta_1, \dots, \beta_{n+1})$. Due to the complexity of such a formulae, we shall not give its explicit form here.

4.1.1. *The symmetric case.* The symmetric case, that is, $\beta_n = 0$, $n \geq 0$, implies simplifications in (53), (54), (55), (56), and all these quantities now depend only on the lattice as well as on the coefficients A, C, D of the difference equation for the Stieltjes function. In such a case, we have the result that follows.

Corollary 2. *Let* $A(x)\mathbb{D}S(x) = C(x)\mathbb{M}S(x) + D(x)$ *with* $\deg(A) \leq 3$, $\deg(C) \leq 2$, $\deg(D) \leq 1$ *subject to the condition (47). Under the previous notation, let* $\beta_n = 0$, $n \geq 0$*. Then, the* γ_n*'s are determined through:*

$$\gamma_{n+2} = \gamma_1 \prod_{j=0}^{n} s_j + \sum_{k=0}^{n} t_k \prod_{j=k+1}^{n} s_j \,, \quad n \geq 0 \,, \tag{72}$$

$$\gamma_1 = \frac{p_1\Theta_{0,1}}{-a_3/p_1 + 2p_1d_1 + c_2} \,, \tag{73}$$

with

$$s_n = \frac{(\Theta_{n-1}/\gamma_n)(x_0)}{(\Theta_{n+1}/\gamma_{n+2})(x_0)}, \quad t_n = \frac{A(x_0) + 2r(x_0)(\pi_{n+1} + \pi_n)(x_0)}{(\Theta_{n+1}/\gamma_{n+2})(x_0)} \,, \quad x_0 = -p_0/p_1 \,,$$

and

$$\Theta_{0,1} = a_1 - r_1d_0 - r_0d_1 + (p_1d_0 + p_0d_1 + c_1)p_0 + (p_0d_0 + c_0)p_1 \,.$$

Proof: Take $\beta_n = 0$, $n \geq 0$. Evaluate (36) at $x_0 = -p_0/p_1$. Thus, as $\mathbb{M}(x_0) = 0$, we get

$$-A(x_0) - 2r(x_0)(\pi_{n+1} + \pi_n)(x_0) + \gamma_{n+2}\frac{\Theta_{n+1}}{\gamma_{n+2}}(x_0) = \gamma_{n+1}\frac{\Theta_{n-1}}{\gamma_n}(x_0)\,. \qquad (74)$$

Therefore, (72) follows. ■

4.1.2. *Condition (47) with* $a_3 = 0$. Let us take the case

$$\deg(A) \leq 2\,,\ \deg(C) = 2\,,\ \deg(D) = 1\,. \qquad (75)$$

In such a case, the quantities given in Lemma 1 are as follow:

$$\begin{aligned}
\ell_{n,2} &= \frac{c_2}{2}\,,\quad \ell_{n,1} = L_{n,1}\,,\\
\frac{\Theta_{n,1}}{\gamma_{n+1}} &= -\frac{c_2}{p_1}\,,\quad \frac{\Theta_{n,0}}{\gamma_{n+1}} = S_{n,0} - \frac{c_2}{p_1^2}\beta_{n+1}\,,\\
\pi_{n,1} &= (n+1)\frac{c_2}{2p_1}\,,\quad \pi_{n,0} = T_{n,0} + \frac{c_2}{2p_1^2}\sum_{k=2}^{n}\beta_k\,,
\end{aligned}$$

with

$$\begin{aligned}
L_{n,1} &= \frac{na_2}{p_1} + \frac{2r_1}{p_1}\frac{nc_2}{p_1}(1+\frac{n}{2}) - p_1 d_0 - (p_0 - \beta_0)d_1 - \frac{c_1}{2}\,,\\
S_{n,0} &= -\frac{(L_{n+1,1} + L_{n,1} + p_0\Theta_{n,1}/\gamma_{n+1})}{p_1}\,,\\
T_{n,0} &= -\frac{d_0}{2} - \frac{1}{2}\frac{\Theta_{0,0}}{\gamma_1} + \frac{1}{2p_1}\sum_{k=1}^{n-1}\left(L_{k+1,1} + L_{k,1} - \frac{c_2 p_0}{p_1}\right)\,.
\end{aligned}$$

Furthermore, by taking the x^2-coefficient of (44), that is,

$$\begin{aligned}
-\ell_{n,1}^2 - 2\ell_{n,2}\ell_{n,0} + \gamma_{n+1}\frac{\Theta_{n,1}}{\gamma_{n+1}}\frac{\Theta_{n-1,1}}{\gamma_{n+1}} &= -8r_1\pi_{n,0}\pi_{n,1}\\
&\quad - 4r_0\pi_{n,1}^2 - (c_1^2 + 2c_2c_0)/4 - 2a_2\pi_{n,0} - 2a_1\pi_{n,1}\,, \qquad (76)
\end{aligned}$$

we get the expression for $\ell_{n,0}$,

$$\ell_{n,0} = \tau_n + \frac{c_2}{p_1^2}\gamma_{n+1} + \left(4r_1(n+1) + 2\frac{a_0}{c_2}\right)\pi_{n,0}\,, \qquad (77)$$

with

$$\tau_n = -\frac{\ell_{n,1}^2}{c_2} + r_0\frac{c_2}{p_1^2}(n+1)^2 + \frac{(c_1^2 + 2c_2c_0)}{4c_2}\,.$$

Theorem 3. *Under the degrees (75) and the previous notations, we have the following difference equations for the recurrence coefficients, for all $n \geq 1$:*

$$\gamma_{n+2} + \gamma_{n+1} + \frac{p_1}{c_2}\left(4r_1(n+2) + 2\frac{a_0}{c_2}\right)\pi_{n+1,0} + \frac{p_1}{c_2}\left(4r_1(n+1) + 2\frac{a_0}{c_2}\right)\pi_{n,0}$$
$$+\frac{p_1}{c_2}(p_0 - \beta_{n+1})\frac{\Theta_{n,0}}{\gamma_{n+1}} + \frac{p_1}{c_2}(\tau_{n+1} + \tau_n) = 0\,, \tag{78}$$

$$\gamma_{n+1} = \frac{2L_{n,1}(\tau_n + (4r_1(n+1) + 2a_0/c_2)\,\pi_{n,0}) + G_n}{\frac{-2c_2}{p_1^2}L_{n,1} - \frac{c_2}{p_1}(\frac{\Theta_{n-1,0}}{\gamma_n} + \frac{\Theta_{n,0}}{\gamma_{n+1}})}\,, \tag{79}$$

with the initial condition γ_1 given by (62), and with $G_n = -4r_1\pi_{n,0}^2 - 8r_0\pi_{n,0}\pi_{n,1} - c_1c_0/2 - 2a_1\pi_{n,0} - 2a_0\pi_{n,1}$.

Proof: To get (78) we use (77) in the equation obtained from the independent term in (35),

$$\ell_{n+1,0} + \ell_{n,0} + (p_0 - \beta_{n+1})\frac{\Theta_{n,0}}{\gamma_{n+1}} = 0\,.$$

To get (79) we take the x-coefficient of (44), that is,

$$-2\ell_{n,1}\ell_{n,0} + \gamma_{n+1}\left(\frac{\Theta_{n,1}}{\gamma_{n+1}}\frac{\Theta_{n-1,0}}{\gamma_n} + \frac{\Theta_{n,0}}{\gamma_{n+1}}\frac{\Theta_{n-1,1}}{\gamma_n}\right) = G_n\,,$$

with $G_n = -4r_1\pi_{n,0}^2 - 8r_0\pi_{n,0}\pi_{n,1} - c_1c_0/2 - 2a_1\pi_{n,0} - 2a_0\pi_{n,1}$. The use of $\ell_{n,0}$ given by (77) into the above equation gives us (79). ∎

4.2. $m = 0$ in (21): classical orthogonal polynomials on quadratic lattices from compatibility relations. Let us take $m = 0$ in (21), that is, $A(x)\mathbb{D}S(x) = C(x)\mathbb{M}S(x) + D(x)$ with

$$\deg(A) \leq 2\,,\ \deg(C) \leq 1\,,\ \deg(D) = 0\,. \tag{80}$$

were we consider, by writing

$$A(x) = a_2x^2 + a_1x + a_0\,,\ C(x) = c_1x + c_0\,,$$

the condition

$$a_2 \neq 0 \ \text{ or } \ c_1 \neq 0\,. \tag{81}$$

In the account of (26)–(27), we have $\deg(l_n) = 1$, $\deg(\Theta_n) = \deg(\pi_n) = 0$. Set

$$l_n(x) = \ell_{n,1}x + \ell_{n,0}\,,\ \Theta_n(x) = \Theta_{n,0}\,,\ \pi_n(x) = \pi_{n,0},.$$

We have $D = d_0 = -(a_2 + c_1p_1)/p_1^2$.

Lemma 2. *Under the previous assumptions and notations, we have, for all* $n \geq 0$,

$$\ell_{n,1} = n\frac{a_2}{p_1} - p_1 d_0 - \frac{c_1}{2}\,, \tag{82}$$

$$\frac{\Theta_{n,0}}{\gamma_{n+1}} = \frac{1}{p_1}\left(-(2n+1)\frac{a_2}{p_1} + 2p_1 d_0 + c_1\right)\,, \tag{83}$$

$$\pi_{n,0} = -\frac{d_0}{2} + \frac{1}{2p_1}\left(\frac{a_2}{p_1}n - 2p_1 d_0 - c_1\right) n\,, \tag{84}$$

and

$$\ell_{n,0} = \frac{2r_1\pi_{n,0}^2 + c_0 c_1/4 + a_1\pi_{n,0}}{\ell_{n,1}}\,, \quad n \geq 1\,, \quad \ell_{0,0} = -(p_0 - \beta_0)d_0 - c_0/2\,. \tag{85}$$

Here, p_1, p_0, r_1 *are the coefficients of* $p(x), r(x)$*, defined in (6).*

Proof: The equations (82)–(84) follow from Lemma 1.
The x-coefficient of (45) gives us (85). ■

Theorem 4. *Let* $A(x)\mathbb{D}S(x) = C(x)\mathbb{M}S(x) + D(x)$ *with* $\deg(A) \leq 2$, $\deg(C) \leq 1$, $\deg(D) \leq 0$ *subject to the condition (81). Consider the notations of the previous lemma. The following holds:*

$$\gamma_{n+1} = \frac{\ell_{n,0}^2 - 4r_0\pi_{n,0}^2 - c_0^2/4 - 2a_0\pi_{n,0}}{\frac{\Theta_{n,0}}{\gamma_{n+1}}\frac{\Theta_{n-1,0}}{\gamma_n}}\,, \quad n \geq 1\,, \tag{86}$$

$$\beta_{n+1} = \frac{\ell_{n+1,0} + \ell_{n,0} + p_0\Theta_{n,0}/\gamma_{n+1}}{\Theta_{n,0}/\gamma_{n+1}}\,, \quad n \geq 0\,, \tag{87}$$

and the initial conditions β_0 *and* γ_1 *given by*

$$\beta_0 = \frac{p_0 d_0 + c_0 + (a_1 - r_1 d_0)/p_1 - a_2 p_0/p_1^2}{d_0 - a_2/p_1^2}\,, \tag{88}$$

$$\gamma_1 = \frac{p_1(a_0 - r_0 d_0 + ((p_0 - \beta_0)d_0 + c_0)(p_0 - \beta_0))}{-a_2/p_1 + 2p_1 d_0 + c_1}\,. \tag{89}$$

Proof: The equation (86) follows from the independent coefficient of (45),

$$-\ell_{n,0}^2 + \gamma_{n+1}\frac{\Theta_{n,0}}{\gamma_{n+1}}\frac{\Theta_{n-1,0}}{\gamma_n} = -4r_0\pi_{n,0}^2 - c_0^2/4 - 2a_0\pi_{n,0}\,.$$

The equation (87) is obtained from the independent term of (35),

$$\ell_{n+1,0} + \ell_{n,0} + (p_0 - \beta_{n+1})\frac{\Theta_{n,0}}{\gamma_{n+1}} = 0\,.$$

To obtain β_0 and γ_1 we equate coefficients in (38) and (39), thus getting

$$\begin{aligned}
\ell_{0,1} &= -p_1 d_0 - c_1/2\,, && (90)\\
\ell_{0,0} &= -(p_0-\beta_0)d_0 - c_0/2\,. && (91)\\
0 &= a_1 - r_1 d_0 - (\ell_{0,1} - c_1/2)(p_0-\beta_0) - (\ell_{0,0} - c_0/2)p_1\,, && (92)\\
\Theta_{0,0} &= a_0 - r_0 d_0 - (\ell_{0,0} - c_0/2)(p_0-\beta_0)\,. && (93)
\end{aligned}$$

The use of (90) and (91) in (92) yields β_0. From (93) we have, using (91),

$$\Theta_{0,0} = a_0 - r_0 d_0 + ((p_0-\beta_0)d_0 + c_0)(p_0-\beta_0)\,. \tag{94}$$

From $\Theta_{0,0}/\gamma_1$ given by

$$\frac{\Theta_{0,0}}{\gamma_1} = \frac{1}{p_1}\left(-a_2/p_1 + 2p_1 d_0 + c_1\right)$$

combined with (94) we get γ_1. ∎

5. Examples

5.1. Dual Hahn polynomials. The Dual Hahn polynomials have the hypergeometric representation [13]

$$P_n(x;\gamma,\delta,N) = {}_3F_2\left(\begin{matrix}-n, n+\gamma+\delta+1, -x\\ \gamma+1, -N\end{matrix}\;;1\right). \tag{95}$$

The lattice $x(s)$ and the polynomials p, r that follow from (7) are

$$x(s) = s(s+\gamma+\delta+1)\,,\quad p(x) = x + \frac{1}{4}\,,\quad r(x) = x + \frac{(\gamma+\delta+1)^2}{4}\,. \tag{96}$$

$\{P_n\}_{n\geq 0}$ is related to a linear functional L that satisfies $\mathbb{D}(\phi L) = \mathbb{M}(\psi L)$, where the polynomials ϕ, ψ are given by [10]

$$\phi(x) = (-1+2N+\delta-\gamma)x + N(1+\gamma)(1+\gamma+\delta)\,,\quad \psi(x) = -2x + 2N(1+\gamma)\,. \tag{97}$$

The Stieltjes function satisfies (18), $A\,\mathbb{D}S = C\,\mathbb{M}S + D$, with A, C given by (20), thus,

$$A = \mathbb{M}\phi + 2r(x) - \frac{1}{2}\mathbb{M}\psi\,,\quad C = -1 - \mathbb{D}\phi + \mathbb{M}\psi\,. \tag{98}$$

where we used $U_1 = 1/2$. The polynomial D is a constant, $D = -c_1/p_1$. As we have $\deg(A) = \deg(C) = 1$, condition (81) of Section 4.2 holds.

From the formulae in Theorem 4 we recover [13, pp. 209], for all $n \geq 1$,

$$\beta_n = (n+\gamma+1)(n-N)+n(n-\delta-N-1)\,, \quad \gamma_n = n(n+\gamma)(n-1-N)(n-\delta-N-1)\,,$$

and $\beta_0 = -N(\gamma+1)$, $\gamma_0 = 1$.

5.2. Modification of Dual Hahn polynomials. We consider the following modifiation of the Dual Hahn polynomials. We take the linear functional [10, Sec. 2.4]

$$\tilde{L} = \left(x + \frac{(\gamma+\delta+1)^2}{4}\right) L\,, \tag{99}$$

being L the linear functional related to the Dual Hahn polynomials. $\tilde{L}$ satisfies

$$\mathbb{D}(\tilde{\phi}\tilde{L}) = \mathbb{M}(\tilde{\psi}\tilde{L})\,,$$

where the polynomials $\tilde{\phi}, \tilde{\psi}$ are given by (see [10, Eq. (40)])

$$\tilde{\phi}(x) = (r(x)+1)\phi(x) + 2r(x)\psi(x)\,, \quad \tilde{\psi}(x) = (r(x)+1)\psi(x) + 2\phi(x)\,, \tag{100}$$

with ϕ, ψ given in (97). Note that (96) holds. Recall that we are taking $\alpha = 1$ and $x - c = r(x)$, with our notation $r(x)$ for the polynomial $U_2(x)$, in [10, Eq. (40)].

Denote by $\{\tilde{P}_n\}_{n\geq 0}$ the SMOP related to $\tilde{L}$, and its recurrence relation coefficients by $\tilde{\beta}_n, \tilde{\gamma}_n$. The corresponding Stieltjes function satisfies (18), $\tilde{A}\,\mathbb{D}S = \tilde{C}\,\mathbb{M}S + \tilde{D}$, with $\tilde{A}, \tilde{C}$ given by (20), thus,

$$\tilde{A} = \mathbb{M}\tilde{\phi} - r\mathbb{D}\tilde{\psi} - \frac{1}{2}\mathbb{M}\tilde{\psi}\,, \quad \tilde{C} = -\mathbb{D}\tilde{\phi} + \mathbb{M}\tilde{\psi} + \frac{1}{2}\mathbb{D}\tilde{\psi}\,. \tag{101}$$

$\tilde{D}$ is a polynomial of degree one, with coefficients given by (48) and (49). As we have $\deg(A) = \deg(C) = 2$, condition (75) of Sub-Section 4.1.2 holds. From Theorem 3, the coefficients $\tilde{\gamma}_n, \tilde{\beta}_n$ are governed through the difference system (78)–(79).

Remark . The modification (99) is related to the Christoffel transformation [23, Sec. 3]. In this case the modified recurrence relation coefficients are known to be given in terms of the non-modified ones [23],

$$\tilde{\beta}_n = \beta_{n+1} - \frac{P_{n+1}(c)}{P_n(c)} + \frac{P_{n+2}(c)}{P_{n+1}(c)}\,, \quad \tilde{\gamma}_n = \gamma_n \frac{P_{n-1}(c)P_{n+1}(c)}{P_n^2(c)}\,, \quad c = -\frac{(\gamma+\delta+1)^2}{4}\,.$$

Note that here the P_n's at c must be evaluated trough (95), whilst our formulae in Theorem 3 give a relation for $\tilde{\beta}_n, \tilde{\gamma}_n$ in terms of the lattice and the polynomials involved in the difference equation for $\tilde{S}$.

References

[1] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Memoirs AMS vol. 54 n. 319, AMS, Providence, 1985.

[2] S. Belmehdi, *On semi-classical linear functionals of class s=1. Classification and integral representations*, Indag Math. 3 (1992), pp. 253–275.

[3] A. Branquinho, Y. Chen, G. Filipuk, and M.N. Rebocho, *A characterization theorem for semi-classical orthogonal polynomials on non uniform lattices*, Appl. Math. Comput. 334 (2018), pp. 356-366.

[4] A. Branquinho and M.N. Rebocho, *Characterization theorem for Laguerre-Hahn orthogonal polynomials on non-uniform lattices*, J. Math. Anal. Appl. 427 (2015), pp. 185–201.

[5] D. Dominici and F. Marcellán, *Discrete semiclassical orthogonal polynomials of class one*, Pacific J. Math. 268 (2014), pp. 389–411.

[6] S. Elaydi, *An introduction to difference equations.* Undergraduate texts in Mathematics. Springer, New York, third ed. (2005).

[7] G. Filipuk and M.N. Rebocho, *Orthogonal polynomials on systems of non-uniform lattices from compatibility conditions*, J. Math. Anal. Appl. 456 (2017), pp. 1380-1396.

[8] M. Foupouagnigni, *On difference equations for orthogonal polynomials on nonuniform lattices*, J. Difference Equ. Appl. 14 (2008), pp. 127–174.

[9] M. Foupouagnigni, M. Kenfack Nangho, and S. Mboutngam, *Characterization theorem for classical orthogonal polynomials on non-uniform lattices: the functional approach*, Integral Transforms Spec. Funct. 22 (2011), pp. 739–758.

[10] S. Mboutngama, M. Foupouagnigni, and P. Njionou Sadjang, *On the modifications of semi-classical orthogonal polynomials on nonuniform lattices*, J. Math. Anal. Appl. 445 (2017), pp. 819-836.

[11] M. Njinkeu Sandjon, A. Branquinho, M. Foupouagnigni, and I. Area, *Characterizations of classical orthogonal polynomials on quadratic lattices*, J. Difference Equ. Appl. 23 (2017), pp. 983-1002

[12] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable, vol. 98 of Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2005.

[13] R. Koekoek and R. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Faculty of Information Technology and Systems*, Delft University of Technology, Netherlands, Report no. 98–17, 1998.

[14] A.P. Magnus, *Painlevé-type differential equations for the recurrence coefficients of semiclassical orthogonal polynomials*, J. Comput. Appl. Math. 57 (1995), pp. 215–237.

[15] A.P. Magnus, *Associated Askey-Wilson polynomials as Laguerre-Hahn orthogonal polynomials*, Springer Lect. Notes in Math. 1329, Springer, Berlin, 1988, pp. 261–278.

[16] A.P. Magnus, *Special nonuniform lattice (snul) orthogonal polynomials on discrete dense sets of points*, J. Comput. Appl. Math. 65 (1995), pp. 253–265.

[17] P. Maroni and M. Mejri, *The symmetric D_ω-semiclassical orthogonal polynomials of class one*, Numer. Algorithms, 49 (2008), pp. 251–282.

[18] A.F. Nikiforov, S.K. Suslov, *Classical Orthogonal Polynomials of a discrete variable on non uniform lattices*, Letters Math. Phys. 11 (1986), pp. 27–34.

[19] A.F. Nikiforov, S.K. Suslov, and V.B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*. Springer, Berlin, 1991.
[20] S.K. Suslov, *On the theory of difference analogues of special functions of hypergeometric type*, Usp. Mat. Nauk 4 (1989), pp. 185–226.
[21] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc. Providence, RI, 1975 (Fourth Edition).
[22] N.S. Witte, *Semi-classical orthogonal polynomial systems on nonuniform lattices, deformations of the Askey table, and analogues of isomonodromy*, Nagoya Math. J. 219 (2015), pp. 127–234.
[23] A. Zhedanov, *Rational spectral transformations and orthogonal polynomials*, J. Comput. Appl. Math. 85 (1997), pp. 67–86.

G. Filipuk
Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, Warsaw, 02-097, Poland.

E-mail address: filipuk@mimuw.edu.pl

M.N. Rebocho
Departamento de Matemática, Universidade da Beira interior, 6201-001 Covilhã, Portugal; CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal.

E-mail address: mneves@ubi.pt