Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–46

CARTESIAN CLOSED EXACT COMPLETIONS IN TOPOLOGY

MARIA MANUEL CLEMENTINO, DIRK HOFMANN AND WILLIAN RIBEIRO

Abstract: Using generalized enriched categories, in this paper we show that Rosický's proof of cartesian closedness of the exact completion of the category of topological spaces can be extended to a wide range of topological categories over **Set**, like metric spaces, approach spaces, ultrametric spaces, probabilistic metric spaces, and bitopological spaces. In order to do so we prove a sufficient criterion for exponentiability of $(\mathbb{T},V)$-categories and show that, under suitable conditions, every $(\mathbb{T},V)$-injective category is exponentiable in $(\mathbb{T},V)$-**Cat**.

Keywords: quantale, enriched category, (probabilistic) metric space, exponentiation, (weakly) cartesian closed category, exact completion.
Math. Subject Classification (2010): 18B30, 18B35, 18D15, 18D20, 54B30, 54E35, 54E70.

1. Introduction

As Lawvere has shown in his celebrated paper [Law73], when V is a closed category the category V-**Cat** of V-enriched categories and V-functors is also monoidal closed. This result extends neither to the cartesian structure nor to the more general setting of $(\mathbb{T},V)$-categories. Indeed, cartesian closedness of V does not guarantee cartesian closedness of V-**Cat**: take for instance the category of (Lawvere's) metric spaces P_+-**Cat**, where P_+ is the complete half-real line, ordered with the $\geq$ relation, and equipped with the monoidal structure given by addition $+$; P_+ is cartesian closed but P_+-**Cat** is not (see [CH06] for details); and, even when the monoidal structure of V is the cartesian one, the category $(\mathbb{T},V)$-**Cat** of $(\mathbb{T},V)$-categories and $(\mathbb{T},V)$-functors (see [CT03]) does not need to be cartesian closed, as it is the case of the

Received November 09, 2018.
Research partially supported by Centro de Matemática da Universidade de Coimbra – UID/MAT/00324/2013, by Centro de Investigação e Desenvolvimento em Matemática e Aplicações da Universidade de Aveiro/FCT – UID/MAT/04106/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020. W. Ribeiro also acknowledges the FCT PhD Grant PD/BD/128059/2016.

category **Top** of topological spaces and continuous maps, that is $(\mathbb{U}, 2)$-**Cat** for $\mathbb{U}$ the ultrafilter monad.

Rosický showed in [Ros99] that **Top** is weakly cartesian closed, and, consequently, that its exact completion is cartesian closed. Weak cartesian closedness of **Top** follows from the existence of enough injectives in its full subcategory $\mathbf{Top}_0$ of $T0$-spaces and the fact that they are exponentiable, and this feature, together with several good properties of **Top**, gives cartesian closedness of its exact completion. More precisely, Rosický has shown in [Ros99] the following theorem.

Theorem 1.1. *Let* $\mathbf{C}$ *be a complete, infinitely extensive and well-powered category with (reg epi, mono)-factorizations such that* $f \times 1$ *is an epimorphism whenever* f *is a regular epimorphism. Then the exact completion of* $\mathbf{C}$ *is cartesian closed provided that* $\mathbf{C}$ *is weakly cartesian closed.*

In this paper we use the setting of $(\mathbb{T}, V)$-categories, for a quantale V and a **Set**-monad $\mathbb{T}$ laxly extended to V-**Rel** to conclude, in a unified way, that several topological categories over **Set** share with **Top** this interesting property, which was recently used by Adámek and Rosický in the study of free completions of categories [AR18]. In fact, the category $(\mathbb{T}, V)$-**Cat** is topological over **Set** [CH03, CT03], hence complete and with (reg epi, mono)-factorizations such that $f \times 1$ is an epimorphism whenever f is, and it is infinitely extensive [MST06]. To assure weak cartesian closedness of $(\mathbb{T}, V)$-**Cat** we consider two distinct scenarios, either restricting to the case when V is a frame – so that its monoidal structure is the cartesian one – or considering the case when the lax extension is determined by a $\mathbb{T}$-algebraic structure on V, as introduced in [Hof07] under the name of topological theory. In the latter case the proof generalizes Rosický's proof for $\mathbf{Top}_0$, after observing that, using the Yoneda embedding of [CH09, Hof11], every separated $(\mathbb{T}, V)$-category can be embedded in an injective one, and, moreover, these are exponentiable in $(\mathbb{T}, V)$-**Cat**. For general $(\mathbb{T}, V)$-categories one proceeds again as in [Ros99], using the fact that the reflection of $(\mathbb{T}, V)$-**Cat** into its full subcategory of separated $(\mathbb{T}, V)$-categories preserves finite products. As observed by Rosický, the exact completion of **Top** relates to the cartesian closed category of equilogical spaces [BBS04]. Analogously, our approach leads to the study of generalized equilogical spaces, as developed in [Rib18].

The paper is organized as follows. In Section 2 we introduce $(\mathbb{T}, V)$-categories and list their properties used throughout the paper. In Section 3 we revisit the exponentiability problem in $(\mathbb{T}, V)$-**Cat**, establishing a sufficient criterion for exponentiability which generalizes the results obtained in [Hof07, HS15]. In Section 4 we study the properties of injective $(\mathbb{T}, V)$-categories which will be used in the forthcoming section to conclude that, under suitable assumptions, injective $(\mathbb{T}, V)$-categories are exponentiable (Theorem 5.5). This result will allow us to conclude, in Theorem 5.8, that $(\mathbb{T}, V)$-**Cat** is weakly cartesian closed, and, finally, thanks to Theorem 1.1, that the exact completion of $(\mathbb{T}, V)$-**Cat** is cartesian closed. We conclude our paper with a section on examples, which include, among others, metric spaces, approach spaces, probabilistic metric spaces, and bitopological spaces.

2. The category of $(\mathbb{T}, V)$-categories

Throughout V *is a commutative and unital quantale*, i.e V is a complete lattice with a symmetric and associative tensor product $\otimes$, with unit k and right adjoint hom, so that $u \otimes v \leq w$ if, and only if, $v \leq \hom(u, w)$, for all $u, v, w \in V$. Further assume that V *is a Heyting algebra*, so that $u \wedge -$ also has a right adjoint, for every $u \in V$. We denote by V-**Rel** the 2-category of V*-relations* (or V-matrices), having as objects sets, as 1-cells V-relations $r : X \nrightarrow Y$, i.e. maps $r : X \times Y \to V$, and 2-cells $\varphi : r \to r'$ given by componentwise order $r(x, y) \leq r'(x, y)$. Composition of 1-cells is given by relational composition. V-**Rel** has an involution, given by transposition: the transpose of $r : X \nrightarrow Y$ is $r^\circ : Y \nrightarrow X$ with $r^\circ(y, x) = r(x, y)$.

We fix a *non-trivial monad* $\mathbb{T} = (T, m, e)$ *on* **Set** *satisfying (BC)*, i.e. T preserves weak pullbacks and the naturality squares of the natural transformation m are weak pullbacks (see [CHJ14]). In general we do not assume that T preserves products. Later we will make use of the comparison map $\mathrm{can}_{X,Y} : T(X \times Y) \to TX \times TY$ defined by $\mathrm{can}_{X,Y}(\mathfrak{w}) = (T\pi_X(\mathfrak{w}), T\pi_Y(\mathfrak{w}))$ for all $\mathfrak{w} \in T(X \times Y)$, where π_X and π_Y are the product projections. Moreover, we assume that $\mathbb{T}$ *has an extension to* V-**Rel**, which we also denote by $\mathbb{T}$, in the following sense:

– there is a functor $T : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ which extends $T : \mathbf{Set} \to \mathbf{Set}$;
– $T(r^\circ) = (Tr)^\circ$ for all V-relations r;

– the natural transformations $e : 1_{V\text{-}\mathbf{Rel}} \to T$ and $m : T^2 \to T$ become op-lax; that is, for every $r : X \to\!\!\!\!\!\!\mapsto Y$,

$$e_Y \cdot r \le Tr \cdot e_X, \qquad\qquad m_Y \cdot TTr \le Tr \cdot m_X.$$

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "r"'] \arrow[dr, phantom, "\le"] & TX \arrow[d, "Tr"] \\ Y \arrow[r, "e_Y"'] & TY \end{tikzcd} \qquad\qquad \begin{tikzcd} TTX \arrow[r, "m_X"] \arrow[d, "TTr"'] \arrow[dr, phantom, "\le"] & TX \arrow[d, "Tr"] \\ TTY \arrow[r, "m_Y"'] & TY \end{tikzcd}$$

We note that our conditions are stronger than the ones used in [HST14].

A $(\mathbb{T}, V)$-*category* is a pair (X, a) where X is a set and $a : TX \to\!\!\!\!\!\!\mapsto X$ is a V-relation such that

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "a"] \arrow[dl, phantom, "\le"] \\ & X \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T^2X \arrow[r, "m_X"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\le"] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd}$$

that is, the map $a : TX \times X \to V$ satisfies the conditions:

(R) for each $x \in X$, $k \le a(e_X(x), x)$;

(T) for each $\mathfrak{X} \in T^2X, \mathfrak{x} \in TX, x \in X$, $Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a(m_X(\mathfrak{X}), x)$.

Given $(\mathbb{T}, V)$-categories (X, a), (Y, b), a $(\mathbb{T}, V)$-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\le"] & TY \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

that is, for each $\mathfrak{x} \in TX$ and $x \in X$, $a(\mathfrak{x}, x) \le b(Tf(\mathfrak{x}), f(x))$; f is said to be *fully faithful* when this inequality is an equality.

$(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors form the category $(\mathbb{T}, V)$-**Cat**. If $(X, a : TX \to\!\!\!\!\!\!\mapsto X)$ satisfies (R) (and not necessarily (T)), we call it a $(\mathbb{T}, V)$-*graph*. The category $(\mathbb{T}, V)$-**Gph**, of $(\mathbb{T}, V)$-graphs and $(\mathbb{T}, V)$-functors, contains $(\mathbb{T}, V)$-**Cat** as a full reflective subcategory.

We chose to present the examples in detail in the last section. We mention here, however, that guiding examples are obtained when one considers the quantale $2 = (\{0, 1\}, \le, \&, 1)$ and the Lawvere's half real line P_+, that is $([0, \infty], \ge, +, 0)$, the identity monad $\mathbb{I}$ and the ultrafilter monad $\mathbb{U}$ on **Set**, obtaining:

- $(\mathbb{I}, V)$-**Cat** is the category of V-categories and V-functors; in particular, $(\mathbb{I}, 2)$-**Cat** is the category **Ord** of (pre)ordered sets and monotone maps, while $(\mathbb{I}, P_+)$-**Cat** is the category **Met** of Lawvere's metric spaces and non-expansive maps (see [Law73]).
- $(\mathbb{U}, 2)$-**Cat** is the category **Top** of topological spaces and continuous maps.
- $(\mathbb{U}, P_+)$-**Cat** is the category **App** of Lowen's approach spaces and non-expansive maps (see [Low97]).

As shown in [CH03] (see also [CT03]).

Theorem 2.1. *The forgetful functors* $(\mathbb{T}, V)$-**Cat** $\to$ **Set** *and* $(\mathbb{T}, V)$-**Gph** $\to$ **Set** *are topological.*

This shows, in particular, that:

- $(\mathbb{T}, V)$-**Cat** is complete and cocomplete.
- Monomorphisms in $(\mathbb{T}, V)$-**Cat** are the morphisms whose underlying map is injective; therefore, since the $(\mathbb{T}, V)$-structures on any set form a set, $(\mathbb{T}, V)$-**Cat** is well-powered.
- Every topological category over **Set** has two factorization systems, (reg epi, mono) and (epi, reg mono); in $(\mathbb{T}, V)$-**Cat** the former one is in general not stable (that is, regular epimorphisms need not be stable under pullback – **Top** is such an example), but the latter one is. Indeed, epimorphisms in $(\mathbb{T}, V)$-**Cat** are the $(\mathbb{T}, V)$-functors which are surjective as maps, the forgetful functor $(\mathbb{T}, V)$-**Cat** $\to$ **Set** preserves pullbacks, and surjective maps are stable under pullback in **Set**. Therefore, as $f \times 1_Z$ is the pullback of $f : X \to Y$ along $p_Y : Y \times Z \to Y$, we conclude that $f \times 1_Z$ is an epimorphism provided f is.

$(\mathbb{T}, V)$-**Cat** has a natural structure of 2-category: for $(\mathbb{T}, V)$-functors $f, g : (X, a) \to (Y, b)$, $f \leq g$ if $g \cdot a \leq b \cdot Tf$. This condition can be equivalently written as $k \leq b(e_Y(f(x)), g(x))$ for every $x \in X$ (see [CT03] for details). We write $f \simeq g$ if $f \leq g$ and $g \leq f$.

Extensivity of $(\mathbb{T}, V)$-**Cat** was studied in [MST06]:

Theorem 2.2. $(\mathbb{T}, V)$-**Cat** *is infinitely extensive.*

In general $(\mathbb{T}, V)$-**Cat** is not cartesian closed, while $(\mathbb{T}, V)$-**Gph** is. In fact, it was proved in [CHT03]:

Theorem 2.3. $(\mathbb{T},V)$-**Gph** *is a quasi-topos.*

Weak cartesian closedness of $(\mathbb{T},V)$-**Cat** needs a thorough study of injective $(\mathbb{T},V)$-categories and some extra conditions. Namely we will use the extension of the **Set**-functor T to V-**Rel** given by a topological theory in the sense of [Hof07]. This is the subject of the following sections.

3. Exponentiable $(\mathbb{T},V)$-categories

Recall that an object C of a category $\mathbf{C}$ with finite products is *exponentiable* whenever the functor $C\times - : \mathbf{C}\to\mathbf{C}$ has a right adjoint. In this section we present a sufficient condition for a $(\mathbb{T},V)$-category X to be exponentiable in $(\mathbb{T},V)$-**Cat**, which generalises [Hof06, Theorem 4.3] and [Hof07, Theorem 6.5]. To start, we recall that $(\mathbb{T},V)$-**Cat** can be fully embedded into the cartesian closed category $(\mathbb{T},V)$-**Gph** of $(\mathbb{T},V)$-graphs and $(\mathbb{T},V)$-functors, see [CHT03] for details. Here, for $(\mathbb{T},V)$-graphs (X,a) and (Y,b), the exponential $\langle (X,a),(Y,b)\rangle$ has as underlying set

$$Z := \{h : (X,a)\times(1,k)\to(Y,b) \mid h \text{ is a } (\mathbb{T},V)\text{-functor}\},$$

which becomes a $(\mathbb{T},V)$-graph when equipped with the largest structure b^a making the evaluation map

$$\mathrm{ev} : Z\times X\to Y,\ (h,x)\mapsto h(x)$$

a $(\mathbb{T},V)$-functor: for $\mathfrak{p}\in TZ$ and $h\in Z$, put $b^a(\mathfrak{p},h)$ as

$$\bigvee\{v\in V \mid \forall \mathfrak{q}\in (T\pi_Z)^{-1}(\mathfrak{p}), x\in X\ a(T\pi_X(\mathfrak{q}),x)\wedge v\le b(T\mathrm{ev}(\mathfrak{q}),h(x))\},$$

where π_X and π_Z are the product projections. Note that the supremum above is even a maximum since $-\wedge-$ distributes over suprema.

Given V-relations $r : X \to\!\!\!\!\!\!\mapsto X'$ and $s : Y \to\!\!\!\!\!\!\mapsto Y'$, we define in V-**Rel** $r\owedge s : X\times Y \to\!\!\!\!\!\!\mapsto X'\times Y'$ by $(r\owedge s)((x,y),(x',y')) = r(x,x')\wedge s(y,y')$.

Theorem 3.1. *Assume that*

$$\begin{tikzcd} T(X\times Y) \arrow[r, "\mathrm{can}_{X,Y}"] \arrow[d, "T(r\owedge s)"', "\shortmid" marking] \arrow[dr, phantom, "\ge"] & TX\times TY \arrow[d, "(Tr)\owedge(Ts)", "\shortmid" marking] \\ T(X'\times Y') \arrow[r, "\mathrm{can}_{X',Y'}"'] & TX'\times TY', \end{tikzcd} \tag{3.i}$$

for all V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$. Then a $(\mathbb{T}, V)$-category (X, a) is exponentiable provided that

$$\bigvee_{\mathfrak{x}\in TX} (Ta(\mathfrak{X}, \mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x}, x) \wedge v) \geq a(m_X(\mathfrak{X}), x) \wedge (u \otimes v), \tag{3.ii}$$

for all $\mathfrak{X} \in TTX$, $x \in X$ and $u, v \in V$.

Proof: We show that the $(\mathbb{T}, V)$-graph structure b^a on Z is transitive, for each $(\mathbb{T}, V)$-category (Y, b). To this end, let $\mathfrak{P} \in TTZ$, $\mathfrak{p} \in TZ$, $h \in Z$, $x \in X$ and $\mathfrak{w} \in T(Z \times X)$ with $T\pi_Z(\mathfrak{w}) = m_Z(\mathfrak{P})$. We have to show that

$$(T(b^a)(\mathfrak{P}, \mathfrak{p}) \otimes b^a(\mathfrak{p}, h)) \wedge a(T\pi_X(\mathfrak{w}), x) \leq b(T\,\mathrm{ev}(\mathfrak{w}), h(x)).$$

Since m has (BC), there is some $\mathfrak{Q} \in TT(Z \times X)$ with $TT\pi_Z(\mathfrak{Q}) = \mathfrak{P}$ and $m_{Z\times X}(\mathfrak{Q}) = \mathfrak{w}$. Hence, $m_X(TT\pi_X(\mathfrak{Q})) = T\pi_X(\mathfrak{w})$, and we calculate:

$$\begin{aligned}
&(T(b^a)(\mathfrak{P}, \mathfrak{p}) \otimes b^a(\mathfrak{p}, h)) \wedge a(T\pi_X(\mathfrak{w}), x)\\
\leq\ & \bigvee_{\mathfrak{x}\in TX} ((T(b^a)(TT\pi_Z(\mathfrak{Q}), \mathfrak{p}) \wedge Ta(TT\pi_X(\mathfrak{Q}), \mathfrak{x})) \otimes (b^a(\mathfrak{p}, h) \wedge a(\mathfrak{x}, x)) && (3.ii)\\
\leq\ & \bigvee_{\mathfrak{x}\in TX} \bigvee_{\mathfrak{q}\in \mathrm{can}^{-1}(\mathfrak{p},\mathfrak{x})} T(b^a \owedge a)(T\,\mathrm{can}_{Z,X}(\mathfrak{Q}), \mathfrak{q}) \otimes (b^a \owedge a)(\mathrm{can}_{Z,X}(\mathfrak{q}), (h, x)) && (3.i)\\
=\ & \bigvee_{\mathfrak{q}\in (T\pi_Z)^{-1}(\mathfrak{p})} T(b^a \owedge a)(T\,\mathrm{can}_{Z,X}(\mathfrak{Q}), \mathfrak{q}) \otimes (b^a \owedge a)(\mathrm{can}_{Z,X}(\mathfrak{q}), (h, x))\\
=\ & \bigvee_{\mathfrak{q}\in (T\pi_Z)^{-1}(\mathfrak{p})} T(b^a \times a)(\mathfrak{Q}, \mathfrak{q}) \otimes (b^a \times a)(\mathfrak{q}, (h, x))\\
\leq\ & \bigvee_{\mathfrak{q}\in (T\pi_Z^{-1})(\mathfrak{p})} Tb(TT\,\mathrm{ev}(\mathfrak{Q}), T\,\mathrm{ev}(\mathfrak{q})) \otimes b(T\,\mathrm{ev}(\mathfrak{q}), h(x))\\
\leq\ & b(m_Y \cdot TT\,\mathrm{ev}(\mathfrak{Q}), h(x)) = b(T\,\mathrm{ev}(\mathfrak{w}), h(x)). \quad \blacksquare
\end{aligned}$$

It is worthwhile to notice that, for $\otimes = \wedge$, the condition above is equivalent to

$$\bigvee_{\mathfrak{x}\in TX} Ta(\mathfrak{X}, \mathfrak{x}) \wedge a(\mathfrak{x}, x) \geq a(m_X(\mathfrak{X}), x),$$

for all $\mathfrak{X} \in TTX$ and $x \in X$; which in turn is equivalent to

$$a \cdot m_X = a \cdot Ta.$$

4. Injective and representable $(\mathbb{T}, V)$-categories

In this section we recall an important class of $(\mathbb{T}, V)$-categories, the so-called *representable* ones. More information on this type of $(\mathbb{T}, V)$-categories can be found in [CCH15, HST14]. We also recall from [CH09, Hof07, Hof11] that every injective $(\mathbb{T}, V)$-category is representable and that every separated $(\mathbb{T}, V)$-category can be embedded into an injective one.

Based on the lax extension of the **Set**-monad $\mathbb{T} = (T, m, e)$ to V-**Rel**, $\mathbb{T}$ admits a natural extension to a monad on V-**Cat**, in the sequel also denoted as $\mathbb{T} = (T, m, e)$ (see [Tho09]). Here the functor $T : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ sends a V-category (X, a_0) to (TX, Ta_0), and with this definition $e_X : X \to TX$ and $m_X : TTX \to TX$ become V-functors for each V-category X. The Eilenberg–Moore algebras for this monad can be described as triples (X, a_0, α) where (X, a_0) is a V-category and (X, α) is an algebra for the **Set**-monad $\mathbb{T}$ such that $\alpha : T(X, a_0) \to (X, a_0)$ is a V-functor. For $\mathbb{T}$-algebras (X, a_0, α) and (Y, b_0, β), a map $f : X \to Y$ is a homomorphism $f : (X, a_0, \alpha) \to (Y, b_0, \beta)$ precisely if f preserves both structures, that is, whenever $f : (X, a_0) \to (Y, b_0)$ is a V-functor and $f : (X, \alpha) \to (Y, \beta)$ is a $\mathbb{T}$-homomorphism.

There are canonical adjoint functors

$$\begin{tikzcd} (V\text{-}\mathbf{Cat})^{\mathbb{T}} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] & (\mathbb{T}, V)\text{-}\mathbf{Cat}. \arrow[l, bend left, "M"] \end{tikzcd}$$

The functor K associates to each $X = (X, a_0, \alpha)$ in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ the $(\mathbb{T}, V)$-category $KX = (X, a)$, where $a = a_0 \cdot \alpha$, and keeps morphisms unchanged. Its left adjoint $M : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (V\text{-}\mathbf{Cat})^{\mathbb{T}}$ sends a $(\mathbb{T}, V)$-category (X, a) to $(TX, Ta \cdot m_X^\circ, m_X)$ and a $(\mathbb{T}, V)$-functor f to Tf. Via the adjunction $M \dashv K$ one obtains a lifting of the **Set**-monad $\mathbb{T} = (T, m, e)$ to a monad on $(\mathbb{T}, V)$-**Cat**, also denoted by $\mathbb{T} = (T, m, e)$.

In this setting we can define 'duals' in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ and carry them into $(\mathbb{T}, V)$-**Cat**. Indeed, since $T : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ commutes with the involution $(-)^\circ$, with $X = (X, a_0, \alpha)$ also (X, a_0°, α) is a $\mathbb{T}$-algebra. Moreover, if (X, a) is a $(\mathbb{T}, V)$-category, we define X^{op} by mapping (X, a) into $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ via M, dualizing the image in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$, and then carrying it back to $(\mathbb{T}, V)$-**Cat**; that is,

$$X^{\mathrm{op}} = K((M(X, a))^{\mathrm{op}}) = (TX, m_X \cdot (Ta)^\circ \cdot m_X).$$

Since the monad $\mathbb{T} = (T, m, e)$ on $(\mathbb{T}, V)$-**Cat** is lax idempotent (i.e, of Kock-Zöberlein type), an algebra structure $\alpha : TX \to X$ on a $(\mathbb{T}, V)$-category X is left adjoint to the unit $e_X : X \to TX$. We call a $(\mathbb{T}, V)$-category X *representable* whenever $e_X : X \to TX$ has a left adjoint in $(\mathbb{T}, V)$-**Cat**; equivalently, whenever there is some $(\mathbb{T}, V)$-functor $\alpha : TX \to X$ with $\alpha \cdot e_X \simeq 1_X$, since then

$$e_X \cdot \alpha = T\alpha \cdot e_{TX} \geq T\alpha \cdot Te_X \simeq 1_{TX}.$$

However, a left adjoint $\alpha : TX \to X$ to e_X is in general only a pseudo-algebra structure on X, that is,

$$\alpha \cdot e_X \simeq 1_X \qquad \text{and} \qquad \alpha \cdot T\alpha \simeq \alpha \cdot m_X.$$

A $(\mathbb{T}, V)$-category X is *injective* whenever, for each fully faithful $h : A \to B$ in $(\mathbb{T}, V)$-**Cat** and each $(\mathbb{T}, V)$-functor $f : A \to X$, there is a $(\mathbb{T}, V)$-functor $g : B \to X$ with $g \cdot h \simeq f$.

Proposition 4.1. *Every injective $(\mathbb{T}, V)$-category is representable.*

Proof: Let X be a $(\mathbb{T}, V)$-category. Since $e_X : X \to TX$ is an embedding (it is easily seen that $a = e_X^\circ \cdot Ta \cdot Te_X$), there is a $(\mathbb{T}, V)$-functor $\alpha : TX \to X$ with $\alpha \cdot e_X = 1_X$, therefore X is representable. ■

In order to obtain a Yoneda embedding, we need to restrict our study to extensions fulfilling our conditions of Section 2 and determined by a $\mathbb{T}$-algebra structure $\xi : TV \to V$ on $(V, \hom)$, so that we are in the setting of a *strict topological theory* in the sense of [Hof07]. The $\mathbb{T}$-algebra $(V, \hom, \xi)$ is mapped by K into the important $(\mathbb{T}, V)$-category $(V, \hom_\xi)$, where $\hom_\xi = \hom \cdot \xi$.

We also note that the tensor product of V induces a canonical structure c on $X \times Y$ defined by

$$c(\mathfrak{w}, (x, y)) = a(T\pi_X(\mathfrak{w}), x) \otimes b(T\pi_Y(\mathfrak{w}), y),$$

where $\mathfrak{w} \in T(X \times Y)$, $x \in X$, $y \in Y$. We put

$$(X, a) \otimes (Y, b) = (X \times Y, c),$$

and this construction is in an obvious way part of a functor

$$\otimes : (\mathbb{T}, V)\text{-}\mathbf{Cat} \times (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Cat}.$$

The proof of the following result can be found in [CH09] and [Hof11].

Theorem 4.2. *For every $(\mathbb{T}, V)$-category (X, a), the V-relation $a : TX \nrightarrow X$ defines a $(\mathbb{T}, V)$-functor of type*

$$a : X^{\mathrm{op}} \otimes X \to (V, \hom_\xi).$$

Moreover, the $\otimes$-exponential mate $y_X = \ulcorner a \urcorner : X \to V^{X^{\mathrm{op}}}$ of a is fully faithful, and the $(\mathbb{T}, V)$-category $PX = V^{X^{\mathrm{op}}}$ is injective. In fact, this construction defines a functor $P : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Cat}$ and $y = (y_X)_X$ is a natural transformation $y : 1_{(\mathbb{T},V)\text{-}\mathbf{Cat}} \to P$.

Since y_X is fully faithful, when X is injective there exists a $(\mathbb{T}, V)$-functor $\mathrm{Sup}_X : PX \to X$ such that $\mathrm{Sup}_X \cdot y_X \simeq 1_X$. Moreover, as shown in [Hof11, Theorem 2.7], $\mathrm{Sup}_X \dashv y_X$.

For each $(\mathbb{T}, V)$-category (X, a), y_X is one-to-one if, and only if, (X, a) is *separated*, i.e. for every $f, g : (Y, b) \to (X, a)$, $f \simeq g$ only if $f = g$ (see [HT10], for example).

Corollary 4.3. *Every separated $(\mathbb{T}, V)$-category embeds into an injective $(\mathbb{T}, V)$-category.*

5. $(\mathbb{T}, V)$-Cat is weakly cartesian closed

I order to achieve the result promised in the title of this section, we shall show that, under certain conditions, every injective $(\mathbb{T}, V)$-category is exponentiable. This problem is considerable easier for V being a frame, that is, assuming that $\otimes = \wedge$, as shown in [Hof14, Proposition 2.7].

Proposition 5.1. *If the quantale V is a frame, i.e. if $\otimes = \wedge$ in V, then every representable $(\mathbb{T}, V)$-category is exponentiable. In particular, in this case every injective $(\mathbb{T}, V)$-category is exponentiable.*

To treat the general case, in this section we consider that both maps

$$V \otimes V \xrightarrow{\ \otimes\ } V \qquad X \xrightarrow{(-,u)} X \otimes V \tag{5.iii}$$

are $(\mathbb{T}, V)$-functors, for all $u \in V$. These morphisms induce an interesting action of V on every injective $(\mathbb{T}, V)$-category (X, a) as follows. The $(\mathbb{T}, V)$-functor

$$TX^{\mathrm{op}} \otimes X \otimes V \xrightarrow{a \otimes 1} V \otimes V \xrightarrow{\ \otimes\ } V$$

induces a $(\mathbb{T}, V)$-functor $\tilde{a} : X \otimes V \to PX$. We denote the composite

$$X \otimes V \xrightarrow{\ \tilde{a}\ } PX \xrightarrow{\mathrm{Sup}_X} X$$

by $\oplus$, and

$$X \xrightarrow{(-,u)} X \otimes V \xrightarrow{\tilde{a}} PX \xrightarrow{\mathrm{Sup}_X} X,$$

assigning to each $x \in X$ an element $x \oplus u$ in X, by $- \oplus u$.

Analogously we will write $\mathfrak{x} \oplus u$ for $T(- \oplus u)(\mathfrak{x})$, for every $\mathfrak{x} \in TX$ and $u \in V$. Note that $(\mathbb{T}, V)$-functoriality of $- \oplus u$ can be written as

$$a(\mathfrak{x}, y) \leq a(\mathfrak{x} \oplus u, y \oplus u),$$

for every $\mathfrak{x} \in TX$ and $y \in X$. Moreover, for every $u \in V$ and V-relation $r : X \nrightarrow Y$, we define the V-relation $r \otimes u : X \nrightarrow Y$ by $(r \otimes u)(x, y) = r(x, y) \otimes u$. We will make use of the following extra condition.

$$T(a \otimes u) = Ta \otimes u \tag{5.iv}$$

for any V-relation a and $u \in V$.

Lemma 5.2. *For an injective $(\mathbb{T}, V)$-category (X, a), with $a = a_0 \cdot \alpha$ as usual, the following holds, for every $x, y \in X$, $\mathfrak{x} \in TX$ and $u \in V$:*

(1) $a_0(x \oplus u, y) = \hom(u, a_0(x, y))$;
(2) $a_0(x, y \oplus u) \geq a_0(x, y) \otimes u$;
(3) $a(\mathfrak{x} \oplus u, y) \geq \hom(u, a(\mathfrak{x}, y))$;
(4) $a(\mathfrak{x}, y \oplus u) \geq a(\mathfrak{x}, y) \otimes u$.

Moreover, if (5.iv) *holds, then, for every $\mathfrak{X} \in T^2X$, $\mathfrak{y} \in TX$, $u \in V$,*

(5) $Ta(\mathfrak{X}, \mathfrak{y} \oplus u) \geq Ta(\mathfrak{X}, \mathfrak{y}) \otimes u$.

Proof: (1) For every $x, y \in X$ and $u \in V$,

$$\begin{aligned}
a_0(x \oplus u, y) &= a_0(\mathrm{Sup}_X(\tilde{a}(x, u)), y) && \text{(by definition of } \oplus) \\
&= [\tilde{a}(x, u), y^*] && \text{(because } \mathrm{Sup}_X \dashv y_X) \\
&= \bigwedge_{\mathfrak{x} \in TX} \hom(\tilde{a}(x, u)(\mathfrak{x}), y^*(\mathfrak{x})) && \text{(by definition of [,])} \\
&= \bigwedge_{\mathfrak{x} \in TX} \hom(a(\mathfrak{x}, x) \otimes u, a(\mathfrak{x}, y)) && \text{(by definition of } \tilde{a} \text{ and } y^*) \\
&= \hom(u, a_0(x, y)),
\end{aligned}$$

because, using the fact that $a = a_0 \cdot \alpha$ and

$$a_0(\alpha(\mathfrak{x}), x) \otimes u \otimes \hom(u, a_0(x, y)) \leq a_0(\alpha(\mathfrak{x}), x) \otimes a_0(x, y) \leq a_0(\alpha(\mathfrak{x}), y),$$

for $\mathfrak{x} \in TX$, we can conclude that

$$\hom(u, a_0(x,y)) \leq \bigwedge_{\mathfrak{x}\in TX} \hom(a_0(\alpha(\mathfrak{x}), x) \otimes u, a_0(\alpha(\mathfrak{x}), y)).$$

Taking $\mathfrak{x} = e_X(x)$, we see that this inequality is in fact an equality as claimed.

(2) Since, by hypothesis, $- \oplus u$ is a $(\mathbb{T}, V)$-functor, and so, in particular, a V-functor $(X, a_0) \to (X, a_0)$,

$$a_0(x,y) \leq a_0(x \oplus u, y \oplus u) = \hom(u, a_0(x, y \oplus u)),$$

and then

$$a_0(x,y) \otimes u \leq \hom(u, a_0(x, y\oplus u)) \otimes u \leq a_0(x, y \oplus u).$$

(3) One has

$$\begin{aligned} k &\leq a_0(\alpha(\mathfrak{x}), \alpha(\mathfrak{x})) = a(\mathfrak{x}, \alpha(\mathfrak{x})) \\ &\leq a(\mathfrak{x} \oplus u, \alpha(\mathfrak{x}) \oplus u) \\ &= a_0(\alpha(\mathfrak{x} \oplus u), \alpha(\mathfrak{x}) \oplus u). \end{aligned}$$

Using (1) we conclude that

$$\begin{aligned} \hom(u, a(\mathfrak{x}, y)) &= a_0(\alpha(\mathfrak{x}) \oplus u, y) \\ &\leq a_0(\alpha(\mathfrak{x} \oplus u), \alpha(\mathfrak{x}) \oplus u) \otimes a_0(\alpha(\mathfrak{x}) \oplus u, y) \\ &\leq a_0(\alpha(\mathfrak{x} \oplus u), y) = a(\mathfrak{x} \oplus u, y). \end{aligned}$$

(4) follows directly from (2), while (5) follows from (4). ■

It was shown in [HR13, Theorem 5.3] that injective V-categories are exponentiable if, and only if, for all $u, v, w \in V$,

$$w \wedge (u \otimes v) = \bigvee \{u' \otimes v' \mid u' \leq u, v' \leq v, u' \otimes v' \leq w\}. \tag{5.v}$$

We have the following obvious fact.

Lemma 5.3. *Let $\varphi : V \to W$ be a surjective quantale homomorphism; that is, φ preserves the tensor, the neutral element, and suprema. Then, if V satisfies condition* (5.v)*, so does W.*

Here we want to study conditions under which every injective $(\mathbb{T}, V)$-category is exponentiable. Therefore this condition is necessary for our result. To summarise, in this section we will typically work under the following

Assumption 5.4. The maps $\otimes : V \otimes V \to V$ and $(-, u) : X \to X \times V$ are $(\mathbb{T}, V)$-functors, $T(a \otimes u) = Ta \otimes u$ for every injective $(\mathbb{T}, V)$-category (X, a) and every $u \in V$, and (5.v) holds.

Theorem 5.5. *Under Assumption 5.4, every injective* $(\mathbb{T},V)$*-category is exponentiable in* $(\mathbb{T},V)$-**Cat**.

Proof: In order to conclude that, for $\mathfrak{X}\in T^2X$, $x\in X$, $u,v\in V$,

$$\bigvee_{\mathfrak{x}\in TX}(Ta(\mathfrak{X},\mathfrak{x})\wedge u)\otimes(a(\mathfrak{x},x)\wedge v)\geq a(m_X(\mathfrak{X}),x)\wedge(u\otimes v),$$

we will show that, with $\mathfrak{y}=T\alpha(\mathfrak{X})\oplus u$,

$$(Ta(\mathfrak{X},\mathfrak{y})\wedge u)\otimes(a(\mathfrak{y},x)\wedge v)\geq a(m_X(\mathfrak{X}),x)\wedge(u\otimes v). \tag{5.vi}$$

First we note that

$$\begin{array}{rcll} Ta(\mathfrak{X},T\alpha(\mathfrak{X})\oplus u)\wedge u & \geq & (Ta(\mathfrak{X},T\alpha(\mathfrak{X}))\otimes u)\wedge u & \text{(by 5.2 (5))}\\ & = & (Ta_0(T\alpha(\mathfrak{X}),T\alpha(\mathfrak{X}))\otimes u)\wedge u & \\ & \geq & (k\otimes u)\wedge u=u. & \end{array}$$

and

$$\begin{array}{rcl} a(T\alpha(\mathfrak{X})\oplus u,x)\wedge v & \geq & \hom(u,a(T\alpha(\mathfrak{X}),x))\wedge v\\ & = & \hom(u,a_0(\alpha(T\alpha(\mathfrak{X})),x))\wedge v\\ & = & \hom(u,a_0(\alpha(m_X(\mathfrak{X})),x))\wedge v\\ & = & \hom(u,a(m_X(\mathfrak{X}),x))\wedge v. \end{array}$$

Hence

$$(Ta(\mathfrak{X},\mathfrak{y})\wedge u)\otimes(a(\mathfrak{y},x)\wedge v)\geq u\otimes(\hom(u,a(m_X(\mathfrak{X}),x)\wedge v).$$

Now, for $v'\in V$ with $v'\leq v$ and $u\otimes v'=v'\otimes u\leq a(m_X(\mathfrak{X}),x)$, we get $v'\leq\hom(u,a(m_X(\mathfrak{X}),x))$, hence

$$u\otimes v'\leq u\otimes(\hom(u,a(m_X(\mathfrak{X}),x))).$$

Using our Hypothesis (5.v) we conclude that

$$u\otimes(\hom(u,a(m_X(\mathfrak{X}),x)))\geq a(m_X(\mathfrak{X}),x)\wedge(u\otimes v),$$

and so (5.vi) follows. ∎

Theorem 5.6. *If every injective* $(\mathbb{T},V)$*-category is exponentiable, then* $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{sep}}$ *is weakly cartesian closed.*

Proof: For X,Y separated $(\mathbb{T},V)$-categories, consider the Yoneda embeddings $y_X:X\to PX$ and $y_Y:Y\to PY$, and the exponential $\langle PX,PY\rangle$. The elements of its underlying set can be identified with $(\mathbb{T},V)$-functors $E\times PX\to PY$ (where E is the generator of $(\mathbb{T},V)$-**Cat** mentioned before), and the universal morphism $\mathrm{ev}:\langle PX,PY\rangle\times PX\to PY$ with the evaluation map: $\mathrm{ev}(\varphi,\mathfrak{x})=\varphi(\mathfrak{x})$ (where, for simplicity, we identify the set $E\times PX$ with PX). We can therefore define

$$\ll X,Y\gg=\{\varphi:E\times PX\to PY\mid \varphi(y_X(X))\subseteq y_Y(Y)\},$$

with the initial structure with respect to the inclusion ι of $\ll X,Y\gg$ in $\langle PX,PY\rangle$. Moreover, the morphism

$$\ll X,Y\gg\times X\xrightarrow{\iota\times y_X}\langle PX,PY\rangle\times PX\xrightarrow{\mathrm{ev}}PY$$

factors through y_Y via a morphism

$$\ll X,Y\gg\times X\xrightarrow{\widetilde{\mathrm{ev}}}Y.$$

Next we show that this is a weak exponential in $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{sep}}$.

Given any separated $(\mathbb{T},V)$-category Z, and a $(\mathbb{T},V)$-functor $f:Z\times X\to Y$, by injectivity of PY there exists a $(\mathbb{T},V)$-functor $f':Z\times PX\to PY$ making the square below commute. Then, by universality of the evaluation map ev, there exists a unique $(\mathbb{T},V)$-functor $\overline{f}:Z\to\langle PX,PY\rangle$ making the bottom triangle commute.

$$\begin{tikzcd}
Z\times X \arrow[r, "f"] \arrow[d, "1_Z\times y_X"'] & Y \arrow[d, "y_Y"] \\
Z\times PX \arrow[r, "f'"] \arrow[d, "\overline{f}\times 1_{PX}"'] & PY \\
\langle PX,PY\rangle\times PX \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

The map $\overline{f}:Z\to\langle PX,PY\rangle$, assigning to each $z\in Z$ a map $\overline{f}(z):PX\to PY$, is such that $\mathrm{ev}(\overline{f}(z),y_X(x))=\overline{f}(z)(y_X(x))=y_Y(f(z,x))$; that is, $\overline{f}(z)(y_X(X))\subseteq y_Y(Y)$, and this means that $\overline{f}(z)\in\ll X,Y\gg$. Hence we can consider the corestriction $\tilde{f}$ of $\overline{f}$ to $\ll X,Y\gg$, which is again a $(\mathbb{T},V)$-functor since $\ll X,Y\gg$ has the initial structure with respect to $\langle PX,PY\rangle$, so that the following diagram commutes.

$$\begin{tikzcd}
\ll X,Y\gg\times X \arrow[r, "\widetilde{\mathrm{ev}}"] & Y \\
Z\times X \arrow[u, "\tilde{f}\times 1_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

■

In order to show that $(\mathbb{T},V)$-**Cat** is weakly cartesian closed, we follow the proof of [Ros99]. Hence, first we show that:

Proposition 5.7. *The reflector* $R:(\mathbb{T},V)\text{-}\mathbf{Cat}\to(\mathbb{T},V)\text{-}\mathbf{Cat}_{\mathrm{sep}}$ *preserves finite products.*

Proof: We recall that, for any $(\mathbb{T},V)$-category (X,a), $R(X,a)=(\tilde{X},\tilde{a})$, with $\tilde{X}=X/\sim$, where $x\sim y$ if $k\le a(e_X(x),y)\wedge a(e_X(y),x)$, and $\tilde{a}=\eta_X\cdot a\cdot\eta_X^\circ$, with $\eta_X:X\to\tilde{X}$ the projection. This structure makes η_X both an initial and a final morphism (see [HST14] for details).

Let $f:R(X\times Y)\to RX\times RY$ be the unique morphism such that $f\cdot\eta_{X\times Y}=\eta_X\times\eta_Y$.

$$\begin{tikzcd}
(X\times Y,c) \arrow[r, "\eta_{X\times Y}"] \arrow[dr, "\eta_X\times\eta_Y"'] & (R(X\times Y),\tilde{c}) \arrow[d, "f"] \\
 & (RX\times RY,d)
\end{tikzcd}$$

From $c(e_{X\times Y}(x,y),(x',y'))=a(e_X(x),x')\wedge b(e_Y(y),y')$ it is immediate that $(x,y)\sim(x',y')$ in $X\times Y$ if, and only if, $x\sim x'$ in X and $y\sim y'$ in Y. Therefore, f is a bijection. Assuming the Axiom of Choice, so that T preserves surjections, we have, for every $\mathfrak{z}\in T(R(X\times Y))$, $(x,y)\in X\times Y$,

$$\begin{array}{rcll}
\tilde{c}(\mathfrak{z},[(x,y)]) & = & c(\mathfrak{w},(x,y)) & (\text{for any } \mathfrak{w}\in(T\eta_{X\times Y})^{-1}(\mathfrak{z}))\\
 & = & d(T(\eta_X\times\eta_Y)(\mathfrak{w}),([x],[y])) & (\text{because } \eta_X\times\eta_Y \text{ is initial})\\
 & = & d(Tf(\mathfrak{z}),([x],[y]); &
\end{array}$$

that is, f is initial and therefore an isomorphism. ■

Theorem 5.8. *If every injective $(\mathbb{T},V)$-category is exponentiable, then $(\mathbb{T},V)$-**Cat** is weakly cartesian closed. In particular,*

*(1) if the quantale V is a frame (that is, $\otimes=\wedge$), then $(\mathbb{T},V)$-**Cat** is weakly cartesian closed;*

*(2) under Assumption 5.4, $(\mathbb{T},V)$-**Cat** is weakly cartesian closed.*

Proof: Given $(\mathbb{T},V)$-categories (X,a), (Y,b), to build the weak exponential $\ll X,Y\gg$ we will show the *cosolution set condition* for the functor $-\times(X,a)$.

For each $(\mathbb{T},V)$-functor $f:(Z,c)\times(X,a)\to(Y,b)$ we consider its reflection $Rf:RZ\times RX\cong R(Z\times X)\to RY$ and we factorise it through the weak evaluation in $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{sep}}$, $Rf=\widetilde{\mathrm{ev}}\cdot(\overline{Rf}\times 1_{RX})$, so that in the diagram below the outer rectangle commutes.

Then we define $Z_f = Z/\sim$ by

$$z \sim z' \text{ if } f(z,x) = f(z',x), \text{ for every } x \in X, \text{ and } \overline{R}f(\eta_Z(z)) = \overline{R}f(\eta_Z(z')),$$

and equip it with the final structure for the projection $q_f : Z \to Z_f$. Then $h_f : Z_f \to \ll RX, RY \gg$, with $h_f([z]) = \overline{R}f(\eta_Z(z))$, is a $(\mathbb{T}, V)$-functor since its composition with q_f is $\overline{R}f \cdot \eta_Z$ and q_f is final. Then we factorise f via the surjection $q_f \times 1_X : Z \times X \to Z_f \times X$ as in the diagram below. Moreover, the map $\hat{f} : Z_f \times X \to Y$, with $\hat{f}([z], x) = f(z,x)$, is a $(\mathbb{T}, V)$-functor because $\eta_Y \cdot \hat{f} = \widetilde{\mathrm{ev}} \cdot (h_f \times \eta_X)$ is and η_Y is initial.

$$\begin{tikzcd}
Z \times X \arrow[rrr, "f"] \arrow[d, "\eta_Z \times 1_X"'] \arrow[dr, "q_f \times 1_X"] & & & Y \arrow[dd, "\eta_Y"] \\
RZ \times X \arrow[d, "\overline{R}f \times 1_X"'] & Z_f \times X \arrow[urr, "\hat{f}"] \arrow[r] \arrow[dl, "h_f \times 1_X"] & (\coprod_g Z_g \times X) \cong (\coprod_g Z_g) \times X \arrow[ur, "\mathrm{ev}"] & \\
\ll RX, RY \gg \times X \arrow[r, "1 \times \eta_X"'] & \ll RX, RY \gg \times RX \arrow[rr, "\widetilde{\mathrm{ev}}"'] & & RY
\end{tikzcd}$$

Since the cardinality of Z_f is bounded by the cardinality of the set $| \ll RX, RY \gg | \times |Y|^{|X|}$, as witnessed by the injective map

$$\begin{array}{rcl} Z_f & \to & | \ll RX, RY \gg | \times |Y|^{|X|}, \\ [z] & \mapsto & (\overline{R}f(\eta_Z(z)), f(z,-)) \end{array}$$

there is only a set of possible $(\mathbb{T}, V)$-categories Z_f. Hence we can form its coproduct, as in the diagram above, and consider the induced $(\mathbb{T}, V)$-functor $\mathrm{ev} : (\coprod_g Z_g) \times X \cong \coprod_g (Z_g \times X) \to Y$ (note that the isomorphism follows from extensivity of $(\mathbb{T}, V)$-**Cat**). ■

6. Examples

In this section we use Theorem 5.8 to present examples of weakly cartesian closed categories. Hence, in conjunction with the following result established in [Ros99], we obtain examples of categories with cartesian closed exact completion since all other conditions are trivially satisfied in these examples.

Theorem 6.1. *Let* **C** *be a complete, infinitely extensive and well-powered category in which every morphism factorizes as a regular epi followed by a* *mono and where $f \times 1$ is an epimorphism, for every regular epimorphism $f : A \to B$ in $\mathbf{C}$. Then, if $\mathbf{C}$ weakly cartesian closed, the exact completion $\mathbf{C}_{\mathrm{ex}}$ of $\mathbf{C}$ is cartesian closed.*

We note that, in order to conclude that $(\mathbb{T}, V)$-**Cat** is weakly cartesian closed, we have to check whether V and $\mathbb{T}$ satisfy conditions (3.i), (5.iii), (5.iv), and (5.v).

First we analyse examples when $\mathbb{T}$ is the identity monad. In this particular setting we only have to check that (5.v) holds. The category V-**Cat** is always monoidal closed, as shown in [Law73]. Therefore, when $\otimes = \wedge$ in V, that is when V is a frame considered as a quantale, then V-**Cat** is cartesian closed. This is the case of 2, and so one concludes that **Ord** *is cartesian closed*. Moreover, for V the lattice $([0,\infty], \geq)$ with $\otimes = \wedge$, V-**Cat** is *the category of ultrametric spaces*, which *is* therefore also *cartesian closed*.

When $V = P_+$, V-**Cat** is the category **Met** of Lawvere's metric spaces [Law73], which is not cartesian closed (see [CH06] for details). However, the quantale P_+ satisfies (5.v), and so from Theorem 5.8 it follows that **Met** *is weakly cartesian closed*.

Metric and ultrametric spaces can be also viewed as categories enriched in a quantale based on the complete lattice $[0,1]$ with the usual "less or equal" relation $\leq$, which is isomorphic to $[0,\infty]$ via the map $[0,1] \to [0,\infty]$, $u \mapsto -\ln(u)$ where $-\ln(0) = \infty$. More in detail, we consider the following quantale operations on $[0,1]$ with neutral element 1.

(1) For $\otimes = *$ being the ordinary multiplication, via the isomorphism $[0,1] \simeq [0,\infty]$, this quantale is isomorphic to the quantale P_+, hence $[0,1]$-**Cat** $\simeq$ **Met**.
(2) For the tensor $\otimes = \wedge$ being infimum, the isomorphism $[0,1] \simeq [0,\infty]$ establishes an equivalence between $[0,1]$-**Cat** and the category of ultrametric spaces and non-expansive maps.
(3) Another interesting multiplication on $[0,1]$ is the *Łukasiewicz tensor* $\otimes = \odot$ given by $u \odot v = \max(0, u+v-1)$. Via the lattice isomorphism $[0,1] \to [0,1]$, $u \mapsto 1-u$, this quantale is isomorphic to the quantale $[0,1]$ with "greater or equal" relation $\geq$ and tensor $u \otimes v = \min(1, u+v)$ truncated addition. Therefore $[0,1]$-**Cat** is equivalent to the *category of bounded-by-1 metric spaces and non-expansive maps.* Moreover, with respect to the "greater or equal" relation and truncated addition

on $[0,1]$, the map

$$[0,\infty] \to [0,1],\ u \mapsto \min(1,u)$$

is a surjective quantale morphism; therefore, by Lemma 5.3, also $[0,1]$ with the Łukasiewicz tensor satisfies (5.v).

(4) More generally, every continuous quantale structure $\otimes$ on the lattice $[0,1]$ (with Euclidean topology and the usual "less or equal" relation) with neutral element 1 satisfies (5.v). This can be shown using the fact, proven in [Fau55] and [MS57], that every such tensor $\otimes : [0,1] \times [0,1] \to [0,1]$ is a combination of the three operations on $[0,1]$ described above. More precise:
  (a) For $u, v \in [0,1]$ and $e \in [0,1]$ idempotent with $u \leq e \leq v$: $u \otimes v = \min(u,v) = u$.
  (b) For every non-idempotent $u \in [0,1]$, there exist idempotents e and f with $e < u < f$ and such that the interval $[e,f]$ (with the restriction of the tensor on $[0,1]$ and with neutral element f) is isomorphic to $[0,1]$ either with multiplication or Łukasiewicz tensor.

Now let $w, u, v \in [0,1]$. We may assume $u \leq v$. If $u \otimes v \leq w$, then clearly

$$w \wedge (u \otimes v) = u \otimes v = \bigvee\{u' \otimes v' \mid u' \leq u, v' \leq v, u' \otimes v' \leq w\}.$$

We consider now $w < u \otimes v \leq u \leq v$. If w is idempotent, then

$$w = w \otimes v, \quad w \leq u, \quad v \leq v;$$

otherwise there are idempotents e and f with $e < w < f$ and $[e,f]$ is isomorphic to $[0,1]$ either with multiplication or Łukasiewicz tensor.

Case 1: $v \leq f$. Then the equation (5.v) holds since $w, u \otimes v, u, v \in [e,f]$.

Case 2: $f < v$. Then $w = w \wedge v = w \otimes v$, $w \leq u$ and $v \leq v$.

We conclude that $[0,1]$-**Cat** *is weakly cartesian closed*, for every continuous quantale structure $\otimes$ on $[0,1]$ with neutral element 1.

Now let $V = \Delta$ be the *quantale of distribution functions* (see [HR13, CH17] for details). As observed in [HR13], it verifies (5.v), and so we can conclude from Theorem 5.8 that *the category* Δ-**Cat** *of probabilistic metric spaces and non-expansive maps is weakly cartesian closed.*

When $\mathbb{T}$ is not the identity monad, Theorem 5.8 applies only when the extension of $\mathbb{T}$ to V-**Rel** is given by a $\mathbb{T}$-algebra structure $\xi : TV \to V$

on V (so that we are dealing with a strict topological theory in the sense of [Hof07]), *which we assume from now on.* In this case, the extension of $T : \mathbf{Set} \to \mathbf{Set}$ to V-$\mathbf{Rel}$ is defined by

$$Tr : TX \times TY \to V$$

$$r(\mathfrak{x},\mathfrak{y}) \mapsto \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_X(\mathfrak{w}) = \mathfrak{x}, T\pi_Y(\mathfrak{w}) = \mathfrak{y} \right\}$$

for each V-relation $r : X \times Y \to V$.

Theorem 6.2. *(1) The tensor product on the quantale V defines a $(\mathbb{T},V)$-functor $\otimes : V \otimes V \to V$.*

(2) Let $u \in V$ satisfying $u\cdot ! \geq \xi \cdot Tu$.

$$\begin{tikzcd} T1 \arrow[r, "Tu"] \arrow[d, "!"'] \arrow[dr, phantom, "\geq"] & TV \arrow[d, "\xi"] \\ 1 \arrow[r, "u"'] & V \end{tikzcd}$$

Then $(-,u) : X \to X \times V$ is a $(\mathbb{T},V)$-functor, for every $(\mathbb{T},V)$-category X.

(3) Let $u \in V$ satisfying $u\cdot ! = \xi \cdot Tu$. Then $T(r \otimes u) = (Tr) \otimes u$, for every V-relation $r : X \nrightarrow Y$.

Proof: The first assertion is [Hof11, Proposition 1.4(1)]. To see (2), assume that $u \in V$ with $u\cdot ! \geq \xi \cdot Tu$. Let (X,a) be a $(\mathbb{T},V)$-category, $\mathfrak{x} \in TX$ and $x \in X$. Considering the map $X \xrightarrow{!} 1 \xrightarrow{u} V$, we have to show that

$$a(\mathfrak{x},x) \leq a(\mathfrak{x},x) \wedge \hom(T(u\cdot !)(\mathfrak{x}),u),$$

which follows immediately from $u\cdot ! \geq \xi \cdot Tu$. Finally, to prove (3), let $r : X \nrightarrow Y$ be a V-relation and $u \in V$ with $u\cdot ! = \xi \cdot Tu$. Note that the V-relation $r \otimes u : X \nrightarrow Y$ is given by

$$X \times Y \xrightarrow{\ r\ } V \xrightarrow{\langle 1_V, u\cdot ! \rangle} V \times V \xrightarrow{\ \otimes\ } V.$$

Hence, applying the $\mathbf{Set}$-functor T to the functions $r : X \times Y \to V$ and $r \otimes u : X \times Y \to V$, we obtain

$$\begin{aligned} \xi \cdot T(r \otimes u) &= \xi \cdot T(\otimes) \cdot T\langle 1_V, u\cdot ! \rangle \cdot Tr \\ &= \otimes \cdot (\xi \times \xi) \cdot \mathrm{can}_{X,Y} \cdot T\langle 1_V, u\cdot ! \rangle \cdot Tr \\ &= \otimes \cdot \langle \xi, u\cdot ! \cdot \xi \rangle \cdot Tr \\ &= \otimes \cdot \langle 1_V, u\cdot ! \rangle \cdot \xi \cdot Tr. \end{aligned}$$

Therefore, returning to V-relations, we conclude that $T(r\otimes u) = (Tr)\otimes u$. ∎

Remark 6.3. If $T1 = 1$, then $u\cdot ! = \xi \cdot Tu$ for every $u \in V$.

In order to guarantee that (3.i) holds we need an extra condition on ξ.

Proposition 6.4. *Assume that*

$$\begin{tikzcd} T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] \arrow[dr, phantom, "\leq"] & TV \arrow[d, "\xi"] \\ V\times V \arrow[r, "\wedge"'] & V. \end{tikzcd}$$

Then, for all V-relations $r : X \to\!\!\!\!\!\!+\ X'$ and $s : Y \to\!\!\!\!\!\!+\ Y'$,

$$\begin{tikzcd} T(X\times Y) \arrow[r, "\mathrm{can}_{X,Y}"] \arrow[d, "T(r\otimes s)"', "\circ" marking] \arrow[dr, phantom, "\geq"] & TX\times TY \arrow[d, "Tr\otimes Ts", "\circ" marking] \\ T(X'\times Y') \arrow[r, "\mathrm{can}_{X',Y'}"'] & TX'\times TY'. \end{tikzcd}$$

Proof: First we note that, from the preservation of weak pullbacks by T, it follows that the commutative diagram

$$\begin{tikzcd} T(A\times B) \arrow[r, "T(f\times g)"] \arrow[d, "\mathrm{can}_{A,B}"'] & T(X\times Y) \arrow[d, "\mathrm{can}_{X,Y}"] \\ TA\times TB \arrow[r, "Tf\times Tg"'] & TX\times TY \end{tikzcd}$$

is also a weak pullback.

Let $\mathfrak{w} \in T(X\times Y)$, $\mathfrak{x}' \in TX'$ and $\mathfrak{y}' \in TY'$. Put $(\mathfrak{x},\mathfrak{y}) = \mathrm{can}_{X,Y}(\mathfrak{w})$. By the definition of the extension of T and since V is a Heyting algebra, $Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}')$ is given by

$$\bigvee \left\{ \xi\cdot Tr(\mathfrak{w}_1)\wedge \xi\cdot Ts(\mathfrak{w}_2) \;\middle|\; \begin{array}{l} \mathfrak{w}_1 \in T(X\times X') : \mathfrak{w}_1 \mapsto \mathfrak{x}, \mathfrak{w}_1 \mapsto \mathfrak{x}' \\ \mathfrak{w}_2 \in T(Y\times Y') : \mathfrak{w}_2 \mapsto \mathfrak{y}, \mathfrak{w}_2 \mapsto \mathfrak{y}' \end{array} \right\}.$$

Note that in

$$\begin{tikzcd}
 & T(X\times Y\times X'\times Y') \arrow[d, "\cong"'] & & \\
T(X\times Y) \arrow[d, "\mathrm{can}"'] & T(X\times X'\times Y\times Y') \arrow[l, "T(\pi_X\times\pi_Y)"'] \arrow[r, "T(r\times s)"] \arrow[d, "\mathrm{can}"'] & T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\mathrm{can}"'] & TV \arrow[dd, "\xi"] \\
TX\times TY & T(X\times X')\times T(Y\times Y') \arrow[l, "T\pi_X\times T\pi_Y"] \arrow[r, "Tr\times Ts"'] & TV\times TV \arrow[d, "\xi\times\xi"'] \arrow[dr, phantom, "\leq"] & \\
 & & V\times V \arrow[r, "\wedge"'] & V
\end{tikzcd}$$

the left hand side is a weak pullback, the middle diagram commutes and in the right hand side we have "lower path" $\leq$ "upper path" as indicated. Therefore, for such $\mathfrak{w}_1 \in T(X\times X')$ and $\mathfrak{w}_2 \in T(Y\times Y')$, there exists some $\mathfrak{v} \in T(X\times X'\times Y\times Y')$ which projects to $\mathfrak{w} \in T(X\times Y)$ and to $(\mathfrak{w}_1,\mathfrak{w}_2) \in T(X\times X')\times T(Y\times Y')$. Hence, taking also into account the definition of the V-relation $T(r \owedge s)$,

$$\begin{aligned}
&Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}')\\
&\leq \bigvee\left\{\xi\cdot T(\wedge)\cdot T(r\times s)(\mathfrak{v}) \;\middle|\; \mathfrak{v}\in T(X\times Y\times X'\times Y');\, \begin{matrix}\mathfrak{v}\mapsto\mathfrak{w}\\ \mathfrak{v}\mapsto\mathfrak{x}',\mathfrak{v}\mapsto\mathfrak{y}'\end{matrix}\right\}\\
&\leq \bigvee\{T(r \owedge s)(\mathfrak{w},\mathfrak{w}') \mid \mathfrak{w}'\in T(X'\times Y'),\ \mathrm{can}_{X',Y'}(\mathfrak{w}') = (\mathfrak{x}',\mathfrak{y}')\}. \quad \blacksquare
\end{aligned}$$

Remark 6.5. We note that the inequality

$$\begin{tikzcd}
T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\langle \xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle"'] \arrow[dr, phantom, "\geq"] & TV \arrow[d, "\xi"] \\
V\times V \arrow[r, "\wedge"'] & V
\end{tikzcd}$$

is always true.

Corollary 6.6. *If the quantale V satisfies* (5.v) *and the diagrams*

$$\begin{tikzcd}
T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\langle \xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle"'] & TV \arrow[d, "\xi"] \\
V\times V \arrow[r, "\wedge"'] & V
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
T1 \arrow[r, "Tu"] \arrow[d, "!"'] & TV \arrow[d, "\xi"] \\
1 \arrow[r, "u"'] & V
\end{tikzcd}$$

commute, for all $u \in V$, then all Assumptions 5.4 are satisfied.

Let $\mathbb{T}$ be the ultrafilter monad $\mathbb{U} = (U, m, e)$. Then, when V is any of the quantales listed above but Δ, all the needed conditions are satisfied. Therefore, in particular we can conclude that:

Examples 6.7. (1) *The category* **Top** $= (\mathbb{U}, 2)$-**Cat** *of topological spaces and continuous maps is weakly cartesian closed* (as shown by Rosický in [Ros99]).

(2) *The category* **App** $= (\mathbb{U}, P_+)$-**Cat** *of approach spaces and non-expansive maps is weakly cartesian closed.*

(3) In fact, for each continuous quantale structure on the lattice $([0,1], \leq) \simeq ([0,\infty], \geq)$, $(\mathbb{U}, [0,1])$-**Cat** is weakly cartesian closed. In particular, *the category of non-Archimedean approach spaces and non-expansive maps studied in* [CVO17] *is weakly cartesian closed.*

(4) If V is a completely distributive complete lattice with $\otimes = \wedge$, then, with

$$\xi : UV \to V,\ \mathfrak{x} \mapsto \bigwedge_{A \in \mathfrak{x}} \bigvee A,$$

all needed conditions are satisfied (see [Hof07, Theorem 3.3]) and therefore $(\mathbb{U}, V)$-**Cat** is weakly cartesian closed. In particular, with $V = P2$ being the powerset of a 2-element set, we obtain that *the category* **BiTop** *of bitopological spaces and bicontinuous maps is weakly cartesian closed* (see [HST14]).

Remark 6.8. For $V = \Delta$ the quantale of distribution functions, we do not know if there is an appropriate compact Hausdorff topology $\xi : UV \to V$ satisfying the conditions of this section.

Now let $\mathbb{T}$ be the free monoid monad $\mathbb{W} = (W, m, e)$. For each quantale V, we consider

$$\xi : WV \to V,\ (v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n,\ () \mapsto k$$

which induces the extension $W : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ sending $r : X \nrightarrow Y$ to the V-relation $Wr : WX \nrightarrow WY$ given by

$$Wr((x_1, \dots, x_n), (y_1, \dots, y_m)) = \begin{cases} r(x_1, y_1) \otimes \dots \otimes r(x_n, y_n) & \text{if } n = m \\ \bot & \text{if } n \neq m. \end{cases}$$

The category $(\mathbb{W}, 2)$-**Cat** is equivalent to the category **MultiOrd** of *multi-ordered sets* and their morphisms (see [HST14]), more generally, $(\mathbb{W}, V)$-categories can be interpreted as multi-V-categories and their morphisms. The representable multi-ordered sets are precisely the ordered monoids, which is a special case of [Her00, Her01] describing monoidal categories as representable multi-categories (see also [CCH15]). We recall that the separated injective multi-ordered sets are precisely the quantales (see [LBKR12] and also [Sea10]), and we conclude:

Proposition 6.9. *Every quantale is exponentiable in* **MultiOrd**.

Theorem 6.10. *If the quantale V is a frame (that is, $\otimes = \wedge$), then* $(\mathbb{W}, V)$-**Cat** *is weakly cartesian closed. In particular,* **MultiOrd** *is weakly cartesian closed.*

Finally, for a monoid $(H, \cdot, h)$, we consider the monad $\mathbb{H} = (- \times H, m, e)$, with $m_X : X \times H \times H \to X \times H$ given by $m_X(x, a, b) = (x, a \cdot b)$ and $e_X : X \to X \times H$ given by $e_X(x) = (x, h)$. Here we consider

$$\xi : V \times H \to V,\ (v, a) \mapsto v,$$

which leads to the extension $- \times H : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ sending the V-relation $r : X \nrightarrow Y$ to the V-relation $r \times H : X \times H \nrightarrow Y \times H$ with

$$r \times H((x, a), (y, b)) = \begin{cases} r(x, y) & \text{if } a = b, \\ \perp & \text{if } a \neq b. \end{cases}$$

In particular, $(\mathbb{H}, 2)$-categories can be interpreted as *H-labelled ordered sets* and *equivariant maps*.

For every quantale V and every $v : 1 \to V$, the diagrams

$$\begin{array}{ccc} V \times V \times H \xrightarrow{\wedge \times 1_H} V \times H \\ {\scriptstyle \pi_{1,2}}\downarrow \qquad\qquad \downarrow{\scriptstyle \xi = \pi_1} \\ V \times V \xrightarrow[\wedge]{} V \end{array} \quad \text{and} \quad \begin{array}{ccc} 1 \times H \xrightarrow{v \times 1_H} V \times H \\ {\scriptstyle !}\downarrow \qquad\qquad \downarrow{\scriptstyle \xi} \\ 1 \xrightarrow[v]{} V \end{array}$$

commute, therefore we obtain:

Theorem 6.11. *For every quantale V satisfying* (5.v)*, the category* $(\mathbb{H}, V)$-**Cat** *is weakly cartesian closed.*

References

[AR18] Jiří Adámek and Jiří Rosický. How nice are free completions of categories? Technical report, 2018, `arXiv:1806.02524 [math.CT]`.

[BBS04] Andrej Bauer, Lars Birkedal, and Dana S. Scott. Equilogical spaces. *Theoretical Computer Science*, 315(1):35–59, 2004.

[CCH15] Dimitri Chikhladze, Maria Manuel Clementino, and Dirk Hofmann. Representable $(\mathbb{T}, \mathcal{V})$-categories. *Applied Categorical Structures*, 23(6):829–858, January 2015, eprint: `http://www.mat.uc.pt/preprints/ps/p1247.pdf`.

[CH03] Maria Manuel Clementino and Dirk Hofmann. Topological features of lax algebras. *Applied Categorical Structures*, 11(3):267–286, June 2003, eprint: `http://www.mat.uc.pt/preprints/ps/p0109.ps`.

[CH06] Maria Manuel Clementino and Dirk Hofmann. Exponentiation in V-categories. *Topology and its Applications*, 153(16):3113–3128, October 2006.

[CH09] Maria Manuel Clementino and Dirk Hofmann. Lawvere completeness in topology. *Applied Categorical Structures*, 17(2):175–210, August 2009, `arXiv:0704.3976 [math.CT]`.

[CH17] Maria Manuel Clementino and Dirk Hofmann. The Rise and Fall of V-functors. *Fuzzy Sets and Systems*, 321:29–49, August 2017, eprint: `http://www.mat.uc.pt/preprints/ps/p1606.pdf`.

[CHJ14] Maria Manuel Clementino, Dirk Hofmann, and George Janelidze. The monads of classical algebra are seldom weakly cartesian. *Journal of Homotopy and Related Structures*, 9(1):175–197, November 2014, eprint: `http://www.mat.uc.pt/preprints/ps/p1246.pdf`.

[CHT03] Maria Manuel Clementino, Dirk Hofmann, and Walter Tholen. Exponentiability in categories of lax algebras. *Theory and Applications of Categories*, 11(15):337–352, 2003, eprint: `http://www.mat.uc.pt/preprints/ps/p0302.pdf`.

[CT03] Maria Manuel Clementino and Walter Tholen. Metric, topology and multicategory—a common approach. *Journal of Pure and Applied Algebra*, 179(1-2):13–47, April 2003.

[CVO17] Eva Colebunders and Karen Van Opdenbosch. Topological properties of non-Archimedean approach spaces. *Theory and Applications of Categories*, 32(41):1454–1484, 2017.

[Fau55] William M. Faucett. Compact semigroups irreducibly connected between two idempotents. *Proceedings of the American Mathematical Society*, 6(5):741–747, May 1955.

[Her00] Claudio Hermida. Representable multicategories. *Advances in Mathematics*, 151(2):164–225, May 2000.

[Her01] Claudio Hermida. From coherent structures to universal properties. *Journal of Pure and Applied Algebra*, 165(1):7–61, November 2001.

[Hof06] Dirk Hofmann. Exponentiation for unitary structures. *Topology and its Applications*, 153(16):3180–3202, October 2006.

[Hof07] Dirk Hofmann. Topological theories and closed objects. *Advances in Mathematics*, 215(2):789–824, November 2007.

[Hof11] Dirk Hofmann. Injective spaces via adjunction. *Journal of Pure and Applied Algebra*, 215(3):283–302, March 2011, `arXiv:0804.0326 [math.CT]`.

[Hof14] Dirk Hofmann. The enriched Vietoris monad on representable spaces. *Journal of Pure and Applied Algebra*, 218(12):2274–2318, December 2014, `arXiv:1212.5539 [math.CT]`.

[HR13] Dirk Hofmann and Carla D. Reis. Probabilistic metric spaces as enriched categories. *Fuzzy Sets and Systems*, 210:1–21, January 2013, `arXiv:1201.1161 [math.GN]`.

[HS15] Dirk Hofmann and Gavin J. Seal. Exponentiable approach spaces. *Houston Journal of Mathematics*, 41(3):1051–1062, 2015, `arXiv:1304.6862 [math.GN]`.

[HST14] Dirk Hofmann, Gavin J. Seal, and Walter Tholen, editors. *Monoidal Topology. A Categorical Approach to Order, Metric, and Topology*, volume 153 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, July 2014. Authors: Maria Manuel Clementino, Eva Colebunders, Dirk Hofmann, Robert Lowen, Rory Lucyshyn-Wright, Gavin J. Seal and Walter Tholen.

[HT10] Dirk Hofmann and Walter Tholen. Lawvere completion and separation via closure. *Applied Categorical Structures*, 18(3):259–287, November 2010, `arXiv:0801.0199 [math.CT]`.

[Law73] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matemàtico e Fisico di Milano*, 43(1):135–166, December 1973. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

[LBKR12] Joachim Lambek, Michael Barr, John F. Kennison, and Robert Raphael. Injective hulls of partially ordered monoids. *Theory and Applications of Categories*, 26(13):338–348, 2012.

[Low97] Robert Lowen. *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 1997.

[MS57] Paul S. Mostert and Allen L. Shields. On the structure of semi-groups on a compact manifold with boundary. *Annals of Mathematics. Second Series*, 65(1):117–143, January 1957.

[MST06] Mojgan Mahmoudi, Christoph Schubert, and Walter Tholen. Universality of coproducts in categories of lax algebras. *Applied Categorical Structures*, 14(3):243–249, June 2006.

[Rib18] Willian Ribeiro. On generalized equilogical spaces. In preparation, 2018.

[Ros99] Jiří Rosický. Cartesian closed exact completions. *Journal of Pure and Applied Algebra*, 142(3):261–270, October 1999.

[Sea10] Gavin J. Seal. Order-adjoint monads and injective objects. *Journal of Pure and Applied Algebra*, 214(6):778–796, June 2010.

[Tho09] Walter Tholen. Ordered topological structures. *Topology and its Applications*, 156(12):2148–2157, July 2009.

Maria Manuel Clementino
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: `mmc@mat.uc.pt`

Dirk Hofmann
CIDMA, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: `dirk@ua.pt`

Willian Ribeiro
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: `willian.ribeiro.vs@gmail.com`