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CONJUGATION SEMIGROUPS AND CONJUGATION MONOIDS WITH CANCELLATION

ANA PAULA GARRÃO, NELSON MARTINS-FERREIRA, MARGARIDA RAPOSO AND MANUELA SOBRAL

Abstract: We show that the category of conjugation semigroups with cancellation is weakly Mal'tsev and give a characterization of all admissible diagrams there. In the subcategory of conjugation monoids with cancellation we describe, for Schreier split epimorphisms with codomain B and kernel X, all morphisms $h : X \to B$ which induce a reflexive graph, an internal category or an internal groupoid there. In this subcategory we prove that a relative version of the so-called "Smith is Huq" condition for Schreier split epimorphisms holds as well as other relative conditions.

Keywords: Admissibility diagrams, weakly Mal'tsev category, conjugation semigroups, internal monoid, internal groupoid.
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1. Introduction

In [11] the concept of weakly Mal'tsev category was introduced to provide a simple axiomatic context where the internal categories and the internal groupoids are particularly simple to describe, the established notion of Mal'tsev category ([7]) being too restrictive for this purpose since there the two notions coincide. Amongst the categories that are weakly Mal'tsev but not Mal'tsev are the categories of distributive lattices ([14]) and the category of commutative monoids with cancellation. In this paper we introduce another class of examples of such categories, that includes the later, characterize there all the admissible diagrams and describe some internal structures. The admissibility of certain type of diagrams is used to go from local to global in a sense we make precise below.

We introduce the category of conjugation semigroups which can be seen as an abstraction of conjugation of complex numbers or of quaternions. A

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conjugation semigroup $(S,\cdot,\overline{(\ )})$ is a semigroup $(S,\cdot)$ equipped with a unary operation $\overline{(\ )}\colon S\to S$ satisfying the following identities: $x\overline{x}=\overline{x}x$, $x\overline{y}y=y\overline{y}x$, and $\overline{xy}=\overline{y}\,\overline{x}$. The quasivariety of conjugation semigroups with cancellation is a weakly Mal'tsev category, that is not Mal'tsev, and we present there a characterization of all admissible diagrams in the sense of [12].

In the subcategory of conjugation monoids with cancellation we describe for Schreier split epimorphisms with codomain B and kernel X (a notion first introduced in [16] for monoids with operations),

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, shift right, dotted, "q_f"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd} ,$$

all morphisms $h\colon X\to B$ which induce a reflexive graph, an internal category or an internal groupoid. That is, all morphisms $h\colon X\to B$ that induce a morphism $\tilde{h}\colon A\to B$, with $\tilde{h}k=h$, such that

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$

gives rise to a reflexive graph, an internal category or an internal groupoid.

We show that a relative version of the so-called "Smith is Huq" condition holds for Schreier split epimorphisms in the category of conjugation monoids with cancellation as well as other relative conditions.

Throughout, for simplicity of exposition, we will use additive notation for monoids and semigroups, though *we do not assume commutativity.*

2. Preliminaries

We recall here definitions and basic properties that will be used throughout.

In a category with pullbacks of split epimorphisms along split epimorphisms we consider the following diagram

$$\begin{tikzcd} A\times_B C \arrow[r, shift right, "\pi_2"'] \arrow[d, shift right, "\pi_1"'] & C \arrow[l, shift right, "e_2"'] \arrow[d, shift right, "g"'] \\ A \arrow[r, shift right, "f"'] \arrow[u, shift right, "e_1"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"'] \end{tikzcd}$$

where $fr = 1_B = gs$, (π_1, π_2) is the pullback of (f, g) and $e_1 =< 1_A, sf >$, $e_2 =< rg, 1_C >$ are the morphisms induced by the universal property of the pullback. Then $e_1 r = e_2 s$.

Any diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[rd, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "\beta"] & C \arrow[l, shift right, "g"'] \arrow[ld, "\gamma"] \\
& D &
\end{tikzcd} \tag{1}$$

with $fr = 1_B = gs$ and $\alpha r = \beta = \gamma s$ will be called an *admissibility diagram.* It induces a diagram

$$\begin{tikzcd}
& C \arrow[ld, shift left, "e_2"] \arrow[rd, shift left, "g"] \arrow[rrd, bend left, "\gamma"] & & \\
A \times_B C \arrow[ru, shift left, "\pi_2"] \arrow[rd, shift right, "\pi_1"'] & & B \arrow[lu, shift left, "s"] \arrow[ld, shift left, "r"] \arrow[r, "\beta"] & D \\
& A \arrow[lu, shift right, "e_1"'] \arrow[ru, shift left, "f"] \arrow[rru, bend right, "\alpha"'] & &
\end{tikzcd}$$

and the existence of a unique morphism $\varphi \colon A \times_B C \longrightarrow D$ such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$ is a way to describe relevant situations and results in categorical algebra as we mention next.

Definition 2.1. The triple (α, β, γ) is admissible with respect to (f, r, g, s) if there exists a unique morphism $\varphi \colon A \times_B C \longrightarrow D$ such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$. Then we say that the diagram (1) is admissible.

The unicity of φ is fundamental and it is achieved in the context of weakly Mal'tsev categories, a notion introduced in [11]. See also [12].

Definition 2.2. A finitely complete category C is weakly Malt'tsev if the morphisms $e_1 =< 1_A, sf >$ and $e_2 =< rq, 1_C >$ are jointly epimorphic.

In a weakly Mal'tsev category, the morphism φ is unique and so the admissibility of diagram (1) is a condition and not an additional structure.

We recall that a finitely complete category is Mal'tsev if (e_1, e_2) is jointly strongly pair ([2]). Hence, all Mal'tsev categories are weakly Mal'tsev but the converse is false. For example, the category of distributive lattices is a weakly Mal'tsev category which is not Mal'tsev ([14]).

In a weakly Mal'tsev category $\mathtt{C}$ the admissibility of diagrams (1) describe several conditions and properties of $\mathtt{C}$. For example, the reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

is an internal category in $\mathtt{C}$ if and only if the diagram

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] \arrow[dr, equal] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] \arrow[d, "e"] & C_1 \arrow[l, shift right, "c"'] \arrow[dl, equal] \\ & C_1 & \end{tikzcd}$$

is admissible. And an internal category in $\mathtt{C}$

$$\begin{tikzcd} C_2 = C_1 \times_{C_0} C_1 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

is a groupoid if and only if the diagram

$$\begin{tikzcd} C_2 \arrow[r, shift left, "m"] \arrow[dr, "\pi_2"'] & C_1 \arrow[l, shift left, "<1_{C_1}{,}ed>"] \arrow[r, shift right, "<ec{,}1_{C_1}>"'] \arrow[d, equal] & C_1 \arrow[l, shift right, "m"'] \arrow[dl, "\pi_1"] \\ & C_1 & \end{tikzcd}$$

is admissible ([12]).

A pair (R, S) of equivalence relations on an object X

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "i_R" description] \arrow[r, "i_S" description] & S \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "s_2"] \end{tikzcd}$$

commute in the sense of Smith-Pedicchio [20], [19] if and only if the diagram

$$\begin{tikzcd}
R \arrow[r, shift left, "r_2"] \arrow[dr, "r_1"'] & X \arrow[l, shift left, "i_R"] \arrow[r, shift right, "i_S"'] \arrow[d, equal] & S \arrow[l, shift right, "s_1"'] \arrow[dl, "s_2"] \\
& X &
\end{tikzcd}$$

is admissible, i.e. if and only if there exists a morphism $\varphi\colon R \times_X S \longrightarrow X$ such that $\varphi < 1_R, i_S r_2 >= r_1$ and $\varphi < i_r s_1, 1_S >= s_2$.

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$$\begin{tikzcd}
K \arrow[r, "k"] & X & L \arrow[l, "l"']
\end{tikzcd}$$

of morphisms in a pointed category commute in the sense of Huq ([8]) if and only if the diagram

$$\begin{tikzcd}
K \arrow[r, shift left] \arrow[dr, "k"'] & 0 \arrow[l, shift left] \arrow[r, shift right] \arrow[d] & L \arrow[l, shift right] \arrow[dl, "l"] \\
& X &
\end{tikzcd}$$

is admissible. This means the existence of a unique $\varphi : K \times L \to X$ such that $\varphi < 1, 0 >= k$ and $\varphi < 0, 1 >= l$.

In [21], Tim Van der Linden observed that this very flexible condition of admissibility is indeed a commutativity condition.

It is well-known that in groups there is an equivalence between the category of split epimorphisms with codomain B and kernel X and group actions of B on X, that is group homomorphisms $\varphi\colon B \to Aut(X)$.

This is not true in the category of monoids. There, the classical notion of monoid action, that is a monoid homomorphism $\varphi\colon B \to End(X)$, corresponds to a special class of split epimorphisms called Schreier split epimorphisms introduced in [16], in the more general context of monoids with operations, and studied in detail in [5] and [4].

Definition 2.3. A Schreier split epimorphism in the category Mon of monoids is a split epimorphism (f, r), $fr = 1_B$, with kernel X, for which there exists a unique set-theoretical map $q : A \to X$, called the Schreier retraction,

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

such that, for every $a \in A$, $a = kq(a) + rf(a)$.

Equivalently, the following conditions should be satisfied

(i) $a = kq(a) + rf(a)$ and
(ii) $x = q(k(x) + r(b)$,

for all $a \in A$, $x \in X$ and $b \in B$, since (ii) gives the unicity of the set map q.

In the following proposition (proved in [4]) we list consequences of the definition that will be used in the sequel.

Proposition 2.4. *Given a Schreier split epimorphism in the category Mon of monoids*

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

we have that, for $a, a' \in A$, $x \in X$ and $b \in B$,

(a) $qk = 1_X$;
(b) $qr = 0$;
(c) $q(0) = 0$;
(d) $kq(r(b) + k(x)) + r(b) = r(b) + k(x)$;
(e) $q(a + a') = q(a) + q(rf(a) + q(a'))$;
(f) the sequence is exact, that is $X = kerf$ and $f = cokerk$, and so we speak of Schreier extension.

To the Schreier extension above corresponds an action $\varphi\colon B \longrightarrow End(X)$ defined by

$$\varphi(b)(x) = q(r(b) + k(x))$$

that we will denote by $b \cdot x$. Then, for example condition (d) can be written as

$$k(b \cdot x) + r(b) = r(b) + k(x).$$

3. Conjugation semigroups with cancellation

Definition 3.1. A conjugation semigroup $(S, +, \overline{()})$ is a semigroup $(S, +)$ equipped with a unary operation $\overline{()} : S \to S$ satisfying the following identities:

(1) $\overline{x} + x = x + \overline{x}$

(2) $x+\overline{y}+y=y+\overline{y}+x$
(3) $\overline{(x+y)}=\overline{y}+\overline{x}$

Examples are groups with $\overline{x}=x^{-1}$, commutative monoids with $\overline{x}=1$ and commutative semigroups with $\overline{x}=x$. The main examples that illustrate our intuition behind the notion of conjugation semigroups are sets of non-zero complex numbers and non-zero quaternions with usual multiplication and conjugation.

The quasivariety $\mathcal{S}$ of conjugation semigroups satisfying the implications

(4) $x+\overline{a}+a=y+\overline{a}+a \Rightarrow x=y$

is a weakly Malt'tsev category. Indeed, this subvariety has a ternary operation

$$p(x,y,z)=x+\overline{y}+z$$

satisfying the identity

$$p(x,y,y)=p(y,y,x)$$

and the quasi-identity

$$p(x,y,y)=p(x',y,y)\Rightarrow x=x'$$

and so it is weakly Mal'tsev as proved in [14].

It is easy to prove that, in presence of (1)-(3), condition (4) is equivalent to cancellation. So, throughout, $\mathcal{S}$ will denote the *category of conjugation semigroups with cancellation.*

Remark 3.2. From the definition of conjugation semigroup it follows that:

(i) $\overline{(x+y)}+(x+y)=\overline{y}+y+x+\overline{x}$ because
$\overline{(x+y)}+(x+y)=\overline{y}+\overline{x}+x+y=\overline{y}+x+\overline{x}+y=\overline{y}+y+\overline{x}+x=\overline{y}+y+x+\overline{x}$.
(ii) $x+y+\overline{y}=\overline{y}+y+x$

that we will use in the sequel.

The following are examples of conjugation semigroups with cancellation that are neither groups nor monoids:

1. $S=\{u\in\mathbb{R}|0<|u|<1\}$ with usual product and $\overline{u}=u$.
2. $S=\{z\in\mathbb{C}|0<\|z\|<1\}$ with usual product and conjugation.
3. $S=\{q\in\mathbb{H}|0<\|q\|<1\}$ with quaternion product and conjugation.

4. Admissibility in $\mathcal{S}$

In the weakly Mal'tsev category $\mathcal{S}$, of conjugation semigroups with cancellation and semigroup homomorphisms preserving conjugation, we are going to give a characterization of all admissible diagrams.

Theorem 4.1. *A diagram*

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "\beta"] & C \arrow[l, shift right, "g"'] \arrow[dl, "\gamma"] \\
 & D &
\end{tikzcd}$$

in $\mathcal{S}$, with $fr = qs = 1_B$ and $\alpha r = \beta = \gamma s$, is admissible if and only if the following conditions hold:

(Ad1) the equation

$$x + \overline{\beta(b)} + \beta(b) = \alpha(a) + \overline{\beta(b)} + \gamma(c)$$

has a unique solution for all $a \in A$ and $c \in C$ such that $f(a) = g(c) = b \in B$.

(Ad2) the equation

$$\alpha(a_1+a_2)+\overline{\beta(b_1+b_2)}+\gamma(c_1+c_2) = \alpha(a_1)+\overline{\beta(b_1)}+\gamma(c_1)+\alpha(a_2)+\overline{\beta(b_2)}+\gamma(c_2)$$

is satisfied for $a_1, a_2 \in A$ and $c_1, c_2 \in C$ such that $f(a_1) = g(c_1) = b_1 \in B$ and $f(a_2) = g(c_2) = b_2 \in B$.

Proof: If there exists a morphism $\varphi\colon A \times_B C \longrightarrow D$ in $\mathcal{S}$ such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$ then

$$\alpha(a) = \varphi e_1(a) = \varphi(a, sf(a) = \varphi(a, s(b))$$

$$\gamma(c) = \varphi e_2(c) = \varphi(rq(c), c) = \varphi(r(b), c)$$

and $\beta(b) = \varphi(r(b), s(b))$, for $f(a) = g(c) = b$.

Then $\varphi(a, c)$ is a solution of $(Ad1)$:

$\varphi(a, c) + \overline{\beta(b)} + \beta(b) = \varphi(a, c) + \overline{\varphi(r(b), s(b))} + \varphi(r(b), s(b))$

$= \varphi(a, c) + \varphi(\overline{r(b)}, \overline{sb)}) + \varphi(r(b), s(b))$

$= \varphi(a + \overline{r(b)} + r(b), c + \overline{s(b)} + s(b)) = \varphi(a + \overline{r(b)} + r(b), s(b) + \overline{s(b)} + c)$

$= \varphi(a, s(b)) + \varphi(\overline{r(b)}, \overline{s(b)}) + \varphi(r(b), c) = \alpha(a) + \overline{\beta(b)} + \gamma(c)$

because φ is a morphism of $\mathcal{S}$ and by 3.1 (2).

To prove $(Ad2)$ we use the previous result and 3.2.

$\alpha(a_1+a_2)+\overline{\beta(b_1+b_2)}+\gamma(c_1+c_2)=$
$=\varphi(a_1+a_2,c_1+c_2)+\overline{\beta(b_1+b_2)}+\beta(b_1+b_2)$
$=\varphi(a_1,c_1)+\varphi(a_2,c_2)+\overline{(\beta(b_1)+\beta(b_2))}+\beta(b_1)+\beta(b_2)$
$=\varphi(a_1,c_1)+\varphi(a_2,c_2)+\overline{\beta(b_2)}+\beta(b_2)+\beta(b_1)+\overline{\beta(b_1)}$
$=\varphi(a_1,c_1)+\overline{\beta(b_1)}+\beta(b_1)+\varphi(a_2,c_2)+\overline{\beta(b_2)}+\beta(b_2)$
$=\alpha(a_1)+\overline{\beta(b_1)}+\gamma(c_1)+\alpha(a_2)+\overline{\beta(b_2)}+\gamma(c_2)$.

Conversely, let us assume that $(Ad1)$ and $(Ad2)$ hold. Then we can define $\varphi(a,c)=x$ where x is the solution of the equation $(Ad1)$, such that $\varphi e_1=\alpha$ and $\varphi e_2=\gamma$. Indeed, $\varphi e_1(a)=\alpha(a)$, since $\varphi e_1(a)=\varphi(a,sf(a))$ satisfies the equation $(Ad1)$:

$\varphi(a,sf(a))+\overline{\beta f(a)}+\beta f(a)=\alpha(a)+\overline{\beta f(a)}+\gamma sf(a)=$
$=\alpha(a)+\overline{\beta f(a)}+\beta f(a)$.

By cancellation, we have that $\varphi(a,sf(a))=\varphi e_1(a)=\alpha(a)$.

Analogously,

$\varphi(rg(c),c)+\overline{\beta g(c)}+\beta g(c)=\alpha rg(c)+\overline{\beta g(c)}+\gamma(c)=$
$=\beta g(c)+\overline{\beta g(c)}+\gamma(c)=\gamma(c)+\overline{\beta g(c)}+\beta g(c)$.

So, $\varphi e_2(c)=\varphi(rg(c),c)=\gamma(c)$.

It remains to prove that φ is a semigroup homomorphism that preserves conjugation:

$\varphi(a_1,c_1)+\varphi(a_2,c_2)+\overline{\beta(b_1+b_2)}+\beta(b_1+b_2)=$
$=\varphi(a_1,c_1)+\varphi(a_2,c_2)+\overline{\beta(b_1)+\beta(b_2)}+\beta(b_1)+\beta(b_2)$
$=\varphi(a_1,c_1)+\varphi(a_2,c_2)+\overline{\beta(b_2)}+\beta(b_2)+\beta(b_1)+\overline{\beta(b_1)}$
$=\varphi(a_1,c_1)+\overline{\beta(b_1)}+\beta(b_1)+\varphi(a_2,c_2)+\overline{\beta(b_2)}+\beta(b_2)$
$=\alpha(a_1)+\overline{\beta(b_1)}+\gamma(c_1)+\alpha(a_2)+\overline{\beta(b_2)}+\gamma(c_2)$
$=\varphi(a_1+a_2,c_1+c_2)+\overline{\beta(b_1+b_2)}+\beta(b_1+b_2)$,
for $f(a_1)=g(c_1)=b_1$ and $f(a_2)=g(c_2)=b_2$.

Thus $\varphi(a_1+a_2,c_1+c_2)=\varphi(a_1,c_1)+\varphi(a_2,c_2)$.

Now we have that

$$\varphi(\overline{a},\overline{c})+\overline{\beta(\overline{b})}+\beta(\overline{b})=\alpha(\overline{a})+\overline{\beta(\overline{b})}+\gamma(\overline{c}),$$

but

$\overline{[\varphi(a,c)]}+\overline{\beta(\overline{b})}+\beta(\overline{b})=\overline{(\beta(b)+\beta(\overline{b})+\varphi(a,c))}=\overline{(\varphi(a,c)+\beta(\overline{b})+\beta(b))}=$
$=\overline{(\alpha(a)+\beta(\overline{b})+\gamma(c))}=\gamma(\overline{c})+\overline{\beta(\overline{b})}+\alpha(\overline{a})$,
being $\alpha(a)+\overline{\beta(b)}+\gamma(c)=\gamma(c)+\overline{\beta(b)}+\alpha(a)$, for all a,b and c such that $f(a)=b=g(c)$, that we prove next, and so $\overline{\varphi(a,c)}=\varphi(\overline{a},\overline{c})$.

Indeed,

$\alpha(a) + \overline{\beta(b)} + \gamma(c) = \varphi(a,c) + \overline{\beta(b)} + \gamma(c) =$
$= \varphi(a,c) + \varphi(\overline{r(b)}, \overline{s(b)}) + \varphi(r(b), s(b))$
$= \varphi(a + \overline{r(b)} + r(b), c + \overline{s(b)} + s(b))$
$= \varphi(r(b) + \overline{r(b)} + a, c + \overline{s(b)} + s(b))$
$= \varphi(r(b), c) + \varphi(\overline{r(b)}, \overline{s(b)}) + \varphi(a, s(b))$
$= \gamma(c) + \overline{\beta(b)} + \alpha(a),$
and this concludes the proof. ■

5. Internal structures in $\mathcal{M}$

We are going to consider the category $\mathcal{M}$ of conjugation monoids with cancellation and monoid homomorphisms that preserve conjugation.

This subcategory of $\mathcal{S}$ is weakly Mal'tsev. Also the characterization 4.1 is still valid for diagrams (1) in $\mathcal{M}$. To prove that it is enough to show that if $(Ad1)$ and $(Ad2)$ hold then $\varphi(0,0) = 0$.

And this indeed is the case, since if
$\varphi(0,0) + \overline{\beta(0)} + \beta(0) = \alpha(0) + \overline{\beta(0)} + \beta(0)$ in $\mathcal{S}$ then
$\varphi(0,0) + \overline{\beta(0)} = 0 + \overline{\beta(0)}$
because $\alpha(0) = \beta(0) = 0$ and so, by cancellation $\varphi(0,0) = 0$.

Let (f, r) be a Schreier split epimorphism in $\mathcal{M}$

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

we are going to investigate which $\mathcal{M}$-morphisms $h : X \to B$ induce in $\mathcal{M}$

(i) an internal reflexive graph,
(ii) an internal category or
(iii) an internal groupoid,

in the sense we make precise below.

Lemma 5.1. *Given a Schreier split epimorphism in $\mathcal{M}$, the Schreier retraction satisfies the equality*

$$q(\overline{a}) = f(\overline{a}) \cdot \overline{q(a)}.$$

Proof: We have that
$f(\overline{a}) \cdot \overline{q(a)} = q(rf(\overline{a}) + k\overline{q(a)} = q(\overline{kq(a) + rf(a)}) = q(\overline{a}).$ ■

So, together with 2.4 (a)-(e), this describes the behaviour of the Schreier retraction in $\mathcal{M}$.

Given $h : X \to B$ and the Schreier split epimorphism (f, r) in $\mathcal{M}$

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] \arrow[rr, bend left, "h"] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

when does h induce a reflexive graph in $\mathcal{M}$? That is, when does it induce a $\mathcal{M}$-morphism $\tilde{h} : A \to B$ such that

$$\tilde{h}k = h \text{ and } \tilde{h}r = 1_B?$$

If such an $\tilde{h}$ exists, since it is a morphism in $\mathcal{M}$ and $a = kq(a) + rf(a)$, we have

(1) $\tilde{h}(a) = \tilde{h}(kq(a) + rf(a)) = hq(a) + f(a)$;
(2) $\tilde{h}(\overline{a}) = \overline{\tilde{h}(a)}$.

Remark 5.2. *If $\tilde{h}$ exists it is defined by (1) and then (2) holds:*

$$\tilde{h}(\overline{a}) = \tilde{h}(\overline{kq(a) + rf(a)}) = \tilde{h}(rf(\overline{a}) + k\overline{q(a)}) = f(\overline{a}) + h\overline{q(a)}$$

and

$$\overline{\tilde{h}(a)} = \overline{hq(a) + f(a)} = f(\overline{a}) + h\overline{q(a)}.$$

Proposition 5.3. *In the category $\mathcal{M}$, given a Schreier split epimorphism*

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

a morphism $h : X \to B$ induces a reflexive graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r"'] \end{tikzcd}$$

if and only if it satisfies the condition

(C1) $h(b \cdot x) + b = b + h(x)$.

Proof: If there exists $\tilde{h}$, such that $\tilde{h}k = h$ and $\tilde{h}r = 1_B$ then, for $b = f(a)$ and $x = q(a)$, we have that

$$
\begin{array}{rcl}
b+h(x) & = & f(a)+hq(a)\\
& = & \tilde{h}rf(a)+\tilde{h}kq(a)\\
& = & \tilde{h}(rf(a)+kq(a))\\
& = & \tilde{h}(kq(rf(a)+kq(a))+rf(rf(a)+kq(a)))\\
& = & \tilde{h}kq(rf(a)+kq(a))+\tilde{h}rf(a)\\
& = & hq(rf(a)+kq(a))+f(a)\\
& = & h(b\cdot x)+b
\end{array}
$$

Conversely, if h satisfies $(C1)$, then we have to prove that the map defined by
$\tilde{h}(a)=hq(a)+f(a)$ preserves addition and conjugation.

We have that

$$
\begin{array}{rcl}
\tilde{h}(a+a') & = & hq(a+a')+f(a+a)\\
& = & h(q(a)+q(rf(a)+q(a)))+f(a)+f(a'), (by\ 2.4(e))\\
& = & hq(a)+hq(rf(a)+q(a))+f(a)+f(a')\\
& = & hq(a)+h(f(a)\cdot q(a'))+f(a)+f(a')
\end{array}
$$

and

$$
\begin{array}{rcl}
\tilde{h}(a)+\tilde{h}(a) & = & hq(a)+f(a)+hq(a)+f(a')\\
(by\ (C1)) & = & hq(a)+h(f(a)\cdot q(a))+f(a)+f(a').
\end{array}
$$

Then, also $\overline{\tilde{h}(a)}=\tilde{h}(\overline{a})$ (see 5.2). ■

We observe that
$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$
is called a Schreier reflexive graph in [4], because (f,r) is a Schreier split epimorphism.

In [4] 3.2.3 it is proved that a Schreier reflexive graph in the category Mon of monoids,

$$\begin{tikzcd} X \arrow[r, shift right=1, "k"'] & X_1 \arrow[l, dashed, shift right=1, "q"'] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "s" description] \end{tikzcd}$$

with $ds=cs=1_{X_0}$ and (d,s) a Schreier split epimorphism in Mon, is an internal category if and only if the condition

$$(C)\ q(sc(x_1)+x)+x_1=x_1+x,$$

for $x_1 \in X_1$ and $x \in X = Ker(d)$, is satisfied. In this case the multiplication

$$X_1 \times_{X_0} X_1 \xrightarrow{m} X_1$$

is given by $m(x_1, x_1') = kq(x_1) + x_1'$.

Let us translate these conditions in terms of the Schreier reflexive graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$

considering f the domain and $\tilde{h}$ the codomain.

Then

$$\begin{tikzcd} X \arrow[r, "k"'] & A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$

is an internal category in Mon if and only if the condition

(C) $kq(r\tilde{h}(a) + k(x)) + a = a + k(x)$

is satisfied for all $x \in X$ and $a \in A$. That is

$$k(\tilde{h}(a) \cdot x) + a = a + k(x).$$

Then for $f(a) = \tilde{h}(a') = hq(a') + f(a')$,

$$m(a, a') = kq(a) + a',$$

where $m : A \times_B A \to A$ is the monoid homomorphism that gives the admissibility of the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[dr, equal] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "r"'] \arrow[d, "r"] & A \arrow[l, shift right, "\tilde{h}"'] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

in the category of monoids.

The fact that $me_1(a) = m(a, rf(a)) = a$ implies that

$$k(x) = m(k(x), rfk(x)) = m(k(x), 0).$$

Also $me_2(a) = m(r\tilde{h}(a), a) = a$.

Then, for (a, a') such that $f(a) = \tilde{h}(a')$,

$$\begin{aligned} m(a,a') &= m(kq(a)+rf(a),a') \\ &= m((ka(a),0)+(rf(a),a')) \\ &= m(kq(a),0)+m(r\tilde{h}(a'),a') \\ &= kq(a)+a'. \end{aligned}$$

So a monoid homomorphism m satisfying the prescribed conditions of admissibility has to be defined by $m(a,a') = kq(a)+a'$, for all $(a,a') \in A \times_B A$ and it is clear that $m(0,0) = kq(0)+0 = 0+0 = 0$.

Let us show that the existence of such a morphism m of monoids implies condition (C): since

$$(r\tilde{h}(a),a)+(x,0) = (r\tilde{h}(a)+x,a)$$

then

$$\begin{aligned} kq(r\tilde{h}(a)+k(x))+a &= m(r\tilde{h}(a)+k(x),a) \\ &= m(r\tilde{h}(a),a)+m(k(x),0) \\ &= a+k(x) \end{aligned}$$

Conversely, if condition (C) holds then

$$\begin{aligned} m((a_1,a_1')+(a_2,a_2')) &= m(a_1+a_2,a_1'+a_2') \\ &= kq(a_1+a_2)+a_1'+a_2'. \end{aligned}$$

$$\begin{aligned} m(a_1,a_1')+m(a_2,a_2') &= kq(a_1)+a_1'+kq(a_2)+a_2' \\ &= kq(a_1)+kq(r\tilde{h}(a_1')\cdot q(a_2))+a_1'+a_2', \ (by\ (C)) \\ &= k(q(a_1)+q(rf(a_1)\cdot q(a_2))+a_1'+a_2' \\ &= kq(a_1+a_2)+a_1'+a_2', \ (by\ 2.4(e)) \end{aligned}$$

and so m preserves addition.

Proposition 5.4. *In the category $\mathcal{M}$, given a Schreier split epimorphism*

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

a morphism $h : X \to B$ induces an internal category if and only if the following conditions hold:

(C_1) $h(b \cdot x)+b = b+h(x), \ \forall x \in X, \forall b \in B$

(C_2) $h(y)\cdot x+y = y+x, \ \forall x,y \in X.$

Proof: We first verify that $(C_2) \Leftrightarrow (C)$ and so that, in presence of (C_1), we can conclude that the reflexive graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$

is an internal category in Mon.

$(C) \Rightarrow (C_2)$

If in $k(\tilde{h}(a) \cdot x) + a = a + k(x)$ we take $a = k(y)$ we get

$$k(\tilde{h}(k(y)) \cdot x) + k(y) = k(y) + k(x)$$

and so, from

$$k(\tilde{h}k(y) \cdot x + y) = k(y + x),$$

we conclude that $\tilde{h}k(y) \cdot x + y = y + x$, because k is injective.

$(C_2) \Rightarrow (C)$

$$\begin{aligned}
\tilde{h}(a) \cdot x + a &= \tilde{h}(kq(a) + rf(a)) \cdot x + kq(a) + rf(a)\\
&= (hq(a) + f(a)) \cdot x + kq(a) + rf(a)\\
&= hq(a) \cdot (f(a) \cdot x) + kq(a) + rf(a), \ (by\ (C_2))\\
&= kq(a) + f(a) \cdot x + rf(a)\\
&= kq(a) + q(rf(a) + k(x)) + rf(a)\\
&= kq(a) + rf(a) + k(x), \ (by\ 2.4(d))\\
&= a + k(x)
\end{aligned}$$

Consequently, if (C_1) and (C_2) are satisfied then h induces an internal category in the category Mon of monoids. And in $\mathcal{M}$? It remains to check that m preserves conjugation: for $(a, a') \in A \times_B A$, $m(\overline{a}, \overline{a'}) = kq(\overline{a})) + \overline{a'}$ and

$$\begin{aligned}
\overline{m(a, a')} &= \overline{kq(a) + a'}\\
&= \overline{a'} + k\overline{q(a)}\\
&= kq(r\tilde{h}(\overline{a'}) + k\overline{q(a)}) + \overline{a'}, \ (by\ (C))\\
&= kq(rf(\overline{a}) + k\overline{q(a)}) + \overline{a'}\\
&= k(f(\overline{a}) \cdot \overline{q(a)}) + \overline{a'}\\
&= kq(\overline{a}) + \overline{a'}, \ (by\ 5.1) \quad \blacksquare
\end{aligned}$$

Proposition 5.5. *In the category $\mathcal{M}$, given a Schreier split epimorphism*

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, shift right, dotted, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \end{tikzcd}$$

a morphism $h : X \to B$ induces an internal groupoid if and only if the following conditions hold:

(C_1) *$h(b \cdot x) + b = b + h(x),\ \forall x \in X, \forall b \in B$*
(C_2) *$h(y) \cdot x + y = y + x,\ \forall x, y \in X$*
(C_3) *X is a group and $-\overline{x} = \overline{(-x)}$.*

Proof: By (C_1) we have a reflexive graph in $\mathcal{M}$, $$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$, with $\tilde{h}(a) = hq(a) + f(a)$. Then condition ($C_2$) is equivalent to the fact that this reflexive graph is an internal category

$$\begin{tikzcd} A \times_B A \arrow[r, "m"] & A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$

with $m(a, a') = kq(a) + a'$, for $f(a)(= dom\ a) = \tilde{h}(a')(= cod\ a')$ and so $m(a, a') = a \circ a'$.

By 3.3.2 in [4] we known that this Schreier category is a groupoid in Mon if and only if X is a group.

Let us analyse how the inverses are defined in the "object of morphisms" A. For

$$\begin{tikzcd} A \times_B A \arrow[r, "m"] & A \arrow[loop above, "t"] \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r" description] \end{tikzcd}$$

$t(a)$ is the inverse of a, i.e.

$$\begin{tikzcd} dom(a) = f(a) \arrow[r, shift right, "a"'] & cod(a) = \tilde{h}(a) \arrow[l, shift right, "t(a)"'] \end{tikzcd}$$

and $t(a) \circ a = m(t(a), a) = 1_{f(a)} = rf(a)$, $a \circ t(a) = m(a, t(a)) = 1_{\tilde{h}(a)} = r\tilde{h}(a)$.

We define t by $t(a) = -kq(a) + r\tilde{h}(a)$. It has the right domain

$$\begin{aligned} dom(t(a)) &= ft(a) \\ &= f(-kq(a) + r\tilde{h}(a)) \\ &= -fkq(a) + fr\tilde{h}(a) \\ &= 0 + \tilde{h}(a) \\ &= \tilde{h}(a) \end{aligned}$$

and codomain

$$\begin{aligned} cod(t(a)) &= \tilde{h}(-kq(a)+r\tilde{h}(a)) \\ &= -\tilde{h}kq(a)+\tilde{h}r\tilde{h}(a) \\ &= -hk(a)+\tilde{h}(a) \\ &= -hq(a)+hq(a)+f(a) \\ &= f(a). \end{aligned}$$

And now we prove that $t(a)$ is the inverse of a:

$$\begin{aligned} m(a,t(a)) &= m(a,-kq(a)+r\tilde{h}(a)) \\ &= kq(a)-kq(a)+r\tilde{h}(a) \\ &= r\tilde{h}(a)=1_{\tilde{h}(a)} \end{aligned}$$

and

$$\begin{aligned} m(t(a),a) &= m(-kq(a)+r\tilde{h}(a),a) \\ &= kq(-kq(a)+r\tilde{h}(a))+a \\ &= kq(k(-q(a))+r(hq(a)+f(a))+a \\ &= k(-q(a))+a \\ &= -kq(a)+kq(a)+rf(a) \\ &= rf(a)=1_{f(a)} \end{aligned}$$

By 3.3.2 in [4] we know that t is a monoid homomorphism. We have to prove that t preserves conjugation and so that it is a morphism of $\mathcal{M}$.

We have that

$$\begin{aligned} t(\overline{a}) &= -kq(\overline{a})+r\tilde{h}(\overline{a}) \\ &= -kq(\overline{a})+r(hq(\overline{a})+f(\overline{a})) \\ &= -k(f(\overline{a})\cdot\overline{q(a)})+r(h(f(\overline{a})\cdot\overline{q(a)})+f(\overline{a}))\ (by\ 5.1) \\ &= k(f(\overline{a})\cdot(-\overline{q(a)}))+r(f(\overline{a})+h\overline{q(a)})\ (by\ (C_1)) \\ \overline{t(a)} &= \overline{-kq(a)+r(hq(a)+f(a))} \\ &= r(\overline{hq(a)+f(a)})+k(\overline{-q(a)}) \\ &= kq(r(\overline{hq(a)+f(a)})+k(\overline{-q(a)}))+rf(r(\overline{hq(a)+f(a)})+k(\overline{-q(a)})) \\ &= k(\overline{hq(a)+f(a)}\cdot(\overline{-q(a)})+r(f(\overline{a})+h\overline{q(a)})) \\ &= k(f(\overline{a})\cdot h\overline{q(a)}\cdot(\overline{-q(a)})+r(f(\overline{a})+h\overline{q(a)})) \\ &= k(f(\overline{a})\cdot(\overline{-q(a)}))+r(f(\overline{a})+h\overline{q(a)}) \end{aligned}$$

because $h(\overline{x})\cdot(\overline{-x})=(\overline{-x})$.

Indeed,

$$\begin{aligned} 0 &= \overline{x}+(-\overline{x}) \\ &= \overline{x}+(\overline{-x})\ (by\ (C_3)) \\ &= h(\overline{x})\cdot(\overline{-x})+\overline{x}\ (by\ (C_2)) \end{aligned}$$

and, since X is a group, $h(\overline{x})\cdot(\overline{-x})=-\overline{x}=(\overline{-x})$.

Conversely, if

$$\begin{tikzcd} A \times_B A \arrow[r, "m"] & A \arrow[loop above, "t"] \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "\tilde{h}"'] & B \arrow[l, "r"' description] \end{tikzcd}$$

is a groupoid in $\mathcal{M}$, and so in Mon, we know that $X = ker\ f$ is a group and that t is defined by

$$t(a) = -kq(a) + r\tilde{h}(a) = -kq(a) + r(hq(a) + f(a)).$$

Since $t(\overline{a}) = \overline{t(a)}$, in particular, when $a = k(x)$,

$$t(\overline{k(x)}) = -k(\overline{x}) + h(\overline{x}),$$

$$\overline{t(k(x))} = k(\overline{-x}) + h(\overline{x})$$

and $t(\overline{k(x)}) = \overline{t(k(x))}$ implies that $-k(\overline{x}) = k(\overline{-x})$, that is $k(-\overline{x}) = k(\overline{-x})$ and so $-\overline{x} = \overline{-x}$.

■

6. From local to global

To study internal structures in several contexts (protomodular, homological, semi-abelian categories) as well as to obtain there strong properties (see [17] and [18]) it is fundamental that they satisfy the so-called "Smith is Huq" condition: any two equivalent relations on an object centralize each other, or commute, in the sense of Smith-Pedicchio (see [20] and [19]), if and only if their normalizations commute in the sense of Huq ([8]).

The category $\mathcal{M}$ of conjugation monoids with cancellation satisfies the Smith is Huq condition with respect to Schreier equivalence relations, that is equivalence relations

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "i_R"' description] \end{tikzcd}$$

where (r_1, i_R), and consequently (r_2, i_R), is a Schreier split epimorphism.

Proposition 6.1. *Consider the following diagram in $\mathcal{M}$*

$$\begin{tikzcd}
& & Y \arrow[d, shift right, "l"'] & \\
& A \times_B C \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "p_1"'] & C \arrow[l, shift right, "e_2"'] \arrow[u, dotted, shift right, "q_g"'] \arrow[d, shift right, "g"'] \arrow[rdd, bend left, "\gamma"] & \\
X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q_f"'] \arrow[u, shift right, "e_1"'] \arrow[r, shift right, "f"'] \arrow[rrd, bend right, "\alpha"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"'] \arrow[rd, "\beta"] & \\
& & & D,
\end{tikzcd}$$

with (f, r) and (g, s) Schreier split epimorphisms with kernels X an Y, respectively, and $\alpha r = \beta = \gamma s$.

If αk and γl commute then there exists a unique morphism $\varphi\colon A \times_B C \to D$ such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$.

Proof: These relies on 5.5 in [15] where the result was proved for monoids with operations. This is not the case here because the unary operation does not preserve addition. The morphisms αk and γl Huq-commute if, for all $x \in X$ and $y \in Y$,

$$\alpha k(x) + \gamma l(y) = \gamma l(y) + \alpha k(x).$$

Then the morphism $\varphi : A \times_B C \to D$ is defined by

$$\varphi(a, c) = \alpha k q_f(a) + \gamma(c),$$

for all $a \in A$ and $c \in C$ such that $f(a) = g(c) = b$.

Indeed,

$$\begin{aligned}
\varphi(a, c) &= \varphi(k q_f(a) + r f(a), c) \\
&= \varphi(k q_f(a), 0) + \varphi(r f(a), c) \\
&= \varphi(k q_f(a), 0) + \varphi(r g(c), c) \\
&= \alpha k q_f(a) + \gamma(c)
\end{aligned}$$

is such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$. Furthermore, it was proved in 5.5 [15] that φ is a monoid homomorphism. It remains to prove that φ preserves conjugation.

For $(a,c) \in A \times_B C$,

$$\begin{array}{rcl}
\varphi(\overline{a},\overline{c}) & = & \alpha k q_f(\overline{a}) + \gamma(\overline{c}) \\
 & = & \alpha k q_f(\overline{k q_f(a) + r f(a)}) + \gamma(\overline{c}) \\
 & = & \alpha k q_f(r f(\overline{a}) + k\overline{q_f(a)}) + \gamma(l q_g(\overline{c}) + s g(\overline{c})) \\
 & = & \alpha k q_f(r f(\overline{a}) + k\overline{q_f(a)}) + \gamma l q_g(\overline{c}) + \gamma s g(\overline{c}) \\
 & = & \gamma l q_g(\overline{c}) + \alpha k q_f(r f(\overline{a}) + k\overline{q_f(a))}) + \alpha r f(\overline{a}), \text{ (}\gamma l \text{ and } \alpha k \text{ commute)} \\
 & = & \gamma l q_g(\overline{c}) + \alpha(k q_f(r f(\overline{a}) + k\overline{q_f(a))}) + r f(\overline{a})) \\
 & = & \gamma l q_g(\overline{c}) + \alpha(r f(\overline{a}) + k\overline{q_f(a))}), \text{ (}by \text{ 1.4 (}d\text{))} \\
 & = & \gamma l q_g(\overline{c}) + \alpha r f(\overline{a}) + \alpha k\overline{q_f(a)} \\
 & = & \gamma l q_g(\overline{c}) + \gamma s g(\overline{c}) + \alpha k\overline{q_f(a)} \\
 & = & \gamma(l q_g(\overline{c}) + s g(\overline{c})) + \alpha k\overline{q_f(a)}, \text{ ((}g,s\text{) is Schreier split epi)} \\
 & = & \gamma(\overline{c}) + \alpha k\overline{q_f(a)}
\end{array}$$

and

$$\overline{\varphi(a,c)} = \overline{\alpha k q_f(a) + \gamma(c)} = \gamma(\overline{c}) + \alpha k\overline{q_f(a)}. \quad \blacksquare$$

Given two equivalence relations R and S on an object X

$$\begin{tikzcd}
R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "i_R" description] \arrow[r, "i_S" description] & S \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "s_2"]
\end{tikzcd} ,$$

they commute if and only the diagram

$$\begin{tikzcd}
R \arrow[r, shift left, "r_2"] \arrow[dr, "r_1"'] & X \arrow[l, shift left, "i_R"] \arrow[r, shift right, "i_S"'] \arrow[d, equal] & S \arrow[l, shift right, "s_1"'] \arrow[dl, "s_2"] \\
 & X &
\end{tikzcd}$$

is admissible. Then if (r_2, i_R) and (s_1, i_S) are Schreier split epimorphisms, by 5.1. we conclude that if $r_1\text{ker}(r_2)$ and $s_2\text{ker}(s_1)$ commute in the sense of Huq then the Schreier equivalence relations R and S commute. Since the converse is always true in a weakly Mal'tsev category we conclude that $\mathcal{M}$ satisfies a relative "Smith is Huq" property:

Theorem 6.2. *In the category $\mathcal{M}$ of conjugation monoids with cancellation two Schreier equivalence relations on an object commute if and only if their normalizations commute.*

We consider now the case where just one of the split epimorphisms is a Schreier split epimorphism.

Proposition 6.3. *Let (f,r) be a Schreier split epimorphism with kernel k and retraction q in the category $\mathcal{M}$. Then for the diagram*

$$\begin{tikzcd}
 & A\times_B C \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "p_1"'] & C \arrow[l, shift right, "e_2"'] \arrow[d, shift right, "g"'] \arrow[rdd, bend left, "\gamma"] & \\
X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q"'] \arrow[u, shift right, "e_1"'] \arrow[r, shift right, "f"'] \arrow[rrd, bend right, "\alpha"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"'] \arrow[rd, "\beta"] & \\
 & & & D,
\end{tikzcd}$$

with $fr = gs = 1_B$, $\alpha r = \gamma s = \beta$, the following conditions are equivalent:

(i) There exists a morphism $\varphi : A\times_B C \to D$ such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$.

(ii) There exists a morphism $\varphi : A\times_B C \to D$ such that $\varphi < k, 0 >= \alpha k$ and $\varphi e_2 = \gamma$.

(iii) For all $x\in X$ and $c\in C$, $\alpha k(g(c)\cdot x)+\gamma(c) = \gamma(c)+\alpha k(x)$.

Proof: *(i)* ⇒ *(ii)* is obvious. To prove the converse we observe that since (f,r) is a Schreier split epimorphism then (k,r) is a jointly strongly epimorphic pair ([4], 2.1.6) and so, since $\varphi e_1 k = \varphi < k, 0 >= \alpha k$ and $\varphi e_1 r = \varphi e_2 s = \gamma s = \alpha r$ then $\varphi e_1 = \alpha$. And $\varphi e_2 = \gamma$.

The morphism φ, if it exists, is defined by

$$\varphi(a,c) = \alpha kq(a)+\gamma(c)$$

because

$$\begin{aligned}
\varphi(a,c) &= \varphi(kq(a)+rf(a),c)\\
&= \varphi(kq(a),0)+\varphi(rf(a),c)\\
&= \varphi(kq(a),0)+\varphi(rg(c),c)\\
&= \alpha kq(a)+\gamma(c).
\end{aligned}$$

Then $\varphi(a_1,c_1)+\varphi(a_2,c_2) = \alpha kq(a_1)+\gamma(c_1)+\alpha kq(a_2)+\gamma(c_2)$ and

$$\begin{aligned}
\varphi(a_1+a_2,c_1+c_2) &= \alpha k(q(a_1)+q(rf(a_1)+kq(a_2)))+\gamma(c_1+c_2)\\
&= \alpha kq(a_1)+\alpha kq(rg(c_1)+kq(a_2))+\gamma(c_1)+\gamma(c_2).
\end{aligned}$$

Thus $\varphi(a_1+a_2, c_1+c_2)=\varphi(a_1,c_1)+\varphi(a_2,c_2)$ if and only if

$$\alpha kq(rg(c_1)+kq(a_2))+\gamma(c_1)=\gamma(c_1)+\alpha kq(a_2)$$

that is if and only if (iii) holds.

And $(iii)\Leftrightarrow(i)$ because φ also preserves conjugation: the fact that $\overline{\varphi(a,c)}=\varphi(\overline{a},\overline{c})$ is equivalent to the identity

$$\alpha k(\overline{f(a)}\cdot\overline{q(a)})+\gamma(\overline{c})=\gamma(\overline{c})+\alpha k\overline{q(a)}$$

that is (iii) for $f(\overline{a})=g(\overline{c})$ and $x=\overline{q(a)}$. ■

Finally, if $A=C$ and $r=s$ we are in the case considered in Proposition 5.3 that is we have a reflexive graph

$$\begin{tikzcd}
X \arrow[r, shift right, "k"'] & X_1 \arrow[l, shift right, dashed, "q"'] \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & X_0 \arrow[l, "r" description]
\end{tikzcd}$$

with (f,r) a Schreier split epimorphism, that is induced by $h=gk$.

Then, taking $c=k(y)$ in (iii) we obtain

$$(iii)'\ \alpha k(h(y)\cdot x)+\gamma k(y)=\gamma k(y)+\alpha k(x)$$

for all $x,y\in X$ and so αk and γk “Huq-commute” up to the action of $h(y)$ on x.

Conversely, if $(iii)'$ holds then, since $c=kq(c)+fr(c)$,

$$\begin{aligned}
\alpha k(g(c)\cdot x)+\gamma(c) &= \alpha k(g(kq(c)+fr(c))\cdot x+\gamma(kq(c)+fr(c)))\\
&= \alpha k(hq(c)\cdot(f(c)\cdot x))+\gamma kq(c)+\gamma rf(c)\\
&= \gamma k(q(c))+\alpha k(f(c)\cdot x)+\alpha rf(c)\ (by (iii)'\ and\ q(c)=y)\\
&= \gamma k(q(c))+\alpha(k(f(c)\cdot x)+rf(c))\\
&= \gamma k(q(c))+\alpha(rf(c)+k(x))\ (by\ 1.4\ (d))\\
&= \gamma k(q(c))+\gamma rf(c)+\alpha k(x)\ (\alpha r=\gamma r)\\
&= \alpha(kq(c)+rf(c))+\alpha k(x)\\
&= \gamma(c)+\alpha k(x).
\end{aligned}$$

Thus we proved the following:

Proposition 6.4. *If, in Proposition 6.3, $A = C$ and $s = r$ then the diagram*

$$\begin{tikzcd}
 & A \times_B A \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "p_1"'] & A \arrow[l, shift right, "e_2"'] \arrow[d, shift right, "g"'] \arrow[ddr, bend left, "\gamma"] & \\
X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q_f"'] \arrow[u, shift right, "e_1"'] \arrow[r, shift right, "f"'] \arrow[drr, bend right, "\alpha"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "r"'] \arrow[dr, "\beta"] & \\
 & & & D,
\end{tikzcd}$$

is admissible if and only if

$$\alpha k(h(y) \cdot x) + \gamma k(y) = \gamma k(y) + \alpha k(x), \ for\ all\ x, y \in X.$$

7. Example

The set $\mathbb{H} \setminus \{0\}$ of non-zero quaternions is a non-commutative group for the usual multiplication. It is cancellative and has conjugation: for $q = a + bi + cj + dk$ the conjugate is $\overline{q} = a - bi - cj - dk$.

We recall that the norm of q is given by $\|q\| = \sqrt{q\overline{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$, is multiplicative, $\|pq\| = \|p\|\|q\|$, and $q^{-1} = \frac{\overline{q}}{\|q\|}$ for $q \neq 0$.

The sets $B = \{q \in \mathbb{H} \mid \|q\| = 1\}$ and $X = \{q \in \mathbb{H} \mid 0 < \|q\| \leq 1\}$ are a conjugation group ($q^{-1} = \overline{q}$) and a conjugation monoid, respectively, with cancellation.

We are going to construct a Schreier split epimorphism in $\mathcal{M}$. For that we consider the monoid action φ of B on X defined by $b \cdot x = bxb^{-1} = bx\overline{b}$, the semidirect product $X \rtimes_\varphi B$ that is the monoid with underlying set $X \times B$ and operation

$$(x_1, b_1)(x_2, b_2) = (x_1(b_1 \cdot x_2), b_1 b_2)$$

and the Schreier split epimorphism of monoids

$$\begin{tikzcd}
X \arrow[r, shift right, "<1{,}0>"'] & X \rtimes_\varphi B \arrow[l, dotted, shift right, "\pi_1"'] \arrow[r, shift right, "\pi_2"'] & B \arrow[l, shift right, "<0{,}1>"']
\end{tikzcd}$$

We obtain a Schreier split epimorphism in $\mathcal{M}$ provide we take $\overline{(x, b)} = (\overline{b} \cdot \overline{x}, \overline{b})$.

Indeed,

(1) $(x,b)\overline{(x,b)} = \overline{(x,b)}(x,b)$

$$\begin{aligned} (x,b)\overline{(x,b)} &= (x,b)(\overline{b}\cdot\overline{x},\overline{b}) \\ &= (x(b\cdot(\overline{b}\cdot\overline{x})),b\overline{b}) \\ &= (x(b\overline{b}\cdot\overline{x}),b\overline{b}) \\ &= (x\overline{x},1)\ (\textit{because } \overline{b}=b^{-1}) \\ \overline{(x,b)}(x,b) &= (\overline{b}\cdot\overline{x},\overline{b})(x,b) \\ &= ((\overline{b}\cdot\overline{x})(\overline{b}\cdot\cdot x),\overline{b}b) \\ &= (\overline{b}\cdot\overline{x}x,\overline{b}b) \\ &= (\overline{x}x,1) \end{aligned}$$

because the center of $\mathbb{H}\backslash\{0\}$ is $\mathbb{R}$ and so $\overline{b}(\overline{x}x)\overline{b}^{-1} = \overline{b}(\overline{x}x)b = \overline{b}b(\overline{x}x) = \overline{x}x$. And $x\overline{x} = \overline{x}x$.

(2) $(x,b)\overline{(y,c)}(y,c) = (y,c)\overline{(y,c)}(x,b)$

$$\begin{aligned} (x,b)\overline{(y,c)}(y,c) &= (x,b)(\overline{y}y,1) \\ &= (x(b\cdot\overline{y}y),b) \\ &= (x\overline{y}y,b)\ (\textit{because } \overline{y}y \textit{ is in the center of } \mathbb{H}) \\ (y,c)\overline{(y,c)}(x,b) &= (y\overline{y},1)(x,b) \\ &= (y\overline{y}(1\cdot x),b) \\ &= (y\overline{y}x,b) \end{aligned}$$

and $x\overline{y}y = y\overline{y}x$ in $\mathbb{H}$.

(3) $\overline{(x,b)(y,c)} = \overline{(y,c)}\ \overline{(x,b)}$

$$\begin{aligned} \overline{(x,b)(y,c)} &= \overline{(x(b\cdot y),bc)} \\ &= (\overline{bc}\cdot\overline{(x(b\cdot y))},\overline{bc}) \\ \overline{(y,c)}\ \overline{(x,b)} &= (\overline{c}\cdot\overline{y},\overline{c})(\overline{b}\cdot\overline{x},\overline{b}) \\ &= ((\overline{c}\cdot\overline{y})(\overline{c}\cdot(\overline{b}\cdot\overline{x})),\overline{c}\overline{b}) \end{aligned}$$

and

$$\begin{aligned} \overline{bc}\cdot\overline{(x(b\cdot y))} &= \overline{c}\overline{b}\cdot(\overline{(b\cdot y)}\overline{x}) \\ &= \overline{c}\overline{b}\cdot(\overline{(by\overline{b})}\overline{x}) \\ &= \overline{c}\overline{b}\cdot(b\overline{y}\overline{b}\overline{x}) \\ &= \overline{c}\overline{b}b\overline{y}\overline{b}\overline{x}bc\ ((\overline{c}\overline{b}^{-1} = \overline{\overline{c}\overline{b}} = bc) \\ &= \overline{c}\overline{y}\overline{b}\overline{x}bc\ (\overline{b}b=1) \end{aligned}$$

$$\begin{aligned}(\overline{c}\cdot\overline{y})(\overline{c}\cdot(\overline{b}\cdot\overline{x})) &= \overline{c}\cdot(\overline{y}(\overline{b}\cdot\overline{x}))\\ &= \overline{c}\cdot(\overline{y}\overline{b}\overline{x}b)\ (since\ b^{-1}=\overline{b})\\ &= \overline{c}\ \overline{y}\overline{b}\overline{x}bc\end{aligned}$$

and so (3) holds.

Thus $X \rtimes_{\varphi} B$ belongs to $\mathcal{M}$ and all monoids homomorphism in the given Schreier split epimorphism in Mon preserve conjugation giving rise to a Schreier split epimorphism in $\mathcal{M}$.

We define $h : X \to B$ by $h(x) = \frac{x}{\|x\|}$ which gives a monoid homomorphism (because the norm is multiplicative) that, furthermore, preserves conjugation.

The morphism h induces a reflexive graph in the sense of Proposition 5.3 because it satisfies condition (C_1):

$$h(b\cdot x)b = \frac{bxb^{-1}}{\|x\|}b = b\frac{x}{\|x\|} = bh(x).$$

But h does not satisfy (C_2) in Proposition 5.4 since, in general,

$$(h(y)\cdot x)y = \frac{y}{\|y\|}x(\frac{y}{\|y})^{-1}y \neq yx,$$

and so does not induce an internal category.

Summing up:

Example 7.1. Given X and B as above, the action φ of B on X defined by $\varphi(b)(x) = bxb^{-1}$ gives rise to a Schreier split epimorphism

$$\begin{tikzcd} X \arrow[r, shift right, "<1,0>"'] & X \rtimes_{\varphi} B \arrow[l, shift right, dotted, "\pi_1"'] \arrow[r, shift right, "\pi_2"'] & B \arrow[l, shift right, "<0,1>"'] \end{tikzcd}$$

in $\mathcal{M}$ defining $\overline{(x,b)} = (\overline{b}\cdot\overline{x},\overline{b})$ in the semidirect product. Then $h : X \to B$ defined by $h(x) = \frac{x}{\|x\|}$ is a morphism in $\mathcal{M}$ that induces a reflexive graph but not an internal category in the sense of Propositions 5.3 and 5.4, respectively.

References

[1] F. Borceux and D. Bourn, *Mal'cev, Protomudular, Homological and Semi-Abelian categories*, in: Mathematics and its Applications, vol. 566, Kluwer Academic Publishers, 2004.

[2] D. Bourn, *Mal'cev categories and fibrations of pointed objects*, Appl. Categ. Structures **4** (6) (1996) 307–327.

[3] D. Bourn and M. Gran, *Centrality and normality in protomodular categories*, Theory Appl. Categ. **9** (8) (2002) 151–165.

[4] D. Bourn, N. Martins-Ferreira, A. Montoli and M. Sobral, *Schreier split epimorphisms in monoids and in semirings* , Textos de Matemática, Universidade de Coimbra, 2013.

[5] D. Bourn, N. Martins-Ferreira, A. Montoli and M. Sobral, *Schreier split epimorphisms between monoids*, Semigroup Forum **88** (2014) 739–752.

[6] D. Bourn, N. Martins-Ferreira, A. Montoli and M. Sobral, *Monoids and pointed S-protomodular categories*, Homology, Homotopy and Applications **18** (1) (2016) 151–172.

[7] A. Carboni, J. Lambek and M. C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra **69** (1991) 271–284.

[8] S. A. Huq, *Commutator, nilpotency and solvability in categories*, Quart. J. Math. Oxford Ser. (2) **19** (1968) 363–389.

[9] G. Janelidze, *Internal crossed modules*, Georgian Math. J. **10** (2003) 99–114.

[10] G. Janelidze, L. Márki and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002) 367–386.

[11] N. Martins-Ferreira, *Low-dimensional internal structures in weacly Mal'cev sesquicategories*, PhD. Thesis, University of Cape Town, 2008.

[12] N. Martins-Ferreira, *Weakly Mal'cev categories*, Theory Appl. Categ. **21** (6) (2008) 91–117.

[13] N. Martins-Ferreira, *New wide classes of weakly Mal'tsev categories*, Appl. Categ. Structures **23** (5) (2015) 741–751.

[14] N. Martins-Ferreira, *On distributive lattices and Weakly Mal'tsev categories*, J. Pure Appl. Algebra **216** (2012) 1961–1963.

[15] N. Martins-Ferreira and A. Montoli, *On the "Smith is Huq" condition in S-protomodular categories*, Appl. Categ. Structures **25** (1) (2017) 59–75.

[16] N. Martins-Ferreira, A. Montoli and M. Sobral, *Semidirect products and crossed modules in monoids with operations*, J. Pure Appl. Algebra, **217** (2013) 334–347.

[17] N. Martins-Ferreira and T. Van der Linden, *A note on the "Smith is Huq" condition*, Appl. Categ. Structures **20** (2) (2012) 175–187.

[18] N. Martins-Ferreira and T. Van der Linden, *Further remarks on the "Smith is Huq" condition*, Appl. Categ. Structures **23** (4) (2015) 527–541.

[19] M. C. Pedicchio, *A categorical approach to commutator theory*, J. Algebra **177** (1995) 647–657.

[20] J. D. H. Smith, *Mal'cev varieties*, Lecture Notes in Mathematics **554**, Springer, 1976.

[21] T. Van der Linden, *Qu'est-ce l'admissibilité de Nelson Martins-Ferreira?*, Scripta-Ingenia **6** (2016) 3–9.

Ana Paula Garrão
Faculdade de Ciências e Tecnologia, Universidade dos Açores, Ponta Delgada, Portugal
E-mail address: `ana.po.garrao@uac.pt`

Nelson Martins-Ferreira
Departamento de Matemática, Escola Superior de Tecnologia e Gestão, and Centre for Rapid and Sustainable Product Development, Instituto Politécnico de Leiria, Portugal
E-mail address: `martins.ferreira@ipleiria.pt`

Margarida Raposo
Faculdade de Ciências e Tecnologia, Universidade dos Açores, Ponta Delgada, Portugal
E-mail address: `margarida.js.raposo@uac.pt`

Manuela Sobral
CMUC, Departmento de Matemática, Universidade de Coimbra, 3001-501 Coimbra, Portugal
E-mail address: `sobral@mat.uc.pt`