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OPTIMAL REGULARITY FOR A TWO-PHASE FREE BOUNDARY PROBLEM RULED BY THE INFINITY LAPLACIAN

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ABSTRACT: In this paper we consider a non-variational two-phase free boundary problem ruled by the infinity Laplacian. Our main result states that normalized viscosity solutions in B_1 are universally Lipschitz continuous in $B_{1/2}$, which is the optimal regularity for the problem. We make a new use of the Ishii-Lions' method, which works as a surrogate for the lack of a monotonicity formula and is bound to be applicable in related problems.

KEYWORDS: Optimal regularity, free boundary problems, infinity Laplacian, viscosity solutions.
AMS SUBJECT CLASSIFICATION (2010): Primary 35B65. Secondary 35R35, 35J60, 35J70, 35D40.

1. Introduction

A celebrated result, due to Alt, Caffarelli and Friedman in [1], states that any local minimizer of the functional,

$$J(u) := \int |Du|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u\leq 0\}} dx, \tag{1.1}$$

with $\lambda_+ > \lambda_- \geq 0$, is universally Lipschitz continuous; that is, the Lipschitz norm in $B_{1/2}$ of a normalized minimizer, $|u| \leq 1$ in B_1, is bounded by a universal constant. The proof is based on a rather powerful monotonicity formula, known in the literature as the Alt-Caffarelli-Friedman (ACF) monotonicity formula.

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Although important advances towards generalizing the ACF monotonicity formula have been carried out, e.g. in [2, 4, 3, 14, 16, 17], it seems that such an impressive tool is indeed restricted to problems governed by linear operators. In particular, it is still wildly open whether local minimizers of

$$J_p(u) := \int |Du|^p + \lambda_+^p \chi_{\{u>0\}} + \lambda_-^p \chi_{\{u\leq 0\}} dx, \tag{1.2}$$

with $p > 2$, are Lipschitz continuous. The best known regularity, obtained in [12], is Log-Lipschitz, that is, minimizers are continuous with a modulus of continuity of the type $\omega(\sigma) = \sigma \log(1/\sigma)$. As

$$\sigma < \sigma \log(1/\sigma) < \sigma^\alpha, \quad 0 < \sigma \ll 1,$$

this is weaker than Lipschitz continuity but implies the $C^{0,\alpha}$ Hölder continuity, for every $0 < \alpha < 1$. In the one-phase case, that is, under the assumption $u \geq 0$, the Lipschitz regularity of minimizers of (1.2) is known to hold, see [7].

Minimizers u_p of (1.2) solve a two-phase free boundary problem, in that they are p-harmonic in their phases, i.e., they solve

$$-\Delta_p u_p := -\mathrm{div}\left(|Du_p|^{p-2} Du_p\right) = 0 \quad \text{in } \{u_p > 0\} \cup \{u_p < 0\},$$

and satisfy the free boundary condition

$$\left((u_p)_\nu^+\right)^p - \left((u_p)_\nu^-\right)^p = \frac{1}{p-1}(\lambda_+^p - \lambda_-^p) \quad \text{on } \partial\{u_p > 0\} \cup \partial\{u_p < 0\}, \tag{1.3}$$

in the sense of measures, and classically along any differentiable piece of the free boundary $\partial\{u_p > 0\} \cup \partial\{u_p < 0\}$. Here $(u_p)_\nu^+$ and $(u_p)_\nu^-$ denote the normal derivatives in the inward direction to $\{u_p > 0\}$ and $\{u_p < 0\}$, respectively. For the non-variational version of this problem, no regularity results are available in the literature.

To understand what happens when $p \to \infty$, let u_p be a local minimizer of the energy functional J_p. Arguing as in [12], it is possible to derive uniform in p local Hölder estimates for $\{u_p\}$. Passing to the limit as $p \to \infty$, it is classical to verify that the limiting function u_∞ satisfies

$$-\Delta_\infty u_\infty := -\left\langle D^2 u_\infty \, Du_\infty, Du_\infty \right\rangle = 0 \quad \text{in } \{u_\infty > 0\} \cup \{u_\infty < 0\},$$

in the viscosity sense. Concerning the free boundary condition (1.3), it converges, at least heuristically, to

$$(u_\infty)_\nu^+ = \max\left\{(u_\infty)_\nu^-, \lambda_+\right\}. \tag{1.4}$$

In fact, we can rewrite (1.3) in the form

$$(u_p)^+_\nu = \max\left\{(u_p)^-_\nu, \lambda_+\right\} \mathcal{G}_p\left((u_p)^-_\nu, \lambda_+, \lambda_-\right),$$

where

$$\mathcal{G}_p(a,b,c) = \left[\left(\frac{a}{\max\{a,b\}}\right)^p + \frac{1}{p-1}\left(\frac{b}{\max\{a,b\}}\right)^p\left(1-\left(\frac{c}{b}\right)^p\right)\right]^{\frac{1}{p}},$$

for $a \geq 0$ and $0 \leq c < b$, and we have

$$\lim_{p\to\infty} \mathcal{G}_p = 1,$$

which is obvious if $a \geq b$ and, if $a < b$, follows from the fact that, for p large enough,

$$\frac{1}{2p} \leq \left(\frac{a}{b}\right)^p + \frac{1}{p-1}\left(1-\left(\frac{c}{b}\right)^p\right) \leq 1 + \frac{1}{p-1}.$$

Now, if at a free boundary point x_0, one has $(u_\infty)^-_\nu(x_0) \geq \lambda_+$, then, by the limiting free boundary condition (1.4) above,

$$(u_\infty)^+_\nu(x_0) = (u_\infty)^-_\nu(x_0),$$

that is, u crosses the free boundary in a differentiable fashion. In the complementary case $(u_\infty)^-_\nu(x_0) < \lambda_+$, it is straightforward to show that (1.4) is equivalent to

$$\max\left\{(u_\infty)^+_\nu, (u_\infty)^-_\nu\right\} = \lambda_+.$$

We are thus naturally led to the singular non-variational free boundary problem, ruled by the infinity Laplacian,

$$\begin{cases} -\Delta_\infty u &= 0 \quad \text{in } \Omega^\pm(u) \\ \max\{u^+_\nu, u^-_\nu\} &= \Lambda \quad \text{on } \mathcal{R}(u), \end{cases} \tag{1.5}$$

where $\Lambda > 0$ is given, and

$$\Omega^\pm(u) := \Omega^+(u) \cup \Omega^-(u); \qquad \mathcal{R}(u) := \partial\Omega^\pm(u) \cap B_1,$$

with

$$\Omega^+(u) := \{u > 0\} \cap B_1 \qquad \text{and} \qquad \Omega^-(u) := \{u < 0\} \cap B_1.$$

The main result we obtain is that *any* normalized viscosity solution of (1.5), in a sense to be detailed, is universally Lipschitz continuous. We stress that Lipschitz estimates are sharp for such a free boundary problem, as simple examples show. It is also timely to note that, since $\mathcal{R}(u)$ is unknown, such a gradient control is far from being obvious or easy to obtain. For related issues, where specific bounds are prescribed on unknown sets and a PDE is given in the complementary regions, we refer for instance to [9].

While it is clear that a function satisfying $\Delta_\infty u = 0$ in $\{u > 0\} \cup \{u < 0\}$ is locally Lipschitz continuous in its phases, the corresponding estimates degenerate as one approaches their (unknown) boundaries. Thus, the main difficulty when proving the optimal regularity for our problem is the Lipschitz regularity across the free boundary. Furthermore, the fact that the infinity Laplacian is elliptic only in the direction of the gradient causes major difficulties in the study of (sharp) regularity estimates for problems ruled by such an operator. Our strategy to obtain Theorem 2.1 relies on doubling variables, in the spirit of the Ishii-Lions' method [10], in a fashion carefully designed to match the structure of the infinity Laplacian. See [8, 11, 13] for more on this highly degenerate operator and also [15], for another free boundary problem involving it.

The paper is organized as follows. In the next section, we define precisely what we mean by a solution of (1.5) and state our main result. The rest of the paper is devoted to its proof; in section 3 we derive pointwise estimates for interior maxima of a certain function, which will be instrumental in the sequel; section 4 brings the definition of an appropriate barrier; the proof is carried out in section 5 and ultimately amounts to the analysis of an alternative.

2. Definition of solution and main result

We will consider very weak solutions of problem (1.5) for which we nevertheless obtain optimal regularity results. The appropriate notion is that of viscosity solution and we need to first recall the definition of jet, given, e.g., in [6].

Let $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $\hat{x} \in \Omega$. Denoting by $\mathcal{S}(n)$ the set of all $n \times n$ symmetric matrices, the second-order superjet of u at $\hat{x}$, $J^{2,+}_\Omega u(\hat{x})$, is the set of all ordered pairs $(p, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ such that

$$u(x) \leq u(\hat{x}) + \langle p, x - \hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x - \hat{x}\rangle + o\left(|x-\hat{x}|^2\right).$$

The subjet is defined by putting $J^{2,-}_\Omega u(\hat{x}) := -J^{2,+}_\Omega(-u)(\hat{x})$. For $\hat{x} \in \overline{\Omega}$, we also denote by $\overline{J}^{2,\pm}_\Omega u(\hat{x})$ the set of all pairs $(p, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ for which there exist sequences $x_n \in \Omega$ and $(p_n, X_n) \in J^{2,\pm}_\Omega u(x_n)$, such that $(x_n, p_n, X_n) \to (\hat{x}, p, X)$.

We are now ready to disclose in what sense the equation and the free boundary condition in (1.5) are to be interpreted.

Definition 2.1. *A lower semi-continuous function u is a viscosity subsolution of* (1.5) *in B_1 if the following two conditions hold:*

(i) *for each $x \in \Omega^{\pm}(u)$ and $(\xi, M) \in \overline{J}^{2,+}_{B_1} u(x)$, we have*

$$-\langle M\xi, \xi\rangle \leq 0;$$

(ii) *for each $x \in \mathcal{R}(u)$, $(\xi, M) \in \overline{J}^{2,+}_{B_1} u(x)$ and $t > 0$, we have*

$$u\left(x - t\frac{\xi}{|\xi|}\right) \geq -\Lambda t + o(t).$$

An upper semi-continuous function u is a viscosity supersolution of (1.5) *in B_1 if the following two conditions hold:*

(i) *for each $x \in \Omega^{\pm}(u)$ and $(\xi, M) \in \overline{J}^{2,-}_{B_1} u(x)$, we have*

$$-\langle M\xi, \xi\rangle \geq 0;$$

(ii) *for each $x \in \mathcal{R}(u)$, $(\xi, M) \in \overline{J}^{2,-}_{B_1} u(x)$ and $t > 0$, we have*

$$u\left(x + t\frac{\xi}{|\xi|}\right) \leq \Lambda t + o(t).$$

A continuous function u is a viscosity solution of (1.5) *in B_1 if is both a viscosity subsolution and a viscosity supersolution.*

Remark 2.1. *The equation is interpreted in the usual way in the context of the infinity Laplacian. Now, if $x \in \partial\{u > 0\}$ is a point of differentiability and, say, $(\xi, M) \in \overline{J}^{2,-}_{B_1} u(x)$, then*

$$\nabla u^+(x) \cdot \frac{\xi}{|\xi|} = \lim_{t\to 0} \frac{u\left(x + t\frac{\xi}{|\xi|}\right)}{t} \geq |\xi|,$$

which yields

$$u^+_\nu(x) = |\nabla u^+(x)| \geq |\xi|.$$

On the other hand, the free boundary condition (ii), along with the subjet estimate, gives

$$t|\xi| + \frac{1}{2}t^2 \left\langle M\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \right\rangle + o\left(t^2\right) \le u\left(x + t\frac{\xi}{|\xi|}\right) \le \Lambda t + o(t).$$

Dividing the above inequality by t and letting $t \to 0$ yields

$$\Lambda \ge |\xi|.$$

Thus, the interpretation of the free boundary condition given above is a (very) weak representative of the corresponding flux balance in (1.5)*.*

We can now state the main theorem of this article, the optimal regularity for viscosity solutions of (1.5).

Theorem 2.1 (Lipschitz regularity)**.** *Any viscosity solution u of* (1.5)*, in the sense of Definition 2.1, is locally Lipschitz continuous. Moreover, there exists a universal constant $C > 0$, depending only on dimension and Λ, such that, for all $x_0 \in B_{1/2}$,*

$$\sup_{B_\rho(x_0)} |u(x) - u(y)| \le C\|u\|_{L^\infty(B_1)}\rho,$$

for all $0 < \rho \le \frac{1}{10}$.

Remark 2.2. *It is worth pointing out that Theorem 2.1 still holds for the slightly more general problem*

$$\begin{cases} -\Delta_\infty u & \ge & 0 & \text{in } \Omega^+(u) \\ -\Delta_\infty u & \le & 0 & \text{in } \Omega^-(u) \\ \max\{u_\nu^+, u_\nu^-\} & = & \Lambda & on\ \mathcal{R}(u), \end{cases}$$

for which a Lipschitz estimate for viscosity solutions still follows from the strategy we put forward. See also [5] *for Lipschitz estimates for viscosity subsolutions of $-\Delta_\infty v = 0$.*

3. Pointwise estimates for interior maxima

In this section we start preparing for the proof of Theorem 2.1, by deriving pointwise estimates involving the intrinsic structure of the infinity Laplacian at interior maximum points of a certain continuous function. Such a powerful analytic tool will be used, so to speak, as a surrogate for the absence of a monotonicity formula in this non-variational two-phase free boundary problem.

Lemma 3.1. *Let $v \in C(B_1)$, $0 \le \omega \in C^2(\mathbb{R}^+)$ and set*

$$w(x,y) := v(x) - v(y) \quad \textit{and} \quad \varphi(x,y) := L\omega(|x-y|) + \varrho\left(|x|^2 + |y|^2\right),$$

with L, ϱ positive constants. If the function $w - \varphi$ attains a maximum at $(x_0, y_0) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}}$, then, for each $\varepsilon > 0$, there exist $M_x, M_y \in \mathcal{S}(n)$, such that

$$(D_x\varphi(x_0,y_0), M_x) \in \overline{J}^{2,+}_{B_{1/2}} v(x_0), \tag{3.1}$$

$$(-D_y\varphi(x_0,y_0), M_y) \in \overline{J}^{2,-}_{B_{1/2}} v(y_0), \tag{3.2}$$

and the estimate

$$\begin{aligned} &\langle M_x D_x\varphi(x_0,y_0), D_x\varphi(x_0,y_0)\rangle - \langle M_y D_y\varphi(x_0,y_0), D_y\varphi(x_0,y_0)\rangle \\ &\qquad \le 4\, L\omega''(\rho)\left(L\omega'(\rho) + \varrho\rho\right)^2 + 16\varrho\left(L^2\omega'(\rho)^2 + \varrho^2\right) \end{aligned} \tag{3.3}$$

holds, where $\rho = |x_0 - y_0|$.

Proof: Under the hypothesis of the lemma, let us consider a local maximum, $(x_0, y_0) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}}$, of $w - \varphi$. By [10, Theorem 3.2], for each $\varepsilon > 0$, there exist matrices $M_x, M_y \in \mathcal{S}(n)$ such that (3.1) and (3.2) hold, and

$$\begin{pmatrix} M_x & 0 \\ 0 & -M_y \end{pmatrix} \le A + \epsilon A^2$$

for

$$A := \begin{pmatrix} M_\omega & -M_\omega \\ -M_\omega & M_\omega \end{pmatrix} + 2\varrho\, I_{2n\times 2n},$$

where

$$\begin{aligned} M_\omega \;&:=\; L\,\omega''(|x_0-y_0|)\frac{(x_0-y_0)\otimes(x_0-y_0)}{|x_0-y_0|^2} \\ &\quad + L\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\left(I - \frac{(x_0-y_0)\otimes(x_0-y_0)}{|x_0-y_0|^2}\right). \end{aligned} \tag{3.4}$$

In particular, we have

$$\begin{aligned}
&\langle M_x D_x\varphi(x_0,y_0), D_x\varphi(x_0,y_0)\rangle - \langle M_y D_y\varphi(x_0,y_0), D_y\varphi(x_0,y_0)\rangle \\
&\le \langle M_\omega(D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0)), D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0)\rangle \\
&\qquad + 2\varrho\left(|D_x\varphi(x_0,y_0)|^2 + |D_y\varphi(x_0,y_0)|^2\right) + \epsilon\lambda, \qquad (3.5)
\end{aligned}$$

where

$$\lambda := \left\langle A^2\left(D_x\varphi(x_0,y_0), D_y\varphi(x_0,y_0)\right), \left(D_x\varphi(x_0,y_0), D_y\varphi(x_0,y_0)\right)\right\rangle.$$

Now, for $\nu := \frac{x_0-y_0}{|x_0-y_0|}$, we have

$$D_x\varphi(x_0,y_0) = L\,\omega'(\rho)\nu + 2\varrho x_0 \qquad (3.6)$$

and

$$-D_y\varphi(x_0,y_0) = L\,\omega'(\rho)\nu - 2\varrho y_0, \qquad (3.7)$$

and thus, with $\iota = 2(L\omega'(\rho)\rho^{-1} + \varrho)$, we have

$$D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0) = \iota(x_0 - y_0).$$

It then follows from (3.4) that

$$\begin{aligned}
&\langle M_\omega(D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0)), D_x\varphi(x_0,y_0) - D_y\varphi(x_0,y_0)\rangle \\
&= \iota^2\langle M_\omega(x_0-y_0), (x_0-y_0)\rangle = \iota^2 L\omega''(\rho)\rho^2 \\
&= 4L\omega''(\rho)\left(L\omega'(\rho) + \varrho\rho\right)^2. \qquad (3.8)
\end{aligned}$$

Moreover, observe that

$$\begin{aligned}
|D_x\varphi(x_0,y_0)|^2 + |D_y\varphi(x_0,y_0)|^2 = 2L^2\omega'(\rho)^2 &+ 4L\varrho\omega'(\rho)\rho \\
&+ 4\varrho^2(|x_0|^2 + |y_0|^2).
\end{aligned}$$

Using Cauchy's inequality, we obtain the estimate

$$4L\varrho\omega'(\rho)\rho \le \frac{(2L\omega'(\rho))^2}{2} + \frac{(2\varrho\rho)^2}{2} = 2L^2\omega'(\rho)^2 + 2\varrho^2\rho^2$$

and then

$$\begin{aligned}
|D_x\varphi(x_0,y_0)|^2 + |D_y\varphi(x_0,y_0)|^2 \le 4L^2\omega'(\rho)^2 &+ 2\varrho^2\rho^2 \\
&+ 4\varrho^2(|x_0|^2 + |y_0|^2).
\end{aligned}$$

Since $\max\{|x_0|, |y_0|, \rho\} \leq 1/2$, we obtain

$$|D_x\varphi(x_0, y_0)|^2 + |D_y\varphi(x_0, y_0)|^2 \leq 4(L^2\omega'(\rho)^2 + \varrho^2). \tag{3.9}$$

Finally, if $\lambda > 0$, choose

$$\epsilon = \frac{8\varrho\left(L^2\omega'(\rho)^2 + \varrho^2\right)}{\lambda},$$

otherwise choose ϵ freely. Using (3.8) and (3.9) in (3.5), together with this choice of ϵ, we obtain (3.3) and the proof is complete.

■

4. Building an appropriate barrier

In this section, we derive an ordinary differential estimate which will be used to derive geometric properties related to problem (1.5). For positive constants κ and θ, to be chosen later, we consider the barrier function

$$\omega(t) = t - \kappa\, t^{1+\theta} \quad \text{for} \quad 0 < t < 1. \tag{4.1}$$

Proposition 4.1. *Given positive parameters a, b, and d, there exist positive constants $\overline{L}$, κ and θ, depending only on such parameters and universal constants, such that*

$$aL^3\omega''(t)\omega'(t)^2 + bL^2\omega'(t)^2 + d < -1, \tag{4.2}$$

for all $L \geq \overline{L}$. Moreover, there holds

$$\omega(t) > 0, \quad \frac{1}{2} \leq \omega'(t) \leq 1 \quad \text{and} \quad \omega''(t) < 0, \tag{4.3}$$

for any $0 < t < 1$.

Proof: By direct computation, one obtains

$$-\,\omega''(t)\,\omega'(t)^2 = \kappa(1+\theta)\theta\left(t^{\theta-1} - 2\kappa(1+\theta)t^{2\theta-1} + \kappa^2(1+\theta)^2t^{3\theta-1}\right).$$

Hence, by choosing (and fixing hereafter) $1/2 \leq \theta \leq 1$, we obtain

$$\begin{aligned} -\omega''(t)\,\omega'(t)^2 &\geq \kappa(1+\theta)\theta\,(1 - 2\kappa(1+\theta)) \\ &\geq \frac{4\kappa}{3}\,(1 - 4\kappa) =: \overline{\kappa} > 0, \end{aligned}$$

provided $\kappa < 1/4$. In view of this and $\omega'(t) \le 1$, we obtain

$$aL^3\omega''(t)\omega'(t)^2 + bL^2\omega'(t)^2 + d < -a\overline{\kappa}L^3 + bL^2 + d.$$

Then, we select $\overline{L}$ large such that estimate (4.2) holds for every $L \ge \overline{L}$. The first and third estimates in (4.3) follow immediately. We conclude the proof by observing that

$$\omega'(t) \ge 1 - \kappa(1+\theta) \ge 1 - 2\kappa \ge \frac{1}{2}.$$

■

5. Proof of the main theorem

In this final section, we prove Theorem 2.1. The strategy is to assume, for the sake of contradiction, that the Lipschitz norm of viscosity solutions to (1.5) cannot be controlled universally. This means that, for a given $\Gamma > 0$, to be chosen universally large, we can find a viscosity solution $u = u_\Gamma$ of (1.5), such that

$$\Gamma \le [u]_{Lip}. \tag{5.1}$$

Hence, for constants L and ϱ, to be chosen later depending only on Γ and $\|u\|_{L^\infty(B_1)}$, assumption (5.1) implies the existence of a pair of points

$$(x_0, y_0) \in \overline{B_{1/2}} \times \overline{B_{1/2}}$$

such that

$$u(x_0) - u(y_0) - L\omega(|x_0 - y_0|) - \varrho(|x_0|^2 + |y_0|^2) > 0. \tag{5.2}$$

It follows from (5.2) that $x_0 \ne y_0$ and

$$\varrho(|x_0|^2 + |y_0|^2) \le 2\|u\|_{L^\infty(B_1)}. \tag{5.3}$$

Thus, in order to guarantee that x_0, y_0 are interior points in $B_{1/2}$, we select

$$\varrho := 9\|u\|_{L^\infty(B_1)}.$$

Next, we note that ω is twice continuously differentiable in a small neighborhood of $\eta := |x_0 - y_0| > 0$, and thus Lemma 3.1 guarantees the existence of

$$(\xi_x, M_x) \in \overline{J}^{2,+}_{B_{1/2}} u(x_0) \qquad \text{and} \qquad (\xi_y, M_y) \in \overline{J}^{2,-}_{B_{1/2}} u(y_0)$$

satisfying

$$\langle M_x\xi_x, \xi_x\rangle - \langle M_y\xi_y, \xi_y\rangle \le aL^3\omega''(\eta)\omega'(\eta)^2 + b\,L^2\omega'(\eta)^2 + d, \tag{5.4}$$

for universal positive parameters a, b and d. We have further used the fact that $\omega''(\eta) < 0$. Hence, by (5.4) and Proposition 4.1, there exists $\overline{L} \gg 1$, such that

$$\langle M_x \xi_x, \xi_x \rangle - \langle M_y \xi_y, \xi_y \rangle < -1, \tag{5.5}$$

for all $L \gg \overline{L}$.

In what follows, we want to prove that x_0 must belong to $\mathcal{R}(u) \cup \Omega^+(u)$ and y_0 has to belong to $\mathcal{R}(u) \cup \Omega^-(u)$. In addition,

$$\{x_0, y_0\} \cap \mathcal{R}(u) \neq \emptyset \quad \text{and} \quad \{x_0, y_0\} \cap \mathcal{R}(u) \neq \{x_0, y_0\}.$$

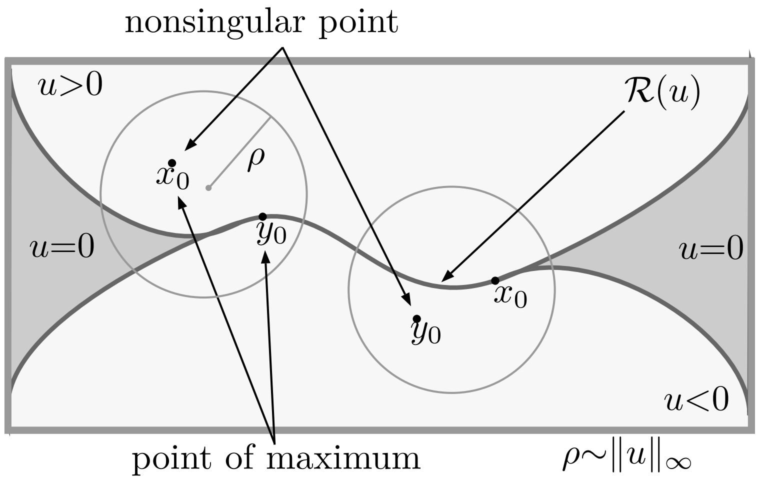
For that purpose, we initially note that (5.2) yields

$$u(x_0) - u(y_0) > 0. \tag{5.6}$$

Hence, if x_0 were to be in $\Omega^-(u)$, then y_0 would necessarily also belong to $\Omega^-(u)$. However, from Definition 2.1, u is infinity sub-harmonic in its negative phase, $\Omega^-(u)$. Thus, the LHS of (5.5) should be non-negative, which yields a contradiction.

Arguing similarly, if one assumes $y_0 \in \Omega^+(u)$, then x_0 would also have to be in $\Omega^+(u)$, and the same reasoning employed above would lead us to a contradiction. Likewise, the case $\{x_0, y_0\} \cap \mathcal{R}(u) = \emptyset$ is ruled out. Finally, from (5.6), we must have $\{x_0, y_0\} \cap \mathcal{R}(u) \neq \{x_0, y_0\}$.

We are now left with two cases to investigate. The following picture gives an impressionistic view of the subsequent analysis.



Case 1. Suppose $x_0 \in \Omega^+(u)$ and $y_0 \in \mathcal{R}(u)$. Since $(\xi_y, M_y) \in \overline{J}^{2,-}u(y_0)$, we have

$$u(x) \geq \langle \xi_y, x - y_0 \rangle + \frac{1}{2}\langle M_y(x-y_0),(x-y_0)\rangle + o\left(|x-y_0|^2\right),$$

for all $x \in B_{1/2}$. Hence, choosing $x = y_0 + t\,\xi_y/|\xi_y|$ and applying the free boundary condition from Definition 2.1, yields

$$\Gamma t + o(t) \geq \frac{t^2}{2|\xi_y|^2}\langle M_y \xi_y, \xi_y\rangle + t|\xi_y| + o(t^2), \tag{5.7}$$

for each $t < 1/2$. In addition, as $x_0 \in \Omega^+(u)$, estimate (5.5) yields

$$-\langle M_y \xi_y, \xi_y \rangle < -1. \tag{5.8}$$

Thus, from (5.7), we can further estimate

$$\Gamma t + o(t) \geq t|\xi_y| + o(t^2), \quad t < 1/2. \tag{5.9}$$

On the other hand, from (3.7), we obtain

$$\xi_y = L\,\omega'(\eta)\eta^{-1}(x_0 - y_0) - 2\varrho\, y_0,$$

and hence, from estimate (4.3), we know there holds

$$|\xi_y| \geq \frac{L}{2} - 2\varrho > 0. \tag{5.10}$$

Thus, dividing (5.9) by t, we reach

$$\Gamma + o(1) \geq O(L) + o(t). \tag{5.11}$$

Finally, if we choose L universally large and let $t \to 0$, a contradiction is obtained in (5.9).

Case 2. Suppose, alternatively, that $y_0 \in \Omega^-(u)$ and $x_0 \in \mathcal{R}(u)$. Arguing similarly as in Case 1, we obtain

$$\langle M_x \xi_x, \xi_x \rangle < -1. \tag{5.12}$$

Since $(\xi_x, M_x) \in \overline{J}^{2,+}u(x_0)$, by selecting $x = x_0 - t\,\xi_x/|\xi_x|$ in Definition 2.1, we obtain the estimate

$$-\Gamma t + o(t) \leq \frac{t^2}{2|\xi_x|^2}\langle M_x \xi_x, \xi_x \rangle - t|\xi_x| + o(t^2) \leq -t|\xi_x| + o(t^2), \tag{5.13}$$

where the last inequality follows from (5.12). In addition, from (3.6), we have

$$\xi_x = L\,\omega'(\eta)\eta^{-1}(x_0 - y_0) + 2\varrho\, x_0$$

and thus, from Proposition 4.1, we can estimate

$$|\xi_x| \leq L + 2\rho. \tag{5.14}$$

Finally, combining (5.13) with (5.14), yields an estimate similar to (5.11). Proceeding as in Case 1, we reach a contradiction. The proof of Theorem 2.1 is now complete.

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