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A CLASSIFICATION OF MONOTONE RIBBONS WITH FULL SCHUR SUPPORT WITH APPLICATION TO THE CLASSIFICATION OF FULL EQUIVALENCE CLASSES

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Abstract: We consider ribbon shapes, not necessarily connected, whose rows, with at least two boxes in each, are in monotone length order. These ribbons are uniquely defined by a pair of partitions: the row partition consisting of the row lengths in decreasing order, and the overlapping partition whose entries count the total number of columns with two boxes in the successive ribbon shapes obtained by sequentially subtracting the longest row. The support of such ribbon Schur functions, considered as a subposet of the dominance order lattice on partitions, has the row partition as bottom element, and, as top element, the partition whose two parts consist of the total number of columns, and the total number of columns of length two respectively. We give a complete system of linear inequalities in terms of the partition pair defining the aforesaid ribbon shape under which the ribbon Schur function attains all the Schur interval when expanded in the basis of Schur functions. We then conclude that the Gaetz-Hardt-Sridhar necessary condition for a connected ribbon to have full equivalence class is equivalent to the condition for a monotone connected ribbon to have full Schur support. That is, the set of partitions with full equivalence class is a subset of those monotone connected ribbons with full Schur support. M. Gaetz, W. Hardt and S. Sridhar conjectured that the necessary condition is also sufficient which translates now to every monotone connected ribbon with full Schur support has full equivalence class. The main tool of our analysis is the structure of the companion tableau of a ribbon Littlewood-Richardson (LR) tableau detected by the descent set defined by the composition whose parts are the ribbon row lengths.

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1. Introduction and statement of results

Littlewood-Richardson (LR) coefficients, non negative integers, arise in a variety of areas of mathematics [Fu00]. Determining its positivity without evaluating its actual value is of importance. There exists a variety of combinatorial models, collectively called Littlewood-Richardson rules (the original model conjectured in [LiRi34] and proved in [Sch77, Tho78]) to compute LR coefficients, and to show their positivity it is enough to exhibit an object in a chosen combinatorial model. Linear inequalities on triples of partitions guaranteeing their positivity have arisen from studying eigenvalues of a sum of Hermitian matrices [Ho62, Kl98, KnTa99, Fu00]. Given the skew partition $A := \lambda/\mu$, with $\mu \subseteq \lambda$ partitions, it is known that it uniquely defines a subposet $[\mathrm{r(A)}, \mathrm{c(A)}']$ in the dominance order lattice of partitions of $|A|$, the number of boxes of A, where the bottom element $\mathrm{r(A)}$ is the partition formed by the row lengths of A, and the top element $\mathrm{c(A)}'$ is the conjugate of the partition $\mathrm{c(A)}$ formed by the column lengths of A. The meaning of this interval is that, given the partition ν of $|A|$, the LR coefficient $c_A^\nu := c_{\mu,\nu}^\lambda > 0$ only if $\nu \in [\mathrm{r(A)}, \mathrm{c(A)}']$ and, in particular, $c_A^{\mathrm{r(A)}} = c_A^{\mathrm{c(A)}'} = 1$ (see,for instance, [Az99, Mc08] and references therein). Indeed it is not enough $\nu \in [\mathrm{r(A)}, \mathrm{c(A)}']$ to guarantee that $c_A^\nu > 0$ [KnTa99, Fu00].

The LR coefficient c_A^ν is a structure coefficient. It arises, for example, as the multiplicity of the Specht module S^ν in the decomposition of the skew Specht module S^A into irreducible representations of the symmetric group $\sum_{|A|}$,

$$S^A \cong \bigoplus_{\nu \in [\mathrm{r(A)}, \mathrm{c(A)}']} (S^\nu)^{\oplus c_A^\nu}; \tag{1.1}$$

and, in the algebra of symmetric functions, as a coefficient of the Schur function s_ν in the expansion of the skew Schur function s_A in the basis of Schur functions s_ν,

$$s_A = \sum_{\nu \in [\mathrm{r(A)}, \mathrm{c(A)}']} c_A^\nu s_\nu. \tag{1.2}$$

The expansion (1.2) is also the image of the character of S^A under the Frobenius characteristic map. Another way to look either at expansions (1.1) or (1.2) is that given $A = \lambda/\mu$, $\mu \subseteq \lambda$, they generate all possible positive LR coefficients $c_A^\nu := c_{\mu,\nu}^\lambda$. In view of these expansions, $[\mathrm{r(A)}, \mathrm{c(A)}']$ is then the Schur interval of the skew shape A, and the *Schur support* $[A]$ of the skew shape A is the set of partition shapes ν where either S^ν appears with positive multiplicity in (1.1) or s_ν appears with nonzero coefficient in (1.2),

$$[A] := \{\nu : c^\nu_A > 0\} \subseteq [\mathrm{r(A)}, \mathrm{c(A)}']. \tag{1.3}$$

The skew shape A is said to have *full Schur support* when in (1.3) the support coincides with the Schur interval.

A very general problem in the calculus of shapes is the classification of skew shapes A whose Schur support consists of the whole interval $[\mathrm{r(A)}, \mathrm{c(A)}']$ in the dominance order lattice of partitions. (See also Question 5.1 in [McWi12, Section 5].) In other words, given the partition ν of $|A|$, we ask under which conditions one has, $c^\nu_A > 0$ if and only if $\nu \in [\mathrm{r(A)}, \mathrm{c(A)}']$. In the special case of requiring all coefficients $c^\nu_A = 1$, the multiplicity free full interval, a classification was given in [ACM17]. We here give, in Theorem 1.7, a full Schur support classification for monotone ribbon shapes, not necessarily connected, with at least two boxes in each row, in terms of linear inequalities (1.8) satisfied by the partition pair (α, p) consisting of the row and overlapping partitions defining the monotone ribbon shape (see Proposition 3.3). The significance of this classification also amounts to the classification of connected ribbons with *full equivalence class* ([GaHaSr17, Definition 7]), that is, connected ribbons whose Schur support is invariant under any order rearrangement of the rows. More precisely, monotone connected ribbons with full equivalence class only exist among those with full Schur support. This is a recent input on our study of monotone ribbons having full Schur support and comes from the work by Gaetz, Hardt, Sridhar and Quoc Tran [GaHaSr17, GaHaSrTr17] where the support equality among connected ribbon Schur functions under any order rearrangement of the rows is addressed. The set of connected ribbons with full equivalence class has partitions as ribbon representatives. Lemma 1.11 shows that the Gaetz-Hardt-Sridhar necessary condition [GaHaSr17, Theorem II.1] for connected ribbons to have full equivalence class is equivalent to our classification, in Theorem 1.7, of monotone connected ribbons with full Schur support. Theorem 1.12 concludes that a monotone connected ribbon with full equivalence class has full Schur support. For monotone connected ribbons with at most four rows, ribbons with full equivalence class coincide with ribbons with full Schur support.

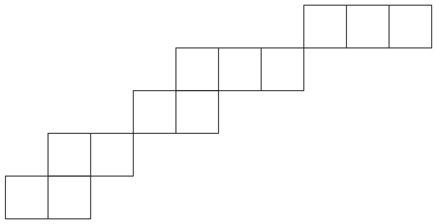
Earlier work on calculus of skew shapes are, for instance, Schur support containments by Pylyavskyy, McNamara and van Willigenburg [DoPy07, McWi12], skew shapes with the same Schur support or skew Schur function equalities by McNamara and van Willigenburg [Mc08, McWi09]. In particular, ribbon Schur functions were already considered by MacMahon [Mac17, 199–202] and Foulkes [Fo76] with representation-theoretic significance by the last. Finally, it is worth noting that Reiner, Shimozono [ReShi98] and R. I. Liu [Liu12] have considered Specht modules and, therefore, Schur functions for more general diagrams than skew shapes. However, apart percentage-avoiding diagrams [ReShi98], the combinatorial description of the coefficients for the Schur expansion is not known in general.

1.1. Overlapping partition of a monotone ribbon and descent set of a SYT. Arbitrary connected ribbons (diagrams corresponding to skew shapes containing no 2×2 rectangle) are in bijection with compositions assigning to the ribbon the row lengths. Thanks to the π-rotation symmetry of LR coefficients [St99, ACM09], the Schur support classification of LR monotone ribbons may be reduced to ribbons with row lengths in monotone decreasing order. Decreasing monotone ribbons with rows in length at least two, have at most columns of length two which occur exactly when two rows overlap: the overlapping partition p, read in reverse order, records sequentially, by accumulation, the number of columns of length two from the bottom to the top rows of the ribbon (see Section 3 and Definition 3.1). Proposition 3.3 shows that monotone ribbons, not necessarily connected, with at least two boxes in each row in monotone length order, are in bijection, up to an antipodal rotation, with partition pairs (α, p) where the $\ell(\alpha)$ parts of the row lengths partition $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$ are in length at least two, and the $\ell(p)$ parts of the overlapping partition $p = (p_1, \ldots, p_{\ell(\alpha)-1}, 0)$ are assigned by a multiset of $\{\ell(\alpha) - k, \ldots, 2, 1\}$ of cardinality $\ell(\alpha) - k \leq \ell(p) \leq \ell(\alpha) - 1$ with $k \in \{1, \ldots, \ell(\alpha)\}$. We often denote these ribbons by R^p_α, or just say the partition α with overlapping partition p to mean that p is the overlapping partition of the ribbon R^p_α. The Schur interval of our ribbon R^p_α, with $k = \ell(\alpha) - p_1$ connected components, is

$$[R^p_\alpha] \subseteq [\alpha, (|\alpha| - \ell(\alpha) + k, \ell(\alpha) - k)]. \tag{1.4}$$

Example 1.1. The partition pair $(\alpha = (3, 3, 2, 2, 2), p = (2, 2, 1, 1, 0))$ where $p_1 = \ell(\alpha) - 3 = 2$ and $\ell(p) = \ell(\alpha) - 1 = 4$, defines the monotone ribbon $R^p_{(3,3,2,2,2)}$, below, with 3 connected components, and Schur interval

$[(3,3,2,2,2),(10,2)]$,



. (1.5)

Our classification is based on the fact that given a monotone ribbon with row lengths at least two, defined by the partition pair (α, p), the existence of a companion tableau [LecLen17, Nak05, Appendix] for an LR filling of R^p_α with content ν, is equivalent to show that the triple of partitions α, p and ν satisfy a certain system of linear inequalities (1.6) in Theorem 1.5. The companion tableau of a LR connected ribbon R_α is detected by the descent set $S(\alpha) = \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{\ell(\alpha)-1}\}$ of its standardization (see sections 2.1 and 2.2). The following alternative description of the LR coefficients in the expansion (1.2) is known [Fo76, Ge84, Ge93], counting exactly standardized companion tableaux of connected LR ribbons.

Theorem 1.2. [Fo76, Ge84, Ge93]. Let α be any composition of N and R_α the corresponding connected ribbon shape. Then

$$s_{R_\alpha} = \sum_\nu d_{\nu,\alpha} s_\nu,$$

where ν runs on the set of partitions of N, and $d_{\nu,\alpha}$ is the number of standard Young tableaux (SYT) of shape ν and descent set $S(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell(\alpha)-1}\}$.

This means that given the connected ribbon R_α, the LR ribbon coefficient $c^\nu_{R_\alpha} = d_{\nu,\alpha}$ is positive if and only if there exists a semistandard Young tableau (SSYT) tableau of shape ν and content α whose standardization has descent set $\mathcal{S}(\alpha) = \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{\ell(\alpha)-1}\}$. For ordered compositions with parts of length at least two, we show, in Theorem 1.5, that the existence of such standard Young tableau guaranteeing the positivity of $c^\nu_{R_\alpha}$ is equivalent to require that the triple of partitions α, ν and $p = (\ell(\alpha)-1, \ldots, 2, 1, 0)$ satisfy a certain system of linear inequalities (1.6). More generally, we prove that the characterization is valid for monotone ribbons with k components by replacing the stair partition p of $\ell(\alpha)-1$ with a multiset of $\{\ell(\alpha)-k, \ldots, 1\}$ of cardinality $\ell(\alpha)-k \le \ell(p) \le \ell(\alpha)-1$, where $k \in \{1, \ldots, \ell(\alpha)\}$. Our method then consists of explicitly identifying in a SSYT of shape ν and content the partition α, the obstructions for being a companion tableau for a monotone LR ribbon, with the goal to remove them through a *rotation procedure* (see Subsection 4.2). This removal is possible whenever linear inequalities (1.6) are satisfied by the triple of partitions (α, p, ν). More precisely, the *effective* obstructions, detected by the overlapping partition p, correspond to some elements in $\mathcal{S}(\alpha)$ which are not in the descent set of the standardized tableau. Thus to exhibit the positivity of a such LR ribbon coefficient one just needs to exhibit a companion tableau for the ribbon LR filling. To minimize the number of obstructions that we have to deal with we work out on a SSYT with canonical filling (see Section 2.4).

1.2. Monotone ribbons: witness vectors and their slacks. Put $x_+ := max\{0, x\}$ where x is a real number. To a monotone ribbon R_α^p, we associate a sequence $\{\tilde{g}^i\}_{i=1}^{\ell(p)-1}$ of $\ell(p) - 1$ *witness vectors*, and to each witness $\tilde{g}^i$ we assign the *slack* $p_{i+1} - 1$, for $i \in \{1, \dots, \ell(p) - 1\}$.

Definition 1.3. Let α be a partition with parts at least two and with overlapping partition p. For each $i \in \{1,\dots, \ell(p)-1\}$, put $\varrho_i - 1 := \sum_{q=i+1}^{\ell(\alpha)} \alpha_q - p_{i+1} > 0$ the *rest of order* i of R_α^p, that is, the total number of columns in the last $\ell(\alpha) - i$ rows of R_α^p. Define the *i-witness vector* of R_α^p to be the nonnegative vector $\tilde{g}^i = (\tilde{g}_1^i, \dots, \tilde{g}_i^i)$ where $\tilde{g}_j^i := [\varrho_i - \alpha_j]_+$, $j = 1, \dots, i$. The *slack* of the i-witness vector is $p_{i+1} - 1$, for $i \in \{1, \dots, \ell(p) - 1\}$. If $\ell(p) = 0, 1$, R_α^p has no witness vectors.

The size $|\tilde{g}^i| := \sum_{j=1}^{i} [\varrho_i - \alpha_j]_+$ of the i-witness vector $\tilde{g}^i$ is said to fit its slack, if $|\tilde{g}^i| \leq p_{i+1} - 1$, otherwise is said to be oversized.

Remark 1.4. For $i \in \{1,\dots, \ell(p) - 1\}$, ϱ_i exceeds the total number of columns in the last $\ell(\alpha) - i$ rows of R_α^p. In any LR filling of R_α^p the $i+1$'s are filled in the last $\ell(\alpha) - i$ rows, and thereby its number is $< \varrho_i$. For $i \in \{1,\dots, \ell(p) - 1\}$, $\tilde{g}^i = 0$ if and only if $\alpha_i \geq \varrho_i$.

1.3. Statement of main results. Our key result is Theorem 1.5 which determines $c_{R_\alpha^p}^\nu > 0$ without determining its actual value. It gives a set of linear inequalities on the partition triple (α, p, ν) as necessary and sufficient conditions for the positivity of $c_{R_\alpha^p}^\nu$. The inequalities are explained by the combinatorial interpretation of $\alpha \preceq \nu$ in the dominance order on partitions (see Remark 2.2), and the obstruction of the overlapping partition p to the partitions dominating α. When $p = 0$, we have no such obstruction, $c^{\nu}_{R_\alpha}$ is a Kostka number, and $\alpha \preceq \nu$ characterizes completely the aforesaid positivity.

Theorem 1.5. Let α be a partition with parts at least two and overlapping partition $p = (p_1, \dots, p_{\ell(\alpha)-1}, 0)$, and ν a partition of $|\alpha|$. Then

$$c^{\nu}_{R^p_\alpha} > 0 \Leftrightarrow \begin{cases} \nu \in [\alpha, (|\alpha| - p_1, p_1)], \\ \nu_i \leq \displaystyle\sum_{q=i}^{\ell(\alpha)} \alpha_q - p_i, \ \text{ for } 1 \leq i \leq \ell(p). \end{cases} \tag{1.6}$$

In particular, when $\ell(p) = \ell(\alpha) - 1$, that is, $p_i = \ell(\alpha) - i$, $1 \leq i \leq \ell(\alpha)$, there exists a SYT of shape ν with descent set $\mathcal{S}(\alpha)$ if and only if the right hand side of (1.6) is satisfied.

The necessary and sufficient condition (1.6) is easily read: $\nu \in [\alpha, (|\alpha| - p_1, p_1)]$ is in the support of R^p_α if and and only if the $\nu_i < \varrho_{i-1}$, with $\varrho_0 := |\alpha| - p_1 + 1$, for $i = 1, \dots, \ell(p)$. With this on hand we give a criterion to decide when R^p_α has full Schur support, that is, when one has $c^{\nu}_{R^p_\alpha} > 0$ if and only if $\nu \in [\alpha, (|\alpha| - p_1, p_1)]$. The test assigns to each $i \in \{1, \dots, \ell(p) - 1\}$ the i-witness vector of R^p_α and compares its size with the slack $p_{i+1} - 1 \geq 0$. The existence of a single witness fitting its slack prevents the full Schur support because it can be used to construct a partition in the Schur interval but not in the support. This is the case of a witness of size zero, that is, when the partition α has $\alpha_i \geq \varrho_i$ for some $1 \leq i \leq \ell(p) - 1$.

Theorem 1.6. Let α be a partition with parts ≥ 2, and overlapping partition $p = (p_1, \dots, p_{\ell(\alpha)-1}, 0)$. Then $[R^p_\alpha] \subsetneq [\alpha, (|\alpha| - p_1, p_1)]$ if and only if $\ell(p) \geq 2$ and, for some $1 \leq i \leq \ell(p) - 1$, the size of the i-witness vector $\tilde{g}^i$ fits its slack, that is,

$$\sum_{j=1}^{i} [\varrho_i - \alpha_j]_+ \leq p_{i+1} - 1. \tag{1.7}$$

In this case, $\alpha_j + \tilde{g}^i_j \geq \varrho_i \geq p_{i+1} - 1 - |\tilde{g}^i|$, $j = 1, \dots, i$, whose decreasing rearrangement is the partition $(\alpha_1 + \tilde{g}^i_1, \dots, \alpha_i + \tilde{g}^i_i, \varrho_i, p_{i+1} - 1 - |\tilde{g}^i|)^+$ of $|\alpha|$ in the Schur interval of R^p_α but not in the support of R^p_α.

The equivalent statement for full Schur support is

Theorem 1.7. Let α be a partition with parts ≥ 2, and overlapping partition $p = (p_1, \dots, p_{\ell(\alpha)-1}, 0)$. Then $[R^p_\alpha] = [\alpha, (|\alpha| - p_1, p_1)]$ if and only if either $\ell(p) < 2$ or $\ell(p) \geq 2$ and, in this case, for every $1 \leq i \leq \ell(p) - 1$, the i-witness vector of R^p_α is oversized with respect to its slack, that is,

$$\sum_{j=1}^{i}[\varrho_i-\alpha_j]_+\geq p_{i+1},\ \ 1\leq i\leq \ell(p)-1. \tag{1.8}$$

Remark 1.8. R^p_α has full support only if

$$\alpha_i<\varrho_i\Leftrightarrow \alpha_i\leq \sum_{q=i+1}^{\ell(\alpha)}\alpha_q-p_{i+1},\quad 1\leq i\leq \ell(p)-1.$$

The following is a generalization of [GaHaSrTr17, Theorem 3.6] to monotone disconnected ribbons with $\ell(p)\leq 3$ which contain the monotone connected ribbons of length ≤ 4.

Corollary 1.9. In particular,
(a) when $p=(2,1,0^{\ell(\alpha)-2})$, $[R^p_\alpha]=[\alpha,(|\alpha|-2,2)]$ if and only if

$$\alpha_1<\varrho_1\Leftrightarrow \alpha_1<\sum_{q=2}^{\ell(\alpha)}\alpha_q. \tag{1.9}$$

(b) when $p=(3,2,1,0^{\ell(\alpha)-3})$, $[R^p_\alpha]=[\alpha,(|\alpha|-3,3)]$ if and only if

$$\alpha_1<\sum_{q=2}^{\ell(\alpha)}\alpha_q-2\ \text{ and }\ \alpha_2<\sum_{q=3}^{\ell(\alpha)}\alpha_q. \tag{1.10}$$

In [GaHaSr17, Theorem II.1], that we reproduce below as Theorem 1.10 for the reader convenience, a necessary condition is given for a connected ribbon with parts at least two, to have full equivalence class [GaHaSr17, Definition 7]. This necessary condition combined with Theorem 1.7 shows that a monotone connected ribbon with parts ≥ 2 has full equivalence class only if it has full Schur support. That is, full equivalence classes only exist among monotone connected ribbons with full Schur support.

Theorem 1.10. [GaHaSr17, Theorem II.1] Let α be a partition with parts ≥ 2 and R_α a connected ribbon. If α has full equivalence class then

$$N_j:=max\{k:\sum_{\substack{1\leq i\leq j\\ \alpha_i<k}}(k-\alpha_i)\leq \ell(\alpha)-j-2\}<\varrho_j,\ 1\leq j\leq \ell(\alpha)-2. \tag{1.11}$$

For monotone connected ribbons, inequality (1.11) is equivalent to inequality (1.8) in Theorem 1.7 characterizing full Schur support.

Lemma 1.11. For all $j \in \{1, \dots, \ell(\alpha) - 2\}$,

$$N_j := max\{k : \sum_{\substack{1 \le i \le j \\ \alpha_i < k}} (k - \alpha_i) \le \ell(\alpha) - j - 2\} < \varrho_j \Leftrightarrow \sum_{\substack{1 \le i \le j \\ \alpha_i < \varrho_j}} (\varrho_j - \alpha_i) \ge \ell(\alpha) - j - 1. \tag{1.12}$$

In addition, combining Theorem 1.7 with [GaHaSrTr17, Theorem 3.6], one has

Theorem 1.12. Let α be a partition with parts ≥ 2 and R_α a connected ribbon. If α has full equivalence class then R_α has full support. When $\ell(\alpha) \le 4$, α has full equivalence class if and only if R_α has full support.

Proofs of main results will be delayed until sections 4, 5 and 6.

1.4. Organization of the paper. This paper is organized in seven sections with the following contents. The next section, divided in seven subsections, contains the basic terminology, definitions and results that we shall be using throughout the paper. We highlight the concepts of descent set of a semistandard Young tableau *versus* SYT and Proposition 2.1 in Subsection 2.2, the combinatorial interpretation of dominance order on partitions, in Subsection 2.3, enlightening inequalities (1.6), and companion tableau of an LR tableau, in Subsection 2.6, our key tool in the proof of the existence of a monotone ribbon LR filling with given shape and content or the positivity of a ribbon LR coefficient.

Section 3 is divided in four subsections. Subsection 3.1 defines (Definition 3.1) and discusses overlapping partition of a ribbon, with row lengths at least two, that we shall use in the (connected or not) monotonic case, and, in the last section, in the connected case with row lengths in any order. It is shown that monotone ribbons not necessarily connected are uniquely defined by the row lengths partition and the overlapping partition. It is recalled in Subsection 3.2 that the descent set of a standard Young tableau detects the companion tableau of a LR connected ribbon. The enumerative characterization of LR connected ribbon coefficients $c^{\nu}_{R_\alpha}$ in Theorem 1.2 is generalized to disconnected ribbons.

Given T a SSYT of shape ν and weight α the descent set of the standardization of T is a subset of $\mathcal{S}(\alpha)$. As our study reduces to ribbons R_α with α a partition, the serious rejection for T to be a companion tableau for a LR ribbon of shape R_α occurs when it leads to a filling of R_α with the same letter in a column of length two. In Subsection 3.3, we translate the numbers in $\mathcal{S}(\alpha)$ and not in the descent set of the standardized T, giving rise to the aforesaid violation, to the *critical numbers set of* T, a subset of $\{2,\dots,\ell(\alpha)\}$. In addition, as our monotone ribbons may be disconnected, the overlapping partition is used to detect the effectiveness of the critical numbers of a companion tableau of a LR ribbon of shape R_α^p, as explained in Subsection 3.4.

Section 4 gives the proof of Theorem 1.5 which determines by means of a set of linear inequalities on the partition triple (α,p,ν), the positivity $c^\nu_{R^p_\alpha}>0$ without determining its actual value. Assuming the linear inequalities on the right hand side of (1.6), the goal is to exhibit a companion tableau for a LR filling of the shape R^p_α. The semistandard tableau of shape ν and weight α with *canonical filling* (Subsection 2.4) is picked, and then if necessary one modifies its filling according to a certain *rotation* procedure to avoid p-effective critical numbers so that the new tableau is a companion tableau of an LR filling with weight ν of the shape R^p_α. The linear inequalities on the right hand side of (1.6) guarantee that our rotation procedure is successful. Section 5 gives the proof of Theorem 1.6 and Theorem 1.7, logically equivalent, which classify the monotone ribbons with full Schur support, and Corollary 1.9 which gives a simple version of those inequalities in the case where the overlapping partition has at most length four. Illustrative examples are also provided.

In section 6, the bridge between the classification of monotone connected ribbons with full Schur support and those with full equivalence class [GaHaSr17] is established. More precisely, Lemma 1.11 shows that for monotone connected ribbons, the inequality (1.11), in Theorem 1.10, [GaHaSr17, Theorem II.1], giving a necessary condition for full equivalence class, is equivalent to the inequality (1.8), in Theorem 1.7, characterizing the full Schur support. The bridge allows to prove Theorem 1.12 which states that every partition with full equivalence class has full Schur support. Instances on the coincidence of these two classifications are provided. More importantly, Corollary 6.4 shows, as observed in Remark 6.5, that a non monotone connected ribbon of length three may have full Schur support while its monotone rearrangement does not have.

Section 7 generalizes, in Theorem 7.1, the necessary condition, in Theorem 1.5, for the LR coefficient $c^\nu_{R_\alpha}$ positivity, with α a partition, to connected ribbons R_β with β a composition. Remark 7.2 shows that if these inequalities on the triple (β,p,ν) with β a composition and p the overlapping partition of R_β, are also sufficient, then the classification on partitions having full equivalence class and full Schur support is the same, and, henceforth, the Gaez-Hardt-Shridar conjecture [GaHaSr17, Conjecture II.4] claiming that the necessary condition (1.11) for a partition to have full equivalence class is also sufficient, is true.

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2. Preliminaries

2.1. Partitions, compositions and tableaux. A partition λ is an ordered list of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0$ where λ_i are the *parts* and $\ell(\lambda)$ the *length* of λ. We say that $|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i$ is the *size* of λ and that λ is a partition of $|\lambda|$. It is convenient to set $\lambda_k = 0$ for $k > \ell(\lambda)$. The *Young diagram* of the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$, or Young diagram of shape λ, is the collection of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left-aligned rows, in the lower right quadrant of the plane, where the ith row has λ_i boxes, for $1 \leq i \leq \ell(\lambda)$. We shall identify a partition with its Young diagram. Given the partition λ, the conjugate or transpose partition λ' is the partition obtained by transposing the Young diagram of λ. A filling T of a Young diagram of shape λ with positive integers is called *semistandard* if the integers increase weakly across rows (row semistandard condition) and strictly down columns (column standard condition). Such a filled-in Young diagram of shape λ is called a *semistandard Young tableau* (SSYT) T of shape λ. The *weight* or *content* of a SSTY is the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$, where α_i is the number of integers i in the filling of the tableau.

A *composition* α with $\ell(\alpha)$ parts is a sequence of $\ell(\alpha)$ positive integers. The partition α^+ is the monotone nonincreasing rearranging of α. The size of α is defined to be $|\alpha| := |\alpha^+|$, in which case we say α is a composition of $|\alpha|$. The length of α is $\ell(\alpha) = \ell(\alpha^+)$. If $\beta = (\beta_1, \ldots, \beta_{\ell(\beta)})$ is another composition, we define the *concatenation of α and β* to be the composition $\alpha.\beta = (\alpha_1, \ldots, \alpha_{\ell(\alpha)}, \beta_1, \ldots, \beta_{\ell(\beta)})$ of length $\ell(\alpha) + \ell(\beta)$.

We denote by $Tab(\lambda, \alpha)$ the set of all SSYTs of shape λ and content the *composition* α. For λ a partition and α a composition of $|\lambda|$, the *Kostka number* $K_{\lambda,\alpha}$ is defined to be $K_{\lambda,\alpha} := \#Tab(\lambda, \alpha)$.

A *skew shape* or (*skew Young diagram*) λ/μ is obtained by removing the Young diagram μ from the top-left corner of the Young diagram λ, when μ is contained in λ as Young diagrams, or equivalently, when $\mu_i \leq \lambda_i$, for all $i \geq 1$. In particular, when μ is the empty partition 0, we have $\lambda/0 = \lambda$. The size of λ/μ is $|\lambda/\mu| := |\lambda| - |\mu|$. An *horizontal strip* is a skew diagram which has at most one box in each column. The *basic form* of a skew shape is the skew diagram obtained by deleting any empty row and any empty column. The skew shape λ/μ in the basic form defines the composition $\lambda - \mu$ that we simply write λ/μ if there is no danger of confusion. A skew shape is said to be *connected* if there exists a path between any two boxes of the diagram using only north, east, south and west steps such that the path is contained in the diagram. A SSYT of skew shape λ/μ and weight ν is a semistandard filling of the the skew-shape λ/μ of weight ν.

2.2. Descent set of a standard tableau. If a SSYT T of size n (n boxes) has entries in $[n] := \{1, 2, \ldots, n\}$, each necessarily appearing exactly once, then T is said to be a *standard Young tableau* (SYT).

A SSYT T in $Tab(\lambda, \alpha)$ may also be regarded as a sequence $0 = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^{\ell(\alpha)} = \lambda$ of partitions such that each skew shape λ^i/λ^{i-1} is an horizontal strip of size α_i. Simply insert an i in each box of the strip λ^i/λ^{i-1} [St99]. The *standard order* on a semistandard Young tableau is the numerical ordering of the labels with priority, in the case of equality, given by the rule southwest=smaller, northeast=larger. The standardization $\widehat{T}$ of a semistandard tableau $T \in Tab(\lambda, \alpha)$ is the enumeration of the labeled boxes according to the standard order of T, that is, the enumeration of the boxes across the sequence $0 = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^{\ell(\alpha)}$ where each horizontal strip λ^i/λ^{i-1} of size α_i is read SW-NE. For instance, the following are SSYT's with shape $\lambda = (4, 3, 2)$ and content $\alpha = (2, 4, 2, 1)$, and their standardizations, respectively:

$$T = \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2&3\\\cline{1-3} 3&4\\\cline{1-2}\end{array} \quad Q = \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2&4\\\cline{1-3} 3&3\\\cline{1-2}\end{array} \in Tab(\lambda, \alpha), \ \widehat{T} = \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&8\\\cline{1-3} 7&9\\\cline{1-2}\end{array} \quad \widehat{Q} = \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&9\\\cline{1-3} 7&8\\\cline{1-2}\end{array}. \tag{2.1}$$

The *descent set* $\mathcal{D}(U)$ of a SYT U of shape λ is defined to be the subset of $[|\lambda|-1]$ formed by those entries i of U for which $i+1$ appears in a strict lower row of U than i. There is a one-to-one natural correspondence between subsets of $[|\lambda|-1]$ and compositions of $|\lambda|$ [St99, Fo76]. The composition $\alpha=(\alpha_1,\ldots,\alpha_{\ell(\alpha)})$ gives rise to the subset $\mathcal{S}(\alpha):=\{\alpha_1,\alpha_1+\alpha_2,\ldots,\alpha_1+\alpha_2+\cdots+\alpha_{\ell(\alpha)-1}\}$, with cardinality $\ell(\alpha)-1$, of $[|\alpha|-1]$, and *vice-versa.* Hence a SYT of shape λ has descent set $\mathcal{S}(\alpha)$ for some composition α of $|\lambda|$. In (2.1), for example,

$$\mathcal{D}(\widehat{T})=\{2,6,8\}=\mathcal{S}(\alpha) \quad \text{and} \quad \mathcal{D}(\widehat{Q})=\{2,6\}\subsetneq\mathcal{S}(\alpha). \tag{2.2}$$

However, $\widehat{Q}$ is also the standardization of $V=\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&2&3\\ \cline{1-3} 3&3\\ \cline{1-2}\end{array}\in Tab(\lambda,\beta=(2,4,3))$ with $\beta=(\alpha_1,\alpha_2,\alpha_3+\alpha_4)$. In particular, $Tab((4,0),(2,2))$ has a sole element whose standardization has descent set the empty set, and $Tab((4,4),(4,2,2))$ has a sole element whose standardization has descent set $\{4\}\subsetneq\mathcal{S}(4,2,2)$.

Given $T\in Tab(\lambda,\alpha)$, the *descent set* $\mathcal{D}(T)$ of the SSYT T is the subset $\mathcal{S}$ of $\{1,\ldots,\ell(\alpha)-1\}$ that consists of $s\in\{1,\ldots,\ell(\alpha)-1\}$ for which there exists a pair of entries s and $s+1$ in T such that $s+1$ appears in a strict lower row of T than s. When T is a SYT, that is, $T\in Tab(\lambda,(1^{|\lambda|}))$, we recover the notion of descent set in a SYT, where $\mathcal{D}(T)$ is a subset $\mathcal{S}$ of $[|\lambda|-1]$. We show next that a SYT of shape λ has descent set $\mathcal{S}(\alpha)$ if and only if it is the standardization of some SSYT in $Tab(\lambda,\alpha)$ with descent set $\mathcal{S}=\{1,\ldots,\ell(\alpha)-1\}$. A SYT of shape λ has descent set $\mathcal{S}(\beta)\subseteq\mathcal{S}(\alpha)$ if and only if it is the standardization of some SSYT in $Tab(\lambda,\alpha)$ with descent set $\mathcal{S}\subseteq\{1,\ldots,\ell(\alpha)-1\}$ and $\mathcal{S}(\beta)=\{\sum_{j=1}^{s}\alpha_j : s\in\mathcal{S}\}\subseteq\mathcal{S}(\alpha)$.

Proposition 2.1. Given a partition λ and a composition α of $|\lambda|$, there exists a bijection between $Tab(\lambda,\alpha)$ and the set of all SYT's of shape λ with descent set a subset of $\mathcal{S}(\alpha)$, defined by the map $T\mapsto\widehat{T}$. Moreover, if $T\in Tab(\lambda,\alpha)$ and $\mathcal{S}=\{s_1<\cdots<s_{|\mathcal{S}|}\}$ then $\mathcal{D}(\widehat{T})=\{\sum_{j=1}^{s}\alpha_j : s\in\mathcal{S}\}=\mathcal{S}(\beta)\subseteq\mathcal{S}(\alpha)$ with $\beta=(\beta_1,\ldots,\beta_{|\mathcal{S}|},|\alpha|-\beta_{|\mathcal{S}|})$ such that $\beta_i-\beta_{i-1}=\sum_{j=1}^{s_i}\alpha_j-\sum_{j=1}^{s_{i-1}}\alpha_j$, $1\le i\le|\mathcal{S}|$ and $\beta_0:=0$.

Proof: Let $0=\lambda^0\subseteq\lambda^1\subseteq\cdots\subseteq\lambda^{\ell(\alpha)}=\lambda$ be the sequence of partitions defining $T\in Tab(\lambda,\alpha)$. The standardization $\widehat{T}$ of a SSYT $T\in Tab(\lambda,\alpha)$ is the enumeration of the boxes across the sequence $0=\lambda^0\subseteq\lambda^1\subseteq\cdots\subseteq\lambda^{\ell(\alpha)}$

defining T where each horizontal strip λ^i/λ^{i-1} of size α_i is read SW-NE. This means that $\widehat{T}$ is a SYT of shape λ and its descent set $\mathcal{D}(\widehat{T}) = \{\sum_{j=1}^{s}\alpha_j : s \in \mathcal{S}\} \subseteq \mathcal{S}(\alpha)$ where $\mathcal{S}$ consists of $s \in \{1,\dots,\ell(\alpha)-1\}$ for which the most SW box in λ^{s+1}/λ^s appears strictly below the most NE box in λ^s/λ^{s-1}.

Given U a SYT of shape λ and with descent set $\mathcal{S}(\beta) \subseteq \mathcal{S}(\alpha)$ for some composition β of $|\lambda|$, the standardization may be reversed to give a SSYT in $Tab(\lambda,\beta)$. A SYT of shape λ with descent set $\mathcal{S}(\beta)$ defines the sequence of partitions $0 = \theta^0 \subseteq \theta^1 \subseteq \cdots \subseteq \theta^{\ell(\beta)} = \lambda$ where each θ^j consists of the $\beta_1 + \cdots + \beta_j$ boxes of U with the entries given by $[\beta_1 + \cdots + \beta_j]$. Therefore filling each horizontal strip θ^j/θ^{j-1} with β_j j's, for all $j \in [\ell(\beta)]$ gives a SSYT in $Tab(\lambda,\beta)$. Because $\mathcal{D}(U) = \mathcal{S}(\beta) \subseteq \mathcal{S}(\alpha)$, given $j \in [\ell(\beta)]$, $\beta_j = \alpha_{k+1} + \cdots + \alpha_{k+d}$ for some $\{k+1,\dots,k+d\} \subseteq [\ell(\alpha)]$. Then we may fill the β_j boxes of the horizontal strip θ^j/θ^{j-1}, from SW-NE, with α_{k+1} $k+1$'s, α_{k+2} $k+2$'s, ..., α_{k+s} $k+d$'s to obtain a SSYT in $Tab(\lambda,\alpha)$. □

2.3. Dominance order on partitions. The *dominance order* on partitions of the same size n, is defined by setting $\lambda \preceq \mu$ if $|\lambda| = |\mu| = n$ and

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i,$$

for $i = 1,\dots,\min\{\ell(\lambda),\ell(\mu)\}$. Equivalently, the Young diagram of μ is obtained by *lifting* at least one box in the Young diagram of λ. Observe that $\lambda \preceq \mu$ if and only if $\mu' \preceq \lambda'$. The pair $(P_n,\preceq)$ with P_n the set of all partitions of n is a lattice with maximum element (n) and minimum element (1^n), and is self dual under the map which sends each partition to its conjugate. The interval $[\lambda,\mu]$ in P_n denotes the set of all partitions ν such that $\lambda \preceq \nu \preceq \mu$.

Remark 2.2. Note that if $\lambda \preceq \mu$, the inequalities $\mu_i \leq \sum_{q=i}^{\ell(\lambda)} \lambda_q = \lambda_i + \sum_{q=i+1}^{\ell(\lambda)} \lambda_q$, for $1 \leq i \leq \ell(\lambda)$, are always satisfied. For $1 \leq i \leq \ell(\lambda)$, either μ_i is obtained by lifting boxes from $(\lambda_{i+1},\dots,\lambda_{\ell(\lambda)})$ to λ_i, in which case, $\lambda_i \leq \mu_i \leq \lambda_i + \sum_{q=i+1}^{\ell(\lambda)} \lambda_q$, or μ_i is obtained by lifting boxes from λ_i to $(\lambda_1,\dots,\lambda_{i-1})$, in which case, $\mu_i \leq \lambda_i \leq \lambda_i + \sum_{q=i+1}^{\ell(\lambda)} \lambda_q$.

2.4. The canonical filling in $Tab(\nu,\alpha)$. Let α be an arbitrary composition and ν a partition such that $|\nu| = |\alpha|$. We exhibit a representative element of $Tab(\nu,\alpha)$, see also [JaVi17]. The proof provides an α-weight canonical filling of a Young diagram of shape ν. The canonical filling enjoys descent properties to be used later, see Section 3.3 and Proposition 3.13.

Lemma 2.3. Let α be any composition and ν a partition such that $\alpha^+ \preceq \nu$. Then, $K_{\nu,\alpha} > 0$ and $Tab(\nu, \alpha)$ has a *canonical* filling representative of the shape ν with weight α. It is constructed by filling horizontal strips greedily, from the bottom to the top of ν, starting with the longest columns, while rows are filled from right to left.

Proof: Assume that $\alpha^+ \preceq \nu$. Then $s := \ell(\alpha) \geq \ell(\nu) = m$, and the shape $\nu = (\nu_1, \ldots, \nu_m)$ has m rows and ν_1 columns. We will show by induction on s that we can construct a SSYT T of shape ν and weight α by filling horizontal strips greedily, from bottom to the top of ν, starting with the longest columns, while rows are filled from right to left. The case $s = 1$ is trivial. So, assume $s \geq 2$. If $\alpha = (\alpha_1, \ldots, \alpha_{s-1}, \alpha_s^+)$, then fill in, as above, α_s^+ entries of the shape ν with letters s. The remaining shape $\widehat{\nu}$ satisfy $(\alpha_1^+, \ldots, \alpha_{s-1}^+) \preceq \widehat{\nu}$ and, by the inductive step, there is a filling as above of the shape $\widehat{\nu}$ with content $(\alpha_1, \ldots, \alpha_{s-1})$. Therefore, there is also a filling of the shape ν as above, with content $(\alpha_1, \ldots, \alpha_{s-1}, \alpha_s^+)$.

Consider now the case $\alpha = (\alpha_1, \ldots, \alpha_s^+, \ldots, \alpha_{s-1}, \alpha_i^+)$, where the entry α_s^+ is in position $1 \leq k < s$. Let $(s; s-1)$ and $(k; s-1)$ be transpositions of the symmetric group S_s. Write $\widetilde{\alpha} = (s; s-1)(k; s-1)\alpha = (\alpha_1, \ldots, \alpha_{s-1}, \ldots, \alpha_i^+, \alpha_s^+)$. From the previous case, there is a filling of the shape ν with content $\widetilde{\alpha}$. Consider now the two bottom row strips filled with α_i^+ letters $s-1$, and α_s^+ letters s. We refill these strips first with α_i^+ letters s, and then with α_s^+ letters $s-1$, to obtain a filling of the shape ν with content $(\alpha_1, \ldots, \alpha_{s-1}, \ldots, \alpha_s^+, \alpha_i^+)$. Subtracting the strip filled with α_i^+, we get a shape $\widetilde{\nu}$ filled with content $(\alpha_1, \ldots, \alpha_{s-1}, \ldots, \alpha_s^+)$. By the inductive hypotheses, it can also be filled in the way described above with content $(\alpha_1, \ldots, \alpha_s^+, \ldots, \alpha_{s-1})$. Rejoining the strip α_i^+ we get the desired filling. $\square$

Example 2.4. Below are examples of SSYT's of partition shape with canonical filling:

$$
\begin{array}{|c|c|c|c|c|}
\hline 1 & 2 & 2 & 3 & 4 \\
\hline 3 & 5 & 5 & 6 & 6 \\
\hline 4 \\
\cline{1-1} 5 \\
\cline{1-1} 6 \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|c|c|}
\hline 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
\hline 2 & 3 & 3 & 4 & 4 & 5 & 5 \\
\hline 4 & 4 & 5 & 5 & 5 & 6 \\
\cline{1-6} 5 & 5 \\
\cline{1-2} 6 & 6 \\
\cline{1-2}
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 5 & 5 & 6 \\
\hline 3 & 4 & 5 & 5 & 5 & 6 & 6 \\
\cline{1-7} 5 & 5 & 6 & 6 & 6 & 7 \\
\cline{1-6} 6 & 6 \\
\cline{1-2} 7 & 7 \\
\cline{1-2}
\end{array}
$$

1	1	1	2	2	2
2	2	3	3	4	4
3	4	5	5	5	
4	5				
5					

The previous lemma gives a constructive proof of the *only if part* of (c) in the next proposition.

Proposition 2.5. [Fu97, Sa01, St99] Let α be a composition and ν a partition of $|\alpha|$. Then
(a) $K_{\nu,\alpha^+} = K_{\nu,\alpha}$,
(b) $K_{\alpha^+,\alpha} = 1$,
(c) $\alpha^+ \preceq \nu$ if and only if $K_{\nu,\alpha} > 0$.

For instance, in (2.1), $\alpha^+ = (4,2,2,1) \preceq \lambda$.

2.5. Skew-Schur functions, LR tableaux and Littlewood-Richardson rule. Let Λ denote the ring of symmetric functions in the variables $x = (x_1, x_2, \ldots)$ over $\mathbb{Q}$, say. The Schur functions s_λ form an orthonormal basis for Λ, with respect to the Hall inner product, and may be defined in terms of SSYT by

$$s_\lambda = \sum_T x^T = \sum_T x_1^{t_1} x_2^{t_2} x_3^{t_3} \cdots \in \Lambda, \tag{2.3}$$

where the sum is over all SSYT of shape λ and $t_i \geq 0$ is the number of occurrences of i in T [St99]. The notion of Schur functions can be generalized to apply to *skew shapes* λ/μ. Replacing λ by λ/μ in (2.3) gives the definition of the skew Schur function $s_{\lambda/\mu} \in \Lambda$ as a sum of monomial weights over all SSYTs of skew shape λ/μ. We identify $s_{\lambda/\mu}$ with the skew Schur function indexed by the skew Young diagram in the basic form.

The *reading word* w of a SSYT T is the word obtained by reading the entries of T from right to left and top to bottom. If, for all positive integers i and j, the first j letters of w includes at least as many i's as $(i+1)'$s, then we say that w is a *Yamanouchi word.* Clearly, the content of a Yamanouchi word is a partition. Yamanouchi words of content ν are in bijection with standard Young tableaux of shape ν [Fu97, Section 5.3]. Each SYT U of shape ν specifies a Yamanouch word $w_U = w_1 \cdots w_{|\nu|}$ of content ν, in the alphabet $[\ell(\nu)]$, where the number $u \in [|\nu|]$ is in the w_uth row of the SYT, and this map is one-to-one. Moreover, one has $w_j \geq w_{j+1}$ unless $j \in \mathcal{D}(U)$

in which case $w_j < w_{j+1}$. In (2.1), for example,

$$w_{\widehat{T}} = 11\,2211\,32\,3 \quad \text{and} \quad w_{\widehat{V}} = 11\,2211\,332 \tag{2.4}$$

are Yamanouchi words of content $\nu = (4,3,2)$, where $\mathcal{D}(\widehat{T}) = \{2,6,8\}$ and $\mathcal{D}(\widehat{V}) = \{2,6\}$.

A Littlewood–Richardson (LR) tableau [LiRi34] is a SSYT whose reading word is Yamanouchi. We denote by $\mathcal{LR}(\lambda/\mu,\nu)$ the set of all LR tableaux of shape λ/μ and content ν. When μ is empty, $\lambda = \nu$ and the LR tableau of shape ν and content ν, denoted $Y(\nu)$, is called the Yamanouchi tableau of shape ν. In fact, $Y(\nu)$ is the unique SSYT of shape and content ν, precisely, the SSYT that is filled with i's in row i. The structure constants $c^{\nu}_{\lambda/\mu}$ in the expansion (1.2) of the skew Schur function $s_{\lambda/\mu}$, in the basis of Schur functions, are given by the *Littlewood–Richardson rule* which states that the *Littlewood–Richardson coefficient* $c^{\nu}_{\lambda/\mu} = \#\mathcal{LR}(\lambda/\mu,\nu)$, the number of LR tableaux with skew shape λ/μ and content ν [LiRi34, St99].

2.6. LR tableaux and companion tableaux. LR tableaux in $\mathcal{LR}(\lambda/\mu,\nu)$ can be replaced by their *companion tableaux* which are certain SSYTs in $Tab(\nu,\lambda/\mu)$ whose standardizations encode the Yamanouchi reading words of the LR tableaux in $\mathcal{LR}(\lambda/\mu,\nu)$. Given $G \in Tab(\nu,\lambda/\mu)$, the containment of the descent set of $\widehat{G}$ in $\mathcal{S}(\lambda/\mu)$ guarantees that the filling of λ/μ with Yamanouchi reading word $w_{\widehat{G}}$ satisfies the row semistandard condition. Thus any tableau $G \in Tab(\nu,\lambda/\mu)$ specifies through $\widehat{G}$ a filling of the skew shape λ/μ with the Yamanouchi reading word $w_{\widehat{G}}$ of content ν with the row semistandard condition satisfied but not necessarily the standard condition of the column filling. In addition, by Proposition 2.1, we know that, a filling of the skew shape λ/μ with a Yamanouchi reading word satisfying the row semistandard condition is encoded by a SYT of shape ν with descent set in $\mathcal{S}(\lambda/\mu)$. For example, the two Yamanouchi words in (2.4) give fillings for the skew shape $\lambda/\mu = (2,4,2,1)$ where all satisfy the row semistandard condition. The word $w_{\widehat{V}}$ does not garantee the column standard condition in the filling

$$w_{\widehat{T}} = 112211323,\ \begin{array}{|c|c|c|c|c|c|} \hline &&&&1&1\\ \hline &1&1&2&2&\\ \hline 2&3&&&&\\ \hline 3&&&&&\\ \hline \end{array}\ ;\ w_{\widehat{V}} = 112211332,\ \begin{array}{|c|c|c|c|c|c|} \hline &&&&1&1\\ \hline &1&1&2&2&\\ \hline 3&3&&&&\\ \hline 2&&&&&\\ \hline \end{array}\ ,$$

$$\begin{ytableau} \none & \none & \none & \none & \none & 1 & 1 \\ \none & \none & 1 & 1 & 2 & 2 \\ \none & 3 & 3 \\ 2 \end{ytableau} \quad .$$

Given $H \in \mathcal{LR}(\lambda/\mu, \nu)$ the *companion tableau* G of H is the SSYT in $Tab(\nu, \lambda/\mu)$ whose ν_i entries of each row i of G are the numbers of the rows of H where the ν_i i's are filled in. This defines a bijection between $\mathcal{LR}(\lambda/\mu, \nu)$ and a subset $\mathrm{LR}_{\nu,\lambda/\mu}$, of $Tab(\nu, \lambda/\mu)$ that sends $H \in \mathcal{LR}(\lambda/\mu, \nu)$ to $G \in \mathrm{LR}_{\nu,\lambda/\mu}$. Therefore, the LR coefficient in (1.2) also satisfies

$$c^{\nu}_{\lambda/\mu} = \#\mathcal{LR}(\lambda/\mu, \nu) = \#\mathrm{LR}_{\nu,\lambda/\mu}. \tag{2.5}$$

The set $\mathrm{LR}_{\nu,\lambda/\mu}$ may be characterized in several ways: by linear inequalities as in [GeZe86]; or observing that $Tab(\nu, \lambda/\mu)$ is a subset of the gl_n-crystal $B(\nu)$ consisting of all SSYTs of shape ν in the alphabet $[n] := \{1, \dots, n\}$, $n \geq \ell(\lambda)$, [Kwo09, BumSch16]. The highest weight element of $B(\nu)$ is $Y(\nu)$ and $\mathrm{LR}_{\nu,\lambda/\mu}$ consists of the vertices G in $B(\nu)$ such that $Y(\mu) \otimes G$ is a highest weight element of weight λ of $B(\mu) \otimes B(\nu)$ [Kwo09, Section 4.3].

Given $G \in Tab(\nu, \alpha)$, for each $1 \leq i \leq \ell(\nu)$, and $j \geq i$, let χ^i_j denote the multiplicity of letter j in row i of G. Note that, for $j = 1, \dots, \ell(\alpha)$, $\chi^i_j = 0$, whenever $1 \leq j < i$. Fix $\mu \subseteq \lambda$ so that $\alpha = \lambda/\mu$. One then has the bijection,

$$\begin{array}{rcl} \phi_{\lambda/\mu} : \mathrm{LR}_{\nu,\lambda/\mu} = \{G \in Tab(\nu, \alpha)\colon\ Y(\mu) \otimes G \approx_{gl_n} Y(\lambda)\ \} & \longrightarrow & \mathcal{LR}(\lambda/\mu, \nu) \\ G & \mapsto & \phi_{\lambda/\mu}(G), \end{array} \tag{2.6}$$

such that $\phi_{\lambda/\mu}(G)$ is the ν-weight semistandard filling of λ/μ by putting χ^i_j letters i, starting from the left, in row j of the skew-shape λ/μ, for $i = 1, \dots, \ell(\nu)$, and $j = 1, \dots, \ell(\alpha)$. The reading word of $\phi_{\lambda/\mu}(G)$ is precisely the Yamanouchi word of weight ν, $w_{\widehat{G}} = w_1 \cdots w_{\alpha_1} w_{\alpha_1+1} \cdots w_{\alpha_1+\alpha_2}\ w_{\alpha_1+\alpha_2+1} \cdots w_{\alpha_1+\cdots+\alpha_{\ell(\alpha)-1}} \cdots w_{|\alpha|}$. That is, $\mathrm{LR}_{\nu,\lambda/\mu}$ consists of those tableaux in $Tab(\nu, \alpha)$ assigning to the skew shape λ/μ a semistandard filling of content ν whose reading word is the Yamanouchi $w_{\widehat{G}}$ (hence an LR filling). Theorem 1.2 characterises $\mathrm{LR}_{\nu,\mathrm{R}_\alpha}$ in the case of connected ribbons R_α.

2.7. Schur support and symmetries. The definition (1.3) of Schur support of the skew shape λ/μ can be rephrased as follows: $\nu \in [\lambda/\mu]$ if and only if $\mathcal{LR}(\lambda/\mu, \nu) \neq \emptyset$, equivalently, $\mathrm{LR}_{\nu,\lambda/\mu} \neq \emptyset$.

LR coefficients satisfy a number of symmetries [St99, ACM09, AKT16], including: $c^{\nu}_{\lambda/\mu} = c^{\mu}_{\lambda/\nu}$, $c^{\nu}_{\lambda/\mu} = c^{\nu}_{(\lambda/\mu)^{\circ}}$ where $(\lambda/\mu)^{\circ}$ is the π-rotation of λ/μ, and $c^{\nu}_{\lambda/\mu} = c^{\nu'}_{\lambda'/\mu'}$. As a consequence $[\lambda/\mu] = [(\lambda/\mu)^{\circ}]$ and $[(\lambda/\mu)'] = [\lambda/\mu]'$ where

$$s_{\lambda/\mu} = s_{(\lambda/\mu)^{\circ}} \text{ and } s_{\lambda'/\mu'} = \sum_{\nu \in [\mathrm{r}(\lambda/\mu), \mathrm{c}(\lambda/\mu)']} c^{\nu}_{\lambda/\mu} s_{\nu'}.$$

The full support of one of the shapes λ/μ, $(\lambda/\mu)'$ or $(\lambda/\mu)^{\circ}$ implies the full support of any of the others. When λ/μ is not connected, and consists of two connected components A and B, and may themselves be either Young diagrams or skew Young diagrams, then the combinatorial definition of (skew) Schur function (2.3) gives [St99] $s_{\lambda/\mu} = s_A s_B = s_B s_A$. This means that a skew Schur function is invariant under permutation and rotation of the connected components.

3. Ribbons

A *ribbon* is a skew shape which does not contain a 2×2 block as a subdiagram and it is connected when each pair of consecutive rows intersects in exactly one column. Thus, any composition $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)})$ determines a unique connected ribbon consisting of $\ell(\alpha)$ rows (or *parts*) $< \alpha_i >$ of length α_i, for $i = 1, \dots, \ell(\alpha)$, from top to bottom.

Given the composition α, R_α will denote a ribbon (not necessarily connected) where row lengths from top to bottom are given by the parts of α and adjacent rows overlap in at most one column. If each row is at least two boxes in length then the column length is at most two otherwise the column length might be bigger than two. If β is another composition, the direct sum $R_\alpha \oplus R_\beta$ of the ribbons R_α and R_β, is the ribbon $R_{\alpha\cdot\beta}$ where the ribbons R_α and R_β have no edge in common. In general, R_α is a direct sum of connected ribbons unless otherwise stated.

3.1. Overlapping partition of a ribbon with parts at least two.

In this subsection, we only consider compositions α with parts ≥ 2, and therefore the ribbon R_α has columns of length at most two.

Definition 3.1. Let α be an arbitrary composition with parts ≥ 2. The *overlapping partition* of R_α is the partition $p = (p_1, p_2, \dots, p_{\ell(\alpha)-1}, 0)$, $\ell(p) \leq \ell(\alpha) - 1$, such that p_i is the number of columns of length two among the smallest $\ell(\alpha) - i + 1$ rows of R_α in lowest position, for $i = 1, \dots, \ell(\alpha)$.

When α is a partition, p_i is the number of columns of length two in the last $\ell(\alpha)-i+1$ rows of R_α for $i=1,\dots,\ell(\alpha)$.

Observe that $\sum_{j=i}^{\ell(\alpha)}\alpha_j^+-p_i$ is the number of columns of $R_\alpha\setminus(\bigcup_{j=1}^{i-1}<\alpha_j^+>)$, for $1\le i\le\ell(\alpha)$. In particular, $|\alpha|-p_1$ is the number of columns of R_α and thus the Schur interval of a ribbon R_α with overlapping partition p is $[\alpha^+,(|\alpha|-p_1,p_1)]$. When α is a partition, one obtains (1.4) as a special case of this interval.

Proposition 3.2. Let α be a composition with parts ≥ 2. For $1\le i\le\ell(\alpha)$, let $k_i\in\{1,\dots,\ell(\alpha)\}$ be the number of connected components (ribbons) of $R_\alpha\setminus(\bigcup_{j=1}^{i-1}<\alpha_j^+>)$. Then $p_i=\ell(\alpha)-(i-1)-k_i$, for $i=1,\dots,\ell(\alpha)-1$, with $0\le p_{\ell(\alpha)-1}\le 1$, and

$$p\subseteq(\ell(\alpha)-1,\dots,2,1,0)\subseteq(|\alpha|-\alpha_1^+=\sum_{j=2}^{\ell(\alpha)}\alpha_j^+,\dots,|\alpha|-\sum_{j=1}^{\ell(\alpha)-1}\alpha_i^+=\alpha_\ell^+(\alpha),0),\tag{3.1}$$

where the set of distinct entries of p is contained in $\{\ell(\alpha)-1,\ell(\alpha)-2,\dots,2,1,0\}$.

Proof: Observe that, $p_1=\ell(\alpha)-k_1\in\{0,1,2,\dots,\ell(\alpha)-1\}$ and by induction on $i\ge 1$, $p_i=\ell(\alpha)-(i-1)-k_i$ is the first entry of the overlapping partition of $R_\alpha\setminus(\bigcup_{j=1}^{i-1}<\alpha_j^+>)$, $1\le i\le\ell(\alpha)$. Henceforth $0\le p_{i+1}\le p_i\le\ell(\alpha)-i\le\sum_{j=i+1}^{s}\alpha_j^+$, for $i=1,\dots,\ell(\alpha)-1$. □

A ribbon R_α is connected if and only if $p_1=\ell(\alpha)-1$, otherwise $p_1\in\{0,1,2,\dots,\ell(\alpha)-2\}$. It is an horizontal strip if $p_1=0$. When $\alpha=\alpha^+$, a ribbon R_{α^+} (not necessarily connected) is uniquely defined by the partition α and its overlapping partition p and hence R_α^p denotes such ribbon. In fact, more can be said. It is shown next that monotone ribbons with at least two boxes in each row are in bijection with pairs of partitions (α,p) where the parts of p are assigned by a multiset of $\{\ell(\alpha)-k,\dots,1\}$ of cardinality $\ell(\alpha)-k\le\ell(p)<\ell(\alpha)$ with $k\in\{1,\dots,\ell(\alpha)\}$. Recall Example 1.1, $R_{(3)}\oplus$

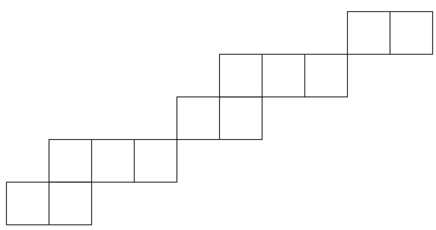
$R_{(3,2)} \oplus R_{(2,2)} = R^{(2,2,1,1,0)}_{(3,3,2,2,2)}$ is defined by the partition pair $(3,3,2,2,2)$ and $p = (2,2,1,1,0)$.

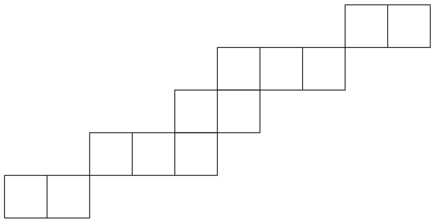
Proposition 3.3. Let α be a partition with parts ≥ 2 and let $k \in \{1, \ldots, \ell(\alpha)\}$. There is a bijection between ribbons R_α with k connected components and multisets of $\{\ell(\alpha) - k, \ldots, 2, 1\}$ of cardinality $\ell(\alpha) - k \leq \ell(p) \leq \ell(\alpha) - 1$ assigning the parts of the overlapping partition p.

Proof: Let R_α with k connected components. One has $p_1 = \ell(\alpha) - k$, and, for $2 \leq i \leq \ell(\alpha)$, $p_i = p_{i-1}$ if rows i and $i-1$ of R_α do not overlap, and $p_i = p_{i-1} - 1$ otherwise. In particular, $0 \leq p_{\ell(\alpha)-1} \leq 1$. Henceforth the parts of p form a multiset of $\{\ell(\alpha)-k, \ell(\alpha)-k-1, \ldots, 2, 1\}$ of cardinality $\ell(\alpha)-k \leq \ell(p) \leq \ell(\alpha) - 1$. Let R_α and $\tilde{R}_\alpha$ be two distinct ribbons (skew shapes do not coincide) with k connected components and overlapping partitions p and $\tilde{p}$ respectively. Let us choose the first $i \in \{2, \ldots, \ell(\alpha)\}$ such that rows i and $i-1$ in one of them overlap and in the other do not. Then $p_q = \tilde{p}_q$, for $1 \leq q \leq i-1$, and $p_i = p_{i-1} - 1$ and $\tilde{p}_i = p_{i-1}$ or reciprocally, and thus $p \neq \tilde{p}$.

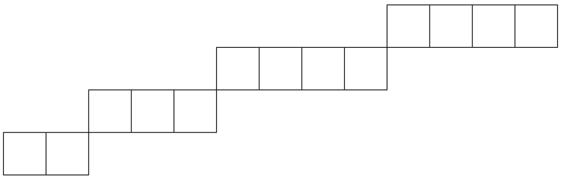
Let us consider a multiset of $\{\ell(\alpha) - k, \ell(\alpha) - k - 1, \ldots, 2, 1\}$ of cardinality $\ell(\alpha) - k \leq \ell(p) \leq \ell(\alpha) - 1$, and $p = (p_1, p_2, \ldots, p_{\ell(p)}, 0^{\ell(\alpha)-\ell(p)})$ the partition where $\{p_1, p_2, \ldots, p_{\ell(p)} = 1\}$ is the given multiset. We have to construct a ribbon R_α with k components and overlapping partition p. Put the last $\ell(\alpha) - \ell(p)$ rows of R_α pairwise disconnected and, observing that $p_{\ell(p)} = 1$, whenever $p_{i+1} = p_i$ rows i and $i+1$ of R_α do not overlap, and $p_{i+1} = p_i - 1$ otherwise for $1 \leq i \leq \ell(p)$. $\square$

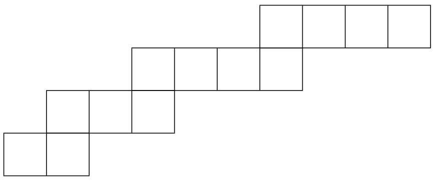
Remark 3.4. Observe that if α is not a partition, in general α and p do not uniquely define a disconnected ribbon with more than two connected components. For instance, below $\alpha = (2).(3,2).(3,2) = (2).(3,2,3).(2)$, and $R_{(2)} \oplus R_{(3,2)} \oplus R_{(3,2)}$, $R_{(2)} \oplus R_{(3,2,3)} \oplus R_{(2)}$ are distinct ribbons with the same overlapping partition $p = (2,1,0,0,0)$,

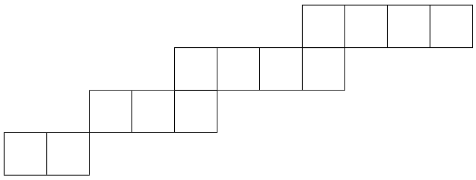
$R_{(2).(3,2).(3,2)} =$ 

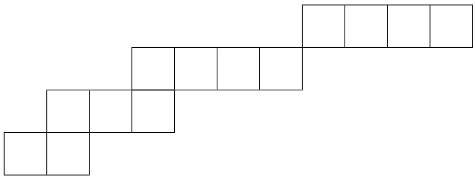
$R_{(2).(3,2,3).(2)} =$  .

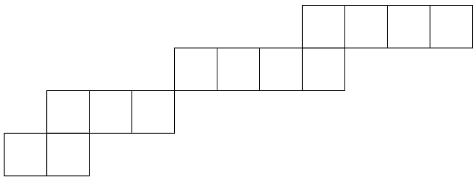
Example 3.5. (*a*) Ribbons with shape R^p_α, for $\alpha = (4,4,3,2) = \alpha^+$, $\ell(\alpha) = 4$:

$R^{(0,0,0,0)}_\alpha =$ 

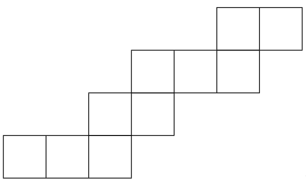
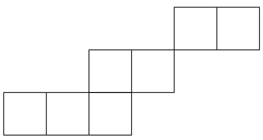
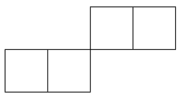
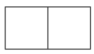
$R^{(3,2,1,0)}_\alpha =$ 

$R^{(2,1,0,0)}_\alpha =$ 

$R^{(2,2,1,0)}_\alpha =$ 

$R^{(2,1,1,0)}_\alpha =$ 

(*b*) $R_\alpha = R_{(2,3,2,3)}$ with $\ell(\alpha) = 4$ and $p = (3,1,0,0) \subseteq (3,2,1,0)$. The sequence of ribbons $R_\alpha \setminus (\cup_{j=1}^{i-1} < \alpha_j^+ >)$, $1 \leq i \leq \ell(\alpha)$, is depicted below

 ,  ,  , .

3.2. LR ribbons and companion tableaux. Let α be an arbitrary composition. As we have seen in (2.6), if one picks $G \in Tab(\nu, \alpha)$ to be the companion tableau of some LR ribbon in $\mathcal{LR}(R_\alpha, \nu)$ the Yamanouchi word $w_{\widehat{G}}$ has to guarantee in the filling of R_α the standard condition in the columns. The overlapping of two consecutive rows reduces to at most one column. Thus for ribbon shapes R_α one has just to avoid the violation of the standard condition on the overlapping row pairs which just occurs in one column. In other words, whenever, in R_α, rows α_k and α_{k+1} overlap then in the reading word $w_{\widehat{G}}$ the subword $w_{\alpha_1+\cdots+\alpha_k} w_{\alpha_1+\cdots+\alpha_k+1}$ is strictly increasing which means $\alpha_1 + \cdots + \alpha_k$ is a descent of $\widehat{G}$. In the case of connected ribbons R_α, this is exactly the content of Theorem 1.2: to avoiding the violation of the semistandard condition on the overlapping row pairs it requires the descent set of the standardization of the companion tableau to be equal to $\mathcal{S}(\alpha)$. To figure out what are the conditions to be imposed on the entries of a SSYT to be the companion of an LR ribbon, we take into account the bijection (2.6), whose domain we now extend to the set $Tab(\nu, \alpha)$. Thanks to Proposition 2.1 we may define the bijection

Definition 3.6. Let ν be a partition and α an arbitrary composition of $|\nu|$. Given $T \in Tab(\nu, \alpha)$, for each $1 \leq i \leq \ell(\alpha)$, and $j \geq i$, let χ^i_j denote the multiplicity of letter j in row i of T. Given a ribbon R_α, define the map

$$\varphi_{R_\alpha} : Tab(\nu, \alpha) \longrightarrow \left\{ \begin{array}{c} \nu\text{-Yamanouchi fillings of } R_\alpha \\ \text{with row semistandard condition} \end{array} \right\}$$

such that $\varphi_{R_\alpha}(T)$ is the filling of R_α by putting χ^i_j letters i in each row strip $< \alpha_j >$, starting from the left, for $i = 1, \ldots, \ell(\nu)$, and $j = 1, \ldots, \ell(\alpha)$, that is, the reading word of $\varphi_{R_\alpha}(T)$ is $w_{\widehat{T}}$.

Remark 3.7. When R_α is an horizontal strip, the map $\varphi_{R_\alpha} : Tab(\nu, \alpha) \longrightarrow \mathcal{LR}(R_\alpha, \nu)$ is a bijection and $c^\nu_{R_\alpha} = K_{\nu,\alpha}$.

Example 3.8. Let $\nu = (6, 4, 2)$ and $\alpha = (4, 2, 2, 2, 2)$.

(*a*) Let $T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 & 5 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} \end{array} \in Tab(\nu, \alpha)$ with $\mathcal{D}(\widehat{T}) = \mathcal{S}(\alpha)$ and $\chi^1_1 = 4, \chi^1_2 = 1, \chi^1_3 = 0, \chi^1_4 = 1, \chi^1_5 = 0, \chi^2_2 = 1, \chi^2_3 = 2, \chi^2_4 = 0, \chi^2_5 = 1, \chi^3_3 = 0, \chi^3_4 = 1, \quad \chi^3_5 = 1.$. Considering the overlapping sequence $p = (4, 3, 2, 1, 0)$ for α, we get the tableau

$$\varphi_{R^p_\alpha}(T) = \begin{array}{cccccccc} & & & & 1 & 1 & 1 & 1 \\ & & & 1 & 2 & & & \\ & & 2 & 2 & & & & \\ & 1 & 3 & & & & & \\ 2 & 3 & & & & & & \end{array} .$$

with Yamanouchi reading word $w_{\widehat{T}}$ satisfying both requirements of semistandard property. Thus, $\varphi_{R^p_\alpha}(T) \in \mathcal{LR}(R_{\alpha^p}, \nu)$, and T is the companion tableau of $\varphi_{R^p_\alpha}(T)$.

(b) Next, one exhibits the violation of the column semistandard condition of $\varphi_{R^p_\alpha}(T)$ in the two possible ways. Consider now $Q = \begin{array}{cccccc} 1 & 1 & 1 & 1 & 2 & 2 \\ 3 & 3 & 4 & 5 & & \\ 4 & 5 & & & & \end{array}$ and $V = \begin{array}{cccccc} 1 & 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 4 & 5 & & \\ 4 & 5 & & & & \end{array}$ in $Tab(\nu, \alpha)$ where $\mathcal{S}(\alpha) = \{4, 6, 8, 10\}$, $\mathcal{D}(\widehat{Q}) = \{6, 8, 10\} = \mathcal{S}(\alpha) \setminus \{4\}$, $w_{\widehat{Q}} = 1111\,11223232$, $w_4 = w_5$, and $\mathcal{D}(\widehat{V}) = \{4, 8, 10\} = \mathcal{S}(\alpha) \setminus \{6\}$, $w_{\widehat{V}} = 111122\,113232$, $w_6 > w_7$.

If $p = (4, 3, 2, 1, 0)$, the strict increasing filling along columns of $\varphi_{R^p_\alpha}(Q)$ and $\varphi_{R^p_\alpha}(V)$ fails in the overlapping of the rows $< \alpha_1 >$ and $< \alpha_2 >$, and $< \alpha_2 >$ and $< \alpha_3 >$, respectively:

$$\varphi_{R^p_\alpha}(Q) = \begin{array}{cccccccc} & & & & 1 & 1 & 1 & 1 \\ & & & 1 & 1 & & & \\ & & 2 & 2 & & & & \\ & 2 & 3 & & & & & \\ 2 & 3 & & & & & & \end{array} , \ \varphi_{R^p_\alpha}(V) = \begin{array}{cccccccc} & & & & 1 & 1 & 1 & 1 \\ & & & 2 & 2 & & & \\ & & 1 & 1 & & & & \\ & 2 & 3 & & & & & \\ 2 & 3 & & & & & & \end{array} \notin \mathcal{LR}(R^p_\alpha, \nu).$$

(i) In the first case, $w_{\alpha_1} = w_{\alpha_1+1}$, if we instead consider the overlapping sequence $\tilde{p} = (3, 3, 2, 1, 0)$, then Q becomes the companion tableau of $\varphi_{R^{\tilde{p}}_\alpha}(Q) \in \mathcal{LR}(R^{\tilde{p}}_\alpha, \nu)$.

(ii) In the second case, $w_{\alpha_1+\alpha_2} > w_{\alpha_1+\alpha_2+1}$, we keep p but change V to $U = \begin{array}{cccccc} 1 & 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & & \\ 4 & 5 & & & & \end{array}$, where $\mathcal{D}(\widehat{U}) = \{4, 6, 8, 10\}$, then U is the companion tableau of $\varphi_{R^p_\alpha}(U) \in \mathcal{LR}(R^p_\alpha, \nu)$,

$$\varphi_{R^{\tilde p}_\alpha}(Q) = \begin{array}{ccccccccc} &&&&&1&1&1&1\\ &&&1&1&&&&\\ &&2&2&&&&&\\ &2&3&&&&&&\\ 2&3&&&&&&& \end{array}, \quad \varphi_{R^p_\alpha}(U) = \begin{array}{cccccccc} &&&&1&1&1&1\\ &&&1&2&&&\\ &&1&2&&&&\\ &2&3&&&&&\\ 2&3&&&&&& \end{array}.$$

Example 3.9. (a) Let $\alpha = (3,3,2,3,3)$ and $\nu = (4,4,1)$. Let

$$Q = \begin{array}{cccc} 1&1&2&2\\ 3&4&4&5\\ 5&&& \end{array}, \quad V = \begin{array}{cccc} 1&2&3&4\\ 5&6&7&9\\ 8&&& \end{array} \in Tab(\nu,\alpha).$$

The descent set $\mathcal{D}(\widehat{Q}) = \{\alpha_1+\alpha_2 = 4, \alpha_1+\alpha_2+\alpha_3+\alpha_4 = 7\} = S(\alpha) \setminus \{\alpha_1, \alpha_1+\alpha_2+\alpha_3\}$; and the descent set $\mathcal{D}(\widehat{V}) = \mathcal{S}(\alpha)$. The tableaux Q and V are companion tableaux of the following LR fillings for $R_{(2).(2,1).(2,2)} = R_{(2)} \oplus R_{(2,1)} \oplus R_{(2,2)}$,

$$\varphi_{R_\alpha}(Q) = \begin{array}{ccccccc} &&&&&1&1\\ &&&1&1&&\\ &&&2&&&\\ &2&2&&&&\\ 2&3&&&&& \end{array} \qquad \varphi_{R_\alpha}(V) = \begin{array}{ccccccc} &&&&&1&1\\ &&&1&2&&\\ &&&2&&&\\ &1&3&&&&\\ 2&2&&&&& \end{array}.$$

From Proposition 2.1 we easily conclude

Proposition 3.10. Let $G \in Tab(\nu,\alpha)$ and R_α a ribbon. Then

(a) $\varphi_{R_\alpha}(G) \in \mathcal{LR}(R_\alpha,\nu)$ if and only if whenever two consecutive rows j and $j+1$ of R_α overlap then $\sum_{k=1}^{j} \alpha_k$ is in the descent set of $\widehat{G}$.

(b) if R_α is connected, $\varphi_{R_\alpha}(G)$ is an LR ribbon if and only if $\mathcal{S}(\alpha) = \mathcal{D}(\widehat{G})$.

(c) if $R_\alpha = \oplus_{i=1}^{k} R_{\tilde\alpha_i}$ is a direct sum of k connected ribbons, $\varphi_{R_\alpha}(G)$ is an LR ribbon if and only if $\mathcal{S}(\alpha) \setminus \{\sum_{i=1}^{r} |\tilde\alpha_i|, 1 \le r \le k\} \subseteq \mathcal{D}(\widehat{G})$.

Corollary 3.11. (a) Let R_α be a connected ribbon and ν a partition such that $|\nu| = |\alpha|$. Then

(1) $\mathrm{LR}_{\nu,\mathrm{R}_\alpha} = \{\mathrm{G} \in \mathrm{Tab}(\nu,\alpha) : \ \mathcal{S}(\alpha) = \mathcal{D}(\widehat{G})\ \}$.
(2) $c^{\nu}_{R_\alpha} = d_{\nu,\alpha}$ the number of standard Young tableaux of shape ν with descent set $\mathcal{S}(\alpha)$.

(b) Let $\alpha = \tilde\alpha_1 \cdot \cdots \cdot \tilde\alpha_s$ and $R_\alpha = \oplus_{i=1}^{k} R_{\tilde\alpha_i}$ a direct sum of k connected ribbons $R_{\tilde\alpha_i}$. Then

(1) $\mathrm{LR}_{\nu,\mathrm{R}_\alpha} = \{\mathrm{G} \in \mathrm{Tab}(\nu,\alpha) :\ \mathcal{S}(\alpha) \setminus \{\sum_{i=1}^{r} |\tilde{\alpha}_i|, 1 \leq r \leq k\} \subseteq \mathcal{D}(\widehat{G})\ \}$.

(2) $c^{\nu}_{R_\alpha}$ is the number of standard Young tableaux of shape ν whose descent set, a subset of $S(\alpha)$, contains $\mathcal{S}(\alpha) \setminus \{\sum_{i=1}^{r} |\tilde{\alpha}_i|, 1 \leq r \leq k\}$.

3.3. The critical set of a SSYT in $Tab(\nu,\alpha)$. We now reduce our study to compositions α with parts ≥ 2. Given $T \in Tab(\nu,\alpha)$, recall that $\mathcal{D}(\widehat{T}) \subseteq \mathcal{S}(\alpha)$. The goal is to identify in the SSYT T the entries of $\widehat{T}$ that are elements of $\mathcal{S}(\alpha) \setminus \mathcal{D}(\widehat{T})$. More precisely, the numbers $j \in \{2,\dots,\ell(\alpha)\}$ in the filling of T such that in the word $w_{\widehat{T}} = w_1 \cdots w_{\ell(\alpha)}$ (Subsection 2.5) either it occurs (1) $w_{\sum_{k=1}^{j-1}\alpha_k} = w_{1+\sum_{k=1}^{j-1}\alpha_k}$; or (2) $w_{\sum_{k=1}^{j-1}\alpha_k} > w_{1+\sum_{k=1}^{j-1}\alpha_k}$. See Example 3.8 (b), (i).

The serious rejection for $T \in Tab(\nu,\alpha)$ to be a companion tableau of a LR ribbon in $\mathcal{LR}(R_\alpha,\nu)$ occurs when one has repeated letters in a column of length 2 of $\varphi_{R_\alpha}(T)$. This means that we are collecting in the filling of T the numbers $j \in \{2,\dots,\ell(\alpha)\}$ verifying (1). This numbers define a subset of $\{2,\dots,\ell(\alpha)\}$ called the *critical set $\mathcal{C}(T)$ of T*. The set $\mathcal{C}(T)$ of critical numbers of T verifies

$$\mathcal{C}(T) =$$

$$= \{j \in \{2,\dots,\ell(\alpha)\} : \sum_{k=1}^{j-1}\alpha_k \text{ and} 1 + \sum_{k=1}^{j-1}\alpha_k \text{are entries in a same row of } \widehat{T}\}$$

$$= \{j \in \{2,\dots,\ell(\alpha)\} : w_{\widehat{T}} = w_1 \cdots w_{\ell(\alpha)} \text{ and } w_{\sum_{k=1}^{j-1}\alpha_k} = w_{1+\sum_{k=1}^{j-1}\alpha_k}\}$$

$$\subseteq \{j \in \{2,\dots,\ell(\alpha)\} : \sum_{k=1}^{j-1}\alpha_k \in \mathcal{S}(\alpha) \setminus \mathcal{D}(\widehat{T})\}.$$

From Proposition 3.10 and Corollary 3.11, we conclude that $\mathcal{C}(T)$ detects the elements j in the alphabet $\{2,\dots,\ell(\alpha)\}$ for which $\sum_{k=1}^{j-1}\alpha_k \in S(\alpha)$ are not in the descent set of $\widehat{T}$ and give rise in $\varphi_{R_\alpha}(T)$ to a filling of a column of length 2 with two repeated letters. This column of length two is obtained in the overlapping of the rows $j-1$ and j of R_α and is filled with a same letter $i < j$. Henceforth, because T is a sequence of partitions $0 = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^{\ell(\alpha)}$, the right most box of the horizontal strip $\lambda^{j-1}/\lambda^{j-2}$ is glued with the left most box of λ^j/λ^{j-1}, and one has

Proposition 3.12. Let $T \in Tab(\nu,\alpha)$ and $j \in \{2,\dots,\ell(\alpha)\}$. The number $j \in \mathcal{C}(T)$ or j is a critical number of T, if for some $i \in \{1,\dots,j-1\}$,

$\chi^i_{j-1}, \chi^i_j \neq 0$, and $\chi^k_{j-1} = \chi^h_j = 0$ for all $k \leq i-1$ and $h \geq i+1$. In this case, we also say that the integer j generates the critical row i of T.

The numbers in $\mathcal{S}(\alpha) \setminus \mathcal{D}(\widehat{T})$ giving rise to the violation (2) by inverting the increasing order in the filling of a column of length two in $\varphi_{R_\alpha}(T)$ are negligible critical numbers, because they may be removed anytime without creating new ones. In the SSYT T we collect the numbers $j \in \{2, \dots, \ell(\alpha)\}$ verifying condition (2). In fact, if, in a such column of $\varphi_{R_\alpha}(T)$, resulting from the overlapping of rows, say, $j-1$ and j of R_α, one has $\begin{array}{|c|c|c|c|c|c|c|}\hline \multicolumn{3}{c|}{} & b & x & \cdots & y \\ \hline w & \cdots & z & a & \multicolumn{3}{c}{} \\ \cline{1-4} \end{array}$, with $x \geq b > a \geq z$, we may easily correct this Yamanouchi filling, without creating new violations in the new Yamanouchi filling, by just reordering the entries of that column, $\begin{array}{|c|c|c|c|c|c|c|}\hline \multicolumn{3}{c|}{} & a & x & \cdots & y \\ \hline w & \cdots & z & b & \multicolumn{3}{c}{} \\ \cline{1-4} \end{array}$, with $y \geq \cdots \geq x > a < b > z \geq \cdots \geq w$, to obtain an LR ribbon. This tells that $j-1$ appears in T only in row b and possibly below, and j only appears in row a and above. (The horizontal strip $\lambda^{j-1}/\lambda^{j-2}$ is strictly below the horizontal strip λ^j/λ^{j-1}.) Henceforth, we should replace in row a of T the left most entry j with $j-1$, and replace in row b of T the rightmost entry $j-1$ with j. One then says j is a negligible critical number of T. See Example 3.8, (b), (ii).

Canonical fillings of SSYTs do not have negligible critical numbers and the critical numbers have an easier formulation. Note that the multiplicity of letter $j \geq i$ in row i of $T \in Tab(\nu, \alpha)$ satisfies $\chi^i_j \leq \alpha_j$.

Proposition 3.13. Let $T \in Tab(\nu, \alpha)$ with canonical filling and $j+1 \in \{2, \dots, \ell(\alpha)\}$. Then $j+1$ is a critical number of T if and only if $\chi^i_j = \alpha_j$ and $\chi^i_{j+1} = \alpha_{j+1}$ for some $i \in \{1, \dots, j\}$.

Proof: Recall $\ell(\nu) \leq \ell(\alpha)$. If T has canonical filling and $\chi^i_j, \chi^i_{j+1} \neq 0$ with $\chi^h_{j+1} = 0$ for all $h \geq i+1$, then below row i the entries are empty or bigger than $j+1$. Therefore there is no need to put $j+1$'s in rows above i because positions of row i have been used to put the letter j, that is, one also has $\chi^h_{j+1} = 0$ for all $h < i$. Hence $\chi^i_{j+1} = \alpha_{j+1}$. Similarly $\chi^h_j = 0$ for all $h > i$ because $j+1$ has to be filled first and there are no $j+1$ below row i. Hence $\chi^i_j = \alpha_j$. $\square$

We then may conclude

Proposition 3.14. Let $T \in Tab(\nu, \alpha)$ without negligible critical numbers. Then $T \notin \mathrm{LR}_{\nu, \mathrm{R}_\alpha}$ if and only if T has a critical number $j+1 \in \{2, \dots, \ell(\alpha)\}$

such that rows j and $j+1$ of R_α overlap. In this case, the column of length two obtained in the overlapping of rows $< \alpha_j >$ and $< \alpha_{j+1} >$ of R_α is filled with a same letter $i < j+1$.

3.4. Effectiveness of critical numbers. The ribbon R_α, with rows of length at least two, is now assumed to be connected or monotone up to a permutation and rotation of the connected components of R_α. Since the ribbon can be monotone and disconnected, the overlapping partition p is used to detect the effectiveness of the critical numbers of a companion tableau in $\mathrm{LR}_{\nu,\mathrm{R}_\alpha^\mathrm{p}}$.

Definition 3.15. Let $T \in Tab(\nu,\alpha)$ and let p be an overlapping partition for α. A critical number j of T is said to be p-effective if rows $j-1$ and j of R_α^p overlap. Otherwise, the critical number j is said to be p-ineffective.

This is a reformulation of Corollary 3.11 for ribbons uniquely determined by α and p.

Theorem 3.16. Let $T \in Tab(\nu,\alpha)$ and p an overlapping partition for α. Then,

(a) $T \in \mathrm{LR}_{\nu,\mathrm{R}_\alpha^\mathrm{p}}$ only if $\#\mathcal{D}(\widehat{T}) \geq p_1$,

(b) if T has no negligible critical numbers and $\mathcal{C}(T) \neq \emptyset$, $T \in \mathrm{LR}_{\nu,\mathrm{R}_\alpha^\mathrm{p}}$ if and only if every critical number of T is p-ineffective.

Proof: (a) The number of columns of length two of R_α^p is p_1. Since T has no negligible critical numbers, to avoid columns of length two filled with the same letter, we need that the descent set of $\widehat{T}$ has at least p_1 elements.

(b) It is the translation of Proposition 3.14 according to the Definition 3.15. □

4. Characterization of monotone ribbon LR coefficients positivity by means of linear inequalities

Throughout this section we consider α a partition with parts of length at least 2, and overlapping partition p. Theorem 3.16 says that $c^\nu_{R_\alpha^p} > 0$ if and only if, whenever there exists $T \in Tab(\nu,\alpha)$ without negligible critical numbers and $\mathcal{C}(T) \neq \emptyset$, then every critical number of T is p-ineffective. Theorem 1.5 gives a set of linear inequalities on the triple of partitions (α, p, ν) as necessary and sufficient conditions for the positivity of $c^\nu_{R_\alpha^p}$. We split the proof of the *only if* and *if* parts of Theorem 1.5 into two subsections respectively.

4.1. Proof of the *only if part* of Theorem 1.5. If $c^{\nu}_{R^{p}_{\alpha}} = \#\mathrm{LR}_{\nu,\mathrm{R}^{\mathrm{p}}_{\alpha}} > 0$ then there exists $T \in \mathrm{LR}_{\nu,\mathrm{R}^{\mathrm{p}}_{\alpha}} \subseteq \mathrm{Tab}(\nu,\alpha)$ and $\alpha \preceq \nu$. Let $p = (p_1,\ldots,p_{\ell(\alpha)-1},0)$ where $\{p_1,\ldots,p_{\ell(\alpha)-1}\}$ is a multiset of $\{\ell(\alpha)-k,\ldots,1\}$ such that $p_{\ell(\tilde\alpha_i)} = p_{\ell(\tilde\alpha_i)+1}$ and $\alpha = \tilde\alpha_1\cdots\tilde\alpha_k$ with $R_{\tilde\alpha_i}$, $1 \le i \le k$, the connected components of R_α. Therefore $T \in Tab(\nu,\alpha)$ with $\mathcal{D}(\widehat{T}) = \{\sum_{j=1}^{s}\alpha_j : s \in S\} \subseteq \mathcal{S}(\alpha)$ for some subset $S = \{s_1 < \cdots < s_{|S|}\} \subseteq \{1,\ldots,\ell(\alpha)-1\}$ satisfying

$$\begin{aligned}&[\{1,\ldots,\ell(\alpha)-1\} \setminus \{\ell(\tilde\alpha_1),\ldots,\ell(\tilde\alpha_1\tilde\alpha_2\cdots\tilde\alpha_{k-1})\}] \subseteq \\ &\qquad \subseteq S = \{s_1 < \cdots < s_{|S|}\} \subseteq \{1,\ldots,\ell(\alpha)-1\}.\end{aligned} \tag{4.1}$$

Observe that $|\{\sum_{j=1}^{s}\alpha_j : s \in \{s_i,\ldots,s_{|S|}\}\}| = |\{s_i,\ldots,s_{|S|}\}| \ge p_{s_i}$, for $1 \le i \le |S| \le \ell(\alpha)-1$. Because $\alpha \preceq \nu$, by Remark 2.2, $\nu_i \le \alpha_i + \cdots + \alpha_{\ell(\alpha)}$, for $i \in \{1,\ldots,\ell(\nu)\}$. On the other hand, the α_i i's constitute the i-th horizontal strip ν^i/ν^{i-1} of T whose rows belong to the first $\min\{i,\ell(\nu)\}$ rows of T, for $i \in \{1,\ldots,\ell(\alpha)\}$. Consider the SYT $\widehat{T}$ and $i \in \{1,\ldots,\ell(\nu)\}$. If $\alpha_1+\cdots+\alpha_i+\cdots+\alpha_s$, $i \le s \in S$, is a descent of $\widehat{T}$ in the ith row of $\widehat{T}$, then $\alpha_1+\cdots+\alpha_s+1$ belongs to a row of $\widehat{T}$ strictly below row i. That is, for $1 \le i \le q \le \ell(\alpha)-1$, if $\alpha_1+\cdots+\alpha_i+\cdots+\alpha_q$ is a descent of $\hat{T}$, then either $\alpha_1+\cdots+\alpha_q$ belongs to a row of $\widehat{T}$ strictly above row i, or $\alpha_1+\cdots+\alpha_q+1$ belongs to a row of $\widehat{T}$ strictly below row i. Observe that $|S\cap\{i,\ldots,\ell(\alpha)-1\}|$ is the maximum number of descents of $\hat{T}$ in row i, and, simultaneously, is at least equal to the overlapping number p_i, the number of columns of length two among the last $\ell(\alpha)-i+1$ rows of R_α,

$$|S \cap \{i,\ldots,\ell(\alpha)-1\}| \ge p_i. \tag{4.2}$$

Hence $\nu_i \le \alpha_i+\cdots+\alpha_{\ell(\alpha)} - |S\cap\{i,\ldots,\ell(\alpha)-1\}| \le \alpha_i+\cdots+\alpha_{\ell(\alpha)} - p_i$, for $i \in \{1,\ldots,\ell(\nu)\}$. $\square$

4.2. Proof of the *if part* of Theorem 1.5. Given the triple of partitions, ν, and α, with parts ≥ 2, and overlapping partition p, satisfying the linear inequalities on the right hand side of (1.6), the goal is now to exhibit a SSYT $T \in \mathrm{LR}_{\mathrm{R}^{\mathrm{p}}_{\alpha},\nu}$. In other words, assuming the linear inequalities on the right hand side of (1.6), we construct a SSYT $T \in Tab(\nu,\alpha)$ without negligible critical numbers and p-effective critical numbers. In more detail, we pick $T \in Tab(\nu,\alpha)$ with the *canonical filling*, thus without negligible critical numbers, and then, if it has p-effective critical numbers, one modifies its filling according to a certain *rotation* procedure to remove them so that the new tableau is in $\mathrm{LR}_{\mathrm{R}^{\mathrm{p}}_{\alpha},\nu}$. The application of rotation procedure does not create negligible critical numbers. The linear inequalities on the right hand side of (1.6) guarantee that our rotation procedure is successful.

Remark 4.1. Let $\alpha \preceq \nu$ and p an overlapping partition for α.

(a) If $\ell(\nu) = \ell(\alpha)$, given $T \in Tab(\nu, \alpha)$, the first entry of each row i of T is i and T has no critical numbers of any kind. The descent set of $\widehat{T}$ is $S(\alpha)$ and every $T \in Tab(\nu, \alpha)$ is a companion tableau for an LR filling of R^p_α. In this case, the linear inequalities (1.6) are trivially satisfied because below each row i of T one has at least $\ell(\alpha) - i \geq p_i$ entries and thereby $\nu_i \leq \alpha_i + \cdots + \alpha_{\ell(\alpha)} - \ell(\alpha) + i \leq \alpha_i + \cdots + \alpha_{\ell(\alpha)} - p_i$. Also $c^\nu_{R^p_\alpha} = K_{\nu,\alpha}$.

(b) If $\ell(\nu) = 1$ then $\nu = (|\alpha|)$, $p = 0$, and $|Tab(\nu, \alpha)| = 1$. The descent set of the sole $\widehat{T}$ is $S(\alpha) = \emptyset$, and linear inequalities (1.6) are trivially satisfied with $p = 0$. One has $c^\nu_{R^p_\alpha} = K_{\nu,\alpha} = 1$.

We shall consider ν with at least two rows and less than $\ell(\alpha)$ rows, $2 \leq \ell(\nu) < \ell(\alpha)$.

We start with the case $\ell(\nu) = \ell(\alpha) - 1$.

Lemma 4.2. Let $\nu \in [\alpha, (|\alpha| - p_1, p_1)]$ with $\ell(\nu) = \ell(\alpha) - 1$, such that

$$\nu_i \leq \sum_{j \geq i} \alpha_j - p_i, \text{ for } 1 \leq i < \ell(\alpha).$$

Then, $c^\nu_{R^p_\alpha} > 0$.

Proof: Let $s := \ell(\alpha)$. Let $T \in Tab(\nu, \alpha)$, with the *canonical filling*, and note that since $\ell(\nu) = s - 1$, then the first column of T has all letters of $[s] \setminus \{j\}$, for some $2 \leq j \leq s$, and necessarily row $j - 1$ contains α_j letters j. That is, the first entry of row i of T is i, for $i = 1, \ldots, j - 1$, and is $i + 1$ for $i = j, \ldots, s - 1$. Thus, $\chi^k_k \neq 0$, $1 \leq k \leq j - 1$, $\chi^{j-1}_j = \alpha_j$, $\chi^k_{k+1} \neq 0$, $j \leq k \leq s - 1$, and $\chi^k_j = 0$, $k \geq j$. The only row of T which can potentially be critical is row $j - 1$, since by Proposition 3.13, $\chi^{j-1}_{j-1} \neq 0$ and $\chi^{j-1}_j = \alpha_j$. That is, the rows $j - 1$ and j of $\varphi_{R_\alpha}(T)$ look like

$$\begin{array}{rccc|c|c|c|c|c|c|}
\cline{5-10}
<\alpha_{j-1}> & & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & x & \cdots & x & j-1 & \cdots & j-1 \\ \cline{2-10}
<\alpha_j> & \multicolumn{1}{|c|}{j-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{} & j-1 & \multicolumn{5}{c}{} \\ \cline{2-5}
\end{array} \qquad (4.3)$$

or

$$\begin{array}{rccc|c|c|c|}
\cline{5-7}
<\alpha_{j-1}> & & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & j-1 & \cdots & j-1 \\ \cline{2-7}
<\alpha_j> & \multicolumn{1}{|c|}{j-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{} & j-1 & \multicolumn{2}{c}{} \\ \cline{2-5}
\end{array} \qquad (4.4)$$

where the word $x \cdots x = (j-2)^r$, $r \geq 0$, may be empty. If j is not a critical number, or if it is a p-ineffective critical number (4.3) then, by Proposition 3.14, $\varphi_{R_\alpha}(T)$ is a skew SSYT.

Assume now that j is p-effective critical number of T (4.4). In particular, this means that $\chi_{j-1}^{j-1} = \alpha_{j-1}$. Notice that if $j = s$, then

$$\nu_{j-1} = \nu_{s-1} = \alpha_{s-1} + \alpha_s \leq \alpha_{s-1} + \alpha_s - p_{s-1},$$

which implies $p_{s-1} = 0$. That is, rows $j-1$ and j of R_α^p do not overlap, which contradicts the fact that $j = s$ is p-effective critical. So, we must have $2 \leq j < s$, and, in particular, row j of T has at least one integer $j+1$. Table 1 depicts rows $j-1$ and j of T, where $*$ denotes $\chi_{j+1}^{j-1} > 0$ boxes with the letter $j+1$, or the empty cell if $\chi_{j+1}^{j-1} = 0$,

row $j-1$	$j-1$	$j-1$	$\cdots$	$j-1$	$\cdots$	$j-1$	$\mathbf{j}$	$\cdots$	$\mathbf{j}$	$\mathbf{j}$	$*$
row j	$\mathbf{j+1}$	$\mathbf{j+1}$	$\cdots$	$\mathbf{j+1}$	$\cdots$						

Table 1. Rows $j-1$ and j of T

Perform the procedure Rotation described in Table 2 with $\ell = j-1$, $a = j$ and $b = j+1$ on the tableau T.

Procedure: Rotation

Data: Tableau T; Integers a and ℓ;

Begin

Let $\ell' > \ell$ be the smallest integer such that row ℓ' of T has an integer b greater or equal to the rightmost letter in row ℓ;

Rotate by one turn in anticlockwise order all letters greater or equal to a in row ℓ, and all letters b of row ℓ' of T;

Stop

Table 2. Procedure: Rotation

That is, rotate the highlight letters j and $j+1$ of T (Table 1) in anticlockwise order to obtain the rows shown in Table 3, and denote by T' the tableau obtained from T by this operation.

row $j-1$	$j-1$	$j-1$	$\cdots$	$j-1$	$\cdots$	$j-1$	$\mathbf{j}$	$\cdots$	$\mathbf{j}$	$\mathbf{j+1}$	$*$
row j	$\mathbf{j}$	$\mathbf{j+1}$	$\cdots$	$\mathbf{j+1}$	$\cdots$						

Table 3. Rows $j-1$ and j of T'

We recall that we are assuming α a partition and thus $\alpha_{j-1} \geq \alpha_j \geq \alpha_{j+1}$. The new tableau T' is still semistandard and the integer j is no longer critical, since $\chi_j^j > 0$. Notice, however, that if $\chi_{j+1}^j = 1$ in T, then in T' the integer $j+1$ is critical. Therefore, if $\chi_{j+1}^j > 1$ in T, or $\chi_{j+1}^j = 1$ in T and $j+1$ is p-ineffective critical, then $\varphi_{R_\alpha^p}(T')$ is a skew SSYT. So, assume $\chi_{j+1}^j = 1$, j p-effective critical in T, and in addition rows j and $j+1$ of R_α^p overlap ($j+1$ is p-effective in T'). If $j+1 = s$, (Table 4) then $\nu_{s-1} = 1$ and $p_{s-2} = 2$,

row $s-2$	$s-2$	$\cdots$	$s-2$	$\mathbf{s-1}$	$\cdots$	$\mathbf{s-1}$	$\mathbf{s}$	$\cdots$	$\mathbf{s}$
row $s-1$	$\mathbf{s-1}$								

TABLE 4. Rows $s-2$ and $s-1$ of T'

and $\nu_{s-2} = \alpha_{s-2} + \alpha_{s-1} + (\alpha_s - 1) \leq \alpha_{s-2} + \alpha_{s-1} + \alpha_s - p_{s-2}$, that is, $p_{s-2} \leq 1$. A contradiction, then the rows $s-1$ and s of R_α^p cannot overlap, and $j+1 = s$ is not p-effective critical in T'.

So we must have $j+1 < s$, and there must be integers other than $j+1$ in row j of T', since otherwise the rows of T' below row j would have only one box, which in turn would imply $2 \leq \alpha_{j+2} = 1$, a contradiction. So there are letters $j+2$ in row j and the number of letters $j+2$ below row $j-1$ is $\alpha_{j+2} \geq 2$ (Table 5). Apply the procedure Rotation with $a = j+1$ and $\ell = j-1$ to T',

row $j-1$	$j-1$	$j-1$	$\cdots$	$j-1$	$\cdots$	$j-1$	j	$\cdots$	j	$\mathbf{j+1}$	$\cdots$	$\mathbf{j+1}$
row j	j	$\mathbf{j+2}$	$\cdots$	$\mathbf{j+2}$	$\cdots$							

TABLE 5. Rows $j-1$ and j of T'

and let T'' be the resulting tableau (Table 6),

row $j-1$	$j-1$	$j-1$	$\cdots$	$j-1$	$\cdots$	$j-1$	j	$\cdots$	j	$\mathbf{j+1}$	$\cdots$	$\mathbf{j+2}$
row j	j	$\mathbf{j+1}$	$\cdots$	$\mathbf{j+2}$	$\cdots$							

TABLE 6. Rows $j-1$ and j of T''

This new tableau is semistandard and $j+1$ is no longer a critical number, since there is now a letter $j+1$ in row j. Also, since $\alpha_{j+2} \geq 2$, there must be integers $j+2$ below row $j-1$. Thus, T'' does not have critical numbers and then $\varphi_{R^p_\alpha}(T'')$ is a skew SSYT. $\square$

Remark 4.3. Notice that when applying the procedures, described in the proof of the result above, to a tableau T with only one critical number j in row $j-1$, we only modify rows $j-1$ and j of T. Moreover, in row j, only the integers $j+1$, and possible $j+2$, are acted upon. The rows above row $j-1$, as well as the letters in row $j-1$ to the left of the letters j, are not considered for the application of the procedure.

Example 4.4. Let $\nu = (8,1)$ and $\alpha = (3,3,3)$, and consider the tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \in Tab(\nu,\alpha).$$

The tableau T has only one critical number: the integer 2, that is, the descent of $\widehat{T}$ is $\{\alpha_1+\alpha_2\}$. If $R_\alpha = R_{\alpha_1} \oplus R_{(\alpha_2,\alpha_3)}$, equivalently, $p=(1,1,0)$, then $\varphi_{R_\alpha}(T)$ is SSYT and the integer 2 is not p-effective critical, and so

$$\varphi_{R_\alpha}(T) = \begin{array}{cccccccc} & & & & & \boxed{1} & \boxed{1} & \boxed{1} \\ & & \boxed{1} & \boxed{1} & \boxed{1} & & & \\ \boxed{1} & \boxed{1} & \boxed{2} & & & & & \end{array}$$

is a skew SSYT. Note also, $\nu_1 = 8 \leq \sum_{i=1}^{3} \alpha_i - 1$, $\nu_2 = 1 \leq \alpha_2 + \alpha_3 - 1$, $\nu_3 \leq \alpha_3 - 0$.

If $p=(1,0,0)$ then $R_\alpha = R_{(\alpha_1,\alpha_2)} \oplus R_{(\alpha_3)}$, 2 is a p-effective critical number and $\varphi_{R_\alpha}(T)$ is not SSYT. Perform the procedure Rotation on T with $a=2$ and $\ell=1$ to get

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} = T'.$$

The tableau T' has no effective critical numbers for the overlapping partition $p=(1,0,0)$, the descent set of $\widehat{T'}$ is $\{\alpha_1\}$, and therefore

$$\varphi_{R^p_\alpha}(T') = \begin{array}{cccccccc} & & & & & \boxed{1} & \boxed{1} & \boxed{1} \\ & & & \boxed{1} & \boxed{1} & \boxed{2} & & \\ \boxed{1} & \boxed{1} & \boxed{1} & & & & & \end{array}$$

is a skew SSYT. There is no connected LR ribbon of shape R_α and content ν: if $p=(2,1,0)$, $\nu_1 = 8 > |\alpha| - 2 = 9-2$.

Example 4.5. Let $\nu = (9,3)$ and $\alpha = (4,3,3,2)$, and consider the tableau

$$T = \begin{ytableau} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 3 & 4 & 4 \end{ytableau} \in Tab(\nu,\alpha).$$

The letter 2 is the only critical number of T, and is effective when we consider the overlapping partition $p = (3,2,1,0)$. So, we apply the procedure Rotation on T with $a = 2$ and $\ell = 1$:

$$T = \begin{ytableau} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 3 & 4 & 4 \end{ytableau} \rightarrow \begin{ytableau} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ 2 & 4 & 4 \end{ytableau} = T'.$$

In T', the number 2 is no longer critical. However, a new critical number was created: the number 3. So we apply Rotation on T' with $a = 3$ and $\ell = 1$ to get the tableau

$$T'' = \begin{ytableau} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 2 & 3 & 4 \end{ytableau},$$

which has no critical numbers. It follows that

$$\varphi_{R_\alpha}(T'') = \begin{ytableau} \none & \none & \none & \none & \none & 1 & 1 & 1 & 1 \\ \none & \none & \none & 1 & 1 & 2 \\ \none & 1 & 1 & 3 \\ 1 & 4 \end{ytableau}$$

is a skew SSYT. Note $\nu_1 = 9 \leq |\alpha| - 3 = 12-3$, $\nu_2 = 3 \leq 8-2$, $\nu_3 = 0 \leq 5-1$.

Lemma 4.6. Let $\nu \in [\alpha, (|\alpha| - p_1, p_1)]$ with $\ell(\nu) = \ell(\alpha) - k$, $1 \leq k \leq \ell(\alpha) - 2$, and satisfying

$$\nu_i \leq \sum_{j \geq i} \alpha_j - p_i, \quad \text{for } 1 \leq i \leq \ell(\nu).$$

If T is the SSYT with canonical filling in $Tab(\nu,\alpha)$ and has $\mathcal{C}(T) = \{j_1, j_2, \ldots, j_k\}$ with $j_{i+1} = j_i + 1$, for $i = 1, \ldots, k-1$, then, $c^{\nu}_{R^p_\alpha} > 0$.

Proof: Let T be the canonical filling in $Tab(\nu,\alpha)$ with $\mathcal{C}(T) = \{j_1, j_2, \ldots, j_k\}$ such that $j_{i+1} = j_i + 1$ for $i = 1, \ldots, k-1$. Then, the first column of T has all letters of $[s] \setminus \{j_1, j_2, \ldots, j_k\}$, and row $j_1 - 1$ has α_i letters j_i, for $i = 1, \ldots, k$. We are assuming that j_1 is critical but $j_k + 1$ is not, row $j_1 - 1$ also has α_{j_1-1} letters $j_1 - 1$ and $0 \leq \chi^{j_1-1}_{j_k+1} < \alpha_{j_k+1}$ letters $j_k + 1$, thus, row $j_1 - 1$ of T satisfy

$$\nu_{j_1-1} = \alpha_{j_1-1} + \alpha_{j_1} + \cdots + \alpha_{j_k} + \left(\alpha_{j_k+1} - \chi^{j_1}_{j_k+1}\right) \leq \alpha_{j_1-1} + \alpha_{j_1} + \cdots + \alpha_s - p_{j_1-1},$$

that is,

$$p_{j_1-1} \leq \alpha_{j_k+2} + \cdots + \alpha_s + \chi_{j_k+1}^{j_1} \tag{4.5}$$

where $0 < \chi_{j_k+1}^{j_1}$. The number of p-effective critical numbers of T, which are at most k, must be less than or equal to p_{j_1-1}. Thus, by (4.5), there are at least p_{j_1-1} integers greater than or equal to $j_k + 1$ below row $j_1 - 1$ of T and we can perform procedure Rotation 1 on T with $\mathcal{C}(T) = \{j_1, \ldots, j_k\}$ and $\overline{\ell} = j_1 - 1$.

Procedure: Rotation 1
Data: Tableau T; Set $\mathcal{C}(T) = \{j_1 < \ldots < j_k\}$; Integer $\overline{\ell}$;
Begin
For $i = 1$ to k do
If j_i is an p-effective critical point of T, perform procedure Rotation (Table 2) with $a = j_i$ and $\ell = \overline{\ell}$;
End If
End For
Stop

TABLE 7. Procedure: Rotation 1

Let T' be the tableau resulting from the application of Procedure Rotation 1 (Table 7) on T. Notice that the assumption of α a partition and the canonical filling of T asserts that T' is semistandard. Moreover, the integers $j_1, \ldots, j_k$ are not critical numbers of T since there are letters $j_1, \ldots, j_k$ below row $j_1 - 1$. However, the operations performed on T to produce T' may create new critical numbers, all of which are in row $j_1 - 1$. This only happens when all letters of an integer, say $r > j_k$, are sent to row $j_1 - 1$. Note that r must be one of the first k letters below row $j_1 - 1$ which are greater or equal to the rightmost letter of row $j_1 - 1$. Let $r_1, \ldots, r_{k'}$ be the new critical numbers created in T'. If they are p-effective, then by (4.5), we must have

$$k + k' \leq p_{j_1-1}.$$

This means that below row $j_1 - 1$ of T' there exist at least k' integers greater or equal to the rightmost letter of row $j_1 - 1$, and we can perform procedure Rotation 1 on T' with $\mathcal{C}(T) = \{r_1, \ldots, r_{k'}\}$ and $\overline{\ell} = j_1 - 1$, obtaining a new tableau T'', where $r_1, \ldots, r_{k'}$ are not critical . Again, new critical numbers $q_1, \ldots, q_{k''}$, with $r_{k'} < q_1, \ldots, q_{k''}$ may occur, in which case we repeat the process. Note that since the number of p-effective critical numbers cannot exceed p_{j_1-1}, this process must terminate.

Therefore, the tableau $\widetilde{T}$ obtained after this procedure is semistandard and has no critical numbers. We can conclude that $\varphi_{R_\alpha}(\widetilde{T})$ is semistandard. $\square$

Remark 4.7. Notice that Lemma 4.2 is a special case of Lemma 4.6. Also, notice that the tableau $\widetilde{T}$ obtained after the process described in the result above only differs from T between the rows j_1-1, the ones having the critical numbers, and some row below it, say j, from the leftmost integer of j until the last integer in row j that has been rotated to row j_1-1.

Example 4.8. Let $\nu=(9,2,2,2)$, $\alpha=(3,2,2,2,2,2,2)$, and consider the overlapping vector $p=(6,5,4,3,2,1,0)$ and the tableau

$$T=\begin{array}{ccccccccc} 1&1&1&2&2&3&3&4&4\\ 5&5&&&&&&&\\ 6&6&&&&&&&\\ 7&7&&&&&&& \end{array} \in Tab(\nu,\alpha).$$

The letters $2,3$ and 4 are consecutive p-effective critical numbers of T. Apply procedure Rotation 1 with $\mathcal{C}(T)=\{2,3,4\}$ and $\bar{\ell}=1$:

$$T\to\begin{array}{ccccccccc} 1&1&1&2&3&3&4&4&5\\ 2&5&&&&&&&\\ 6&6&&&&&&&\\ 7&7&&&&&&& \end{array} \quad \to\begin{array}{ccccccccc} 1&1&1&2&3&4&4&5&5\\ 2&3&&&&&&&\\ 6&6&&&&&&&\\ 7&7&&&&&&& \end{array}$$

$$\to\begin{array}{ccccccccc} 1&1&1&2&3&4&5&5&6\\ 2&3&&&&&&&\\ 4&6&&&&&&&\\ 7&7&&&&&&& \end{array} = T'.$$

Now, the letter 5 is the only critical number of the resulting tableau T'. So, we apply Rotation 1 again on T' with $\mathcal{C}(T)=\{5\}$ and $\bar{\ell}=1$:

$$T'=\begin{array}{ccccccccc} 1&1&1&2&3&4&5&5&6\\ 2&3&&&&&&&\\ 4&6&&&&&&&\\ 7&7&&&&&&& \end{array} \quad \to\begin{array}{ccccccccc} 1&1&1&2&3&4&5&6&6\\ 2&3&&&&&&&\\ 4&5&&&&&&&\\ 7&7&&&&&&& \end{array} = T''.$$

Now, the letter 6 is the only critical number of the resulting tableau T''. So, we apply Rotation 1 again on T' with $\mathcal{C}(T) = \{6\}$ and $\overline{\ell} = 1$:

$$T'' = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&4&5&6&6\\\hline 2&3\\\cline{1-2} 4&5\\\cline{1-2} 7&7\\\cline{1-2}\end{array} \quad \to \quad \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&4&5&6&7\\\hline 2&3\\\cline{1-2} 4&5\\\cline{1-2} 6&7\\\cline{1-2}\end{array} = \widetilde{T}.$$

The tableau $\widetilde{T}$ has no critical numbers and thus $\varphi_{R_\alpha}(\widetilde{T})$ is a skew SSYT.

We now can prove the general case.

Theorem 4.9. Let $\nu \in [\alpha, (|\alpha| - p_1, p_1)]$ where $\ell(\nu) = \ell(\alpha) - k$, $1 \leq k \leq \ell(\alpha) - 2$, and satisfying

$$\nu_i \leq \sum_{j \geq i} \alpha_j - p_i, \ \text{ for } 1 \leq i \leq \ell(\alpha).$$

Then, $c^{\nu}_{R^p_\alpha} > 0$.

Proof: Let $T \in Tab(\nu, \alpha)$ with the canonical filling, and $\mathcal{C}(T) = \{j_1, \ldots, j_k\}$. Write

$$\mathcal{C}(T) = A_1 \cup A_2 \cup \cdots \cup A_r$$

the set partition of $\mathcal{C}(T)$ such that in each set A_i all critical numbers are consecutive, and if $a \in A_i$ and $b \in A_{i+q}$, for some $q > 0$, then $a < b$ and $b - a \geq 2$.

Notice that in this case, the α_a letters a must be all in some row ℓ, and the α_b letters b must be all in some row ℓ' of T, with $\ell < \ell'$.

Apply the procedure described in Lemma 4.6 to the set of consecutive critical numbers in A_1. This procedure may use some integers from $A_2 \cup \cdots \cup A_r$ in its Rotation routines. If this is the case, then in the resulting tableau T', some of the critical numbers in $A_2 \cup \cdots \cup A_r$ may no longer be critical numbers, since some of them may have been brought, by rotation, to a higher row of the tableau. Nevertheless, no new critical numbers are created by this process. So, in T', the critical numbers can be partitioned as

$$A'_2 \cup \cdots \cup A'_r,$$

where $A'_i \subseteq A_i$ for all $i = 2, \ldots, r$.

Repeating the process, until no more critical points remain, we obtain a tableau $\widetilde{T}$ such that $\varphi_{R_\alpha}(\widetilde{T})$ is a skew SSYT. $\square$

Example 4.10. Let $\nu = (13, 13, 2)$, $\alpha = (4, 3^7)$ and $p = (8, 7, 6, 5, 4, 3, 2, 1, 0)$. The tableau

$$T = \begin{array}{ccccccccccccc} 1&1&1&1&2&2&2&3&3&3&4&4&4\\ 5&5&5&6&6&6&7&7&7&8&8&8&9\\ 9&9&&&&&&&&&&& \end{array} \in Tab(\nu, \alpha)$$

has the critical points $2, 3, 4, 6, 7, 8$, which can be partitioned as

$$A_1 = \{2, 3, 4\} \cup A_2 = \{6, 7, 8\},$$

according to the proof of the theorem above. We start by removing the critical numbers in A_1:

$$T \to \begin{array}{ccccccccccccc} 1&1&1&1&2&2&3&3&3&4&4&4&5\\ 2&5&5&6&6&6&7&7&7&8&8&8&9\\ 9&9&&&&&&&&&&& \end{array} \to \begin{array}{ccccccccccccc} 1&1&1&1&2&2&3&3&4&4&4&5&5\\ 2&3&5&6&6&6&7&7&7&8&8&8&9\\ 9&9&&&&&&&&&&& \end{array}$$

$$\to \begin{array}{ccccccccccccc} 1&1&1&1&2&2&3&3&4&4&5&5&5\\ 2&3&4&6&6&6&7&7&7&8&8&8&9\\ 9&9&&&&&&&&&&& \end{array} \to$$

$$\to \begin{array}{ccccccccccccc} 1&1&1&1&2&2&3&3&4&4&5&5&6\\ 2&3&4&5&6&6&7&7&7&8&8&8&9\\ 9&9&&&&&&&&&&& \end{array} = T'.$$

After the application of the procedure described in Lemma 4.6 to the critical numbers in A_1, we get the tableau T', whose only critical olga number is

$$\{8\} = A'_2 \subset A_2.$$

So, we apply the procedure described in Lemma 4.6 again to the critical number in A'_2:

$$T' \to \begin{array}{ccccccccccccc} 1&1&1&1&2&2&3&3&4&4&5&5&6\\ 2&3&4&5&6&6&7&7&7&8&8&9&9\\ 8&9&&&&&&&&&&& \end{array} = \widetilde{T}.$$

The resulting tableau $\widetilde{T}$ has no critical numbers and thus, the skew tableau

$$\varphi_{R_\alpha}(\widetilde{T}) = \begin{array}{cccccccccccccccccccc}
&&&&&&&&&&&&&&&&1&1&1&1\\
&&&&&&&&&&&&&&1&1&2&&&\\
&&&&&&&&&&&&1&1&2&&&&&\\
&&&&&&&&&&1&1&2&&&&&&&\\
&&&&&&&&1&1&2&&&&&&&&&\\
&&&&&&1&2&2&&&&&&&&&&&\\
&&&&2&2&2&&&&&&&&&&&&&\\
&&2&2&3&&&&&&&&&&&&&&&\\
2&2&3&&&&&&&&&&&&&&&&&
\end{array}$$

is a skew SSYT.

5. Classification of monotone ribbons with full Schur support

Theorem 1.5, characterizing the positivity of monotone ribbon LR coefficients, $c^{\nu}_{R^p_\alpha} > 0$, by means of linear inequalities, may be rephrased in the language of the Schur support of R^p_α. Let $\nu \in [\alpha, (|\alpha| - p_1, p_1)]$, α a partition with parts ≥ 2. Then

$$\nu \in [R^p_\alpha] \text{ if and only if } \nu_i \leq \sum_{q=i}^{\ell(\alpha)} \alpha_q - p_i, \;\; 1 \leq i \leq \ell(\alpha). \tag{5.1}$$

By Remark 2.2, if $\alpha \preceq \nu$ one has $\nu_i \leq \displaystyle\sum_{q=i}^{\ell(\alpha)} \alpha_q$, for $1 \leq i \leq \ell(\alpha)$. Hence, if $\nu \in [\alpha, (|\alpha| - p_1, p_1)]$ then $\nu_1 \leq \sum_{q=1}^{\ell(\alpha)} \alpha_q - p_1$, and because one has $p_i = 0$, for $\ell(p) < i \leq \ell(\alpha)$, the inequalities (5.1) are always satisfied for $\ell(p) < i \leq \ell(\alpha)$. Note that, when $\ell(p) \geq 2$, $p_{i+1} - 1 \geq 0$, $i \in \{1, \dots, \ell(p) - 1\}$. Recall Definition 1.3 and $\varrho_i = \displaystyle\sum_{q=i+1}^{\ell(\alpha)} \alpha_q - p_{i+1} + 1 > 0$, where $\varrho_i - 1$ is the total number of columns in the last $\ell(\alpha) - i$ rows of R^p_α, for $1 \leq i \leq \ell(p) - 1$.

Remark 5.1. Because the parts of α are ≥ 2, and $p_i = p_{i+1}$ or $p_i = p_{i+1} + 1$, $|\alpha| - p_1 > \varrho_1 > \cdots > \varrho_{\ell(p)-1}$.

Thus, the negation of (5.1) characterizes the partitions in the interval $[\alpha, (|\alpha| - p_1, p_1)]$ which are not in the support of R^p_α.

Corollary 5.2. Let $\nu \in [\alpha, (|\alpha| - p_1, p_1)]$ and α a partition with parts ≥ 2. Then, if $\ell(p) = 0, 1$, $[R^p_\alpha] = [\alpha, (|\alpha| - p_1, p_1)]$, and, if $\ell(p) \geq 2$, the following are equivalent

(*a*) $\nu \notin [R^p_\alpha]$ if and only if there exists $i \in \{1, \dots, \ell(p) - 1\}$ such that

$$\nu_{i+1} \geq \sum_{q \geq i+1} \alpha_q - p_{i+1} + 1 \Leftrightarrow \nu_{i+1} \geq \varrho_i.$$

(*b*) $\nu \notin [R^p_\alpha]$ if and only if, for some $i \in \{1, \dots, \ell(p) - 1\}$, ν_{i+1} exceeds the number of columns in the last $\ell(\alpha) - i$ rows of R^p_α.

(*c*) [ACM17, Lemma 4.8] $\nu \notin [R^p_\alpha]$ if and only if, there exists $i \in \{1, \dots, \ell(p) - 1\}$ such that after placing α_j j's, in row j of R^p_α, for $j = 1, \dots, i$, there is no space to place ν_{i+1} $i+1$'s in the remain $\ell(\alpha) - i$ rows of R^p_α without avoiding the violation of the column standard condition of the filling.

(*d*) $\nu \notin [R^p_\alpha]$, if, for every $T \in Tab(\nu, \alpha)$, there exists $i \geq 1$ such that $|\mathcal{D}(\widehat{T}) \cap \{\sum_{q \geq 1}^{j} \alpha_q : i + 1 \leq j \leq \ell(\alpha)\}| < p_{i+1}$.

Example 5.3. Consider the partition $\alpha = (7, 6, 6, 2, 2, 2, 2)$ with the overlapping partition $p = (6, 5, 4, 3, 2, 1, 0)$. The partition $\nu = (8, 7, 6, 6)$ is in the Schur interval $[\alpha, (27 - 6, 6)]$ of R^p_α, but not in its support since $\nu_4 = 6 \geq \varrho_3 = \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - p_4 + 1 = 2 + 2 + 2 + 2 - 3 + 1 = 6$. Therefore, $[R^p_\alpha] \subsetneqq [\alpha, (27 - 6, 6)]$.

Theorem 1.6 characterizes the monotone ribbons R^p_α with full Schur support in terms of their partition skew shape α and the overlapping partition p. In Definition 1.3 a sequence of $\ell(p) - 1$ witness vectors $\tilde{g}^i = \{\tilde{g}^i_j\}_{j=1}^i = \{[\varrho_i - \alpha_j]_+\}_{j=1}^i$ with its slack $p_{i+1} - 1$, $1 \leq i \leq \ell(p) - 1$, is introduced to test the fullness of the Schur support of R^p_α. Theorem 1.6 says that if, for some $1 \leq i \leq \ell(p) - 1$, the size of the witness vector $\tilde{g}^i$ fits the slack $p_{i+1} - 1$, that is $\sum_{j=1}^i \tilde{g}^i_j \leq p_{i+1} - 1$, then R^p_α has not full Schur support. In this case the vector $\tilde{g}^i$ witnesses that the Schur support R^p_α is not full in the sense that it can be used to exhibit a partition in the Schur interval that is not in $[R^p_\alpha]$. More precisely, $(\alpha_1 + \tilde{g}^i_1, \dots, \alpha_i + \tilde{g}^i_i, \varrho_i, p_{i+1} - 1 - |\tilde{g}^i|)^+$, with $\varrho_i - 1$ the total number of columns in the last $\ell(\alpha) - i$ rows of R^p_α, is a partition of $|\alpha|$ in the Schur interval of R^p_α but not in the support of R^p_α.

5.1. Proof of Theorem 1.6. The *"only if"* part. Let $\nu \in [\alpha, (|\alpha| - p_1, p_1)]$ such that $\nu \notin [R^p_\alpha]$. Then, on one hand, since $\alpha \preceq \nu$, $\sum_{q=1}^k (\nu_q - \alpha_q) \geq 0$, $k = 1, \dots, \ell(\alpha)$, and on the other hand, since $\nu \notin [R^p_\alpha]$, by Corollary 5.2,

$\ell(\alpha) \geq 3$, $\ell(p) \geq 2$, and there exists $1 \leq i \leq \ell(\alpha) - 2$ with $p_{i+1} \geq 1$ such that

$$\nu_{i+1} \geq \varrho_i = \sum_{q \geq i+1} \alpha_q - p_{i+1} + 1.$$

We want to show that the i-witness vector $\widetilde{g}^i = (\widetilde{g}^i_1, \ldots, \widetilde{g}^i_i)$ of R^p_α fits its slack $p_{i+1} - 1$. It follows that $0 \leq \sum_{q=1}^{i} (\nu_q - \alpha_q) \leq p_{i+1} - 1$, otherwise, we would have

$$\sum_{q=1}^{i+1} \nu_q = \sum_{q=1}^{i} \nu_q + \nu_{i+1} > \sum_{q=1}^{i} \alpha_q + p_{i+1} - 1 + \sum_{q=i+1}^{\ell(\alpha)} \alpha_q - p_{i+1} + 1 = \sum_{q=1}^{\ell(\alpha)} \alpha_q,$$

contradicting the equality $|\alpha| = |\nu|$.

Let $U := \{j \in \{2, \ldots, i\} : \nu_j - \alpha_j < 0\}$ (indeed $\nu_1 \geq \alpha_1$) and $u := \max U$. Put $u := 0$ if $U = \emptyset$.

Claim: There exist $\mu_j \geq \alpha_j$, $j = 1, \ldots, u$, such that

$$\mu_1 \geq \cdots \geq \mu_{u-1} \geq \alpha_{u-1} \geq \mu_u = \alpha_u > \nu_u \geq \nu_{u+1} \geq \cdots \geq \nu_i \geq \nu_{i+1}, \text{ and} \tag{5.2}$$

$$\sum_{j=1}^{u} (\mu_j - \alpha_j) = \sum_{j=1}^{u} (\nu_j - \alpha_j) \geq 0. \tag{5.3}$$

In these conditions, defining $g_j := \mu_j - \alpha_j \geq 0$, $j = 1, \ldots, u$, and $g_j := \nu_j - \alpha_j \geq 0$, $j = u+1, \ldots, i$, one has $\sum_{j=1}^{i} g_j = \sum_{j=1}^{i} (\nu_j - \alpha_j) \leq p_{i+1} - 1$,

$$\alpha_j + g_j = \mu_j \geq \alpha_u > \nu_u \geq \nu_{i+1} \geq \sum_{q \geq i+1} \alpha_q - p_{i+1} + 1 = \varrho_i,\ j = 1, \ldots, u,$$

and

$$\alpha_j + g_j = \nu_j \geq \nu_i \geq \nu_{i+1} \geq \sum_{q \geq i+1} \alpha_q - p_{i+1} + 1 = \varrho_i,\ j = u+1, \ldots, i,$$

so that $g_j \geq \varrho_i - \alpha_j$ for $j = 1, \ldots, i$. It follows that the witness vector $\widetilde{g}^i = (\widetilde{g}^i_1, \ldots, \widetilde{g}^i_i)$, with $\widetilde{g}^i_j = \varrho_i - \alpha_j$ for $j = 1, \ldots, i$, fits its slack:

$$|\widetilde{g}^i| = \sum_{j=1}^{i} \widetilde{g}^i_j \leq \sum_{j=1}^{i} g_j \leq p_{i+1} - 1.$$

Proof of the Claim: We prove the claim by double induction on $|U| \geq 0$ and $i \geq 2$.

For $|U| = 0$ there is nothing to prove whatever is $i \geq 2$. Let $|U| \geq 1$. For $i = 2$, one has $\nu_1 - \alpha_1 \geq 0$ and $u = 2$ with $\nu_2 < \alpha_2$. Since $(\nu_1 - \alpha_1) + (\nu_2 - \alpha_2) \geq 0$ and $\nu_2 = \alpha_2 - \epsilon_2$, for some $\epsilon_2 > 0$, we may write

$$(\nu_1 - \alpha_1) + (\nu_2 - \alpha_2) = [(\nu_1 - \epsilon_2) - \alpha_1] + (\alpha_2 - \alpha_2) = (\nu_1 - \epsilon_2) - \alpha_1 \geq 0.$$

Thus $\mu_1 := \nu_1 - \epsilon_2 \geq \alpha_1 \geq \mu_2 := \alpha_2 > \nu_2 \geq \nu_3$.

Let $i = m + 1 \geq 3$, and $u \in \{2, \ldots, m+1\}$ where $\nu_u = \alpha_u - \epsilon_u$, for some $\epsilon_u > 0$, and $\nu_v - \alpha_v \geq 0$, $u < v \leq m + 1$. We distinguish two situations:

(a) $u = 2$: $\nu_1 > \alpha_1$, $\nu_2 = \alpha_2 - \epsilon$, for some $\epsilon > 0$, and $\nu_j \geq \alpha_j$, for $3 \leq j \leq i$. We have $\alpha \preceq \nu$ and we may write

$$(\nu_1 - \alpha_1) + (\nu_2 - \alpha_2) = (\mu_1 - \alpha_1) + (\alpha_2 - \alpha_2) = (\mu_1 - \alpha_1) + (\mu_2 - \alpha_2) \geq 0,$$

where $\mu_1 := \nu_1 - \epsilon \geq \alpha_1$, $\mu_2 := \alpha_2$. Also $\mu_1 \geq \alpha_1 \geq \mu_2 = \alpha_2 > \nu_2 \geq \nu_3 \geq \cdots \geq \nu_i \geq \mu_{i+1}$.

(b) $u > 2$: $\nu = \alpha_u - \epsilon_u$ for some $\epsilon_u > 0$ and $\nu_j \geq \alpha_j$, $u < j \leq i$. One has $\alpha \preceq \nu$, henceforth

$$\sum_{j=1}^{u} (\nu_j - \alpha_j) = \left[\left(\sum_{j=1}^{u-1} (\nu_j - \alpha_j) \right) - \epsilon_u \right] + (\alpha_u - \alpha_u) \geq 0.$$

Thus $\mu_u := \alpha_u > \nu_u \geq \nu_{u+1} \geq \cdots \geq \nu_i \geq \nu_{i+1}$ and $\sum_{j=1}^{u-1} (\nu_j - \alpha_j) \geq \epsilon_u > 0$.

Since $2 \leq u - 1 \leq i - 1 \leq m$, by induction, there exist $\nu'_1 \geq \cdots \geq \nu'_{u-1}$ with $\nu'_j \geq \alpha_j$, $j = 1, \ldots, u-1$, such that

$$\sum_{j=1}^{u-1} (\nu_j - \alpha_j) = \sum_{j=1}^{u-1} (\nu'_j - \alpha_j) \geq \epsilon_u.$$

Indeed, one has $\nu'_j = \alpha_j + \epsilon_j$, with $\epsilon_j \geq 0$, $j = 1, \ldots, u-1$, such that $\sum_{j=1}^{u-1} \epsilon_j \geq \epsilon_u$. Define recursively the non negative integers

$$\delta_j := min(\epsilon_j, \epsilon_u - \sum_{j+1 \leq q \leq u-1} \delta_q), \quad \text{for} \quad j = u-1, \ldots, 1,$$

and put $\mu_j := \nu'_j - \delta_j = \alpha_j + (\epsilon_j - \delta_j) \geq 0, \quad$ for $\quad j = u - 1, \ldots, 1$. Therefore, there exists $1 \leq u_0 < u$ such that $0 < \delta_{u_0} \leq \epsilon_{u_0}$ and

$$\mu_j = \begin{cases} \alpha_j & u_0 < j < u \\ \alpha_{u_0} \; + \; (\epsilon_{u_0} - \delta_{u_0}) & \\ \nu'_j & 1 \leq j < u_0. \end{cases}$$

Hence,

$$\mu_1 \geq \cdots \geq \mu_{u_0+1} \geq \nu'_{u_0} > \mu_0 \geq \alpha_{u_0} \geq \mu_{u_0+1} =$$
$$= \alpha_{u_0+1} \geq \cdots \geq \mu_{u-1} = \alpha_{u-1} \geq \mu_u = \alpha_u,$$

as required.

The *"if"* part. Let $\varrho_i = \sum_{q=i+1}^{\ell(\alpha)} \alpha_q - p_{i+1} + 1 > 0$, $1 \leq i \leq \ell(p) - 1$. Suppose now that there is an i-witness vector $\tilde{g}^i = (\tilde{g}^i_1, \ldots, \tilde{g}^i_i)$ of R^p_α for some $1 \leq i \leq \ell(p) - 1$, with $\tilde{g}^i_j := [\varrho_i - \alpha_j]_+$, $j = 1, \ldots, i$, such that $|\tilde{g}^i| \leq p_{i+1} - 1$. Let $\nu = (\nu_1, \ldots, \nu_{i+1}, \nu_{i+2})$ be the partition of $|\alpha|$ formed by the rearrangement of the composition

$$(\alpha_1 + \tilde{g}^i_1, \ldots, \alpha_i + \tilde{g}^i_i, \varrho_i, p_{i+1} - 1 - |\tilde{g}^i|), \tag{5.4}$$

where $\alpha_1 + \tilde{g}^i_1, \ldots, \alpha_i + \tilde{g}^i_i \geq \nu_{i+1} = \varrho_i \geq \nu_{i+2} = p_{i+1} - 1 - |\tilde{g}^i|$.

We will show that ν is a partition in the Schur interval of the ribbon R^p_α that is not in its support. Indeed, the inequality $|\tilde{g}^i| \leq p_{i+1} - 1$ shows that all entries in (5.4) are non negative, and $\displaystyle\sum_{q=1}^{i+2} \nu_q = \sum_{q=1}^{\ell(\alpha)} \alpha_q = |\alpha|$. Thus, ν is well defined and is a partition of $|\alpha|$.

Recall that $\varrho_i - 1 = \displaystyle\sum_{q \geq i+1} \alpha_q - p_{i+1}$ is the total number of columns of $R^p_\alpha \setminus \left(\cup_{q=1}^{i} < \alpha_q >\right)$ and that p_{i+1} is the number of columns of length two in this same ribbon. Therefore, we have $\varrho_i > p_{i+1} - 1 - |\tilde{g}^i|$. Moreover, for each $1 \leq j \leq i$, we have

$$\alpha_j + \tilde{g}^i_j = \begin{cases} \varrho_i, & \text{if } \varrho_i > \alpha_j \\ \alpha_j, & \text{if } \varrho_i \leq \alpha_j \end{cases} . \tag{5.5}$$

It follows that $\alpha_j + \tilde{g}^i_j \geq \varrho_i$. This means that the last two entries of ν are $\nu_{i+1} = \varrho_i$ and $\nu_{i+2} = p_{i+1} - 1 - |\tilde{g}^i|$. In particular, it follows from Corollary 5.2, (b), that ν is not in the Schur support of R^p_α.

It remains to prove that ν is a partition in the Schur interval $[\alpha, (|\alpha| - p_1, p_1)]$. We start by showing that $\alpha \preceq \nu$. From (5.5), we find that for each $1 \leq k \leq i$,

$$\sum_{j=1}^{k} \nu_j = \sum_{j=1}^{k} (\alpha_j + \tilde{g}^i_j) \geq \sum_{j=1}^{k} \alpha_j$$

and since $\varrho_i \geq \alpha_{i+1}$, $\sum_{j=1}^{i+1} \nu_j = \sum_{j=1}^{i}(\alpha_j + \tilde{g}^i_j) + \varrho_i \geq \sum_{j=1}^{i} \alpha_j + \varrho_i \geq \sum_{j=1}^{i+1} \alpha_j$. Finally, since ν is a partition of $|\alpha|$, we get $\alpha \preceq \nu$. To prove that we also have $\nu \preceq (|\alpha| - p_1, p_1)$, notice that by (5.5) and Remark 5.1, ν_1 is either equal to ϱ_1 or to α_1, and $\varrho_1 \leq |\alpha| - p_1$. Therefore, we have $\nu_1 \leq |\alpha| - p_1$. Clearly, $\nu_1 + \nu_2 \leq |\alpha|$, from which it follows that $\nu \preceq (|\alpha| - p_1, p_1)$. $\square$

Remark 5.4. Let α be a partition with parts ≥ 2 and overlapping partition p with $\ell(p) \geq 2$. Recall Definition 1.3, $\varrho_i = 1 + \sum_{q=i+1}^{\ell(\alpha)} \alpha_q - p_{i+1} > 0$, and $\tilde{g}^i = \{\tilde{g}^i_j\}_{j=1}^i = \{[\varrho_i - \alpha_j]_+\}_{j=1}^i$, for $1 \leq i \leq \ell(p) - 1$. Observe that the following are equivalent:

(a) for some $1 \leq i \leq \ell(p) - 1$, the size of the i-witness vector $\tilde{g}^i$ fits its slack, that is,

$$|\tilde{g}^i| = \sum_{j=1}^{i} [\varrho_i - \alpha_j]_+ \leq p_{i+1} - 1. \tag{5.6}$$

(b) for some $1 \leq i \leq \ell(p) - 1$, there exist integers $g_1, \dots, g_i \geq 0$ with $\sum_{j=1}^{i} g_j \leq p_{i+1} - 1$, such that

$$\alpha_j + g_j \geq 1 + \sum_{q=i+1}^{\ell(\alpha)} \alpha_q - p_{i+1} \Leftrightarrow g_j \geq \varrho_i - \alpha_j, \quad j = 1, \dots, i. \tag{5.7}$$

Indeed, (5.7) says that, for $1 \leq i \leq \ell(p) - 1$, other "witness vectors" $g = \{g_j\}_{j=1}^i$ can be found depending on how big is the slack $p_{i+1} - 1$. Simultaneously (5.7) tells that the selected witness $\tilde{g}^i$ in Definition 1.3 is entrywise the smallest,

$$\tilde{g}^i_j \leq g_j,\ j = 1, \dots, i, \Rightarrow |\tilde{g}^i| \geq |g|.$$

If our selected witness $\tilde{g}^i$ does not fit (is over the size of) its slack, no other (any other) choice for the witness vector will fit (oversize) it.

In the conditions of (b), it can be shown that $(\alpha_1 + g_1, \dots, \alpha_i + g_i, \varrho_i)^+$ with $\sum_{j=1}^{i} g_j = p_{i+1} - 1$ (g has the possible biggest size) is a partition of $|\alpha|$ in the Schur interval of R^p_α but not in the support of R^p_α.

Example 5.5. (a) Consider the same example as before, $\alpha = (7, 6, 6, 2, 2, 2, 2)$ and the ribbon R^p_α with $p_1 = 6$. Applying Theorem 1.6 with $i = 3$, one has $p_{i+1} = 3$, $\varrho_3 = 6$ and the 3-witness vector $\tilde{g}^3 = (\tilde{g}^3_1, \tilde{g}^3_2, \tilde{g}^3_3) = (0, 0, 0)$, satisfy $|\tilde{g}^3| \leq p_{i+1} + 1 = 4$. Therefore, the support $[R^p_\alpha]$ is not the full Schur interval.

The partition

$$(7-\tilde{g}_1^3, 6-\tilde{g}_2^3, 6-\tilde{g}_3^3, \varrho_3, p_4-1-|\tilde{g}^3|) = (7,6,6,6,2)$$

is in the Schur interval $[\alpha, (27-6,6)]$ but not in the support $[R_\alpha^p]$.

(b) Furthermore, considering $g_1+g_2+g_3 = 2 = p_4-1$, with $g_i \geq 0$, $i=1,2,3$, the partitions $\nu_1 = (6+2,7,6,2+2+2+2-2)$, $\nu_2 = (6+1,7,6+1,2+2+2+2-2)$ and $\nu_3 = (7+2,6,6,2+2+2+2-2)$ are in the interval $[\alpha,(27-6,6)]$ but not in the support of R_α^p.

5.2. Proof of Remark 1.8 and Corollary 1.9. . Theorem 1.7 is logically equivalent to Theorem 1.6 and says that if every i-witness $\tilde{g}^i$ vector of R_α^p, for $i=1,\dots,\ell(p)-1$, is oversized, with respect to its slack $p_{i+1}-1$, then R_α^p has full Schur support. In particular, R_α^p has full support only if $\alpha_i < \varrho_i$ for every $1 \leq i \leq \ell(p)-1$. In fact, if, for some $k \in \{1,\dots,\ell(p)-1\}$, $\alpha_k \geq \varrho_k$, then $\alpha_1 \geq \cdots \geq \alpha_k \geq \varrho_k$ and $|\tilde{g}^k| = \sum_{j=1}^k [\varrho_k - \alpha_j]_+ = 0 \leq p_{k+1}-1$. This implies that $(\alpha_1,\dots,\alpha_k,\varrho_k,p_{k+1}-1) \in [\alpha,(|\alpha|-p_1,p_1)]$ is not in $[R_\alpha^p]$ which is absurd.

(a) When $\ell(p)=2$, one has $p=(2,1,0^{\ell(\alpha)-2})$, and $[R_\alpha^p] = [\alpha,(|\alpha|-2,2)]$ if and only if $\alpha_1 < \varrho_1 \Leftrightarrow \alpha_1 < \sum_{q=2}^{\ell(\alpha)} \alpha_q$. In fact, if $\ell(p)=2$, (1.9) means

$$[\varrho_1-\alpha_1]_+ \geq 1 \Leftrightarrow \varrho_1-\alpha_1 > 0 \Leftrightarrow \varrho_1 > \alpha_1 \Leftrightarrow \alpha_1 < 1+\sum_{q=2}^{\ell(\alpha)}\alpha_q - 1 = \sum_{q=2}^{\ell(\alpha)}\alpha_q.$$

(b) When $\ell(p)=3$, one has $p=(3,2,1,0^{\ell(\alpha)-3})$, and $[R_\alpha^p] = [\alpha,(|\alpha|-3,3)]$ if and only if $\alpha_1 < \sum_{q=2}^{\ell(\alpha)}\alpha_q - 2$ and $\alpha_2 < \sum_{q=3}^{\ell(\alpha)}\alpha_q$. In fact, if $\ell(p)=3$, (1.10) means

$$\varrho_1-\alpha_1 \geq 2 \Leftrightarrow \varrho_1 > \alpha_1+1 \Leftrightarrow 1+\sum_{q=2}^{\ell(\alpha)}\alpha_q - 2 > \alpha_1+1 \Leftrightarrow \sum_{q=2}^{\ell(\alpha)}\alpha_q - 2 > \alpha_1,$$

$$[\varrho_2-\alpha_1]_+ + [\varrho_2-\alpha_2]_+ \geq 1 \Leftrightarrow \varrho_2-\alpha_2 \geq 1 \Leftrightarrow \varrho_2 > \alpha_2 \Leftrightarrow \alpha_2 < 1+\sum_{q=3}^{\ell(\alpha)}\alpha_q - 1$$
$$= \sum_{q=3}^{\ell(\alpha)}\alpha_q.$$

□

Example 5.6. Let $\alpha = (4,3,2,2)$ with $p = (3,2,1,0)$. We use the characterization given by the Theorem 1.7 (b) to prove that R^p_α has full support $[\alpha,(8,3)]$. Since $\ell(\alpha) = 4$ and $\ell(p) = 3$, we have two inequalities to check:

$$\alpha_2 + \alpha_3 + \alpha_4 - 2 > \alpha_1 \Leftrightarrow 7 - 2 > 4, \quad \alpha_2 < \alpha_3 + \alpha_4 \Leftrightarrow 3 < 4.$$

6. Connected ribbons with full equivalence class and full Schur support

Building on [Mc08], M. Gaetz, W. Hardt and S. Sridhar have introduced in [GaHaSr17] the family of connected ribbons with full equivalence class.

Definition 6.1. [GaHaSr17, Definition 7] Let α be a partition with parts ≥ 2 and $\ell(\alpha) \geq 3$. The connected ribbon R_α is said to have *full equivalence class* if $[R_\alpha] = [R_\beta]$, for any rearrangement β of the entries of α.

Definition 6.2. [GaHaSr17] Three integers $x \leq y \leq z$ are said to satisfy the *strict triangle inequality* if $z < x + y$. In this case, the multiset $\{x, y, z\}$ is said to satisfy the strict triangle inequality.

The set of connected ribbons with full equivalence class have partitions as representatives. For monotone connected ribbons, the inequality (1.11), in Theorem 1.10, [GaHaSr17, Theorem II.1], giving a necessary condition for full equivalence class, is equivalent to inequality (1.8), in Theorem 1.7, characterizing the full Schur support.

Proof of Lemma 1.11 Let $j \in \{1, \dots, \ell(\alpha) - 2\}$ and $N_j := max\{k : \sum\limits_{\substack{1\leq i\leq j\\ \alpha_i<k}} (k - \alpha_i) \leq \ell(\alpha) - j - 2\}$. From the definition of N_j, one has $\sum\limits_{\substack{1\leq i\leq j\\ \alpha_i<N_j}} (N_j - \alpha_i) \leq \ell(\alpha) - j - 2$. Then $N_j < \varrho_j \Leftrightarrow \sum\limits_{\substack{1\leq i\leq j\\ \alpha_i<\varrho_j}} (\varrho_j - \alpha_i) \geq \ell(\alpha) - j - 1$. $\square$

Proof of Theorem 1.12. Because R_α is connected, $p = (\ell(\alpha)-1, \dots, 2, 1, 0)$ and, in Definition 1.3, $\varrho_j = \sum_{q=j+1}^{\ell(\alpha)} \alpha_q - (\ell(\alpha) - j - 2)$, for $j \in \{1, \dots, \ell(\alpha) - 2\}$. Suppose that R_α does not have full Schur support. Then Theorem 1.7 says that for some $t \in \{1, \dots, \ell(\alpha) - 2\}$,

$$\sum_{1\leq i\leq t} [\varrho_t - \alpha_i]_+ = \sum_{\substack{1\leq i\leq t\\ \alpha_i<\varrho_t}} (\varrho_t - \alpha_i) \leq \ell(\alpha) - t - 2. \tag{6.1}$$

Inequality (6.1) implies in the definition of N_t, (1.11), that $N_t \geq \varrho_t$ with $t \in \{1, \dots, \ell(\alpha) - 2\}$. Henceforth, by Theorem 1.10, [GaHaSr17, Theorem 1.2], one concludes that α does not have full equivalence class. When $\ell(\alpha) = 3$, by Theorem 1.6, R_α has full support $[\alpha, (|\alpha| - 2, 2)]$ if and only if $\alpha_1 < \alpha_2 + \alpha_3$ (strict triangle inequality). Theorem 3.4, in [GaHaSrTr17], also shows that R_α has full equivalence class if and only if $\alpha_1 < \alpha_2 + \alpha_3$. When $\ell(\alpha) = 4$, by Theorem 1.6 (b), R_α has full support $[\alpha, (|\alpha| - 3, 3)]$ if and only if (1.10) are satisfied. Theorem 3.6, in [GaHaSrTr17], also shows that R_α has full equivalence class if and only if (1.10) are satisfied. $\square$

Next theorem gives a sufficient condition for a monotone connected ribbon to have full equivalence class [GaHaSr17, Corollary1.4] which in turn, thanks to Theorem 1.12, also gives a sufficient condition for monotone connected ribbons to have full Schur support.

Theorem 6.3. Let $\beta = (\beta_1, \ldots, \beta_{\ell(\beta)})$ be a composition with parts ≥ 2 and $\ell(\beta) \geq 3$. If all 3-multisets contained in $\{\beta_1, \ldots, \beta_{\ell(\beta)}\}$ satisfy the strict triangle inequality then the connected ribbon R_β has

(a) [GaHaSr17, Corollary 1.4] full equivalence class; and

(b) full Schur support $[\beta^+, (|\beta| - \ell(\beta) + 1, \ell(\beta) - 1)]$.

The strict triangle inequality condition given by the previous theorem is sufficient for a connected ribbon to have full support, but it is not necessary. For instance, not all 3-subsets of the partition $\alpha = (4, 3, 2, 2)$ satisfy the strict triangular inequality $(4 = 2 + 2)$, but as we have seen in Example 5.6, the connected ribbon R^p_α has full support. Nevertheless, for partitions α with length 3 the connected ribbon R^p_α has full support (full support) if and only if α satisfy the strict triangular inequality (1.9).

Next statement classifies arbitrary compositions with length 3 with respect to the full support where we may verify that for non monotone compositions the strict triangular inequality is not a necessary condition. This means that the full Schur support classification for non monotone compositions and for partitions is not the same.

Corollary 6.4. Let β be a composition of length 3 with each part ≥ 2. Then, the connected ribbon R_β has full support except when $\beta = (\beta_1^+, \beta_2, \beta_3)$ or $\beta = (\beta_2, \beta_3, \beta_1^+)$ with $\beta_1^+ \geq \beta_2 + \beta_3$, in which cases, the partition $\nu = (\beta_1^+, \beta_2 + \beta_3)$ is in the Schur interval but not in the support of R_β.

Proof: By the previous theorem, we know that if β satisfies the strict triangle inequality, $\beta_1^+ < \beta_2 + \beta_3$, then R_β has full support. There remains three cases to analyse: $\beta = (\beta_1^+, \beta_2, \beta_3)$, or $\beta = (\beta_2, \beta_3, \beta_1^+)$, or $\beta = (\beta_2, \beta_1^+, \beta_3)$, with $\beta_1^+ \geq \beta_2 + \beta_3$. Since the support of R_β is invariant under 180 degrees rotation of the ribbon R_β, the first two cases can be reduced to the first one. Suppose that $\beta = (\beta_1^+, \beta_2, \beta_3)$ satisfies $\beta_1 \geq \beta_2 + \beta_3$ and $\beta_2 \geq \beta_3$, and recall that the overlapping partition is $p = (2, 1, 0)$. Applying Theorem 1.6 with $i = 1$ one has $\varrho_1 = \beta_2 + \beta_3$ and $\tilde{g}_1^1 = 0 \leq p_2 - 1 = 0$, and henceforth, the support of R_β^p is not the entire Schur interval, since the partition $\nu = (\beta_1, \beta_2 + \beta_3)$ is in the Schur interval but not in the support of R_β. The same partition ν proves the result when $\beta_3 \geq \beta_2$. Note that an LR filling of $R_{(\beta_1^+, \beta_2, \beta_3)}$ with content ν would oblige to fill the first row with β_1 1's and the last two rows with $\beta_2 + \beta_3$ 2's. Since the two last rows of R_β overlap such a filling violates the column semistandard condition.

Finally, in the case of the connected ribbon $R_{(\beta_2, \beta_1^+, \beta_3)}$ satisfying $\beta_1^+ \geq \beta_2 + \beta_3$, it is easy to show that $\mathrm{LR}_{\mathrm{R}_\beta, \nu} \neq \emptyset$ for any partition in the Schur interval $[\beta^+, (|\beta| - 2, 2)]$. Indeed if $\ell(\nu) = 3$, any $T \in Tab(\nu, \beta)$ is such that $\mathcal{D}(\widehat{T}) = \mathcal{S}(\beta) = \{\beta_2, \beta_1^+ + \beta_2\}$. If $\ell(\nu) = 2$, consider the canonical filling $T \in Tab(\nu, \beta)$, $\nu = (\nu_1, \nu_2)$. Then the second row of T has 3's and because $\beta_1^+ \geq \beta_2 + \beta_3$, $\nu_2 < \beta_1^+ + \beta_3$ (otherwise $\nu_2 > \nu_1 = \beta_2$), the first row of T has β_2 1's and at least one two. In case, $\nu_2 = \beta_3 \geq 2$, the 2nd row of T has β_3 3's and the $\beta_1^+ \geq 2$, 2's are all in the first row of T, in which case we swap the rightmost 2 in the first row with the leftmost 3 in the second row to get a new tableau in $Tab(\nu, \beta)$. The descent set of this new tableau is $\mathcal{S}(\beta) = \{\beta_2, \beta_1 + \beta_2\}$. □

Remark 6.5. If the composition $\beta = (\beta_2, \beta_1^+, \beta_3)$ satisfies $\beta_1^+ \geq \beta_2 + \beta_3$ the connected ribbon R_β has full support while R_{β^+} does not have full support because $\beta_1^+ \geq \beta_2 + \beta_3$.

Corollary 6.6. [McWi12, Theorem 1.5.] Let β be an arbitrary composition with parts ≥ 1. Connected ribbons R_β whose column and row lengths differ at most one have full support. They also have full equivalence class except when $\beta = (2^{\ell(\beta)-1}, 1)$, $\ell(\beta) \geq 3$.

Proof: Let $\beta = (\beta_1, \ldots, \beta_{\ell(\beta)})$ and R_β a connected ribbon in the conditions of the statement. Observe that the transpose of R_β is still in the conditions of the statement. If R_β or its transpose consists only of one or two rows is trivial. Suppose that R_β has at least three rows. If $\beta_i \geq 2$ for all $1 \leq i \leq \ell(\beta)$, then $|\beta_i - \beta_j| \leq 1$, for all $1 \leq i, j \leq \ell(\beta)$, and any three parts $\beta_i \leq \beta_j \leq \beta_k$ of β satisfy the strict triangle inequality $\beta_k < \beta_i + \beta_j$. By Theorem 6.3, (b), R_β has full support and full equivalence class. If $\beta_1 = \beta_{\ell(\beta)} = 1$ then $\beta_i = 2$, $1 < i < \ell(\beta)$, and transposing R_β we fall in one of the previous cases:

$\beta = (2^{\ell(\beta)})$ with $\ell(\beta) \geq 2$, and again R_β has full support and full equivalence class. If $\beta_1 = 1 < \beta_{\ell(\beta)}$ or $\beta_1 > \beta_{\ell(\beta)} = 1$, by 180 degrees-rotation, we may assume the last inequality and we have $\beta = (2^{\ell(\beta)-1}, 1)$ with $\ell(\beta) \geq 3$. Put $s := \ell(\beta) - 1$ and let $I := [(2^s), (s, s)]$ be the Schur interval of $R_{(2^s)}$, $s \geq 2$. By the previous cases, the support of $R_{(2^s)}$ is the full interval I.

The Schur interval of $R_{(2^s,1)}$ is $[(2^s, 1), (s + 1, s)]$ and it is self conjugate. Its partitions are obtained using one extra box in the construction of the elements of I. There are three possible positions to put the extra box in one element of I and obtain $\nu \in [(2^s, 1), (s + 1, s)]$: (a) far right of the first row; (b) below the last row; or (c) far right of the last row.

Because $R_\beta = R_{(2^s,1)} = (R_\beta)'$ and $c^{\nu}_{R_\beta} = c^{\nu'}_{R_\beta}$, by transposition of ν, we may reduce (a) to (b). Hence if $T \in LR_{R_{(2^s)},\mu}$ then the SSYT $T_\square$, obtained by adding one box filled with $s+1$ below the last row of T, is in $LR_{R_{(2^s,1)},\nu}$ with $\nu = (\mu, 1)$. Note that $\mathcal{D}(\widehat{T_\square}) = \mathcal{S}(2^s, 1) = \mathcal{S}(2^s) \cup \{2s\}$. It remains to prove that $\nu = (\mu_1, \ldots, \mu_{\ell(\mu)-1}, \mu_{\ell(\mu)} + 1)$ obtained in (c) is in $[R_{(2^s,1)}]$. If the last row of T has at most one s then just add one box filled with $s + 1$ at the end of the this row to obtain $T_\square$. If the last row of T has two s's also add one box filled with $s + 1$ at the end of this row. At least one entry in the row above is not in the last row and choose that in the rightmost position: it is the far right entry:

$(i)\ s-1,\ T = \begin{array}{cccc} \cdots & & & \\ \cdots & a & b & (s-1) \\ \cdots & s & s & (s+1) \end{array} \to T_\square = \begin{array}{cccc} \cdots & & & \\ \cdots & a & b & s \\ \cdots & s-1 & s & (s+1) \end{array}, a < s-1, b \leq s-1$

$(ii)\ a < s-1, T = \begin{array}{cccc} \cdots & & & \\ \cdots & d & c & a \\ \cdots & s & s & (s+1) \end{array} \to T_\square = \begin{array}{cccc} \cdots & & & \\ \cdots\cdots & d & c & s \\ \cdots a & \cdots & s & (s+1) \end{array}, c \leq a < s-1, d < a$, a enters in the last row of T bumping to the right the left most strictly bigger entry; otherwise, $T = \begin{array}{ccccc} \cdots & & & & \\ \cdots & d & c & a & x \\ \cdots & x & s & s & (s+1) \end{array} \to T_\square = \begin{array}{ccccc} \cdots & & & & \\ \cdots & d & c & x & s \\ \cdots a & \cdots & x & s & (s+1) \end{array}, d < a < x \leq s-1, c \leq a$, a enters in the last row of T bumping to the right the left most strictly bigger entry. In any case and $\mathcal{D}(\widehat{T_\square}) = \mathcal{S}(2^s, 1)$.

Indeed, $\beta = (2^{\ell(\beta)-1}, 1)$ and $\gamma = (2, 1, 2^{\ell(\beta)-2})$, $\ell(\beta) \geq 3$, do not have the same Schur interval. The Schur interval of the latter is $[(2^s, 1); (3, 2^{s-2}, 1^2)]$ with $s = \ell(\beta) - 1$ and henceforth $\beta = (2^{\ell(\beta)-1}, 1)$, $\ell(\beta) \geq 3$ does not have full equivalence class. $\square$

7. Towards to a coincidence between full Schur support monotone connected ribbons and full equivalence classes

In this section we consider connected ribbons with parts ≥ 2 arranged in any order. The necessary condition, given by Theorem 1.5, for the LR coefficient $c^{\nu}_{R_\alpha}$ to be positive, with α a partition, is generalized to a connected ribbon R_{α_π} where α_π, $\pi \in \Sigma_{\ell(\alpha)}$, is a π-permutation of the entries of α. Thanks to the 180°-rotation symmetry of LR coefficients, $c^{\nu}_{R_\alpha} = c^{\nu}_{(R_\alpha)^\circ}$, it is sufficient to consider partitions α of length ≥ 3. That is, we already know that $c^{\nu}_{R_{(\alpha_1,\alpha_2)}} = c^{\nu}_{R_{(\alpha_2,\alpha_1)}} > 0 \Leftrightarrow \nu_i \leq \sum_{q=i}^{2} \alpha_q - p_i$, $1 \leq i \leq 2$. Recall the definition of overlapping partition of a connected ribbon with row lengths in arbitrary order, Definition 3.1, and that the overlapping partition $p^\pi = (p^\pi_1, p^\pi_2, \ldots, p^\pi_{\ell(\alpha)}, 0)$ of the connected ribbon R_{α_π} satisfies (3.1), $p^\pi \subseteq (\ell(\alpha) - 1, \ldots, 1, 0)$, that is, $p^\pi_1 = \ell(\alpha) - 1$, and $p^\pi_i \leq \ell(\alpha) - i$, $2 \leq i \leq \ell(\alpha)$.

Theorem 7.1. Let α be a partition with parts ≥ 2, and R_{α_π} a connected ribbon with overlapping partition p^π. Let $\nu \in [\alpha, (|\alpha| - \ell(\alpha) + 1, \ell(\alpha) - 1)]$. Then

$$\nu \in [R_{\alpha_\pi}](c^{\nu}_{R_{\alpha_\pi}} > 0) \Rightarrow \nu_i \leq \sum_{q=i}^{\ell(\alpha)} \alpha_q - p^\pi_i, \quad 1 \leq i \leq \ell(p^\pi). \tag{7.1}$$

Proof: We prove the contrapositive assertion: if there exists $i \in \{1, \ldots, \ell(\alpha) - 2\}$ such that $\nu_{i+1} \geq \sum_{q \geq i+1}^{\ell(\alpha)} \alpha_q - p^\pi_{i+1} + 1$ then $c^{\nu}_{R_{\alpha_\pi}} = 0$. (Indeed $\nu_1 \leq \sum_{q\geq 1}^{\ell(\alpha)} \alpha_q - p^\pi_1 + 1$ and $\nu_{\ell(\alpha)} \leq \alpha_{\ell(\alpha)}$.)

Let $\alpha_\pi = (\beta_1, \ldots, \beta_{\ell(\alpha)})$ and let i be the smallest element in $\{1, \ldots, \ell(\alpha) - 2\}$ such that $\nu_{i+1} \geq \sum_{q\geq i+1}^{\ell(\alpha)} \alpha_q - p^\pi_{i+1} + 1$. Since $|\nu| = |\alpha|$ and $\alpha \preceq \nu$, one has

$$\sum_{q=1}^{i} \beta_q \leq \sum_{q=1}^{i} \alpha_q \leq \sum_{q=1}^{i} \nu_q = |\alpha| - \sum_{q\geq i+1}^{\ell(\alpha)} \nu_q \leq \sum_{q=1}^{i} \alpha_q + p^\pi_{i+1} - 1. \tag{7.2}$$

If we place ν_1 1's, ν_2 2's, $\ldots, \nu_i$ i's in R_β to obtain an LR filling then at least the first i rows of R_β are completely filled because one can not place in them numbers $\geq i+1$. Henceforth, in the best case one has $\sum_{q=1}^{i}\beta_q = \sum_{q=1}^{i}\alpha_q = \sum_{q=1}^{i}\nu_q$, so that it remains $\ell(\alpha)-i$ rows of R_β to place ν_{i+1} $i+1$'s. Because R_β is connected the number of columns of length two among them is $\ell(\alpha)-i-1 \geq p_{i+1}^{\pi}$. (In fact, in this case, one has the equality $\ell(\alpha)-i-1 = p_{i+1}^{\pi}$. Because one has the equality of the multisets $\{\beta_i,\ldots,\beta_{\ell(\alpha)}\} = \{\alpha_i,\ldots,\alpha_{\ell(\alpha)}\}$ and by definition p_{i+1}^{π} is the number of columns of length two among the rows $\alpha_i,\ldots,\alpha_{\ell(\alpha)}$ of the ribbon R_β which in this the same as among the rows of $R_{(\beta_i,\ldots,\beta_{\ell(\alpha)})}$.) It means that in the best case the number of available boxes to fill with ν_{i+1}, $i+1$'s, is in fact

$$\begin{aligned}
\sum_{q\geq i+1}^{\ell(\alpha)}\beta_q-(\ell(\alpha)-i-1) &= |\beta|-\sum_{q=1}^{i}\beta_q-(\ell(\alpha)-i-1)=\\
&= |\alpha|-\sum_{q=1}^{i}\alpha_q-(\ell(\alpha)-i-1)\\
&= \sum_{q\geq i+1}^{\ell(\alpha)}\alpha_q-(\ell(\alpha)-i-1)\leq \sum_{q\geq i+1}^{\ell(\alpha)}\alpha_q-p_{i+1}^{\pi}\\
&< \sum_{q=1}^{i}\alpha_q+p_{i+1}^{\pi}-1\leq \nu_{i+1},
\end{aligned}$$

which is not enough. Therefore $c_{R_{\alpha_\pi}}^{\nu}=0$. □

Remark 7.2. (1) Under the assumption that row lengths are ≥ 2, R_{α_π} and R_α have the same the Schur interval, $[\alpha,(|\alpha|-\ell(\alpha)+1,\ell(\alpha)-1)]$, for all $\pi\in\Sigma_{\ell(\alpha)}$.

(2) Assuming in Theorem 7.1 that inequalities (7.1) are also sufficient for $\nu\in[R_{\alpha_\pi}]$, we have the following result. If $\nu\in[R_\alpha]$ with α a partition, and $\pi\in\Sigma_{\ell(\alpha)}$ then

$$\nu_i\leq\sum_{q=i}^{\ell(\alpha)}\alpha_q-(\ell(\alpha)-i)\leq\sum_{q=i}^{\ell(\alpha)}\alpha_q-p_i^{\pi},\;\;1\leq i\leq\ell(\nu)\Rightarrow\nu\in[R_{\alpha_\pi}].$$

Therefore, $[R_\alpha]\subseteq[R_{\alpha_\pi}]$, for any $\pi\in\sum_{\ell(\alpha)}$. If R_α has full Schur support, $[R_{\alpha_\pi}]=[R_\alpha]$, for any $\pi\in\sum_{\ell(\alpha)}$, and R_α has full equivalence class. Thereby,

R_α does not have full equivalence class if and only if $[R_\alpha] \subsetneqq [R_{\alpha_\pi}]$, for some $\pi \in \sum_{\ell(\alpha)}$.

In other words, the connected ribbon R_α with α a partition with parts ≥ 2 has full support only if α has full equivalence class. This implies that the Gaetz-Hardt-Sridhar conjecture [GaHaSr17, Conjecture II.4] claiming that the necessary condition on full equivalence classes (1.11) is also sufficient, is true.

Conjecture. Let α be a partition with parts ≥ 2 and R_α a connected ribbon. Then the following are equivalent

(a) R_α has full Schur support, that is, $[R_\alpha] = [\alpha, (|\alpha| - \ell(\alpha) + 1, \ell(\alpha) - 1]$;
(b) α has full equivalence class;
(c) For all $j \in \{1, \ldots, \ell(\alpha) - 2\}$,

$$N_j := max\{k : \sum_{\substack{1 \leq i \leq j \\ \alpha_i < k}} (k - \alpha_i) \leq \ell(\alpha) - j - 2\} < \varrho_j \Leftrightarrow \sum_{\substack{1 \leq i \leq j \\ \alpha_i < \varrho_j}} (\varrho_j - \alpha_i) \geq \ell(\alpha) - j - 1.$$

References

[Az99] O. Azenhas, 'The admissible interval for the invariant factors of a product of matrices', *Linear and Multilinear Algebra* **46** (1999), 51–99.

[ACM09] O. Azenhas, A. Conflitti and R. Mamede. *Linear time equivalent Littlewood–Richardson coefficient maps*, Discrete Math. Theor. Comput. Sci. Proceedings, 21st International Conference on Formal Power Series and Algebraic Combinatorics, (2009), 127–144.

[ACM17] O. Azenhas, A. Conflitti and R. Mamede. Multiplicity–free Skew Schur functions with full interval support, arXiv:1009.4170v2 (2011).

[AKT16] O. Azenhas, R. C. King, I. Terada, The involutive nature of the Littlewood-Richardson commutativity bijection, arXiv:1603.05037

[BumSch16] D. Bump and A. Schilling. *Crystal Bases. Representations and Combinatorics*, World Scientific Publishing Co. Ptc. Ltd, 2016.

[DoPy07] G. Dobrovolska, P. Pylyavskyy, On products of sl_n characters and support containment, J. Algebra 316 (2) (2007) 706–714.

[Fo76] H. O. Foulkes, Enumeration of permutations with prescribed up-down and inversion sequences, Discrete Math. 15 (1976), 235–252.

[Fu97] W. Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, USA, 1997.

[Fu00] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, Bull. Amer. Math. Soc. (NS) 37 (3) (2000) 209–249.

[GaHaSr17] M. Gaetz, W. Hardt and S. Sridhar, Support Equalities Among Ribbon Schur Functions, arXiv:1709.03011v1 (and arXiv:1709.03011v2).

[GaHaSrTr17] M. Gaetz, W. Hardt, S. Sridhar and A. Quoc Tran, Support Equalities of Ribbon Schur Functions, October, 2017, http://www-users.math.umn.edu/ reiner/REU/REU.html.

[Ge84] I. M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. In Combinatorics and algebra (Boulder, Colo., 1983), volume 34 of Contemp. Math., pages 289–301. Amer. Math. Soc., Providence, RI, 1984.

[Ge93] I. Gessel and C. Reutenauer, Counting Permutations with Given Cycle Structure and Descent Set, J. Combin., Ser A 64, 189–215 (1993).

[GeZe86] I.M. Gelfand and A.V. Zelevinsky, Multiplicities and proper bases for gl_n, in *Group Theoretical Methods in Physics, Vol II, Proc. 3rd Yurmala Seminar 1985*, VNU Sci. Press, Utrecht, (1986), 147–159.

[Ho62] A. Horn, Eigenvalues of sums of Hermitian matrices, Pacific J. Math., 12 (1962), 225–241.

[Kl98] A. A. Klyachko, Stable bundles, representation theory and Hermitian operators, Sel. math., New ser. 4 (1998) 419–445.

[KnTa99] A. Knutson and T. Tao. The honeycomb model of GLn(C) tensor products. I. Proof of the saturation conjecture, J. Amer. Math. Soc., 12(4):1055–1090, 1999.

[Kwo09] J.–H. Kwon. *Cystal Graphs and the Combinatorics of Young Tableaux* in M. Hazewinkel (ed.), Handbook of Algebra Vol 6, pp. 473–504, North-Holland 2009.

[JaVi17] J. Janopaul-Naylor, C. R. Vinroot, Kostka multiplicity one for multipartitions, Austr. J. of Combinatorics, Vol 68(2), 2017, 153–185.

[LecLen17] C. Lecouvey, C. Lenart. *Combinatorics of generalized exponents* , arXiv:1707.03314, 2017.

[Nak05] T. Nakashima. *Crystal base and a generalization of the Littlewood-Richardson rule for classical Lie Algebras*, Commun. Math. Phy, 154 (1993), 215–243.

[LiRi34] D.E. Littlewood and A. R. Richardson, 'Group characters and algebra', *Phil. Trans. Royal Soc. A (London)* **233** (1934), 99–141.

[Liu12] R. I. Liu, Matching polytopes and Specht modules, Trans. of the American Math. Soc., Volume 364, Number 2, February 2012, pp. 1089–1107.

[Mac17] P.A. MacMahon, Combinatory Analysis, Cambridge University Press, 1917 (Vol. I), 1918 (Vol. II), reprint, Chelsea, New York, USA, 1960.

[Mc08] P. R. W. McNamara, Necessary conditions for Schur–positivity, *J. Algebraic Combin.* **28** (2008), 495–507.

[McWi09] P. R. W. McNamara and S. van Willigenburg, Towards a Combinatorial Classification of Skew Schur Functions, Trans. American Math. Soc. Vol. 361, No. 8 (Aug., 2009), pp. 4437–4470

[McWi12] P. R. W. McNamara and S. van Willigenburg. Maximal supports and Schur positivity among connected skew shapes. European J. of Combin. V. 33 (6), 2012, pp. 1190–1206.

[ReShi98] V. Reiner and M. Shimozono. Percentage-avoiding, northwest shapes and peelable tableaux. J. Combin. Theory Ser. A, 82(1):1–73, 1998.

[Sa01] B. Sagan. *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric functions*, Springer Verlag, New York, 2001.

[Sch77] M.-P. Schützenberger. La correspondance de Robinson. In Combinatoire et representation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976), 59–113. Lecture Notes in Math., Vol. 579. Springer, Berlin, 1977.

[St99] R. P. Stanley, *Enumerative Combinatorics. Vol. 2*, Cambridge University Press, Cambridge 1999.

[Tho78] G.P. Thomas, On Schensted's construction and the multiplication of Schur functions, *Adv. Math.* **30** (1978) 8–32.

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