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ON UNKNOTTING TUNNEL SYSTEMS OF SATELLITE CHAIN LINKS

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Abstract: We prove that the tunnel number of a satellite chain link with a number of components higher than or equal to twice the bridge number of the companion is as small as possible among links with the same number of components. We prove this result to be sharp for satellite chain links over a 2-bridge knot. We also observe that the links in the main result satisfy the genus versus rank conjecture.

Keywords: Chain links, Tunnel number, Heegaard genus, Rank.
AMS Subject Classification (2010): 57M25, 57N10.

1. Introduction

An unknotting tunnel system for a link L in S^3 is a collection of properly embedded disjoint arcs $\{t_1, \dots, t_n\}$ in the exterior of L, such that the exterior of $L \cup t_1 \cup \dots \cup t_n$ is a handlebody. The minimal cardinality of an unknotting tunnel system of L is the *tunnel number* of L, denoted $t(L)$. The boundary surface of this handlebody defines an Heegaard decomposition of $E(L)$. We recall that a *Heegaard decomposition* of a compact 3-manifold M is a decomposition of M into two compression bodies H_1 and H_2 along a surface F. The genus of F is referred to as the genus of the Heegaard decomposition. If one of the compression bodies of a Heegaard decomposition of genus g is a handlebody, we can naturally present the fundamental group $\pi_1(M)$ with g generators: the core of the handlebody defines g generators, and the compressing disks of the compression body give a set of relators. In this case, the rank $r(M)$ of $\pi_1(M)$, referred to as the rank of M, which is the minimal number of elements needed to generate $\pi_1(M)$, is at most g. Within this context, we define the *Heegaard genus* of M, denoted by $g(M)$, as the minimal genus over all Heegaard decompositions splitting M into one handlebody and a compression body. Hence, as observed before, we have $r(M) \leq g(M)$, and if M is a exterior of some link L in S^3, $E(L)$, we also have $t(L) = g(E(L)) - 1$. Note that when M is closed or has connected boundary, any Heegaard splitting of M consists of at least one handlebody;

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so, in this case, the Heegaard genus of M is the minimal genus among all Heegaard decompositions of M. However, if the boundary of M has more than one component, as a link exterior can have, then a Heegaard decomposition of M might not decompose M into a handlebody and a compression body; so, in this case the Heegaard genus of M, as defined above, might not be the minimal genus among all Heegaard decompositions of M.
Under this setting, Waldhausen [10] asked whether $r(M)$ can be realized geometrically as the genus of a Heegaard decomposition splitting M into one handlebody and a compression body, that is if $r(M) = g(M)$, for every compact 3-manifold M. This question became to be known as the Rank versus Genus Conjecture. In [2], Boileau–Zieschang provided the first counter-examples by showing that there are Seifert manifolds where the rank is strictly smaller than the Heegaard genus. Later, Schultens and Weidman [9] generalized these counter-examples to graph manifolds. Very recently, Li [7] proved that the conjecture also doesn't hold true for hyperbolic 3-manifolds. As far as we know, the conjecture remains open for link exteriors in S^3. The first author [4] proved this conjecture to be true for augmented links. In this paper, we show that this is also the case for "most" of *chain links*, which we proceed to define.

An *satellite n-chain link* is a link L defined by a sequence of $n \geq 2$ unknotted linked components where each component bounds a disk such that each disk D of these intersects the other disks at exactly two arcs, each of which with only one end point in ∂D. Note that if two such disks D and D' intersect at an arc, then this arc has one end point in ∂D and the other end point in $\partial D'$. The regular neighborhood of the union of these disks is a regular neighborhood of a knot or link K. We also refer to L as an *n-chain link over K*. When K is the unknot, L is known in the literature simply as an *n-chain link* [8, 1, 6]. When K is a non-trivial knot, L is a satellite link with companion K and pattern an n-chain link (over the unknot).

The n-chain links over the unknot have been subject of recent attention for the study of hyperbolic structures. For instance, Neumann and Reid [8] showed that, for $n \geq 5$, the complement of an n-chain link over the unknot admits a hyperbolic structure. Agol [1] conjectures that, for $n \leq 10$, an n-chain over the unknot is the smallest volume hyperbolic 3-manifold with n cusps. In [6] Kaiser, Purcell and Rollins proved that, for $n \geq 60$, an n-chain over the unknot cannot be the smallest volume hyperbolic 3-manifold with n cusps.

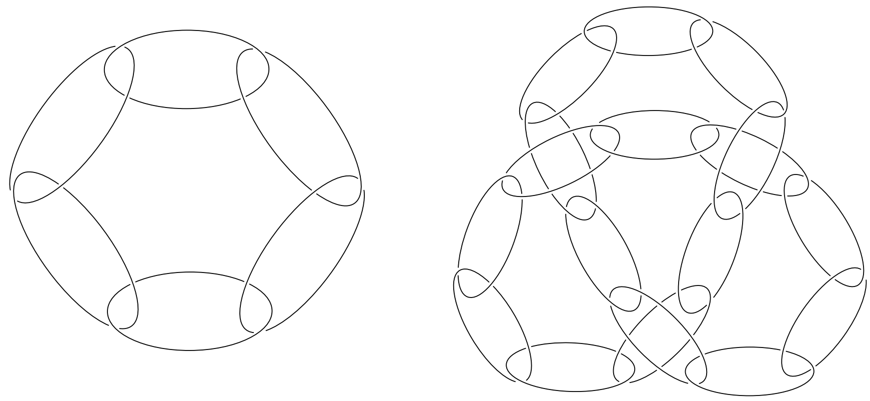

FIGURE 1. Left: a chain link over the unknot; Right: a chain link over the trefoil.

In this paper we study unknotting tunnel systems of satellite chain links L, and observe on the relation between Heegaard genus and rank of their exteriors. We know that if the companion of L is the unknot then the Heegaard genus of $E(L)$ is n (and equal to its rank). In the following theorem we prove that this is also the case for satellite chain links with non-trivial companion as long as the number of components of the link is sufficiently large.

Theorem 1. *Let L be a n-chain link over a b-bridge knot K. If $n \geq 2b$, then the tunnel number of L is $n-1$.*

An immediate consequence of this theorem is that the rank versus genus conjecture holds true for chain links with sufficiently high number of components: Let L be an n-chain link over a b-bridge knot K. If $n \geq 2b$, then $r(E(L)) = g(E(L))$. In fact, from Theorem 1, we have $g(E(L)) = n$, and from the "half lives, half dies" theorem ([5], Lemma 3.5) applied to $E(L)$, we have $r\big(E(L)\big) \geq n$. Then $n = |L| \leq r\big(E(L)\big) \leq g\big(E(L)\big) = n$, and $r\big(E(L)\big) = g\big(E(L)\big) = n$.

We also prove the following theorem for satellite 3-chain links.

Theorem 2. *Let L be a satellite 3-chain link. Then the tunnel number of L is greater than or equal to 3.*

Hence, for chain links over 2-bridge knots, Theorem 1 is sharp. That is, if L is a n-chain link over a 2-bridge knot K and $t(L) = n-1$, then $n \geq 4$ (two times the bridge number of K). This is a consequence of satellite 2-chain links not having tunnel number one, as proved in [3] by Eudave-Muñoz and

Uchida (or by following an argument as in Lemma 3), and of Theorem 2. The authors wouldn't be surprised Theorem 1 to be sharp for any number of bridges of K.

This paper is organized into two sections, one for the proof of each theorem mentioned above. Throughout the paper we assume all manifolds to be in general position.

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Dedicatory

While this paper was under preparation Darlan Girão was diagnosed with cancer. After a prolounged corageous and dignifying battle with his condition, Darlan died before we could finish this work together. Darlan has been a very good friend and colleague, who we will miss. This paper is dedicated to his memory.

2. Proof of Theorem 1

Let $\mathcal{A}$ the collection of arcs of intersection between the disks bounded by the components of L, as in the definition of satellite chain link. Let R be a regular neighborhood of the union of these disks, such that R is also a regular neighborhood of K. Consider also a b-bridge sphere for K, denoted by S, intersecting R in a collection of meridional disks. Denote by B and B' the balls bounded by S in S^3.

Since $n \geq 2b$, we can perform an ambient isotopy so that each component of $B \cap R$ contains exactly one arc of $\mathcal{A}$, and each component of $B' \cap R$ contains at least one arc of $\mathcal{A}$. In the exterior of L, we start by adding $n-b$ tunnels to $N(L)$, denoted $t_1, \ldots, t_{n-b}$, corresponding to regular neighborhoods of the arcs of $\mathcal{A}$ in $B' \cap R$. (See Figure 2.)

After an ambient isotopy of $N(L \cup t_1 \cup \cdots \cup t_{n-b})$, we obtain in B' a regular neighborhood $N(\Gamma)$ of a graph Γ obtained from the b components of $K \cap B'$,

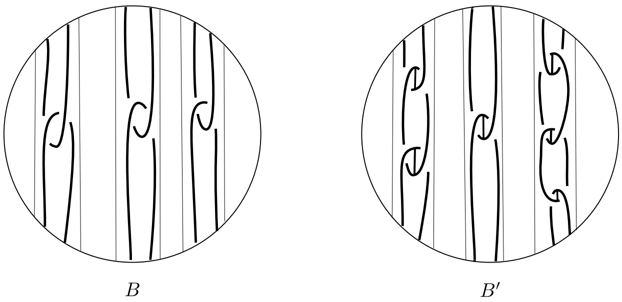

FIGURE 2. An illustration of B and B', with $n-b$ tunnels in B'.

denoted $c_1 \cup \cdots \cup c_b$, by adding $n-2b$ arcs parallel to K. Note that after the isotopy, S intersects $N(L \cup t_1 \cup \cdots \cup t_{n-b})$ in $2b$ disks. (See Figure 3.)

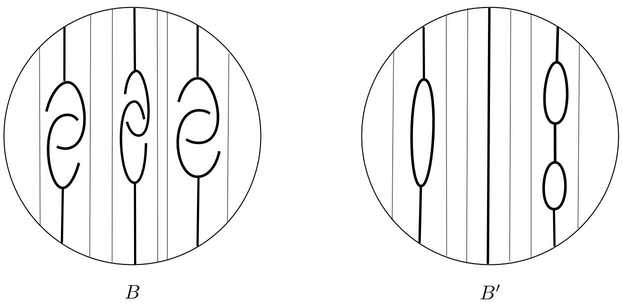

FIGURE 3. The graph Γ in B and B'.

As $(B'; c_1, \ldots, c_b)$ is a trivial tangle, we add $b-1$ tunnels, denoted $t_{n-b+1}, \ldots, t_{n-1}$, to $N(\Gamma)$ in its exterior in B', such that $N(\Gamma \cup t_{n-b+1} \cup \cdots \cup t_{n-1})$ can be isotoped to become the whole B' with $n-2b$ trivial 1-handles. (See Figure 4.)

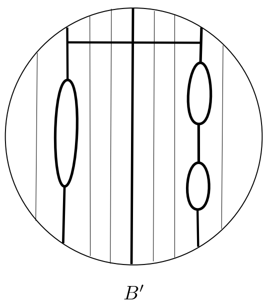

FIGURE 4. The graph $\Gamma \cup t_{n-b+1} \cup \ldots \cup t_{n-1}$ in B'.

Hence, the resulting space of the exterior of $L \cup t_1 \cup \cdots \cup t_{n-1}$ is ambient isotopic to the exterior in B of the union of $B \cap L$ with some trivial arcs in $B - L$. This means that the exterior of $L \cup t_1 \cup \cdots \cup t_{n-1}$ is a handlebody if and only if the exterior of $B \cap L$ in B is a handlebody. The components of $B \cap R$ cobound, each with an arc in S, mutually disjoint disks. That is, $B \cap R$ is a collection of trivial 1-handles added to B'. The components of L in each cylinder of $B \cap R$ define a trivial tangle in the respective cylinder. And together with these cylinders being trivial 1-handles added to B', we have that the exterior of $B' \cup L$ is a handlebody. That is, the exterior of $B \cap L$ in B is a handlebody. Therefore, the tunnel number of L is at most $n-1$ and, as L has n components, it is also at least $n-1$. Hence, the tunnel number of L is $n-1$.

3. Unknotting tunnel systems of satellite 3-chain links

Let L be a satellite 3-chain link over a non-trivial knot K. We will show that $t(L) \geq 3$.

Lemma 3. *Let L be a 3-chain link over a non-trivial knot K. If the tunnel number of L is 2, then there is a minimal unknotting tunnel system such that one of the arcs is in one of the disks bounded by the components of L.*

Proof: Denote the components of L by L_i, for $i = 1, 2, 3$, and, respectively, by D_i the disks they bound, as in the definition of satellite chain link. We denote also by L_i a regular neighborhood of the components of L. The regular neighborhood R of $\mathcal{D} = D_1 \cup D_2 \cup D_3$ is a solid torus, with K its core, and we denote its boundary by T.

Let $\tau = \tau_1 \cup \tau_2 \cup \tau_3$ be a system of disjoint arcs τ_i from the same point in the exterior of R to L_i, such that the exterior H of $L \cup \tau$ is a handlebody. Denote a regular neighborhood of $L \cup \tau$ by G. Note that such a system exists, since the tunnel number of L is 2.

We denote also by $\tau = \tau_1 \cup \tau_2 \cup \tau_3$ a regular neighborhood of the these arcs. As regular neighborhoods, the boundary of τ is a sphere meeting each L_i at a single disk (in τ_i). (See Figure 5).

We will show that there is some τ for which at least two τ_i are disjoint from T. From the definition of T, this implies that one of the arcs from the unknotting tunnel system can be isotoped into some disk D_j, as in the statement of the lemma.

Hence, suppose that two τ_i intersect T, for any τ. Consider τ such that the number of intersections with T, $|\tau \cap T|$, is minimal. Note that $\tau \cap T$ is non-empty, as there is no incompressible torus in a handlebody.

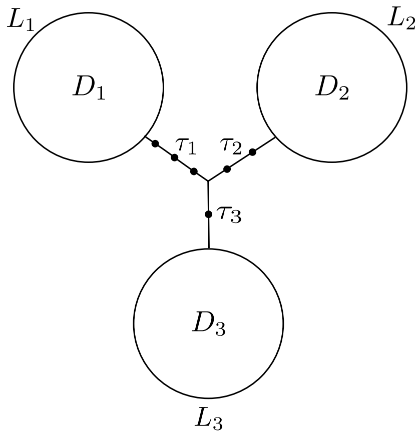

FIGURE 5. A tunnel system of L (the dots represent the intersections of τ and T).

For the chosen τ, consider a complete system of meridian disks $\mathcal{E} = E_1 \cup E_2 \cup E_3$ of H, and assume that the number of intersections of $\mathcal{E}$ with T, $|\mathcal{E} \cap T|$, is minimal among all choices of $\mathcal{E}$. Note that $\mathcal{E} \cap T$ is non-empty, as there is no incompressible torus, or punctured torus, in a 3-ball. Furthermore, no component of $\mathcal{E} \cap T$ is a closed curve. Otherwise, considering an innermost one in $\mathcal{E}$, we obtain a compressing disk for T or its boundary also bounds a disk in T and we can reduce $|\mathcal{E} \cap T|$, contradicting its minimality.

As at least two τ_i intersect T, the components of $G - T$ are balls (intersecting T in two or three disks) and solid tori (intersecting T in one or two disks). (See Figure 6.) Note that the second and fourth types cannot coexist.

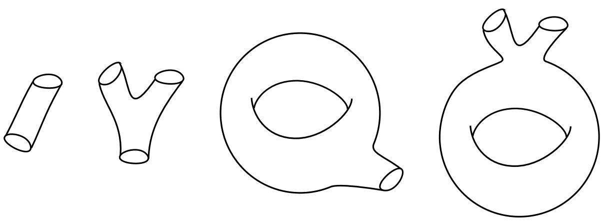

FIGURE 6. The components of $G - T$.

Let δ be an outermost arc of $\mathcal{E} \cap T$ in $\mathcal{E}$. The ends of δ are in the (disk) components of $G \cap T$. Let Δ be the corresponding outermost disk in $\mathcal{E}$, σ the arc $\partial\Delta - \delta$ and Q the component of $G - T$ that contains σ.

Case 1. Q is a ball.
Suppose that the ends of δ are in different disks of $G \cap T$, say α and β.

Then, we can stabilize the Heegaard decomposition by adding a tunnel over δ, and as α aned β are now primitive with respect to Δ, we can destabilize the resulting Heegaard decomposition by cutting along α or β. The resulting tunnel system can also be described from three arcs from a point connecting to the components of L. Hence, we obtain a τ with smaller $|\tau \cap T|$, contradicting its minimality.
Suppose that the ends of δ are in the same disk α of $G \cap T$. If Q has three disks of intersection with T, then σ bounds a disk O in ∂Q with $\partial\alpha$. In case O is disjoint from T, using this disk, and $\mathcal{E} \cap T$ not having closed curves, we can reduce $|\mathcal{E} \cap T|$, contradicting its minimality. In case O intersects T, then σ co-bounds a disk Σ in Q, intersecting T only in its boundary. Hence, if $\partial\Sigma$ bounds a disk in T we can reduce $|\mathcal{E} \cap T|$, contradicting its minimality, otherwise Σ is a compressing disk for T, which contradicts T being incompressible.

Case 2. Q is a solid torus.
Suppose that the ends of δ are in different disks of $G \cap T$. Then, we are in the situation similar to the first part of case 1.
Suppose now that the ends of δ are in the same disk α of $G \cap T$. Without loss of generality, suppose that Q contains the regular neighborhood of L_1. As L_1 is unknotted there is a ball in the solid torus bounded by T containing L_1. Let S be its boundary and, after a small isotopy if needed, suppose that S intersects Q at a disk. Then, considering $\Delta \cap S$, with an innermost curve, outermost arc argument, we have that Δ intersects S at a single arc δ' with both ends in σ, cutting a disk Δ' from Δ. The arc δ' co-bounds a disk O in S with $S \cap Q$. Hence, considering Δ' and O, we have that σ co-bounds a disk with α in the exterior of Q. As we are working in S^3, either σ is parallel to α in the boundary of Q, and using an argument as in Case 1 we can reduce $|\mathcal{E} \cap T|$ contradicting its minimality, or σ intersects a meridian of Q geometrically once. Therefore, L_1 is parallel to $\sigma \cup \alpha$, and to T. As the other components of L are trivial in the torus bounded by T, this means that L_1 is unlinked from each of them, which is a contradiction with the definition of chain link. ■

Suppose now that $\{t, t'\}$ is an unknotting tunnel system for L, with t' over the arc of intersection of D_1 and D_2. Let R be a regular neighborhood of $L_1 \cup t' \cup L_2$, which we consider as a genus two handlebody. For convenience in the argument, we interchange, through an isotopy, the intersection of L_1 with t' with the intersection of L_2 with t', as represented in Figure 7, and keep denoting the resulting circles by L_1 and L_2.

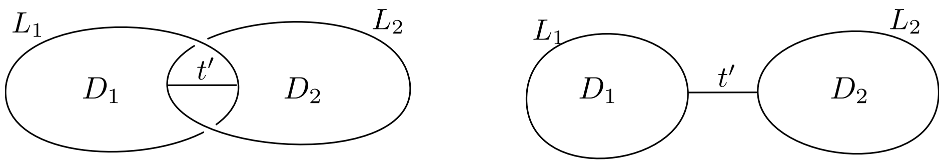

FIGURE 7. A new tunnel system of L.

Let B be a ball intersecting t' at a single arc with the pattern of K', where K' is a prime component of K, and disjoint from L_1, L_2 and L_3. Let S be the boundary of B. We consider t, connecting R and L_3, such that the intersection with S has minimal number of components, denoted $|S \cap t|$.

Lemma 4. *The arc t, as above, is disjoint from S.*

Proof: Let G be a regular neighborhood of $R \cup t \cup L_3$. As $\{t, t'\}$ is an unknotting tunnel system for L, we have that the exterior of G is an handlebody, which we denote by H. Let $\mathcal{E} = E_1 \cup E_2 \cup E_3$ be a complete system of meridian disks of H, with minimal number of intersections with S, denoted $|\mathcal{E} \cap S|$. Following a similar argument as in Lemma 3 we have that there are no closed curves in $E \cap S$, all components are arcs.

The components of $G - S$ are balls (intersecting S in two or three disks) and solid tori (intersecting S in one or two disks). Let δ be an outermost arc of $\mathcal{E} \cap S$ in $\mathcal{E}$, Δ a outermost disk δ co-bounds in $\mathcal{E}$ with an arc σ of $\partial\mathcal{E}$, and Q the component of $G - S$ that contains σ. As in Lemma 3 we separate the argument into two cases.

Case 1. Q is a ball.
Suppose that δ has ends in different components of $G \cap S$. If the ends of δ correspond to the intersection of S with t', then the arc $B \cap t'$ is unknotted, contradicting K' being non-trivial. If δ has at most one end corresponding to the intersection of S with t' then, by an isotopy of S along δ through Δ, we reduce $|G \cap S|$, contradicting its minimality.
Suppose that δ has ends in the same component α of $G \cap S$. If σ co-bounds a disk in ∂Q with α disjoint from T, using this disk for an isotopy of $\mathcal{E}$, and eliminating any originated closed curves, we reduce $|\mathcal{E} \cap S|$, contradicting its minimality. Note that this is always the case when Q intersects T at

two components. If Q intersects T at three disks, then two correspond to the intersection of S with t', and we assume that σ separate the other two components of $Q \cap T$. If α corresponds to a intersection of S with t' then σ co-bounds a disk O in Q disjoint from t' together with α. The disk $O \cup \Delta$ is a properly embedded disk in B that cuts a ball B' from B disjoint from $B \cap t'$. By cutting B' from B, we obtain a ball, say also denoted B, intersecting t' at an arc with the same pattern of K', and intersecting t in fewer components, contradicting the minimality of $|G \cap S|$. If α corresponds to a intersection of S with t then σ co-bounds with alpha a disk O in Q intersecting t' at a single point. Hence, the disk $O \cup \Delta$ in B cuts the string $B \cap t'$ at a single point. Using $O \cup \Delta$ we decompose the 1-string tangle $(B, B \cap t')$ into two 1-string tangles. As K' is prime, one of the tangles is trivial and the other has a string with the pattern of K'. If we consider the ball of the latter, denoted also by B, and consider its boundary, we obtain a contradiction with the minimality of $|G \cap S|$.

Case 2. Q is a solid torus.
Suppose that δ has ends in different components of $G - S$. Then, following a similar argument as in Case 1 of Lemma 3, we obtain an tunnel t in $\{t, t'\}$ with smaller $|S \cap t|$, contradicting its minimality.
Suppose now that the ends of δ are in the same disk α of $G \cap S$. We an argument as in Case 2 of Lemma 3. If Q contains L_1, or L_2, then we show that L_1, or L_2, is unlinked from L_3, a contradiction. If Q contains L_3m then we show that L_3 is unlinked from L_1 and L_2, a contradiction. ∎

Proof of Theorem 2: Suppose that the tunnel number of L is at most two. From Lemmas 3 and 4, we have that L has an unknotting tunnel disjoint from the satellite torus. This torus is then essential in a handlebody, defined by the exterior of L together with the unknotting tunnel system, which is a contradiction as there are no embedded closed surfaces essential in a handlebody. Hence, the tunnel number of L is at least 3. ∎

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