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SEMANTIC FACTORIZATION AND DESCENT

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ABSTRACT: Let $\mathbb{A}$ be a 2-category with suitable opcomma objects and pushouts. We give a direct proof that, provided that the codensity monad of a morphism p exists and is preserved by a suitable morphism, the factorization given by the lax descent object of the higher cokernel of p is up to isomorphism the same as the semantic factorization of p, either one existing if the other does. The result can be seen as a counterpart account to the celebrated Bénabou-Roubaud theorem. This leads in particular to a monadicity theorem, since it characterizes monadicity via descent. It should be noted that all the conditions on the codensity monad of p trivially hold whenever p has a left adjoint and, hence, in this case, we find monadicity to be a 2-dimensional exact condition on p, namely, to be a 2-effective monomorphism of the 2-category $\mathbb{A}$.

KEYWORDS: formal monadicity theorem, formal theory of monads, codensity monads, semantic factorization, descent theory, higher cokernel, opcomma object, descent factorization, 2-effective monomorphism, Bénabou-Roubaud theorem, two dimensional limits, lax descent category, effective descent morphism.
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Introduction

Descent theory, originally developed by Grothendieck [18], has been generalized from a solution of the problem of understanding the image of the functors $\mathsf{Mod}(f)$ in which $\mathsf{Mod} : \mathsf{Ring} \to \mathsf{Cat}$ is the usual pseudofunctor between the category of rings and the 2-category of categories that associates each ring $\mathcal{R}$ with the category $\mathsf{Mod}(\mathcal{R})$ of right $\mathcal{R}$-modules [19, 29].

It is often more descriptive to portray descent theory as a higher dimensional counterpart of *sheaf theory* [27]. In this context, the analogy can be roughly stated as follows: the *descent condition* and the *descent data* are respectively 2-dimensional counterparts of the *sheaf condition* and the *gluing condition.*

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The most fundamental constructions in descent theory are the lax descent category and its variations [28, 56, 41]. Namely, given a truncated pseudocosimplicial category

$$\mathcal{A} : \Delta_3 \to \mathsf{Cat}$$

$$\begin{tikzcd} \mathcal{A}(1) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{A}(2) \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{A}(3) \end{tikzcd}$$

we construct its *lax descent category* [42] or *descent category* [39, 43]. An object of the lax descent category/descent category is an object x of the category $\mathcal{A}(1)$ endowed with a descent data which is a morphism/invertible morphism $\mathcal{A}(d^1)(x) \to \mathcal{A}(d^0)(x)$ satisfying the usual *cocycle/associativity* and *identity* conditions. Morphisms are morphisms of $\mathcal{A}(1)$ that respect the descent data.

Another perspective, which highlights descent theory's main role in Galois theory [22, 23, 24, 25], is that, given a bifibred category, the lax descent category of the truncated pseudocosimplicial category induced by an internal category generalizes the notion of the category of internal actions [28]. If the bifibration is the basic one, we actually get the notion of internal actions. The simplest example is the category of actions (functors) of a small category in Set: a small category a is just an internal category in Set and the category of actions (functors) $a \to \mathsf{Set}$ coincides with the lax descent category of the composition of the (image by $\mathrm{op} : \mathsf{Cat}^{\mathrm{co}} \to \mathsf{Cat}$ of the) internal category a, $\mathrm{op}(a) : \Delta_3 \to \mathsf{Set}^{\mathrm{op}}$, with the pseudofunctor $\mathsf{Set}/- : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Cat}$ that comes from the basic fibration.

Given a pseudofunctor $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$ that comes from a bifibred category with pullbacks and a morphism $q : w \to w'$ of $\mathbb{C}$, Bénabou and Roubaud [5] showed that the lax descent category of the truncated pseudocosimplicial object

$$\begin{tikzcd} \mathcal{F}(w) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{F}(w \times_q w) \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{F}(w \times_q w \times_q w) \end{tikzcd}$$

given by the composition of $\mathcal{F}$ with the internal groupoid induced by q, in which $w \times_q w$ denotes the pullback of q along itself and the functors of the diagram are induced by the usual canonical morphisms (projections and diagonal), is equivalent to the Eilenberg-Moore category of the monad induced by the adjunction $\mathcal{F}(q)! \dashv \mathcal{F}(q)$, provided that the bifibred category satisfies the so called Beck-Chevalley condition [41, 43]. In particular, in this case, q is of effective $\mathcal{F}$-descent (which means that $\mathcal{F}(q)$ gives the lax descent category of the above) if and only if $\mathcal{F}(q)$ is monadic.

Since *monad theory* [21, 16, 32, 3] already was a established subfield of category theory, the Bénabou-Roubaud theorem gave an insightful connection between the theories, motivating what is nowadays often called *monadic approach to descent* [7, 43] by giving a characterization of *descent via monadicity* in several cases of interest [23, 27, 48, 36, 8, 10, 11, 12, 13].

The main contribution of the present article can be seen as a counterpart account to the Bénabou-Roubaud theorem. We give the *semantic factorization* via descent, hence giving, in particular, a characterization of *monadicity via descent*. Although the Bénabou-Roubaud theorem is originally a result on the setting of the 2-category Cat, our contribution takes place in the more general context of 2-*dimensional category theory* [34, 31], or in the so called *formal category theory* [17, 51], as briefly explained below.

In his pioneering work on bicategories, Bénabou [4] observed that the fundamental notion of *monad*, formerly called *standard construction* or *triple*, coincides with the notion of a lax functor $\mathsf{1} \to \mathsf{Cat}$ and can be pursued in any bicategory, giving convincing examples to the generalization of the notion.

Taking Bénabou's point in consideration [57], Street [52, 51] gave a formal account and generalization of the former established theory of monads by developing the theory within the general setting of 2-categories. *The formal theory of monads* [51] is a celebrated example of how 2-dimensional category theory can give insight to 1-dimensional category theory, since, besides generalizing several notions, it conceptually enriches the formerly established theory of monads [3, 15]. Street [51] starts showing that, when it exists, the Eilenberg-Moore construction of a monad in a 2-category $\mathbb{A}$ is given by a right 2-reflection of the monad along a 2-functor between the 2-category $\mathbb{A}$ and the 2-category of monads in $\mathbb{A}$. From this point, making good use of the four dualities of a 2-category, he develops the formal account of aspects of monad theory, including distributive laws [2], comonads, Kleisli construction, and a generalization of the *semantics-structure adjunction* [38, 15].

The theory of 2-dimensional limits [53, 54, 30, 40], or weighted limits in 2-categories, also provides a great account of formal category theory, since it shows that several constructions previously introduced in 1-dimensional category theory are actually examples of weighted limits and, hence, are universally defined and can be pursued in the general context of a 2-category.

Examples of the constructions that are particular weighted limits are: the lax descent category and variations, the Eilenberg-Moore category [16] and the comma category [35]. Duality also plays important role in this context:

it usually illuminates or expands the original concepts of 1-dimensional category theory. For instance:

- The dual of the notion of descent object gives the notion of codescent object, which is important, for instance, in 2-dimensional monad theory [33, 39, 42];
- The dual notion of the *Eilenberg-Moore object* in Cat gives the Kleisli category [32] of a monad, while the codual gives the category of the coalgebras of a comonad.

Despite receiving less attention in the literature than the notion of *comma object*, the dual notion, called *opcomma object*, was already considered in [53] and it is fundamental to the present work. More precisely, given a morphism $p : e \to b$ of a 2-category $\mathbb{A}$, if $\mathbb{A}$ has suitable opcomma objects and pushouts, on one hand, we can consider the *higher cokernel*

$$\mathcal{H}_p : \Delta_{\mathrm{Str}} \to \mathbb{A}$$

$$\begin{tikzcd} b \arrow[r, shift left=2, "\delta^0_{p\uparrow p}"] \arrow[r, shift right=2, "\delta^1_{p\uparrow p}"'] & b \uparrow_p b \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & b \uparrow_p b \uparrow_p b \end{tikzcd}$$

of p, defined in 2.3, whose dual was firstly defined for Cat in [56]. By the universal property of the lax descent object, we get a factorization

$$\begin{tikzcd} e \arrow[rr, "p"] \arrow[dr] & & b \\ & \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{H}_p) \arrow[ur] & \end{tikzcd}$$

of p, provided that $\mathbb{A}$ has the lax descent object of $\mathcal{H}_p$. If the comparison morphism $e \to \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{H}_p)$ is an equivalence, we say that p is a 2-*effective monomorphism.* This concept is actually self-codual, meaning that its codual notion coincides with the original one.

On the other hand, if such a morphism p has a *codensity monad* $\mathtt{t}$ [38, 15, 51], which means that the right Kan extension of p along itself exists in $\mathbb{A}$, we have the semantic factorization [15, 51]

$$\begin{tikzcd} e \arrow[rr, "p"] \arrow[dr] & & b \\ & b^{\mathtt{t}} \arrow[ur] & \end{tikzcd}$$

provided that $\mathbb{A}$ has the Eilenberg-Moore object $b^{\mathtt{t}}$ of $\mathtt{t}$. In this case, if the comparison $e \to b^{\mathtt{t}}$ is an equivalence, we say that p is *monadic*. The codual notion is that of *comonadicity*.

The main theorem of the present article relates both the factorizations above. More precisely, Theorem 4.12 states the following:

Main Theorem: *Assume that the* 2*-category* $\mathbb{A}$ *has the higher cokernel of* p *and* $\operatorname{ran}_p p$ *exists and is preserved by the universal morphism* $\delta^0_{p\uparrow p}$ *of the opcomma object* $b \uparrow_p b$*.*

There is an isomorphism between the Eilenberg-Moore object $b^{\mathtt{t}}$ *and the lax descent object* $\text{lax-}\mathcal{D}\text{esc}\,(\mathcal{H}_p)$*, either one existing if the other does. In this case, the semantic factorization is isomorphic to the factorization induced by the higher cokernel and the lax descent object.*

In particular, this gives a *formal monadicity theorem* as a corollary, since it shows that, assuming that a morphism p of $\mathbb{A}$ satisfies the conditions above on the codensity monad, p is monadic if and only if p is a 2-effective monomorphism. Moreover, since this result holds for any 2-category, we can consider the duals of this formal monadicity theorem. Namely, we also get characterizations of comonadic morphisms, Kleisli and co-Kleisli morphisms.

By the *Dubuc-Street formal adjoint-functor theorem* [15, 58, 17], if p has a left adjoint, the codensity monad is the monad induced by the adjunction and $\operatorname{ran}_p p$ is absolute. Thus, in this case, assuming the existence of the higher cokernel, our theorem trivially holds and both the factorizations above coincide, either one existing if the other does. Therefore, as a corollary of our main result, we get the following monadicity result:

Monadicity Theorem: *Assume that the* 2*-category* $\mathbb{A}$ *has the higher cokernel of* $p : e \to b$*.*

- *The morphism* p *is monadic if and only if* p *is a* 2*-effective monomorphism and has a left adjoint morphism*
- *The morphism* p *is comonadic if and only if* p *is a* 2*-effective monomorphism and has a right adjoint morphism.*

Recall that, in the particular case of $\mathbb{A} = \mathsf{Cat}$ (and other 2-categories, such as the 2-category of enriched categories), we have Beck's monadicity theorem [3, 14, 15]. It states that: a functor is monadic if and only if it creates absolute coequalizers and it has a left adjoint. Hence, by our main result, we can conclude that: provided that the functor p has a left adjoint, p creates absolute coequalizers if and only if it is a 2-effective monomorphism.

The fact above suggests the following question: are 2-effective monomorphisms in Cat characterized by the property of creating absolute coequalizers? In Remark 5.13 we show that the answer to this question is negative by the self coduality of the concept of 2-effective monomorphism and non-self duality of the concept of functor that creates absolute coequalizers.

This work was motivated by three main aims. Firstly, to get a formal monadicity theorem given by a 2-*dimensional exact condition*. Secondly, to better understand the relation between descent and monadicity in a given 2-category and, together with [43], get alternative guiding templates for the development of higher descent theory and monadicity (see, for instance, [20, 56]). Thirdly, to improve the understanding of aspects on *descent theory* related to Janelidze-Galois theory [24].

Although we do not make these connections in this paper, the results on 2-dimensional category theory of the present work already establish framework and have applications to the author's ongoing work on descent theory in the context of [26, 27, 43].

The main aim of Section 1 is to set up basic terminology related to the category of the finite nonempty ordinals Δ and its strict replacement Δ_{Str}. As observed above, this work is meant to be applicable in the classical context of descent theory and, hence, we should consider lax descent categories of pseudofunctors $\Delta \to \mathsf{Cat}$. In order to do so, we consider suitable replacements $\Delta_{\mathrm{Str}} \to \mathsf{Cat}$.

The main results (Theorem 4.11 and Theorem 4.12) can be seen as theorems on 2-dimensional limits and colimits. For this reason, we recall basics on 2-dimensional limits [53, 54, 30, 33, 42, 40] in Section 2. We give an explicit definition of the weights and universal properties of the 2-dimensional limits related to the definition of higher cokernel. This helps to establish terminology and framework for the rest of the paper.

We also give an explicit definition of the weight for the lax descent object [53, 33, 42] for 2-functors $\Delta_{\mathrm{Str}} \to \mathbb{A}$ in 2.4 that agrees with the usual setting of [27, 28], useful to establish the *lax descent factorization induced by the higher cokernel* of a morphism p and to future work on giving further applications in descent theory within the context of [26, 43].

In Section 3, we recall basic aspects of Eilenberg-Moore objects in a 2-category $\mathbb{A}$. Given a tractable morphism p in $\mathbb{A}$, it induces a monad and, in the presence of the Eilenberg-Moore objects, it also induces a factorization, called herein semantic factorization (see [38, 51] or, more particularly, pages

74 and 75 of [15]). We are only interested in morphisms p that have codensity monads, that is to say, the right Kan extension of p along itself. We recall the basics of this setting, including the definitions of right Kan extensions and codensity monads in Section 3, most of which can be found in [51, 58].

We do not present more than very basic toy examples of codensity monads. We refer to [15] for the classical theory on codensity monads, while [37, 1] are recent considerations that can be particularly useful to understand interesting examples.

Still in Section 3, Lemma 3.3 states a straightforward connection between opcomma objects and right Kan extensions. The statement is particularly useful for establishing an important adjunction (see Propositions 4.1 and 4.2) and proving the main results. Also important for these proofs, the Dubuc-Street formal adjoint-functor theorem [15, 58, 17] and a proof of it are recalled in Theorem 3.10.

The *mate correspondence* [31, 41, 50] or *calculus of mates* is a useful framework in 2-dimensional category theory that states an isomorphism between two special double categories that come from each 2-category $\mathbb{A}$. We refer to [31, 50] for more structured considerations. The *mate correspondence* plays a central role in the proof of Theorem 4.11, but we only need it in very basic terms as recalled in Remark 3.11, with which we finish Section 3.

In Section 4 we establish an important condition to the main theorem, which arises from Propositions 4.1 and 4.2: the condition of preservation of $\mathrm{ran}_p p$ by the universal morphism $\delta^0_{p\uparrow p}$, characterizing the existence of a left adjoint to the universal morphism $\delta^0_{p\uparrow p}$. We then illustrate this condition with examples and counterexamples in Remarks 4.3, 4.4, 4.5 and 4.6. Finally, we go towards the proof of the main result, constructing another adjunction in Proposition 4.8, defining particularly useful 2-cells for our proof in Lemma 4.10 and finally proving Theorem 4.11.

The final section is mostly intended to apply our main result in order to get our monadicity theorem using the concept of 2-*effective monomorphism*. We finish the article with a remark on the self-coduality of this concept, in opposition to the non-self duality of the property of creating absolute coequalizers. This gives a comparison between the Beck monadicity theorem and ours, showing in particular that 2-effective monomorphisms in Cat are not characterized by the property of creating absolute coequalizers.

1. Two categories delta

Let Cat be the cartesian closed category of categories (see, for instance, Section 1 of [42, 40]) in some universe. We denote the internal hom by

$$\mathsf{Cat}[-,-] : \mathsf{Cat}^{\mathrm{op}} \times \mathsf{Cat} \to \mathsf{Cat}.$$

A 2-category $\mathbb{A}$ herein is the same as a Cat-enriched category. As usual, the composition of 1-cells (morphisms) are denoted by $\circ$, $\cdot$ or omitted whenever it is clear from the context. The vertical composition of 2-cells is denoted by $\cdot$ or omitted when it is clear, while the horizontal composition is denoted by $*$. Recall that, from the vertical and horizontal compositions, we construct the fundamental operation of *pasting* [47, 55], introduced in [4, 31].

As mentioned in the introduction, duality is one of the most fundamental aspects of theories on 2-categories. Unlike 1-dimensional category theory, 2-dimensional category theory has four duals [34, 41]. More precisely, any 2-category $\mathbb{A}$ gives rise to four 2-categories: $\mathbb{A}$, $\mathbb{A}^{\mathrm{op}}$, $\mathbb{A}^{\mathrm{co}}$, $\mathbb{A}^{\mathrm{coop}}$ which are respectively related to *inverting the directions* of nothing, morphisms, 2-cells, morphisms and 2-cells. Hence every concept/result gives rise to four (not necessarily different) duals: the concept/result itself, the dual, the codual, the codual of the dual.

Although it is important to keep in mind the importance of duality, we usually leave to the interested reader the straightforward exercise of stating precisely the four duals of most of the dualizable aspects of the present work.

In this section, we fix notation related to the categories of ordinals and the strict replacement Δ_{Str}. We denote by Δ the locally discrete 2-category of finite nonempty ordinals and order preserving functions between them. Recall that Δ is generated by the degeneracy and face maps. That is to say, Δ is generated by the diagram

$$\begin{tikzcd}
\mathsf{1} \arrow[r, shift left=3, "d^0"] \arrow[r, shift right=3, "d^1"'] & \mathsf{2} \arrow[l, "s^0" description] \arrow[r, shift left=3, "d^0"] \arrow[r, "d^1" description] \arrow[r, shift right=3, "d^2"'] & \mathsf{3} \arrow[l, bend right=50, "s^0"'] \arrow[l, bend left=50, "s^1"] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[l, phantom] & \cdots \arrow[l, bend right=40] \arrow[l, bend left=30] \arrow[l, bend left=50]
\end{tikzcd}$$

with the following relations:

$$\begin{aligned} d^k d^i &= d^i d^{k-1}, \text{if } i < k; \\ s^k s^i &= s^i s^{k+1}, \text{if } i \leqslant k; \\ s^k d^i &= d^i s^{k-1}, \text{if } i < k; \end{aligned}$$

$$\begin{aligned} s^k d^i &= \mathrm{id}, \text{ if } i = k \text{ or } i = k+1; \\ s^k d^i &= d^{i-1} s^k, \text{ if } i > k+1. \end{aligned}$$

We are particularly interested in the sub-2-category Δ_3 of Δ with the objects $\mathsf{1},\mathsf{2}$ and $\mathsf{3}$ generated by the morphisms below.

$$\begin{tikzcd} \mathsf{1} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & \mathsf{2} \arrow[l, "s^0" description] \arrow[r, shift left=3, "d^0"] \arrow[r, "d^1" description] \arrow[r, shift right=3, "d^2"'] & \mathsf{3} \end{tikzcd}$$

For simplicity, we use the same notation to the objects and morphisms of Δ and the image by the usual inclusion $\Delta \to \mathsf{Cat}$ which is locally bijective on objects. It should be noted that the image of the faces and degeneracy maps by $\Delta \to \mathsf{Cat}$ are given by:

$$\begin{aligned} d^k : \mathsf{n}-\mathsf{1} &\to \mathsf{n} \\ \mathsf{t} &\mapsto \begin{cases} \mathsf{t}+\mathsf{1}, & \text{if } t \geqslant k \\ \mathsf{t}, & \text{otherwise} \end{cases} \end{aligned} \qquad \begin{aligned} s^k : \mathsf{n}+\mathsf{1} &\to \mathsf{n} \\ \mathsf{t} &\mapsto \begin{cases} \mathsf{t}, & \text{if } t \leqslant k \\ \mathsf{t}-\mathsf{1}, & \text{otherwise.} \end{cases} \end{aligned}$$

Furthermore, in order to deal with the original setting of *descent theory*, we consider the 2-category Δ_{Str}, which is the *strict replacement* of Δ_3.

Definition 1.1. [Δ_{Str}] We denote by Δ_{Str} the 2-category freely generated by the diagram

$$\begin{tikzcd} \underline{1} \arrow[r, shift left=3, "\boldsymbol{\mathbb{d}}^0"] \arrow[r, shift right=3, "\boldsymbol{\mathbb{d}}^1"'] & \underline{2} \arrow[l, "\mathbb{s}^0" description] \arrow[r, shift left=3, "\mathbb{d}^0"] \arrow[r, "\mathbb{d}^1" description] \arrow[r, shift right=3, "\mathbb{d}^2"'] & \underline{3} \end{tikzcd}$$

with the invertible 2-cells:

$$\begin{aligned} \sigma_{01} &: \ \mathbb{d}^1\boldsymbol{\mathbb{d}}^0 \Rightarrow \mathbb{d}^0\boldsymbol{\mathbb{d}}^0, \\ \sigma_{02} &: \ \mathbb{d}^2\boldsymbol{\mathbb{d}}^0 \Rightarrow \mathbb{d}^0\boldsymbol{\mathbb{d}}^1, \\ \sigma_{12} &: \ \mathbb{d}^2\boldsymbol{\mathbb{d}}^1 \Rightarrow \mathbb{d}^1\boldsymbol{\mathbb{d}}^1, \end{aligned} \qquad \begin{aligned} \mathfrak{n}_0 &: \ \mathbb{s}^0\boldsymbol{\mathbb{d}}^0 \Rightarrow \mathrm{id}_{\underline{1}}, \\ \mathfrak{n}_1 &: \ \mathbb{s}^0\boldsymbol{\mathbb{d}}^1 \Rightarrow \mathrm{id}_{\underline{1}}. \end{aligned}$$

Definition 1.2. [$\mathsf{e}_{\Delta_{\mathrm{Str}}}$] There is a biequivalence $\mathsf{e}_{\Delta_{\mathrm{Str}}} : \Delta_{\mathrm{Str}} \approx \Delta_3$ which is bijective on objects, defined by:

$$\underline{1} \mapsto \mathsf{1},\ \underline{2} \mapsto \mathsf{2},\ \underline{3} \mapsto \mathsf{3}, \qquad \boldsymbol{\mathbb{d}}^k \mapsto d^k,\ \mathbb{s}^0 \mapsto s^0,\ \mathbb{d}^k \mapsto d^k, \qquad \sigma_{ki} \mapsto \mathrm{id}_{d^i d^k},\ \mathfrak{n}_k \mapsto \mathrm{id}_{\mathrm{id}_{\mathsf{1}}}.$$

Remark 1.3. It should be noted that, given a 2-category $\mathbb{A}$ and a pseudo-functor $\mathcal{B} : \Delta_3 \to \mathbb{A}$, we can replace it by a 2-functor $\mathcal{A} : \Delta_{\mathrm{Str}} \to \mathbb{A}$ defined by

$$\begin{aligned} \mathcal{A}(\boldsymbol{\mathbb{d}}^k) &:= \mathcal{B} \circ \mathsf{e}_{\Delta_{\mathrm{Str}}}(\boldsymbol{\mathbb{d}}^k) \\ \mathcal{A}(\mathbb{d}^k) &:= \mathcal{B} \circ \mathsf{e}_{\Delta_{\mathrm{Str}}}(\mathbb{d}^k) \\ \mathcal{A}(\mathbb{s}^0) &:= \mathcal{B} \circ \mathsf{e}_{\Delta_{\mathrm{Str}}}(\mathbb{s}^0) \end{aligned} \qquad \begin{aligned} \mathcal{A}(\sigma_{ki}) &:= (\mathfrak{b}_{d^i d^{k-1}})^{-1} \cdot \mathfrak{b}_{d^k d^i} \\ \mathcal{A}(\mathfrak{n}_k) &:= \mathfrak{b}_{s^0 d^k} \end{aligned}$$

in which, for each pair of morphisms (v, v') of Δ_3, $\mathfrak{b}_{vv'}$ is the invertible 2-cell

$$\mathfrak{b}_{vv'} : \mathcal{B}(v)\mathcal{B}(v') \Rightarrow \mathcal{B}(vv')$$

component of the pseudofunctor $\mathcal{B}$ (see, for instance, Definition 2.1 of [39]). Whenever we refer to a pseudofunctor (truncated pseudocosimplicial category) $\Delta_3 \to \mathsf{Cat}$ in the introduction, we actually consider the replacement 2-functor $\Delta_{\mathrm{Str}} \to \mathsf{Cat}$. See 4.12 and Proposition 4.17 of [43] for proofs and further observations in this direction and see [46, 6, 33, 39, 42] for the general coherence theorems and strict replacements.

2. Weighted colimits and the higher cokernel

The main result of this paper relates the factorization given by the *lax descent object* of the *higher cokernel* of a morphism with the *semantic factorization*, in the presence of *opcomma objects* and *pushouts* inside a 2-category $\mathbb{A}$. In other words, it relates the *lax descent objects*, the *Eilenberg-Moore objects*, the opcomma objects and pushouts. These are known to be examples of 2-dimensional limits and colimits. Hence, in this section, before defining the *higher cokernel* and the factorization induced by its lax descent object, we recall the basics of the special weighted (co)limits related to the definitions.

Two dimensional limits are the same as weighted limits in the Cat-enriched context. Assuming that $\mathbb{S}$ is a small 2-category, let $\mathcal{W} : \mathbb{S} \to \mathsf{Cat}, \mathcal{D} : \mathbb{S} \to \mathsf{Cat}$ and $\mathcal{D}' : \mathbb{S}^{\mathrm{op}} \to \mathbb{A}$ be 2-functors. If it exists, we denote the *weighted limit* of $\mathcal{D}$ with weight $\mathcal{W}$ by $\lim(\mathcal{W}, \mathcal{D})$. Dually, we denote by $\mathsf{colim}(\mathcal{W}, \mathcal{D}')$ the *weighted colimit* of $\mathcal{D}'$ provided that it exists. Recall that such a weighted colimit exists if and only if we have a 2-natural isomorphism (in z)

$$\mathbb{A}(\mathsf{colim}(\mathcal{W}, \mathcal{D}'), z) \cong [\mathbb{S}^{\mathrm{op}}, \mathsf{Cat}](\mathcal{W}, \mathbb{A}(\mathcal{D}'-, z)) \cong \lim(\mathcal{W}, \mathbb{A}(\mathcal{D}'-, z))$$

in which $[\mathbb{S}^{\mathrm{op}}, \mathsf{Cat}]$ denotes the 2-category of 2-functors $\mathbb{S}^{\mathrm{op}} \to \mathsf{Cat}$, 2-natural transformations and modifications. By the Yoneda embedding of 2-categories, if a two dimensional (co)limit exists, it is unique up to isomorphism. It is also important to keep in mind the fact that existing weighted limits in $\mathbb{A}$ are created by the Yoneda embedding $\mathbb{A} \to [\mathbb{A}^{\mathrm{op}}, \mathsf{Cat}]$, since it preserves weighted limits and is fully faithful.

Recall that Cat has all weighted colimits and all weighted limits. More generally, every weighted colimit can be constructed from some special 2-colimits: tensor coproducts (with 2), coequalizers and (conical) coproducts. Dually, weighted limits can be constructed from cotensor products, equalizers and products.

2.1. Tensorial coproducts. *Tensorial products* and *tensorial coproducts* are weighted limits and colimits with the domain/shape $\mathsf{1}$. So, in this case, the weight of a tensorial coproduct is entirely defined by a category a in Cat. In this case, if b is an object of $\mathbb{A}$, assuming its existence, we usually denote by $a \otimes b$ the tensorial coproduct, while the dual, the cotensorial product, is denoted by $a \pitchfork b$. Clearly, the tensorial coproduct $a \otimes b$ in Cat is isomorphic to the product $a \times b$.

2.2. Pushouts and coproducts. Two dimensional conical (co)limits are just weighted limits with a weight constantly equal to the terminal category $\mathsf{1}$. Hence two dimensional conical (co)limits are entirely defined by the domain (or shape) of the diagram.

The existence of a 2-dimensional conical (co)limit of a 2-functor $\mathcal{D} : \mathbb{S} \to \mathbb{A}$ defined in a locally discrete 2-category $\mathbb{S}$ (*i.e.* a diagram defined in a category $\mathbb{S}$) in a 2-category $\mathbb{A}$ is stronger than the existence of the 1-dimensional conical (co)limit of the underlying functor of the 2-functor $\mathcal{D}$ in the underlying category of $\mathbb{A}$. However, in the presence of the former, by the Yoneda lemma for 2-categories, both are isomorphic.

As in the 1-dimensional case, the conical 2-colimits of diagrams shaped by discrete categories are the coproducts, while the conical 2-colimits of diagrams with the domain being the opposite of the category S below gives the notion of pushout.

$$
\begin{tikzcd}
 & \mathsf{2} & \\
\mathsf{0} \arrow[ur, "\mathsf{d}_1"] & & \mathsf{1} \arrow[ul, "\mathsf{d}_0"']
\end{tikzcd}
$$

Recall that, if $p_0 : e \to b_0$, $p_1 : e \to b_1$ are morphisms of a 2-category $\mathbb{A}$, assuming its existence, the *pushout* of p_1 along p_0 is an object $p_0 \sqcup_e p_1$ satisfying the following: there are 1-cells $\mathfrak{d}^0_{p_0 \sqcup_e p_1} : b_1 \to p_0 \sqcup_e p_1$ and $\mathfrak{d}^1_{p_0 \sqcup_e p_1} : b_0 \to p_0 \sqcup_e p_1$ making the diagram

$$
\begin{tikzcd}
 & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
b_0 \arrow[dr, "\mathfrak{d}^1_{p_0 \sqcup_e p_1}"'] & & b_1 \arrow[dl, "\mathfrak{d}^0_{p_0 \sqcup_e p_1}"] \\
 & p_0 \sqcup_e p_1 &
\end{tikzcd}
$$

commutative and, for every object y and every pair of 2-cells $(\xi_0 : h_0 \Rightarrow h'_0 : b_1 \to y, \xi_1 : h_1 \Rightarrow h'_1 : b_0 \to y)$, such that

$$\begin{tikzcd}[column sep=small] & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ b_0 \arrow[dr, "h_1"'] \arrow[rr, phantom, "="] & & b_1 \arrow[dl, bend right, "h_0"'] \arrow[dl, bend left, "h'_0"] \arrow[dl, phantom, "\overset{\xi_0}{\Longrightarrow}"] \\ & y & \end{tikzcd} = \begin{tikzcd}[column sep=small] & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ b_0 \arrow[dr, bend right, "h_1"'] \arrow[dr, bend left, "h'_1"] \arrow[dr, phantom, "\overset{\xi_1}{\Longrightarrow}"] \arrow[rr, phantom, "="] & & b_1 \arrow[dl, "h'_0"] \\ & y & \end{tikzcd}$$

holds, there is a unique 2-cell $\xi : h \Rightarrow h' : p_0 \sqcup_e p_1 \to y$ satisfying the equations

$$\xi * \mathrm{id}_{\mathfrak{d}^0_{p_0 \sqcup_e p_1}} = \xi_0 \text{ and } \xi * \mathrm{id}_{\mathfrak{d}^1_{p_0 \sqcup_e p_1}} = \xi_1.$$

2.3. Opcomma objects. We consider the 2-category $\mathtt{S}$ defined above and the weight $P : \mathtt{S} \to \mathsf{Cat}$, defined by $P(\mathtt{1}) := P(\mathtt{0}) := \mathsf{1}$, $P(\mathtt{2}) := \mathsf{2}$, and $P(\mathtt{d}_0) = d^0, P(\mathtt{d}_1) = d^1$. That is to say, the weight

$$\begin{tikzcd}[column sep=small] & \mathsf{2} & \\ \mathsf{1} \arrow[ur, "d^1"] & & \mathsf{1} \arrow[ul, "d^0"'] \end{tikzcd}$$

in which d^0 and d^1 are respectively the inclusion of the codomain and the inclusion of the domain of the non-trivial morphism of $\mathsf{2}$ (as defined in Section 1).

Limits weighted by P are the well known *comma objects*, while the colimits weighted by P are called *opcomma objects*. By definition, if $p_0 : e \to b_0$, $p_1 : e \to b_1$ are morphisms of a 2-category $\mathbb{A}$ and $p_0 \uparrow p_1$ is the opcomma object of p_1 along p_0, then $\mathbb{A}(p_0 \uparrow p_1, -)$ is the comma object of $\mathbb{A}(p_1, -)$ along $\mathbb{A}(p_0, -)$. This means that: there are 1-cells $\delta^0_{p_0 \uparrow p_1} : b_1 \to p_0 \uparrow p_1$ and $\delta^1_{p_0 \uparrow p_1} : b_0 \to p_0 \uparrow p_1$ and a 2-cell

$$\begin{tikzcd} & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ b_0 \arrow[dr, "\delta^1_{p_0 \uparrow p_1}"'] \arrow[rr, phantom, "\overset{\alpha^{p_0 \uparrow p_1}}{\Longrightarrow}"] & & b_1 \arrow[dl, "\delta^0_{p_0 \uparrow p_1}"] \\ & p_0 \uparrow p_1 & \end{tikzcd}$$

satisfying the following:

(1) For every triple $(h_0 : b_1 \to y, h_1 : b_0 \to y, \beta : h_1 p_0 \Rightarrow h_0 p_1)$ in which h_0, h_1 are morphisms and β is a 2-cell of $\mathbb{A}$, there is a unique morphism $h : p_0 \uparrow p_1 \to y$ such that the equations $h_0 = h \cdot \delta^0_{p_0 \uparrow p_1}$, $h_1 = h \cdot \delta^1_{p_0 \uparrow p_1}$ and

$$\begin{tikzcd}
 & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
b_0 \arrow[dr, "\delta^1_{p_0\uparrow p_1}"'] \arrow[rr, phantom, "\overset{\alpha^{p_0\uparrow p_1}}{\Longrightarrow}"] & & b_1 \arrow[dl, "\delta^0_{p_0\uparrow p_1}"] \\
 & p_0 \uparrow p_1 \arrow[d, "h"'] & \\
 & y &
\end{tikzcd}
=
\begin{tikzcd}
 & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
b_0 \arrow[ddr, "h_1"'] \arrow[rr, phantom, "\overset{\beta}{\Longrightarrow}"] & & b_1 \arrow[ddl, "h_0"] \\
 & & \\
 & y &
\end{tikzcd}$$

hold.

(2) For every pair of 2-cells $(\xi_0 : h_0 \Rightarrow h'_0 : b_1 \to y, \xi_1 : h_1 \Rightarrow h'_1 : b_0 \to y)$ such that

$$\begin{tikzcd}
 & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
b_0 \arrow[dr, "h_1"'] \arrow[rr, phantom, "\overset{\mathrm{id}_h * \alpha^{p_0\uparrow p_1}}{\Longrightarrow}"] & & b_1 \arrow[dl, bend right, "h_0"'] \arrow[dl, bend left, "h'_0"] \arrow[dl, phantom, "\overset{\xi_0}{\Longrightarrow}"] \\
 & y &
\end{tikzcd}
=
\begin{tikzcd}
 & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
b_0 \arrow[dr, bend left, "h'_1"] \arrow[dr, bend right, "h_1"'] \arrow[dr, phantom, "\overset{\xi_1}{\Longrightarrow}"] \arrow[rr, phantom, "\overset{\mathrm{id}_{h'} * \alpha^{p_0\uparrow p_1}}{\Longrightarrow}"] & & b_1 \arrow[dl, "h'_0"] \\
 & y &
\end{tikzcd}$$

holds, there is a unique 2-cell $\xi : h \Rightarrow h' : p_0 \uparrow p_1 \to y$ such that $\xi * \mathrm{id}_{\delta^0_{p_0\uparrow p_1}} = \xi_0$ and $\xi * \mathrm{id}_{\delta^1_{p_0\uparrow p_1}} = \xi_1$.

Remark 2.1. Since Cat has all weighted colimits and limits, it has opcomma objects. More generally, assuming the existence of tensorial coproducts and pushouts, one can always construct opcomma objects out of them.

Assuming that the tensorial coproduct $\mathsf{2} \otimes e$ exists in $\mathbb{A}$, we have the universal 2-cell $d^1 \otimes e \Rightarrow d^0 \otimes e : e \to \mathsf{2} \otimes e$ given by the image of the identity $\mathsf{2} \otimes e \to \mathsf{2} \otimes e$ by the isomorphism

$$\mathbb{A}(\mathsf{2} \otimes e, \mathsf{2} \otimes e) \cong \mathsf{Cat}\left[\mathsf{2}, \mathbb{A}(e, \mathsf{2} \otimes e)\right].$$

If it exists, the conical colimit of the diagram below is the opcomma object $p_0 \uparrow p_1$ of $p_1 : e \to b_1$ along $p_0 : e \to b_0$.

$$\begin{tikzcd}
 & e \arrow[dl, "p_1"'] \arrow[dr, "d^1 \otimes e"] & & e \arrow[dl, "d^0 \otimes e"'] \arrow[dr, "p_0"] & \\
b_1 & & \mathsf{2} \otimes e & & b_0
\end{tikzcd}$$

2.4. Lax descent objects. We consider the 2-category Δ_{Str} of Definition 1.1 and we define the weight $\mathfrak{D} : \Delta_{\mathrm{Str}} \to \mathsf{Cat}$ by

$$\begin{tikzcd}[column sep=huge]
\Delta_{\mathrm{Str}}(\underline{1}, \underline{1}) \times \mathsf{1} \arrow[r, shift right=6, "\mathrm{id}_{\Delta_{\mathrm{Str}}(\underline{1},\underline{1})} \times d^1"'] \arrow[r, shift left=6, "\mathrm{id}_{\Delta_{\mathrm{Str}}(\underline{1},\underline{1})} \times d^0"] & \Delta_{\mathrm{Str}}(\underline{1}, \underline{1}) \times \mathsf{2} \arrow[l, "\overline{S}" description] \arrow[r, shift left=6, "\mathrm{id}_{\Delta_{\mathrm{Str}}(\underline{1},\underline{1})} \times \overline{D}^0"] \arrow[r, "\mathrm{id}_{\Delta_{\mathrm{Str}}(\underline{1},\underline{1})} \times \overline{D}^1" description] \arrow[r, shift right=6, "\mathrm{id}_{\Delta_{\mathrm{Str}}(\underline{1},\underline{1})} \times \overline{D}^2"'] & \Delta_{\mathrm{Str}}(\underline{1}, \underline{1}) \times \langle \mathsf{3} \rangle
\end{tikzcd}$$

in which:

– The functor $\overline{S} : \Delta_{\mathrm{Str}}(\underline{1}, \underline{1}) \times \mathsf{2} \to \Delta_{\mathrm{Str}}(\underline{1}, \underline{1}) \times \mathsf{1}$ is defined by

$$\overline{S}(\overline{v} : v \cong v', \mathsf{0} \to \mathsf{1}) = \left(\left(\mathfrak{n}_0^{-1} \cdot \mathfrak{n}_1 \right) * \overline{v}, \mathrm{id}_{\mathsf{0}} \right) = \left(s^0 d^1 v \cong s^0 d^0 v', \mathrm{id}_{\mathsf{0}} \right).$$

– $\langle \mathsf{3} \rangle$ is the category corresponding to the set

$$\left\{ (\mathsf{i}, \mathsf{k}) \in \{ \mathsf{0}, \mathsf{1}, \mathsf{2} \}^2 : \mathsf{i} \neq \mathsf{k} \right\}$$

with the preorder induced by the first coordinate, that is to say, $(\mathsf{i}, \mathsf{k}) \leqslant (\mathsf{i}', \mathsf{k}')$ if $\mathsf{i} \leqslant \mathsf{i}'$. In other words, the category $\langle \mathsf{3} \rangle$ is defined by the preordered set below.

$$\begin{tikzcd}
 & (\mathsf{2}, \mathsf{0}) \arrow[r, leftrightarrow, "\cong"] & (\mathsf{2}, \mathsf{1}) \\
(\mathsf{1}, \mathsf{2}) \arrow[r, leftrightarrow, "\cong"] & (\mathsf{1}, \mathsf{0}) \arrow[u] & \\
(\mathsf{0}, \mathsf{2}) \arrow[u] \arrow[rr, leftrightarrow, "\cong"'] & & (\mathsf{0}, \mathsf{1}). \arrow[uu]
\end{tikzcd}$$

– The functors $\overline{D}^0, \overline{D}^1, \overline{D}^2 : \mathsf{2} \to \langle \mathsf{3} \rangle$ are defined by

$$\overline{D}^0(\mathsf{0} \to \mathsf{1}) = ((\mathsf{1}, \mathsf{0}) \to (\mathsf{2}, \mathsf{0})), \ \overline{D}^2(\mathsf{0} \to \mathsf{1}) = ((\mathsf{0}, \mathsf{2}) \to (\mathsf{1}, \mathsf{2}))$$
and
$$\overline{D}^1(\mathsf{0} \to \mathsf{1}) = ((\mathsf{0}, \mathsf{1}) \to (\mathsf{2}, \mathsf{1})).$$

– The natural transformations $\mathfrak{D}(\sigma_{01})$, $\mathfrak{D}(\sigma_{02})$ and $\mathfrak{D}(\sigma_{12})$ are defined by
$$\mathfrak{D}(\sigma_{ij}) := \mathrm{id}_{\mathrm{id}_{\Delta_{\mathrm{Str}}(\underline{1},\underline{1})}} \times \overline{\mathfrak{D}(\sigma_{ij})},$$
in which
$$\overline{\mathfrak{D}(\sigma_{01})}_{\mathsf{0}} := ((\mathsf{2},\mathsf{1}) \cong (\mathsf{2},\mathsf{0})),\ \overline{\mathfrak{D}(\sigma_{02})}_{\mathsf{0}} := ((\mathsf{1},\mathsf{2}) \cong (\mathsf{1},\mathsf{0}))$$
and
$$\overline{\mathfrak{D}(\sigma_{12})}_{\mathsf{0}} := ((\mathsf{0},\mathsf{2}) \cong (\mathsf{0},\mathsf{1})).$$
– The natural transformation
$$\mathfrak{D}(\mathfrak{n}_i) : \overline{S} \circ \left(\mathrm{id}_{\Delta_{\mathrm{Str}}(\underline{1},\underline{1})} \times d^i\right) \Rightarrow \mathrm{id}_{\Delta_{\mathrm{Str}}(\underline{1},\underline{1})\times 1}$$
is defined by $\mathfrak{D}(\mathfrak{n}_i)_{(v,\mathsf{0})} := (\mathfrak{n}_i * \mathrm{id}_v, \mathrm{id}_{\mathsf{0}})$.

Remark 2.2. [[42]] Since Cat has all weighted limits, it has lax descent objects. More precisely, if $\mathcal{A} : \Delta_{\mathrm{Str}} \to \mathsf{Cat}$ is a 2-functor,
$$\mathsf{lim}(\mathfrak{D}, \mathcal{A}) \cong [\Delta_{\mathrm{Str}}, \mathsf{Cat}]\,(\mathfrak{D}, \mathcal{A})$$
is the category in which:

(1) Objects are 2-natural transformations $\overline{\psi} : \mathfrak{D} \longrightarrow \mathcal{A}$. We have a bijective correspondence between such 2-natural transformations and pairs (w, ψ) in which w is an object of $\mathcal{A}(\underline{1})$ and $\psi : \mathcal{A}(\mathbb{d}^1)(w) \to \mathcal{A}(\mathbb{d}^0)(w)$ is a morphism in $\mathcal{A}(\underline{2})$ satisfying the following equations:

Associativity:
$$\mathcal{A}(\mathbb{d}^0)(\psi) \cdot \mathcal{A}(\sigma_{02})_w \cdot \mathcal{A}(\mathbb{d}^2)(\psi) = \mathcal{A}(\sigma_{01})_w \cdot \mathcal{A}(\mathbb{d}^1)(\psi) \cdot \mathcal{A}(\sigma_{12})_w;$$
Identity:
$$\mathcal{A}(\mathfrak{n}_0)_w \cdot \mathcal{A}(\mathbb{s}^0)(\psi) = \mathcal{A}(\mathfrak{n}_1)_w.$$
If $\overline{\psi} : \mathfrak{D} \longrightarrow \mathcal{A}$ is a 2-natural transformation, we get such pair by the correspondence
$$\overline{\psi} \mapsto (\overline{\psi}_{\underline{1}}(\mathrm{id}_{\underline{1}}, \mathsf{0}), \overline{\psi}_{\underline{2}}(\mathrm{id}_{\underline{1}}, \mathsf{0} \to \mathsf{1})).$$

(2) The morphisms are modifications. In other words, a morphism $\mathfrak{m} : (w, \psi) \to (w', \psi')$ is determined by a morphism $\mathfrak{m} : w \to w'$ in $\mathcal{A}(\underline{1})$ such that
$$\mathcal{A}(\mathbb{d}^0)(\mathfrak{m}) \cdot \psi = \psi' \cdot \mathcal{A}(\mathbb{d}^1)(\mathfrak{m}).$$

By definition, if $\mathcal{B} : \Delta_{\mathrm{Str}} \to \mathbb{A}$ is a 2-functor, an object lax-$\mathcal{D}$esc $(\mathcal{B})$ is the lax descent object $\mathsf{lim}(\mathfrak{D}, \mathcal{B})$ of $\mathcal{B}$ if and only if there is a 2-natural isomorphism (in y)
$$\mathbb{A}(y, \text{lax-}\mathcal{D}\mathrm{esc}\,(\mathcal{B})) \cong \mathsf{lim}(\mathfrak{D}, \mathbb{A}(y, \mathcal{B}-)).$$

Equivalently, $\lim(\mathfrak{D},\mathcal{B})$ is defined by the following universal property: there are a morphism $\mathbb{d}^{(\mathfrak{D},\mathcal{B})} : \lim(\mathfrak{D},\mathcal{B}) \to \mathcal{B}(\underline{1})$ and a 2-cell

$$\begin{tikzcd}
 & \lim(\mathfrak{D},\mathcal{B}) \arrow[dl, "\mathbb{d}^{(\mathfrak{D},\mathcal{B})}"'] \arrow[dr, "\mathbb{d}^{(\mathfrak{D},\mathcal{B})}"] & \\
\mathcal{B}(\underline{1}) \arrow[rr, phantom, "\xRightarrow{\Psi^{(\mathfrak{D},\mathcal{B})}}"] \arrow[dr, "\mathcal{B}(\mathbb{d}^1)"'] & & \mathcal{B}(\underline{1}) \arrow[dl, "\mathcal{B}(\mathbb{d}^0)"] \\
 & \mathcal{B}(\underline{2}) &
\end{tikzcd}$$

satisfying the following:

(1) For each pair $(h : y \to \mathcal{B}(\underline{1}), \beta : \mathcal{B}(\mathbb{d}^1) \cdot h \Rightarrow \mathcal{B}(\mathbb{d}^0) \cdot h)$ in which h is a morphism and β is a 2-cell of $\mathbb{A}$ such that the equations

$$\begin{tikzcd}
y \arrow[r, "h"] \arrow[d, "h"'] \arrow[dr, phantom, "\xRightarrow{\beta}"] & \mathcal{B}(\underline{1}) \arrow[r, "\mathcal{B}(\mathbb{d}^0)"] \arrow[d, "\mathcal{B}(\mathbb{d}^0)"] \arrow[dr, phantom, "\xRightarrow{\mathcal{B}(\sigma_{01})}"] & \mathcal{B}(\underline{2}) \arrow[d, "\mathcal{B}(\mathbb{d}^0)"] \\
\mathcal{B}(\underline{1}) \arrow[r, "\mathcal{B}(\mathbb{d}^1)"] \arrow[d, "\mathcal{B}(\mathbb{d}^1)"'] \arrow[drr, phantom, "\xRightarrow{\mathcal{B}(\sigma_{12})}"] & \mathcal{B}(\underline{2}) \arrow[r, "\mathcal{B}(\mathbb{d}^1)"] & \mathcal{B}(\underline{3}) \arrow[d, "\mathcal{B}(\mathrm{id}_{\underline{3}})"] \\
\mathcal{B}(\underline{2}) \arrow[rr, "\mathcal{B}(\mathbb{d}^2)"'] & & \mathcal{B}(\underline{3})
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{B}(\underline{3}) & \mathcal{B}(\underline{2}) \arrow[l, "\mathcal{B}(\mathbb{d}^0)"'] \arrow[r, equal] & \mathcal{B}(\underline{2}) \\
\mathcal{B}(\underline{2}) \arrow[u, "\mathcal{B}(\mathbb{d}^2)"] \arrow[ur, phantom, "\xRightarrow{\mathcal{B}(\sigma_{02})}"] & \mathcal{B}(\underline{1}) \arrow[l, "\mathcal{B}(\mathbb{d}^0)"'] \arrow[u, "\mathcal{B}(\mathbb{d}^1)"] \arrow[r, phantom, "\xRightarrow{\beta}"] & \\
\mathcal{B}(\underline{1}) \arrow[u, "\mathcal{B}(\mathbb{d}^1)"] \arrow[ur, phantom, "\xRightarrow{\beta}"] & y \arrow[l, "h"] \arrow[r, "h"'] \arrow[u, "h"] & \mathcal{B}(\underline{1}) \arrow[uu, "\mathcal{B}(\mathbb{d}^0)"']
\end{tikzcd}$$

(*descent associativity*)

$$\begin{tikzcd}
y \arrow[r, "h"] \arrow[d, "h"'] \arrow[dr, phantom, "\xRightarrow{\beta}"] & \mathcal{B}(\underline{1}) \arrow[d, "\mathcal{B}(\mathbb{d}^0)"] \arrow[ddr, bend left, equal] \arrow[dr, phantom, "\xRightarrow{\mathcal{B}(\mathfrak{n}_0)}"] & \\
\mathcal{B}(\underline{1}) \arrow[r, "\mathcal{B}(\mathbb{d}^1)"] \arrow[drr, bend right, equal] \arrow[dr, phantom, "\xRightarrow{\mathcal{B}(\mathfrak{n}_1)^{-1}}"] & \mathcal{B}(\underline{2}) \arrow[dr, "\mathcal{B}(\mathfrak{s}^0)"] & \\
 & & \mathcal{B}(\underline{1})
\end{tikzcd}
=
\begin{tikzcd}
y \arrow[d, bend right, "h"'] \arrow[d, bend left, "h"] \arrow[d, phantom, "="] \\
\mathcal{B}(\underline{1})
\end{tikzcd}$$

(*descent identity*)

hold, there is a unique morphism $h^{(\mathcal{B},\beta)} : y \to \lim(\mathfrak{D},\mathcal{B})$ such that

$$
\begin{tikzcd}
 & y \arrow[d, "h^{(\mathcal{B},\beta)}"] & \\
 & \lim(\mathfrak{D},\mathcal{B}) \arrow[dl, "\mathbb{d}^{(\mathfrak{D},\mathcal{B})}"'] \arrow[dr, "\mathbb{d}^{(\mathfrak{D},\mathcal{B})}"] & \\
\mathcal{B}(\underline{1}) \arrow[dr, "\mathcal{B}(\mathbb{d}^1)"'] \arrow[rr, phantom, "\overset{\Psi^{(\mathfrak{D},\mathcal{B})}}{\Longrightarrow}"] & & \mathcal{B}(\underline{1}) \arrow[dl, "\mathcal{B}(\mathbb{d}^0)"] \\
 & \mathcal{B}(\underline{2}) &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & y \arrow[dl, "h"'] \arrow[dr, "h"] & \\
\mathcal{B}(\underline{1}) \arrow[dr, "\mathcal{B}(\mathbb{d}^1)"'] \arrow[rr, phantom, "\overset{\beta}{\Longrightarrow}"] & & \mathcal{B}(\underline{1}) \arrow[dl, "\mathcal{B}(\mathbb{d}^0)"] \\
 & \mathcal{B}(\underline{2}) &
\end{tikzcd}
$$

and $h = \mathbb{d}^{(\mathfrak{D},\mathcal{B})} \cdot h^{(\mathcal{B},\beta)}$.

Moreover, the pair $(\mathbb{d}^{(\mathfrak{D},\mathcal{B})}, \Psi^{(\mathfrak{D},\mathcal{B})})$ satisfies the *descent associativity* and *descent identity* equations above. In this case, the unique morphism induced is clearly the identity on $\lim(\mathfrak{D},\mathcal{B})$, that is to say,

$$\left(\mathbb{d}^{(\mathfrak{D},\mathcal{B})}\right)^{(\mathcal{B},\Psi^{(\mathfrak{D},\mathcal{B})})} = \mathrm{id}_{\lim(\mathfrak{D},\mathcal{B})}.$$

(2) Assuming that (h_1,β_1) and (h_0,β_0) are pairs satisfying the *descent associativity* and *descent identity* equations, for each 2-cell $\xi : h_1 \Rightarrow h_0 : y \to \mathcal{B}(\underline{1})$ satisfying the equation

$$
\begin{tikzcd}
 & y \arrow[dl, bend right=40, "h_1"'] \arrow[dl, bend left=20, "h_0"] \arrow[dr, "h_0"] & \\
\mathcal{B}(\underline{1}) \arrow[dr, "\mathcal{B}(\mathbb{d}^1)"'] \arrow[rr, phantom, "\overset{\beta_0}{\Longrightarrow}"] & & \mathcal{B}(\underline{1}) \arrow[dl, "\mathcal{B}(\mathbb{d}^0)"] \\
 & \mathcal{B}(\underline{2}) &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & y \arrow[dl, "h_1"'] \arrow[dr, bend left=40, "h_0"] \arrow[dr, bend right=20, "h_1"'] & \\
\mathcal{B}(\underline{1}) \arrow[dr, "\mathcal{B}(\mathbb{d}^1)"'] \arrow[rr, phantom, "\overset{\beta_1}{\Longrightarrow}"] & & \mathcal{B}(\underline{1}) \arrow[dl, "\mathcal{B}(\mathbb{d}^0)"] \\
 & \mathcal{B}(\underline{2}) &
\end{tikzcd}
$$

there is a unique 2-cell $\xi^{(\mathcal{B},\beta_1,\beta_0)} : h_1^{(\mathcal{B},\beta_0)} \Rightarrow h_0^{(\mathcal{B},\beta_1)} : \mathcal{B}(\underline{1}) \to \lim(\mathfrak{D},\mathcal{B})$ such that

$$\mathrm{id}_{\mathbb{d}^{(\mathfrak{D},\mathcal{B})}} * \xi^{(\mathcal{B},\beta_1,\beta_0)} = \xi.$$

2.5. The higher cokernel. Let $p : e \to b$ be a morphism of a 2-category $\mathbb{A}$, $\mathbb{A}$ *has the higher cokernel of* p if $\mathbb{A}$ has the opcomma object

$$\begin{tikzcd}
 & e \arrow[dl, "p"'] \arrow[dr, "p"] & \\
b \arrow[rr, phantom, "\xRightarrow{\alpha}"] \arrow[dr, "\delta^1"'] & & b \arrow[dl, "\delta^0"] \\
 & b \uparrow_p b &
\end{tikzcd}$$

of p along itself and the pushout of δ^0 along δ^1 as below.

$$\begin{tikzcd}
 & b \arrow[dl, "\delta^0"'] \arrow[dr, "\delta^1"] & \\
b \uparrow_p b \arrow[dr, "\partial^2"'] & & b \uparrow_p b \arrow[dl, "\partial^0"] \\
 & b \uparrow_p b \uparrow_p b &
\end{tikzcd}$$

Henceforth, in this section, we assume that $\mathbb{A}$ has the higher cokernel of p as above. We denote by $\partial^1 : b \uparrow_p b \to b \uparrow_p b \uparrow_p b$ the unique morphism such that the equations

$$\partial^1 \delta^1 = \partial^2 \delta^1, \ \partial^1 \delta^0 = \partial^0 \delta^0$$

$$\begin{tikzcd}
 & e \arrow[dl, "p"'] \arrow[dr, "p"] & \\
b \arrow[rr, phantom, "\xRightarrow{\alpha}"] \arrow[dr, "\delta^1"'] & & b \arrow[dl, "\delta^0"] \\
 & b \uparrow_p b \arrow[d, "\partial^1"'] & \\
 & b \uparrow_p b \uparrow_p b &
\end{tikzcd}
=
\begin{tikzcd}
 & e \arrow[d, "p"] \arrow[ddl, "p"'] \arrow[ddr, "p"] & \\
 & b \arrow[dd, bend right, "\partial^2\delta^0 = \partial^0\delta^1"'] \arrow[dd, bend left] & \\
b \arrow[r, phantom, "\xRightarrow{\mathrm{id}_{\partial^2} * \alpha}"] \arrow[dr, "\partial^2\delta^1"'] & {} \arrow[r, phantom, "\xRightarrow{\mathrm{id}_{\partial^0} * \alpha}"] & b \arrow[dl, "\partial^0\delta^0"] \\
 & b \uparrow_p b \uparrow_p b &
\end{tikzcd}$$

(*higher cokernel associativity*)

hold, while we denote by $\mathfrak{s}^0 : b \uparrow_p b \to b$ the unique morphism such that $\mathfrak{s}^0 \cdot \delta^1 = \mathfrak{s}^0 \cdot \delta^0 = \mathrm{id}_b$ and the equation

$$
\begin{tikzcd}
 & e \arrow[dl, "p"'] \arrow[dr, "p"] & \\
b \arrow[rr, phantom, "\overset{\alpha}{\Longrightarrow}"] \arrow[dr, "\delta^1"'] & & b \arrow[dl, "\delta^0"] \\
 & b \uparrow_p b \arrow[d, "\mathfrak{s}^0"'] & \\
 & b &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
e \arrow[d, bend right, "p"'] \arrow[d, bend left, "p"] \arrow[d, phantom, "="] \\
b
\end{tikzcd}
\qquad (\textit{higher cokernel identity})
$$

holds. In this case, we have:

Definition 2.3. [Higher cokernel] Consider the 2-functor $\mathcal{H}'_p : \Delta_3 \to \mathbb{A}$ defined by $\mathcal{H}'_p(d^i : \mathsf{1} \to \mathsf{2}) = \delta^i$, $\mathcal{H}'_p(d^i : \mathsf{2} \to \mathsf{3}) = \partial^i$ and $\mathcal{H}'_p(s^0) = \mathfrak{s}^0$. The 2-functor

$$\mathcal{H}_p := \mathcal{H}'_p \circ \mathsf{e}_{\Delta_{\mathrm{Str}}} : \Delta_{\mathrm{Str}} \to \mathbb{A}$$

$$
\begin{tikzcd}[column sep=large]
b \arrow[r, shift left=2, "\delta^0"] \arrow[r, shift right=2, "\delta^1"'] & b \uparrow_p b \arrow[l, "\mathfrak{s}^0" description] \arrow[r, shift left=3, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=3, "\partial^2"'] & b \uparrow_p b \uparrow_p b
\end{tikzcd}
$$

is called the *higher cokernel of* p.

Remark 2.4. The 2-category of categories Cat has the higher cokernel of any functor. In particular, the higher cokernel of id_{1} is just the usual inclusion

$$
\begin{tikzcd}
\mathsf{1} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & \mathsf{2} \arrow[l, "s^0" description] \arrow[r, shift left=3, "d^0"] \arrow[r, "d^1" description] \arrow[r, shift right=3, "d^2"'] & \mathsf{3}
\end{tikzcd}
\qquad (\mathcal{H}_{\mathrm{id}_1})
$$

of the locally discrete 2-category Δ_3 in Cat, which is actually the weight for lax descent objects of 2-functors $\mathcal{B} : \Delta_3 \to \mathbb{A}$, that is to say, 2-functors $\mathcal{A} : \Delta_{\mathrm{Str}} \to \mathbb{A}$ that can be written as $\mathcal{A} = \mathcal{B} \circ \mathsf{e}_{\Delta_{\mathrm{Str}}}$. This is the case of the diagrams of higher cokernels of morphisms.

By the *higher cokernel associativity* and the *higher cokernel identity* equations of the definition of ∂^1 and $\mathfrak{s}^0$, the pair $(p : e \to b, \alpha : \delta^1 \cdot p \Rightarrow \delta^0 \cdot p : e \to b \uparrow_p b)$ satisfies the *descent associativity* and *descent identity* equations w.r.t. $\mathcal{H}_p : \Delta_{\mathrm{Str}} \to \mathbb{A}$. Hence, if $\mathbb{A}$ has the lax descent object $(\mathbb{d}^p, \Psi^p) := (\mathbb{d}^{(\mathfrak{D}, \mathcal{H}_p)}, \Psi^{(\mathfrak{D}, \mathcal{H}_p)})$ of $\mathcal{H}_p$, we say that the *lax descent factorization induced by the higher cokernel* exists. In this case, by the universal property of the lax descent object, there is a unique morphism $p^{\mathcal{H}} := p^{(\mathcal{H}_p, \alpha)} : e \to \mathsf{lim}(\mathfrak{D}, \mathcal{H}_p)$ such that

$$\begin{tikzcd} e \arrow[rr, "p"] \arrow[dr, "p^{\mathcal{H}}"'] & & b \\ & \mathsf{lim}(\mathfrak{D}, \mathcal{H}_p) \arrow[ur, "\mathbb{d}^p"'] & \end{tikzcd}$$

(*lax descent factorization induced by the higher cokernel*)

commutes and $\Psi^p * \mathrm{id}_{p^{\mathcal{H}}} = \alpha$. Moreover, assuming that the *lax descent factorization induced by the higher cokernel* exists, we have that, given an object x of $\mathbb{A}$, the diagram of the factorization induced by the pair $(\mathbb{A}(x,p), \mathbb{A}(x,\alpha))$ and by the universal property of the lax descent category $\mathsf{lim}(\mathfrak{D}, \mathbb{A}(x, \mathcal{H}_p-))$, that is to say, the commutative diagram

$$\begin{tikzcd} \mathbb{A}(x,e) \arrow[rr, "\mathbb{A}(x,p)"] \arrow[dr, "\mathbb{A}(x,p)^{(\mathbb{A}(x,\mathcal{H}_p-), \mathbb{A}(x,\alpha))}"'] & & \mathbb{A}(x,b) \\ & \mathsf{lim}(\mathfrak{D}, \mathbb{A}(x, \mathcal{H}_p-)) \arrow[ur, "\mathbb{d}^{(\mathfrak{D}, \mathbb{A}(x,\mathcal{H}_p-))}"'] & \end{tikzcd}$$

(*factorization of* $\mathbb{A}(x,p)$ *induced by the pair* $(\mathbb{A}(x,p), \mathbb{A}(x,\alpha))$ *and* $\mathsf{lim}(\mathfrak{D}, \mathbb{A}(x, \mathcal{H}_p-)))$

which is given by

$$\begin{array}{llll} \mathbb{A}(x,p)^{(\mathbb{A}(x,\mathcal{H}_p-), \mathbb{A}(x,\alpha))} : & \mathbb{A}(x,e) & \to \mathsf{lim}(\mathfrak{D}, \mathbb{A}(x, \mathcal{H}_p-)) \cong \mathbb{A}(x, \mathsf{lim}(\mathfrak{D}, \mathcal{H}_p)) \\ & g & \mapsto (p \cdot g, \alpha * \mathrm{id}_g) \\ & \chi : g \Rightarrow g' & \mapsto \mathrm{id}_p * \chi \end{array}$$

$$\begin{array}{lll} \mathbb{d}^{(\mathfrak{D}, \mathbb{A}(x,\mathcal{H}_p-))} : & \mathsf{lim}(\mathfrak{D}, \mathbb{A}(x, \mathcal{H}_p-)) & \to \mathbb{A}(x,b) \\ & (f, \psi) & \mapsto f \\ & \xi & \mapsto \xi \end{array}$$

is isomorphic to the factorization $\mathbb{A}(x,p) = \mathbb{A}(x, \mathbb{d}^p) \circ \mathbb{A}(x, p^{\mathcal{H}})$, since the Yoneda embedding creates any existing weighted limit and, in particular, existing lax descent objects in $\mathbb{A}$.

Remark 2.5. It should be noted that, assuming that we can construct $\mathcal{H}_p$, the *factorization of* $\mathbb{A}(x,p)$ *induced by the pair* $(\mathbb{A}(x,p), \mathbb{A}(x,\alpha))$ *and* $\mathsf{lim}(\mathfrak{D}, \mathbb{A}(x, \mathcal{H}_p-))$ above always exists, since Cat has lax descent objects (lax descent categories).

Moreover, the definition of the *factorization of* $\mathbb{A}(x,p)$ *induced by the pair* $(\mathbb{A}(x,p), \mathbb{A}(x,\alpha))$ *and* $\mathsf{lim}(\mathfrak{D}, \mathbb{A}(x, \mathcal{H}_p-))$ above does not coincide with the definition of *lax descent factorization induced by the higher cokernel* of $\mathbb{A}(x,p)$. Indeed, opcomma objects (weighted colimits in general) might not be preserved by the Yoneda embedding.

For instance, consider the example of Remark 2.4. For any object x of Cat, clearly the opcomma object of $\mathsf{Cat}[x, \mathrm{id}_1]$ along itself is isomorphic to the opcomma object of id_1 along itself, that is to say, 2. Hence, since there is a category x such that $\mathsf{Cat}[x, 2]$ is not isomorphic to 2, this shows that the Yoneda embedding does not preserve the opcomma object $\mathrm{id}_1 \uparrow \mathrm{id}_1$.

Remark 2.6. [Duality: higher kernel and lax codescent factorization] The codual notion of that of the higher cokernel gives the same notion of factorization (assuming the existence of the suitable lax descent object): that is to say, the *lax descent factorization induced by the higher cokernel* of p.

The dual notion of the higher cokernel, the higher kernel of $l : b \to e$, if it exists, is a 2-functor

$$\mathcal{H}^l : \Delta_{\mathrm{Str}}^{\mathrm{op}} \to \mathbb{A}$$

$$\begin{tikzcd}
b \downarrow^l b \downarrow^l b \arrow[r, shift left=4, "\partial_0^{l\downarrow l}"] \arrow[r, "\partial_1^{l\downarrow l}" description] \arrow[r, shift right=4, "\partial_2^{l\downarrow l}"'] & b \downarrow^l b \arrow[r, shift left=4, "\delta_0^{l\downarrow l}"] \arrow[r, shift right=4, "\delta_1^{l\downarrow l}"'] & b \arrow[l, "\mathfrak{s}_0^{l\downarrow l}" description]
\end{tikzcd}$$

constructed from suitable comma objects and pullbacks. In this case, in the presence of the lax *codescent object* [33, 42] of $\mathcal{H}^l$, we get the *lax codescent factorization induced by the higher kernel* of l.

$$\begin{tikzcd}
e \arrow[rr, "l"] \arrow[dr, "\mathbbm{d}_l"'] & & b \\
& \mathsf{colim}(\mathfrak{D}, \mathcal{H}^l) \arrow[ur, "l_{\mathcal{H}}"'] &
\end{tikzcd}$$

(*lax codescent factorization induced by the higher kernel*)

3. Semantic factorization

Assuming that $\mathbb{A}$ has suitable Eilenberg-Moore objects, the *semantics-structure* adjunction (see [51]) gives rise to what is called herein the semantic factorization of a tractable morphism p [15]. In this section, we recall the semantic factorization of morphisms that have codensity monads. Before doing so, we recall the definition of the Eilenberg-Moore object of a given monad.

3.1. Eilenberg-Moore object. Recall that a monad in a 2-category $\mathbb{A}$ is a quadruple

$$\mathtt{t} = (b, t : b \to b, m : t^2 \Rightarrow t, \eta : \mathrm{id}_b \Rightarrow t)$$

in which b is an object, t is a morphism and m, η are 2-cells in $\mathbb{A}$ such that the equations

$$m \cdot (\mathrm{id}_t * m) = m \cdot (m * \mathrm{id}_t) \qquad \text{and} \qquad m \cdot (\eta * \mathrm{id}_t) = \mathrm{id}_t = m \cdot (\mathrm{id}_t * \eta)$$

hold. A monad can be seen as a 2-functor $\mathtt{t} : \mathsf{mnd} \to \mathbb{A}$ from the free monad 2-category mnd to $\mathbb{A}$ (see [49, 54, 43]). If it exists, the *Eilenberg-Moore object*, also called the *object of algebras*, is a special weighted limit of $\mathtt{t}$. More precisely, given a monad $\mathtt{t}$ in $\mathbb{A}$, the object $b^{\mathtt{t}}$ is the Eilenberg-Moore object of $\mathtt{t}$ if and only if there is a 2-natural isomorphism (in y)

$$\mathbb{A}(y, b^{\mathtt{t}}) \cong \mathbb{A}(y, b)^{\mathbb{A}(y,\mathtt{t})}$$

in which $\mathbb{A}(y, b)^{\mathbb{A}(y,\mathtt{t})}$ is the Eilenberg-Moore category of the monad

$$(\mathbb{A}(y, b), \mathbb{A}(y, t), \mathbb{A}(y, m), \mathbb{A}(y, \eta))$$

in Cat. This means that the Eilenberg-Moore object $b^{\mathtt{t}}$ of $\mathtt{t} = (b, t, m, \eta)$ is characterized by the following universal property: there is a pair $(\mathtt{u}^{\mathtt{t}} : b^{\mathtt{t}} \to b, \mu^{\mathtt{t}} : t \cdot \mathtt{u}^{\mathtt{t}} \Rightarrow t)$ in which $\mathtt{u}^{\mathtt{t}}$ is a morphism and $\mu^{\mathtt{t}}$ is a 2-cell in $\mathbb{A}$ satisfying the following:

(1) For each pair $(h : y \to b, \beta : t \cdot h \Rightarrow h)$ in which h is a morphism and β is a 2-cell in $\mathbb{A}$ such that the equations

$$\beta \cdot (m * \mathrm{id}_h) = \beta \cdot (\mathrm{id}_t * \beta) \quad \text{and} \quad \beta \cdot (\eta * \mathrm{id}_h) = \mathrm{id}_h$$

(*algebra associativity and identity*)

hold, there is a unique morphism $h^{(\mathtt{t},\beta)} : y \to b^{\mathtt{t}}$ such that $\mu^{\mathtt{t}} * \mathrm{id}_{h^{(\mathtt{t},\beta)}} = \beta$ and $\mathtt{u}^{\mathtt{t}} \cdot h^{(\mathtt{t},\beta)} = h$.

Moreover, the pair $(\mathtt{u}^{\mathtt{t}}, \mu^{\mathtt{t}})$ satisfies the *algebra associativity and identity* equations above. In this case, the unique morphism induced is clearly the identity on $b^{\mathtt{t}}$.

(2) Assuming that (h_0, β_0) and (h_1, β_1) are pairs satisfying the *algebra associativity and identity* equations, for each 2-cell $\xi : h_0 \Rightarrow h_1 : y \to b$ such that $\beta_1 \cdot (\mathrm{id}_t * \xi) = \xi \cdot \beta_0$, there is a unique 2-cell $\xi_{(\mathtt{t},\beta_0,\beta_1)} : h_0^{(\mathtt{t},\beta_0)} \Rightarrow h_1^{(\mathtt{t},\beta_1)} : b \to b^{\mathtt{t}}$ such that $\mathrm{id}_{\mathtt{u}^{\mathtt{t}}} * \xi_{(\mathtt{t},\beta_0,\beta_1)} = \xi$.

Remark 3.1. [Duality: Kleisli objects and co-Eilenberg-Moore objects] The dual to the notion of Eilenberg-Moore object of a monad is called the *Kleisli object* of a monad, while the codual is called the *co-Eilenberg-Moore object*, or *object of coalgebras*, of a comonad. These notions coincide with the usual notions in Cat, as it is carefully explained in [51].

3.2. Kan extensions. Let $f : z \to y$ and $g : z \to x$ be morphisms of a 2-category $\mathbb{A}$. The right Kan extension of f along g is, if it exists, the right reflection $\mathrm{ran}_g f$ of f along the functor

$$\mathbb{A}(g, y) : \mathbb{A}(x, y) \to \mathbb{A}(z, y).$$

This means that the right Kan extension is actually a pair

$$\left(\mathrm{ran}_g f : x \to y, \gamma^{\mathrm{ran}_g f} : (\mathrm{ran}_g f) \cdot g \Rightarrow f\right)$$

of a morphism $\mathrm{ran}_g f$ and a 2-cell $\gamma^{\mathrm{ran}_g f}$, called the universal 2-cell, such that, for each morphism $h : x \to y$ of $\mathbb{A}$,

$$\begin{tikzcd}
x \arrow[rd, "\mathrm{ran}_g f"] \arrow[rd, bend right=60, "h"'] \arrow[rd, phantom, bend right=30, "\overset{\beta}{\Longrightarrow}"] & \\
& y
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}
x \arrow[rd, "\mathrm{ran}_g f"] \arrow[rd, bend right=60, "h"'] \arrow[rd, phantom, bend right=30, "\overset{\beta}{\Longrightarrow}"] & z \arrow[l, "g"'] \arrow[d, "f"] \arrow[ld, phantom, "\overset{\gamma^{\mathrm{ran}_g f}}{\Longrightarrow}"] \\
& y
\end{tikzcd}$$

defines a bijection $\mathbb{A}(x, y)(h, \mathrm{ran}_g f) \cong \mathbb{A}(z, y)(h \cdot g, f)$.

Remark 3.2. [Duality: right lifting and left Kan extension] The dual notion to that of a right Kan extension is called *right lifting* (see [58]), while the codual notion is called the *left Kan extension.* Finally, of course, we also have the codual notion of the right lifting: the *left lifting.*

Let $p_0 : e \to b_0, p_1 : e \to b_1$ be morphisms of a 2-category $\mathbb{A}$. Assume that $\mathbb{A}$ has the opcomma object $p_0 \uparrow p_1$ and

$$\alpha^{p_0 \uparrow p_1} : \delta^1_{p_0 \uparrow p_1} \cdot p_0 \Rightarrow \delta^0_{p_0 \uparrow p_1} \cdot p_1$$

is the universal 2-cell that gives $p_0 \uparrow p_1$ as the opcomma object of p_1 along p_0, as in 2.3. In this case, we have:

Lemma 3.3. *Given a morphism $h : p_0 \uparrow p_1 \to y$, the following statements are equivalent.*

i) The pair $(h, \mathrm{id}_{h \cdot \delta^0_{p_0 \uparrow p_1}})$ is the right Kan extension of $h \cdot \delta^0_{p_0 \uparrow p_1}$ along $\delta^0_{p_0 \uparrow p_1}$.

ii) The pair $(h \cdot \delta^1_{p_0 \uparrow p_1}, \mathrm{id}_h * \alpha^{p_0 \uparrow p_1})$ is the right Kan extension of $h \cdot \delta^0_{p_0 \uparrow p_1} \cdot p_1$ along p_0.

Proof: Assuming i), given a 2-cell

$$\beta : h'_1 \cdot p_0 \Rightarrow h \cdot \delta^0_{p_0 \uparrow p_1} \cdot p_1,$$

we have, by the universal property of the opcomma object, that there is a unique morphism $h' : p_0 \uparrow p_1 \to y$ such that $\mathrm{id}_{h'} * \alpha^{p_0\uparrow p_1} = \beta$, $h' \cdot \delta^1_{p_0\uparrow p_1} = h'_1$ and $h' \cdot \delta^0_{p_0\uparrow p_1} = h \cdot \delta^0_{p_0\uparrow p_1}$.

By the universal property of the Kan extension, there is a unique 2-cell $\underline{\beta} : h' \Rightarrow h$ such that $\underline{\beta} * \mathrm{id}_{\delta^0_{p_0\uparrow p_1}}$ is the identity $h' \cdot \delta^0_{p_0\uparrow p_1} = h \cdot \delta^0_{p_0\uparrow p_1}$. By the universal property of the opcomma object, this means that $\underline{\beta} * \mathrm{id}_{\delta^1_{p_0\uparrow p_1}}$ is the unique 2-cell such that

$$\left(\mathrm{id}_h * \alpha^{p_0\uparrow p_1}\right) \cdot \left(\underline{\beta} * \mathrm{id}_{\delta^1_{p_0\uparrow p_1} \cdot p_0}\right) = \beta.$$

This proves ii). Reciprocally, assuming ii), by the universal property of the Kan extension, we have that, given any 2-cell

$$\beta_0 : h' \cdot \delta^0_{p_0\uparrow p_1} \Rightarrow h \cdot \delta^0_{p_0\uparrow p_1},$$

there is a unique 2-cell $\beta_1 : h' \cdot \delta^1_{p_0\uparrow p_1} \Rightarrow h \cdot \delta^1_{p_0\uparrow p_1}$ such that

$$\begin{tikzcd}
 & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
b \arrow[dr, "h'_1"'] \arrow[rr, phantom, "\xRightarrow{\mathrm{id}_{h'} * \alpha^{p_0\uparrow p_1}}"] & & b \arrow[dl, bend right, "h'_0"'] \arrow[dl, bend left, "h_0"] \arrow[dl, phantom, "\xRightarrow{\beta_0}"] \\
 & y &
\end{tikzcd}
=
\begin{tikzcd}
 & e \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
b \arrow[dr, bend left, "h_1"] \arrow[dr, bend right, "h'_1"'] \arrow[dr, phantom, "\xRightarrow{\beta_1}"] \arrow[rr, phantom, "\xRightarrow{\mathrm{id}_{h} * \alpha^{p_0\uparrow p_1}}"] & & b \arrow[dl, "h_0"] \\
 & y &
\end{tikzcd}$$

holds, in which, for each $i \in \{1, 2\}$, $h'_i := h' \cdot \delta^i_{p_0\uparrow p_1}$ and $h_i := h \cdot \delta^i_{p_0\uparrow p_1}$. This implies that there is a unique $\beta : h' \Rightarrow h$ such that $\beta * \mathrm{id}_{\delta^0_{p_0\uparrow p_1}} = \beta_0$. ■

Definition 3.4. [Codensity monad] A morphism $p : e \to b$ of a 2-category $\mathbb{A}$ *has the codensity monad* if the right Kan extension $(\mathrm{ran}_p p, \gamma)$ of p along itself exists. Assuming that $\mathbb{A}$ has the codensity monad of p and denoting $\mathrm{ran}_p p$ by t, we consider:

– the 2-cell $m : t^2 \Rightarrow t$ such that the equation

$$\begin{tikzcd}
b \arrow[dd, bend right=60, "t^2"'] \arrow[dd, "t"] \arrow[dd, phantom, shift right=3ex, "\xRightarrow{m}"] & e \arrow[l, "p"'] \arrow[dd, "p"] \arrow[ddl, phantom, "\xRightarrow{\gamma}"] \\
 & \\
 & b
\end{tikzcd}
=
\begin{tikzcd}
b \arrow[dr, "t"'] \arrow[rr, phantom, shift right=2ex, "\xRightarrow{\gamma}"] & & e \arrow[ll, "p"'] \arrow[dl, "p"] \arrow[dd, "p"] \\
 & b \arrow[dr, "t"'] \arrow[r, phantom, "\xRightarrow{\gamma}"] & \\
 & & b
\end{tikzcd}
\qquad (\textit{codensity multiplication})$$

holds;
– the 2-cell $\eta : \mathrm{id}_b \Rightarrow t$ defined by the equation below.

$$
\begin{tikzcd}
b \arrow[dr, "t"] \arrow[dr, bend right=60, "\mathrm{id}_b"'] \arrow[dr, phantom, bend right=30, "\Rightarrow\eta"] & e \arrow[l, "p"'] \arrow[d, "p"] \arrow[dl, phantom, "\Rightarrow\gamma"] \\
 & b
\end{tikzcd}
=
\begin{tikzcd}
b \arrow[dr, "\mathrm{id}_b"'] & e \arrow[l, "p"'] \arrow[d, "p"] \arrow[dl, phantom, "\Rightarrow\mathrm{id}_p"] \\
 & b
\end{tikzcd}
\qquad (\textit{codensity unit})
$$

In this case, by the universal property of the right Kan extension of p along itself, the quadruple $\mathtt{t} = (b, t, m, \eta)$ is a monad called the *codensity monad* of p.

In the situation above, by the *codensity multiplication* and the *codensity unit* equations of the definitions of m and η, it is clear that the pair $(p : e \to b, \gamma : t \cdot p \Rightarrow p)$ satisfies the *algebra associativity and identity* equations w.r.t. the monad $\mathtt{t}$. Hence, assuming that $\mathbb{A}$ has the Eilenberg-Moore object $(\mathtt{u}^{\mathtt{t}} : b^{\mathtt{t}} \to b, \mu^{\mathtt{t}})$ of the monad $\mathtt{t}$, by the universal property, there is a unique $p^{\mathtt{t}} := p^{(\mathtt{t},\gamma)}$ such that

$$
\begin{tikzcd}
e \arrow[rr, "p"] \arrow[dr, "p^{\mathtt{t}}"'] & & b \\
 & b^{\mathtt{t}} \arrow[ur, "\mathtt{u}^{\mathtt{t}}"'] &
\end{tikzcd}
\qquad (\textit{semantic factorization})
$$

commutes and $\mu^{\mathtt{t}} * \mathrm{id}_{p^{\mathtt{t}}} = \gamma$. In this case, we can consider, for each object x, the image of the *semantic factorization* by the representable 2-functor $\mathbb{A}(x, -) : \mathbb{A} \to \mathsf{Cat}$, getting the factorization

$$\mathbb{A}(x, p) = \mathbb{A}(x, \mathtt{u}^{\mathtt{t}}) \circ \mathbb{A}(x, p^{\mathtt{t}}),$$

which coincides up to isomorphism with the factorization $\mathbb{A}(x, p)$ induced by the pair $(\mathbb{A}(x, p), \mathbb{A}(x, \gamma))$ and the universal property of the Eilenberg-Moore category $\mathbb{A}(x, b)^{\mathbb{A}(x,\mathtt{t})}$ of the monad $\mathbb{A}(x, \mathtt{t})$. That is to say, the commutative triangle

$$\begin{tikzcd}
\mathbb{A}(x,e) \arrow[rr, "\mathbb{A}(x,p)"] \arrow[dr, "\mathbb{A}(x,p)^{(\mathbb{A}(x,\mathtt{t}),\mathbb{A}(x,\gamma))}"'] & & \mathbb{A}(x,b) \\
& \mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})} \arrow[ur, "\mathtt{u}^{\mathbb{A}(x,\mathtt{t})}"'] &
\end{tikzcd}$$

(*factorization of* $\mathbb{A}(x,p)$ *induced by* $(\mathbb{A}(x,p),\mathbb{A}(x,\gamma))$ *and* $\mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})}$) which is given by

$$\begin{aligned}
\mathbb{A}(x,p)^{(\mathbb{A}(x,\mathtt{t}),\mathbb{A}(x,\gamma))} : \quad \mathbb{A}(x,e) &\to \mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})} \\
g &\mapsto (p\cdot g, \gamma * \mathrm{id}_g) \\
\chi : g \Rightarrow g' &\mapsto \mathrm{id}_p * \chi
\end{aligned}$$

$$\begin{aligned}
\mathtt{u}^{\mathbb{A}(x,\mathtt{t})} : \quad \mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})} &\to \mathbb{A}(x,b) \\
(f,\beta) &\mapsto f \\
\xi &\mapsto \xi
\end{aligned}$$

Remark 3.5. It is clear that, if p is a morphism of $\mathbb{A}$ that has the codensity monad $\mathtt{t}$, the *factorization of* $\mathbb{A}(x,p)$ *induced by* $(\mathbb{A}(x,p),\mathbb{A}(x,\gamma))$ *and* $\mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})}$ exists, even if $\mathbb{A}$ does not have the Eilenberg-Moore object of $\mathtt{t}$, since Cat has Eilenberg-Moore objects (categories of algebras).

Remark 3.6. [Duality: op-codensity monad] The codual notion of the notion of codensity monad is that of *density comonad*, which is induced by the left Kan extension of the morphism along itself, assuming its existence.

The dual notion is herein called *op-codensity monad*. Notice that, if it exists, the op-codensity monad of a morphism is induced by the right lifting of the morphism through itself. Finally, of course, we have also the codual notion of the op-codensity monad, called herein the *op-density comonad*.

Therefore, we also have factorizations: assuming the existence of the Kleisli object of the op-codensity monad of a morphism, we get the *op-semantic factorization*. Codually, we have the *co-semantic factorization* of a morphism that has the density comonad, provided that the 2-category has its co-Eilenberg-Moore object.

3.3. Right adjoint morphism. Recall that an adjunction inside a 2-category $\mathbb{A}$ is a quadruple

$$(l : b \to e, p : e \to b, \varepsilon : lp \Rightarrow \mathrm{id}_e, \eta : \mathrm{id}_b \Rightarrow pl)$$

in which l, p are 1-cells and ε, η are 2-cells of $\mathbb{A}$ satisfying the *triangle identities.* This means that

$$\begin{tikzcd}
b \arrow[r, "l"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\eta}{\Longrightarrow}\quad", near start] & e \arrow[dl, "p"] \arrow[d, equal] \\
b \arrow[r, "l"'] \arrow[ur, phantom, "\quad\overset{\varepsilon}{\Longrightarrow}", near start] & e
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
b \arrow[d, equal] \arrow[dr, phantom, "\overset{\eta}{\Longrightarrow}\quad", near start] & e \arrow[l, "p"'] \arrow[d, equal] \\
b \arrow[ur, "l"] & e \arrow[l, "p"] \arrow[ul, phantom, "\quad\overset{\varepsilon}{\Longrightarrow}", near start]
\end{tikzcd}$$

are the identities $\mathrm{id}_l : l \Rightarrow l$ and $\mathrm{id}_p : p \Rightarrow p$. In this case, p is right adjoint to l and we denote the adjunction by $(l \dashv p, \varepsilon, \eta) : b \to e$.

If $(l \dashv p, \varepsilon, \eta) : b \to e$ is an adjunction in a 2-category $\mathbb{A}$, p has the codensity monad and the op-density comonad. More precisely, in this case, the pair $(pl, \mathrm{id}_p * \varepsilon)$ is the right Kan extension of p along itself and $(lp, \eta * \mathrm{id}_p)$ is the left lifting of p through itself. Hence, the codensity monad of p coincides with the monad $\mathtt{t} = (b, pl, \mathrm{id}_p * \varepsilon * \mathrm{id}_l, \eta)$ induced by the adjunction, while the op-density comonad coincides with the comonad $(e, lp, \mathrm{id}_l * \eta * \mathrm{id}_p, \varepsilon)$ induced by the adjunction. Codually, if $(l \dashv p, \varepsilon, \eta) : b \to e$ is an adjunction, the density comonad and the op-codensity monad induced by $l : b \to e$ are the same of those induced by the adjunction.

Assuming the existence of the Eilenberg-Moore object of the monad (codensity monad $\mathtt{t}$) induced by the adjunction $(l \dashv p, \varepsilon, \eta)$, the semantic factorization is the usual factorization of the right adjoint morphism through the object of algebras. Dually and codually, assuming the existence of the suitable weighted limits and colimits, we get all the four usual factorizations of l and p. For instance, the op-semantic factorization of $l : b \to e$ gives the factorization

$$\begin{tikzcd}
b \arrow[rr, "l"] \arrow[dr, "\mathbb{l}_{(pl, \mathrm{id}_p * \varepsilon * \mathrm{id}_l, \eta)}"'] & & e \\
& b_{(pl, \mathrm{id}_p * \varepsilon * \mathrm{id}_l, \eta)} \arrow[ur, "l_{\mathtt{t}}"'] &
\end{tikzcd}$$

induced by the universal property of the Kleisli object $b_{\mathtt{t}}$ of the monad $(b, pl, \mathrm{id}_p * \varepsilon * \mathrm{id}_l, \eta)$, while the co-semantic factorization and the coop-semantic factorization give the usual factorizations w.r.t. the object of coalgebras and the co-Kleisli object of the comonad $(e, lp, \mathrm{id}_l * \eta * \mathrm{id}_p, \varepsilon)$.

Definition 3.7. [Preservation of a Kan extension] Let $\left(\mathrm{ran}_g f, \gamma^{\mathrm{ran}_g f}\right)$ be the right Kan extension of $f : z \to y$ along g in a 2-category $\mathbb{A}$. A morphism $\delta : y \to y'$ *preserves the right Kan extension* $\mathrm{ran}_g f$ if $\left(\delta \cdot \mathrm{ran}_g f, \mathrm{id}_\delta * \gamma^{\mathrm{ran}_g f}\right)$ gives the right Kan extension of the morphism $\delta \cdot f$ along g. Furthermore, the right Kan extension $\left(\mathrm{ran}_g f, \gamma^{\mathrm{ran}_g f}\right)$ is *absolute* if it is preserved by any morphism with domain in y.

Remark 3.8. [Duality: respecting liftings] The dual notion of that of preservation of a Kan extension is that of *respecting* a lifting. If a pair $\left(\mathrm{rlift}_g f, \boldsymbol{\gamma}^{\mathrm{rlift}_g f}\right)$ is the right lifting of f through g, a morphism $\delta : y' \to y$ *respects the right lifting of f through g* if $\left((\mathrm{rlift}_g f) \cdot \delta, \boldsymbol{\gamma}^{\mathrm{rlift}_g f} * \mathrm{id}_\delta\right)$ is the right lifting of $f \cdot \delta$ through g.

Remark 3.9. In some contexts, such in the case of 2-categories endowed with Yoneda structures [58], we have a stronger notion of Kan extensions: pointwise Kan extensions [15, 58]. Although this concept plays a fundamental role in the theory of Kan extensions, we do not use this notion in our main theorem. However, we mention them in our examples and, herein, *a pointwise Kan extension in* Cat *is just a Kan extension that is preserved by any representable functor.* See [45, 15] for basic aspects of pointwise Kan extensions in Cat and their constructions via conical (co)limits.

If $(l \dashv p, \varepsilon, \eta) : b \to e$ is an adjunction in a 2-category $\mathbb{A}$, p preserves any right Kan extension with codomain in b. Furthermore:

Theorem 3.10 (Dubuc-Street [15, 58, 17])**.** *If $p : e \to b$ is a morphism in a 2-category $\mathbb{A}$, the following statements are equivalent.*

i) The pair (l, ε) is the right Kan extension of id_e along p and it is preserved by p.

ii) The pair (l, ε) is the right Kan extension of id_e along p and it is absolute.

iii) The morphism p has a left adjoint l, with the counit $\varepsilon : lp \Rightarrow \mathrm{id}_e$.

In particular, if $p : e \to b$ has a left adjoint, then it has the codensity monad and the right Kan extension of p along p is absolute.

Proof: Given an adjunction $(l \dashv p, \varepsilon, \eta) : b \to e$, we have that $\mathbb{A}(p, -) \dashv \mathbb{A}(l, -)$. From this fact, by the definition of right Kan extension, we get that the right Kan extension of any $f : e \to y$ along p is given by $(f \cdot l, \mathrm{id}_f * \varepsilon)$. In particular, we get that (l, ε) is the right Kan extension of id_e along p and it is absolute. This proves that iii) implies ii). In order to complete the proof, since ii) obviously implies i), it is enough to prove i) implies iii).

Assuming that (l, ε) is the right Kan extension of id_e along p and it is preserved by p, we have that $\mathrm{ran}_p p$ exists and is given by $(pl, \mathrm{id}_p * \varepsilon)$. Hence it has the codensity monad and, by the definition of the unit η of the codensity monad (see Definition 3.4), we already have that $(\mathrm{id}_p * \varepsilon) \cdot (\eta * \mathrm{id}_p)$ is the identity on id_p. The other triangle identity follows from the universal property of the right Kan extension (l, ε) of id_e along p and the fact that the 2-cell $\beta = (\varepsilon * \mathrm{id}_l) \cdot (\mathrm{id}_l * \eta)$ is such that $\varepsilon \cdot (\beta * \mathrm{id}_p) = \varepsilon$. ∎

Remark 3.11. [Mate correspondence] Given adjunctions $(l_1 \dashv p_1) := (l_1 \dashv p_1, \varepsilon_1, \eta_1) : b_1 \to e_1$ and $(l_0 \dashv p_0) := (l_0 \dashv p_0, \varepsilon_0, \eta_0) : b_0 \to e_0$ in a 2-category $\mathbb{A}$, recall that we have the *mate correspondence* [31, 41, 40]. More precisely, given 1-cells $h_b : b_0 \to b_1$ and $h_e : e_0 \to e_1$ of $\mathbb{A}$, there is a bijection

$$\mathbb{A}(e_0, b_1)(h_b \cdot p_0, p_1 \cdot h_e) \cong \mathbb{A}(b_0, e_1)(l_1 \cdot h_b, h_e \cdot l_0)$$

defined by

$$\begin{tikzcd}
& e_0 \arrow[dl, "p_0"'] \arrow[d, "h_e"] & & b_0 \arrow[r, "l_0"] \arrow[d, equal] \arrow[dr, phantom, "\xRightarrow{\eta_0}"] & e_0 \arrow[dl, "p_0"'] \arrow[d, "h_e"] \\
b_0 \arrow[r, phantom, "\xRightarrow{\beta}"] \arrow[d, "h_b"'] & e_1 \arrow[dl, "p_1"] & \mapsto & b_0 \arrow[r, phantom, "\xRightarrow{\beta}"] \arrow[d, "h_b"'] & e_1 \arrow[dl, "p_1"'] \arrow[d, equal] \\
b_1 & & & b_1 \arrow[r, "l_1"'] \arrow[ur, phantom, "\xRightarrow{\varepsilon_1}"] & e_1
\end{tikzcd}$$

whose inverse is given by $\beta' \mapsto (\mathrm{id}_{p_1 h_e} * \varepsilon_0) \cdot (\mathrm{id}_{p_1} * \beta' * \mathrm{id}_{p_0}) \cdot (\eta_1 * \mathrm{id}_{h_b p_0})$. The image of a 2-cell $\beta : h_b \cdot p_0 \Rightarrow p_1 \cdot h_e$ by the isomorphism $\mathbb{A}(e_0, b_1)(h_b \cdot p_0, p_1 \cdot h_e) \cong \mathbb{A}(b_0, e_1)(l_1 \cdot h_b, h_e \cdot l_0)$ above is called the mate of β under the adjunction $l_0 \dashv p_0$ and $l_1 \dashv p_1$.

4. Main theorems

Let $\mathbb{A}$ be a 2-category and $p : e \to b$ a morphism of $\mathbb{A}$. Throughout this section, we assume that p has the codensity monad $\mathtt{t} = (b, t, m, \eta)$ and $\mathbb{A}$ has the higher cokernel $\mathcal{H}_p : \Delta_{\mathrm{Str}} \to \mathbb{A}$ of p. We use the same notation of 3.4 for the codensity monad of p, while we use the same notation of 2.5 and 2.3 for the higher cokernel of p.

Proposition 4.1. *The morphism $\delta^0 : b \to b \uparrow_p b$ is such that the right Kan extension of* id_b *along δ^0 exists. Moreover, it is given by the pair $(\ell, \mathrm{id}_{\mathrm{id}_b})$ in which ℓ is the unique morphism such that the equations*

$$\ell \cdot \delta^0 = \mathrm{id}_b, \quad \ell \cdot \delta^1 = t, \quad \mathrm{id}_\ell * \alpha = \gamma$$

hold.

Proof: Indeed, by the universal property of the opcomma object $b \uparrow_p b$, there is a unique morphism $\ell : b \uparrow_p b \to b$ such that the equations above hold.

By Lemma 3.3, since (t, γ) is the right Kan extension of p along itself, we get that (ℓ, γ) is the right Kan extension of $\ell \cdot \delta^0 = \mathrm{id}_b$ along δ^0. ∎

Proposition 4.2 (Condition)**.** *The right Kan extension (t, γ) of p along itself is preserved by $\delta^0 : b \to b \uparrow_p b$ if and only if ℓ is left adjoint to δ^0. In this case, we have an adjunction*

$$(\ell \dashv \delta^0, \mathrm{id}_{\mathrm{id}_b}, \overline{\eta}) : b \uparrow_p b \to b.$$

Proof: Indeed, by Lemma 3.3, $(\delta^0 \cdot \ell, \mathrm{id}_{\delta^0})$ is the right Kan extension of δ^0 along δ^0 if and only if $(\delta^0 \cdot \mathrm{ran}_p p, \mathrm{id}_{\delta^0} * \gamma)$ is the right Kan extension of $\delta^0 \cdot p$ along p.

By Proposition 4.1 and Theorem 3.10, we know that δ^0 preserves the right Kan extension $(\ell, \mathrm{id}_{\mathrm{id}_b})$ of id_b along δ^0 if and only if $\ell \dashv p$.

In this case, the counit is indeed given by the universal 2-cell of $\mathrm{ran}_{\delta^0}\mathrm{id}_b$: that is to say, $\mathrm{id}_{\mathrm{id}_b}$. ∎

Remark 4.3. [Right adjoint morphism] By the Dubuc-Street Theorem (Theorem 3.10), if $\mathfrak{p} : \mathfrak{e} \to \mathfrak{b}$ is a right adjoint morphism of the 2-category $\mathbb{A}$ and $\mathbb{A}$ has the higher cokernel of $\mathfrak{p}$, then $\mathfrak{p}$ satisfies the condition of Proposition 4.2, since, in this case, the right Kan extension $\mathrm{ran}_{\mathfrak{p}}\mathfrak{p}$ exists and is absolute. More particularly, since Cat has the higher cokernel of any functor, any right adjoint functor satisfies the condition of Proposition 4.2.

Remark 4.4. [Example of a left adjoint functor that does not satisfy the condition] Even Cat has morphisms that do not satisfy the condition of Proposition 4.2. For instance, the inclusion of the domain $d^1 : \mathsf{1} \to \mathsf{2}$ has the codensity monad which is actually given by a pointwise right Kan extension:

more precisely $\mathrm{id}_2 : 2 \to 2$ with the unique 2-cell (natural transformation) $d^1 \Rightarrow d^1$. However, in this case, $\delta^0_{d^1 \uparrow d^1}$ is the inclusion

$$\begin{tikzcd} 0 \arrow[d] & & 0 \arrow[d] & 0' \arrow[l] \arrow[d] \\ 1 & \mapsto & 1 & 1' \end{tikzcd}$$

which does not preserve the terminal object, since 2 has terminal object and $d^1 \uparrow d^1$ does not. Hence $\delta^0_{d^1 \uparrow d^1}$ does not have a left adjoint. Actually, it even does not have a codensity monad.

It should be noted that d^1 is left adjoint to s^0 and, hence, it does satisfy the codual of the condition of Proposition 4.2. Explicitly, we have that $\mathrm{lan}_{d^1} d^1$ is given by the functor $d^1 s^0$ with the only natural transformation $d^1 s^0 d^1 \Rightarrow d^1$ and this left Kan extension is absolute.

Remark 4.5. [Example of a functor that does satisfy the condition] By Remark 4.3, any right adjoint morphism satisfies Proposition 4.2. The converse is false, that is to say, the condition of Proposition 4.2 does not imply the existence of a left adjoint.

There are simple counterexamples in Cat. In order to construct such an example, recall that any functor $\iota_e : e \to 1$ has the codensity monad given by a pointwise Kan extension. But it does have a left (right) adjoint if and only if e has initial (terminal) object.

Hence, for instance, if we consider the *thin* category $\mathbb{R}$ corresponding to the usual preordered set of real numbers, the only functor $\iota_{\mathbb{R}} : \mathbb{R} \to 1$ does not have any adjoint. However, it is clear that every functor $1 \to b$ preserves the (conical) limit of $\mathbb{R} \to 1$ and, hence, any such a functor does preserve $\mathrm{ran}_{\iota_{\mathbb{R}}} \iota_{\mathbb{R}}$ (see [9, 44]). In particular, $\iota_{\mathbb{R}}$ does satisfy Proposition 4.2.

Remark 4.6. [Counterexample for the dual and the codual] Neither the condition of Proposition 4.2 nor the codual is satisfied by the only functor $\iota_{1 \sqcup 1} : 1 \sqcup 1 \to 1$. More precisely, this functor has a codensity monad and a density monad: both are given by pointwise Kan extensions and they are of course the identity on 1. Therefore it is clear that a functor $1 \to b$ preserves $\mathrm{ran}_{\iota_{1\sqcup 1}} \iota_{1 \sqcup 1}$ ($\mathrm{lan}_{\iota_{1\sqcup 1}} \iota_{1 \sqcup 1}$) if and only if $1 \to b$ preserves binary products (binary coproducts) which does happen if and only if the image of $1 \to b$ is a preterminal (preinitial) object (see [9, 44] for instance). The opcomma category of $\iota_{1 \sqcup 1}$ along itself is the category with two distinct objects and two parallel arrows between them: hence it does not have any preterminal or preinitial objects. This shows that neither $\mathrm{ran}_{\iota_{1\sqcup 1}}\iota_{1\sqcup 1}$ nor $\mathrm{lan}_{\iota_{1\sqcup 1}}\iota_{1\sqcup 1}$ is preserved by any functor $\mathbf{1} \to \iota_{1\sqcup 1} \uparrow \iota_{1\sqcup 1}$.

Remark 4.7. [Explicit definition of the unit $\overline{\eta}$] Assuming that p satifies the condition of Proposition 4.2 and denoting by $\overline{\alpha} : t \cdot \delta^1 \Rightarrow t \cdot \delta^0$ the unique 2-cell of $\mathbb{A}$ such that the equation

$$\begin{tikzcd}
b \arrow[d, "t"'] \arrow[dr, "t"] & e \arrow[l, "p"'] \arrow[d, "p"] \arrow[dl, phantom, "\xRightarrow{\gamma}"] \\
b \arrow[r, phantom, "\xRightarrow{\overline{\alpha}}"] \arrow[dr, "\delta^1"'] & b \arrow[d, "\delta^0"] \\
 & b \uparrow_p b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[d, "t"'] \arrow[dr, phantom, "\xRightarrow{\gamma}"] & e \arrow[l, "p"'] \arrow[dl, "p"] \arrow[d, "p"] \\
b \arrow[r, phantom, "\xRightarrow{\alpha}"] \arrow[dr, "\delta^1"'] & b \arrow[d, "\delta^0"] \\
 & b \uparrow_p b
\end{tikzcd}$$

holds, we have that the unit $\overline{\eta} : \mathrm{id}_{b\uparrow_p b} \Rightarrow \delta^0 \cdot \ell$ of the adjunction $(\ell \dashv \delta^0, \mathrm{id}_{\mathrm{id}_b}, \overline{\eta})$ is such that

$$\overline{\eta} * \mathrm{id}_{\delta^1} = \overline{\alpha} \cdot (\mathrm{id}_{\delta^1} * \eta)\,.$$

We prove this equality as follows: firstly, by the universal property of the opcomma object $b \uparrow_p b$ of p along itself, there is a unique 2-cell $\overline{\eta}' : \mathrm{id}_{b\uparrow_p b} \Rightarrow \delta^0 \cdot \ell$ such that the equations

$$\overline{\eta}' * \mathrm{id}_{\delta^0} = \mathrm{id}_{\mathrm{id}_{\delta^0}} \quad \text{and} \quad \overline{\eta}' * \mathrm{id}_{\delta^1} = \overline{\alpha} \cdot (\mathrm{id}_{\delta^1} * \eta) \qquad (\textit{definition of } \overline{\eta}')$$

are satisfied, since the equation

$$\begin{tikzcd}
 & e \arrow[dl, "p"'] \arrow[dr, "p"] & \\
b \arrow[rr, phantom, "\xRightarrow{\alpha}"] \arrow[dr, "\delta^1"'] & & b \arrow[dl, "\delta^0"] \\
 & b \uparrow_p b &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & e \arrow[dl, "p"'] \arrow[dr, "p"] & \\
b \arrow[rr, phantom, "\xRightarrow{\mathrm{id}_{\delta^0} * \gamma}"] \arrow[dr, bend left, "\delta^0 \cdot t"] \arrow[dr, bend right, "\delta^1"'] \arrow[dr, phantom, "\xRightarrow{\overline{\alpha} \cdot \left(\mathrm{id}_{\delta^1} * \eta\right)}"'] & & b \arrow[dl, "h_0"] \\
 & b \uparrow_p b &
\end{tikzcd}$$

holds. Indeed, by the definition of $\overline{\alpha}$, the right side of the equation above is equal to

$$\begin{tikzcd}
b \arrow[d, "t"] \arrow[d, bend right=70, equal] \arrow[d, phantom, shift right=5ex, "\overset{\eta}{\Longrightarrow}"] \arrow[dr, phantom, shift left=2ex, "\overset{\gamma}{\Longrightarrow}"] & e \arrow[l, "p"'] \arrow[dl, "p"] \arrow[d, "p"] \\
b \arrow[r, phantom, "\overset{\alpha}{\Longrightarrow}"] \arrow[dr, "\delta^1"'] & b \arrow[d, "\delta^0"] \\
& b \uparrow_p b
\end{tikzcd}$$

which, by the definition of η, is equal to $\alpha : \delta^1 \cdot p \Rightarrow \delta^0 \cdot p$.

Secondly, by Theorem 3.10 and Proposition 4.1, since we are assuming the condition of Proposition 4.2, we have that $\mathrm{id}_{\mathrm{id}_b}$ is the counit of the adjunction $\ell \dashv p$. Thus the unit $\overline{\eta}$ of $\ell \dashv p$ is the unique 2-cell $\mathrm{id}_{b\uparrow_p b} \Rightarrow \delta^0 \cdot \ell$ such that $\overline{\eta} * \mathrm{id}_{\delta^0} = \mathrm{id}_{\delta^0}$.

Finally, since $\overline{\eta}' : \mathrm{id}_{b\uparrow_p b} \Rightarrow \delta^0 \cdot \ell$ is such that $\overline{\eta}' * \mathrm{id}_{\delta^0} = \mathrm{id}_{\delta^0}$, we get that $\overline{\eta} = \overline{\eta}'$. Therefore $\overline{\eta} * \mathrm{id}_{\delta^1} = \overline{\alpha} \cdot (\mathrm{id}_{\delta^1} * \eta)$ and $\overline{\eta}$ is defined by the equations of the *definition of* $\overline{\eta}'$.

Proposition 4.8. *Assume that $p : e \to b$ satisfies the condition of Proposition 4.2. There is an adjunction $\left(\ell_* \dashv \partial^0 \cdot \delta^0, \mathrm{id}_{\mathrm{id}_b}, \rho\right) : b \uparrow_p b \uparrow_p b \to b$. Moreover, the equations*

$$\ell_* \cdot \partial^2 = t \cdot \ell, \qquad \ell_* \cdot \partial^0 = \ell, \qquad \rho * \mathrm{id}_{\partial^0} = \mathrm{id}_{\partial^0} * \overline{\eta}$$

are satisfied. Furthermore, the 2-cell ρ_2 given by the pasting

$$\begin{tikzcd}[column sep=huge]
b \uparrow_p b \arrow[d, bend right=80, "\partial^2"'] \arrow[d, bend right=20] \arrow[d, bend left=20] \arrow[d, bend left=80, "\partial^0\delta^0\ell\delta^1\ell = \partial^0\,\delta^0\,t\,\ell"] \arrow[d, phantom, "\partial^2\delta^0\ell = \partial^0\delta^1\ell"] \arrow[d, phantom, shift right=14ex, "\overset{\mathrm{id}_{\partial^2} * \overline{\eta}}{\Longrightarrow}"] \arrow[d, phantom, shift left=14ex, "\overset{\mathrm{id}_{\partial^0} * \overline{\eta} * \mathrm{id}_{\delta^1 \ell}}{\Longrightarrow}"] \\
b \uparrow_p b \uparrow_p b
\end{tikzcd}$$

is equal to $\rho * \mathrm{id}_{\partial^2}$.

Proof: Firstly, in fact, by the universal property of the pushout $b \uparrow_p b \uparrow_p b$ of δ^0 along δ^1, we have that:

- there is a unique morphism $\ell_* : b \uparrow_p b \uparrow_p b \to b$ such that $\ell_* \cdot \partial^2 = t \cdot \ell$ and $\ell_* \cdot \partial^0 = \partial^0\,\delta^0 \cdot \ell$, since $\ell \cdot \delta^1 = t = t \cdot \mathrm{id}_b = t \cdot \ell \cdot \delta^0$;

– there is a unique $\rho : \mathrm{id}_{b\uparrow_p b\uparrow_p b} \Rightarrow \partial^0\delta^0 \cdot \ell_*$ such that $\rho * \mathrm{id}_{\partial^0} = \mathrm{id}_{\partial^0} * \overline{\eta}$ and $\rho * \mathrm{id}_{\partial^2} = \rho_2$, because $\rho_2 * \mathrm{id}_{\delta^0}$ is equal to the composition of 2-cells

$$\partial^2 \cdot \delta^0 = \partial^0 \cdot \delta^1 \xRightarrow{\mathrm{id}_{\partial^0} * \overline{\eta} * \mathrm{id}_{\delta^1}} \partial^0\,\delta^0 \cdot \ell\,\delta^1 = \partial^0\,\delta^0 \cdot t$$

since $\overline{\eta} * \mathrm{id}_{\delta^0} = \mathrm{id}_{\partial^2\,\delta^0}$.

Secondly, $\rho * \mathrm{id}_{\partial^0\delta^0} = \rho * \mathrm{id}_{\partial^0} * \mathrm{id}_{\delta^0} = \mathrm{id}_{\partial^0} * \overline{\eta} * \mathrm{id}_{\delta^0}$, which, since $\overline{\eta} * \mathrm{id}_{\delta^0} = \mathrm{id}_{\delta^0}$, is the horizontal composition of identities and, hence, it is an identity. This proves one of the triangle identities for the adjunction $\ell_* \dashv \partial^0\delta^0$.

Finally, by the universal property of the pushout $b \uparrow_p b \uparrow_p b$ of δ^0 along δ^1, the 2-cell $\mathrm{id}_{\ell_*} * \rho$ is the identity on ℓ_*, since:

– $(\mathrm{id}_{\ell_*} * \rho) * \mathrm{id}_{\partial^0}$ is equal to

$$\mathrm{id}_{\ell_*} * \rho * \mathrm{id}_{\partial^0} = \mathrm{id}_{\ell_*} * \mathrm{id}_{\partial^0} * \overline{\eta} = \mathrm{id}_{\ell} * \overline{\eta} = \mathrm{id}_{\ell} = \mathrm{id}_{\ell_*\partial^0};$$

– $(\mathrm{id}_{\ell_*} * \rho) * \mathrm{id}_{\partial^2} = \mathrm{id}_{\ell_*} * \rho_2$ is, by the definition of ℓ_* and ρ_2, equal to

$$\begin{tikzcd}[column sep=huge, row sep=huge]
b \uparrow_p b \arrow[d, bend right=80, "t\cdot\ell"', ""{name=L}] \arrow[d, "t\cdot\ell" description, ""{name=M}] \arrow[d, bend left=80, "t\cdot\ell", ""{name=R}] \\
b
\arrow[Rightarrow, from=L, to=M, "\mathrm{id}_t * \mathrm{id}_\ell * \overline{\eta}"]
\arrow[Rightarrow, from=M, to=R, "\mathrm{id}_\ell * \overline{\eta} * \mathrm{id}_{\delta^1 \ell}"]
\end{tikzcd}$$

which is a vertical composition of identities, since $\mathrm{id}_\ell * \overline{\eta}$ is equal to the identity by the triangle identity of the adjunction $(\ell \dashv \delta^0, \mathrm{id}_{\mathrm{id}_b}, \overline{\eta})$.

This completes the proof that $(\ell_* \dashv \partial^0\delta^0, \mathrm{id}_{\mathrm{id}_b}, \rho)$ is an adjunction. ■

In order to prove Theorem 4.11, we consider the 2-cells defined in Lemma 4.10. Before defining them, it should be noted that:

Lemma 4.9 ($\ell_* \cdot \partial^1$)**.** *Assume that p satisfies the condition of Proposition 4.2. The morphism $\ell_* \cdot \partial^1$ is the unique morphism such that the equations*

$$\left(\ell_*\partial^1\right) \cdot \delta^1 = t^2, \quad \left(\ell_*\partial^1\right) \cdot \delta^0 = \mathrm{id}_b \quad \textit{and} \quad \mathrm{id}_{\ell_*\partial^1} * \alpha = (\mathrm{id}_t * \gamma) \cdot \gamma$$

hold.

Proof: In fact, by the definitions of ℓ (see Proposition 4.1) and ℓ_* (see Proposition 4.8), the equations

$$\left(\ell_*\partial^1\right) \cdot \delta^0 = \ell_* \cdot \partial^0\delta^0 = \ell \cdot \delta^0 = \mathrm{id}_b, \qquad \left(\ell_*\partial^1\right) \cdot \delta^1 = \ell_* \cdot \partial^2\delta^1 = t\ell \cdot \delta^1 = t^2,$$

$$\mathrm{id}_{\ell_*} * \mathrm{id}_{\partial^0} * \alpha = \mathrm{id}_\ell * \alpha = \gamma \quad \text{and} \quad \mathrm{id}_{\ell_*} * \mathrm{id}_{\partial^2} * \alpha = \mathrm{id}_{t\ell} * \alpha = \mathrm{id}_t * \gamma$$

hold. Therefore, by the definition of ∂^1 (see 2.5) and by the universal property of the opcomma object $b \uparrow_p b$ of p along itself, we get the result. ∎

Lemma 4.10 (θ and λ). *Assume that $p : e \to b$ satisfies the condition of Proposition 4.2. There are 2-cells*

$$\theta : \mathfrak{s}^0 \Rightarrow \ell\ : b \uparrow_p b \to b, \qquad \lambda : \ell_* \cdot \partial^1 \Rightarrow \ell\ : b \uparrow_p b \to b$$

such that the equations

$$\theta * \mathrm{id}_{\delta^1} = \eta, \quad \theta * \mathrm{id}_{\delta^0} = \mathrm{id}_{\mathrm{id}_b}, \quad \lambda * \mathrm{id}_{\delta^1} = m, \quad \lambda * \mathrm{id}_{\delta^0} = \mathrm{id}_{\mathrm{id}_b}$$

are satisfied.

Proof: In fact, by the universal property of the opcomma object $b \uparrow_p b$ of p along p:

- there is a unique 2-cell $\theta : \mathfrak{s}^0 \Rightarrow \ell$ such that $\theta * \mathrm{id}_{\delta^1} = \eta$ and $\theta * \mathrm{id}_{\delta^0} = \mathrm{id}_{\mathrm{id}_b}$, since

  $$(\mathrm{id}_\ell * \alpha) \cdot (\eta * \mathrm{id}_p) = \gamma \cdot (\eta * \mathrm{id}_p) = \mathrm{id}_p = \mathrm{id}_{\mathfrak{s}^0} * \alpha$$

  by the definitions of ℓ, η and $\mathfrak{s}^0$;
- there is a unique 2-cell $\lambda : \ell_* \cdot \partial^1 \Rightarrow \ell$ such that $\lambda * \mathrm{id}_{\delta^1} = m$ and $\lambda * \mathrm{id}_{\delta^0} = \mathrm{id}_{\mathrm{id}_b}$, since

  $$(\mathrm{id}_\ell * \alpha) \cdot (m * \mathrm{id}_p) = \gamma \cdot (m * \mathrm{id}_p) = \gamma \cdot (\mathrm{id}_t * \gamma) = \mathrm{id}_{\ell_* \cdot \partial^1} * \alpha$$

  by the definition of $m : t^2 \Rightarrow t$ and Lemma 4.9.

∎

Theorem 4.11. *Assume that p satisfies the condition of Proposition 4.2. We have that the diagrams*

$$\begin{tikzcd}
\mathbb{A}(x,e) \arrow[rr, "\mathbb{A}(x,p)"] \arrow[dr, "\mathbb{A}(x,p)^{(\mathbb{A}(x,\mathcal{H}_p-),\mathbb{A}(x,\alpha))}"'] & & \mathbb{A}(x,b) \\
& \lim(\mathfrak{D}, \mathbb{A}(x,\mathcal{H}_p-)) \arrow[ur, "\mathfrak{d}^{(\mathfrak{D},\mathbb{A}(x,\mathcal{H}_p-))}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}(x,e) \arrow[rr, "\mathbb{A}(x,p)"] \arrow[dr, "\mathbb{A}(x,p)^{(\mathbb{A}(x,\mathtt{t}),\mathbb{A}(x,\gamma))}"'] & & \mathbb{A}(x,b) \\
& \mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})} \arrow[ur, "\mathfrak{u}^{\mathbb{A}(x,\mathtt{t})}"'] &
\end{tikzcd}$$

are isomorphic (2-naturallly in x), in which the first diagram is the factorization of $\mathbb{A}(x,p)$ induced by the pair $(\mathbb{A}(x,p), \mathbb{A}(x,\alpha))$ and $\lim(\mathfrak{D}, \mathbb{A}(x,\mathcal{H}_p-))$, while the second diagram is the factorization of $\mathbb{A}(x,p)$ induced by $(\mathbb{A}(x,p), \mathbb{A}(x,\gamma))$ and $\mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})}$.

Proof: Recall that: (1) the first diagram is the factorization of $\mathbb{A}(x, p)$ induced by the pair $(\mathbb{A}(x, p), \mathbb{A}(x, \alpha))$ and the universal property of the lax descent category of

$$\mathbb{A}(x, \mathcal{H}_p-) : \Delta_{\mathrm{Str}} \to \mathsf{Cat};$$

and (2) the second diagram is the factorization of $\mathbb{A}(x, p)$ induced by the pair $(\mathbb{A}(x, p), \mathbb{A}(x, \gamma))$ and the universal property of the Eilenberg-Moore category $\mathbb{A}(x, b)^{\mathbb{A}(x,\mathtt{t})}$ of the monad $\mathbb{A}(x, \mathtt{t})$.

Firstly, observe that, since $(\ell \dashv \delta^0, \mathrm{id}_{\mathrm{id}_b}, \overline{\eta}) : b \uparrow_p b \to b$ is an adjunction by Proposition 4.2, for each morphism $h : x \to b$ of $\mathbb{A}$ (*i.e.* for each object of $\mathbb{A}(x, b)$), there is a bijection

$$\mathbb{A}(x, b \uparrow_p b)(\delta^1 \cdot h, \delta^0 \cdot h) \cong \mathbb{A}(x, b)(\ell \cdot \delta^1 \cdot h, \ell \cdot \delta^0 \cdot h) = \mathbb{A}(x, b)(t \cdot h, h)$$

defined by $\beta \mapsto \mathrm{id}_\ell * \beta$, that is to say, the mate correspondence under the identity adjunction $\mathrm{id}_x \dashv \mathrm{id}_x$ and the adjunction $(\ell \dashv \delta^0, \mathrm{id}_{\mathrm{id}_b}, \overline{\eta})$, see Remark 3.11.

Secondly, given an object h of $\mathbb{A}(x, b)$, we prove below that a 2-cell $\beta : \delta^1 \cdot h \Rightarrow \delta^0 \cdot h$ satisfies the *descent associativity* and the *descent identity* w.r.t. $\mathcal{H}_p$ if and only if its corresponding 2-cell $\mathrm{id}_\ell * \beta$ satisfies the *algebra associativity and identity* w.r.t. $\mathtt{t}$.

(1) Observe that, given a 2-cell $\beta : \delta^1 \cdot h \Rightarrow \delta^0 \cdot h$, by the definition of θ and Lemma 4.10, we get that

$$\begin{aligned}
\mathrm{id}_{\mathfrak{s}^0} * \beta &= \mathrm{id}_{\mathfrak{s}^0 \cdot \delta^0 \cdot h} \cdot (\mathrm{id}_{\mathfrak{s}^0} * \beta) \\
&= (\mathrm{id}_{\mathrm{id}_b} * \mathrm{id}_h) \cdot (\mathrm{id}_{\mathfrak{s}^0} * \beta) \\
&= ((\theta * \mathrm{id}_{\delta^0}) * \mathrm{id}_h) \cdot (\mathrm{id}_{\mathfrak{s}^0} * \beta) \\
&= (\theta * \mathrm{id}_{\delta^0 \cdot h}) \cdot (\mathrm{id}_{\mathfrak{s}^0} * \beta) \\
&= \theta * \beta
\end{aligned}$$

which, by the interchange law, is equal to the left side of the equation

$$(\mathrm{id}_\ell * \beta) \cdot (\theta * \mathrm{id}_{\delta^1} * \mathrm{id}_h) = (\mathrm{id}_\ell * \beta) \cdot (\eta * \mathrm{id}_h)$$

which holds by Lemma 4.10. Thus, of course, $(\mathrm{id}_\ell * \beta) \cdot (\eta * \mathrm{id}_h)$ is the identity on h if and only if $\mathrm{id}_{\mathfrak{s}^0} * \beta = (\mathrm{id}_\ell * \beta) \cdot (\eta * \mathrm{id}_h)$ is the identity on h as well. That is to say, β satisfies the *descent identity* w.r.t. $\mathcal{H}_p$ if and only if $\mathrm{id}_\ell * \beta$ satisfies the algebra identity equation w.r.t. $\mathtt{t}$.

(2) Recall the adjunction $(\ell_* \dashv \partial^0\delta^0, \mathrm{id}_{\mathrm{id}_b}, \rho)$ of Proposition 4.8. Given a 2-cell $\beta : \delta^1 \cdot h \Rightarrow \delta^0 \cdot h$, consider the 2-cells defined by the pastings below.

$$\beta_1 \quad = \quad \begin{tikzcd}
 & & x \arrow[dll, "h"'] \arrow[drr, "h"] & & \\
b \arrow[rrrr, phantom, "\overset{\beta}{\Rightarrow}"] \arrow[drr, "\delta^1"] \arrow[ddrr, bend right, "\partial^2\cdot\delta^1"'] & & & & b \arrow[dll, "\delta^0"'] \arrow[ddll, bend left, "\partial^0\cdot\delta^0"] \\
 & = & b \uparrow_p b \arrow[d, "\partial^1"] & = & \\
 & & b \uparrow_p b \uparrow_p b & &
\end{tikzcd}
\qquad
\beta_c \quad = \quad \begin{tikzcd}
 & & x \arrow[dll, "h"'] \arrow[d, "h"] \arrow[drr, "h"] & & \\
b \arrow[rr, phantom, "\overset{\beta}{\Rightarrow}"] \arrow[drr, "\delta^1"'] & & b \arrow[rr, phantom, "\overset{\beta}{\Rightarrow}"] \arrow[d, "\delta^0"] \arrow[drr, "\delta^1"] & & b \arrow[d, "\delta^0"] \\
 & & b \uparrow_p b \arrow[rr, phantom, "="] \arrow[d, "\partial^2"] & & b \uparrow_p b \arrow[dll, "\partial^0"] \\
 & & b \uparrow_p b \uparrow_p b & &
\end{tikzcd}$$

We have that the mates of β_c and β_1 under the identity adjunction $\mathrm{id}_x \dashv \mathrm{id}_x$ and the adjunction $(\ell_* \dashv \partial^0\delta^0, \mathrm{id}_{\mathrm{id}_b}, \rho)$ are respectively equal to $\mathrm{id}_{\ell_*} * \beta_c$ and $\mathrm{id}_{\ell_*} * \beta_1$.

On one hand, since $\ell_* \cdot \partial^2 = t \cdot \ell$ and $\ell_* \cdot \partial^0 = \ell$, the 2-cell $\mathrm{id}_{\ell_*} * \beta_c$ is equal to

$$(\mathrm{id}_\ell * \beta) \cdot (\mathrm{id}_t * (\mathrm{id}_\ell * \beta)) \, .$$

On the other hand, by Lemma 4.10

$$\begin{aligned}
\mathrm{id}_{\ell_*} * \beta_1 &= (\mathrm{id}_{\ell_*\partial^1} * \beta) \\
&= (\mathrm{id}_{\mathrm{id}_b} * \mathrm{id}_h) \cdot (\mathrm{id}_{\ell_*\partial^1} * \beta) \\
&= (\lambda * \mathrm{id}_{\delta^0} * \mathrm{id}_h) \cdot (\mathrm{id}_{\ell_*\partial^1} * \beta)
\end{aligned}$$

which, by the interchange law and Lemma 4.10, is equal to

$$\lambda * \beta = (\mathrm{id}_\ell * \beta) \cdot (\lambda * \mathrm{id}_{\delta^1}) = (\mathrm{id}_\ell * \beta) \cdot m.$$

In order to complete the proof, it should be noted that the 2-cell β satisfies the *descent associativity* w.r.t. $\mathcal{H}_p$ if and only if $\beta_c = \beta_1$ which holds if and only if the mates of β_c and β_1 under the identity adjunction $\mathrm{id}_x \dashv \mathrm{id}_x$ and the adjunction $(\ell_* \dashv \partial^0\delta^0, \mathrm{id}_{\mathrm{id}_b}, \rho)$ are equal.

From what we proved above, this means that the 2-cell β satisfies the *descent associativity* if and only if $(\mathrm{id}_\ell * \beta) \cdot (\mathrm{id}_t * (\mathrm{id}_\ell * \beta)) = (\mathrm{id}_\ell * \beta) \cdot m$, which is precisely the algebra associativity equation w.r.t. $\mathtt{t}$ for $\mathrm{id}_\ell * \beta$.

This implies that the association $(h,\beta)\mapsto(h,\mathrm{id}_\ell{*}\beta)$ gives a bijection between the objects of $\lim(\mathfrak{D},\mathbb{A}(x,\mathcal{H}_p-))$ and $\mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})}$.

Thirdly, given objects (h_1,β_1) and (h_0,β_0) of $\lim(\mathfrak{D},\mathbb{A}(x,\mathcal{H}_p-))$, by the mate under the identity adjunction and $\ell\dashv\delta^0$ correspondence, a 2-cell

$$\xi : h_1 \Rightarrow h_0 : x \to b$$

satisfies the equation

$$
\begin{tikzcd}
 & x \arrow[dl, bend right=50, "h_1"'] \arrow[dl, bend left=20, "h_0"] \arrow[dl, phantom, "\overset{\xi}{\Longrightarrow}"] \arrow[dr, "h_0"] & & & & x \arrow[dl, "h_1"'] \arrow[dr, bend left=50, "h_0"] \arrow[dr, bend right=20, "h_1"'] \arrow[dr, phantom, "\overset{\xi}{\Longrightarrow}"] & \\
b \arrow[dr, "\delta^1"'] \arrow[rr, phantom, "\overset{\beta_0}{\Longrightarrow}"] & & b \arrow[dl, "\delta^0"] & = & b \arrow[dr, "\delta^1"'] \arrow[rr, phantom, "\overset{\beta_1}{\Longrightarrow}"] & & b \arrow[dl, "\delta^0"] \\
 & b\uparrow_p b & & & & b\uparrow_p b &
\end{tikzcd}
$$

if and only if the mate of the left side is equal to the mate of the right side, which means

$$
\begin{tikzcd}
 & x \arrow[dl, bend right=50, "h_1"'] \arrow[dl, bend left=20, "h_0"] \arrow[dl, phantom, "\overset{\xi}{\Longrightarrow}"] \arrow[dr, "h_0"] & & & & x \arrow[dl, "h_1"'] \arrow[dr, bend left=50, "h_0"] \arrow[dr, bend right=20, "h_1"'] \arrow[dr, phantom, "\overset{\xi}{\Longrightarrow}"] & \\
b \arrow[dr, "\delta^1"'] \arrow[rr, phantom, "\overset{\beta_0}{\Longrightarrow}"] & & b \arrow[dl, "\delta^0"] & = & b \arrow[dr, "\delta^1"'] \arrow[rr, phantom, "\overset{\beta_1}{\Longrightarrow}"] & & b \arrow[dl, "\delta^0"] \\
 & b\uparrow_p b \arrow[d, "\ell"] & & & & b\uparrow_p b \arrow[d, "\ell"] & \\
 & b & & & & b &
\end{tikzcd}
$$

which is precisely the condition of being a morphism of algebras in $\mathbb{A}\left(x,\lim(\mathfrak{D},\mathcal{H}_p-)\right)$. In other words, this proves that ξ gives a morphism between (h_1,β_1) and (h_0,β_0) in $\lim(\mathfrak{D},\mathbb{A}(x,\mathcal{H}_p-))$ if and only if it gives a morphism between $(h_1,\mathrm{id}_\ell*\beta_1)$ and $(h_0,\mathrm{id}_\ell*\beta_0)$ in $\mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})}$.

Finally, given the facts above, we can conclude that we actually can define

$$
\begin{array}{rcl}
\lim(\mathfrak{D},\mathbb{A}(x,\mathcal{H}_p-)) & \to & \mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})} \\
(h,\beta) & \mapsto & (h,\mathrm{id}_\ell*\beta) \\
\xi & \mapsto & \xi
\end{array}
$$

which is clearly functorial and, hence, it defines an invertible functor (since it is bijective on objects and fully faithful as proved above). This invertible functor is 2-natural in x giving an isomorphism between the *factorization of*

$\mathbb{A}(x,p)$ *induced by the pair* $(\mathbb{A}(x,p), \mathbb{A}(x,\alpha))$ *and* $\lim(\mathfrak{D}, \mathbb{A}(x, \mathcal{H}_p-))$ and the *factorization of* $\mathbb{A}(x,p)$ *induced by* $(\mathbb{A}(x,p), \mathbb{A}(x,\gamma))$ *and* $\mathbb{A}(x,b)^{\mathbb{A}(x,\mathtt{t})}$. ■

Theorem 4.12 (Main Theorem)**.** *Assume that* $\mathrm{ran}_p p$ *exists and is preserved by the morphism* $\delta^0 : b \to b \uparrow_p b$*. We have that the semantic factorization of* p *is isomorphic to the lax descent factorization induced by the higher cokernel of* p*, either one existing if the other does.*

Proof: It is clearly a direct consequence of Theorem 4.11. ■

Recall that , since the result above works for any 2-category, we have dual results. For instance, we have Theorem 4.13 and Theorem 4.14.

Theorem 4.13 (Codual)**.** *Let* $l : b \to e$ *be a morphism of* $\mathbb{A}$ *satisfying the following conditions:*

(1) $\mathbb{A}$ *has the higher cokernel of* l*;*
(2) the left Kan extension $\mathrm{lan}_l l$ *of* l *along* l *exists (that is to say,* l *has the density comonad);*
(3) the left Kan extension $\mathrm{lan}_l l$ *is preserved by* $\delta^1_{l\uparrow l} : e \to l \uparrow l$*.*

The co-semantic factorization of l *is isomorphic to the lax descent factorization induced by the higher cokernel of* l*, either one existing if the other does.*

Theorem 4.14 (Dual)**.** *Let* $l : b \to e$ *be a morphism of* $\mathbb{A}$ *satisfying the following conditions:*

(1) $\mathbb{A}$ *has the higher kernel of* l*;*
(2) the right lifting of l *through* l *exists (that is to say,* l *has the op-codensity monad);*
(3) the right lifting of l *through* l *is respected by the arrow* $\delta_0^{l\downarrow l} : l \downarrow l \to b$*.*

The op-semantic factorization of l *is isomorphic to the lax codescent factorization induced by the higher kernel of* l *(of Remark 2.6), either one existing if the other does.*

As a consequence of Theorem 4.12 and its duals, by Remark 4.3, we get:

Theorem 4.15 (Adjunction)**.** *Let* $(\mathfrak{l} \dashv \mathfrak{p}, \varepsilon, \eta) : \mathfrak{b} \to \mathfrak{e}$ *be an adjunction in* $\mathbb{A}$*. We have the following:*

(1) if $\mathbb{A}$ *has the higher cokernel of* $\mathfrak{p}$*, then the lax descent factorization induced by the higher cokernel of* $\mathfrak{p}$ *coincides up to isomorphism with*

the usual factorization of $\mathfrak{p}$ through the Eilenberg-Moore object of the induced monad, either one existing if the other does;

(2) if $\mathbb{A}$ has the higher kernel of $\mathfrak{l}$, then the lax codescent factorization induced by the higher kernel of $\mathfrak{l}$ (Remark 2.6) coincides up to isomorphism with the usual factorization of $\mathfrak{l}$ through the Kleisli object of the induced monad, either one existing if the other does;

(3) if $\mathbb{A}$ has the higher cokernel of $\mathfrak{l}$, then the lax descent factorization induced by the higher cokernel of $\mathfrak{l}$ coincides up to isomorphism with the usual factorization of $\mathfrak{l}$ through the co-Eilenberg-Moore object of the induced comonad, either one existing if the other does;

(4) if $\mathbb{A}$ has the higher kernel of $\mathfrak{p}$, then the lax codescent factorization induced by the higher kernel of $\mathfrak{p}$ coincides up to isomorphism with the usual factorization of $\mathfrak{p}$ through the co-Kleisli object.

Remark 4.16. [Examples] Clearly, since $d^1 : \mathsf{1} \to \mathsf{2}$ is a left adjoint functor (morphism of Cat), it satisfies the hypothesis of 3 of Theorem 4.15 (see Remark 4.4). Hence the co-semantic factorization (usual factorization through the category of coalgebras) coincides with the *lax descent factorization induced by the higher cokernel* of d^1. These factorizations are given by $d^1 = d^1 \circ \mathrm{id}_{\mathsf{1}}$.

Although the morphism $\iota_{\mathbb{R}} : \mathbb{R} \to \mathsf{1}$ of Cat (see Remark 4.5) does not satisfy any of the versions of Theorem 4.15, it does satisfy the conditions of Theorem 4.12. Hence the *lax descent factorization induced by the higher cokernel* of $\iota_{\mathbb{R}}$ coincides with the semantic factorization of $\iota_{\mathbb{R}}$. In this case, both the factorizations are given by $\iota_{\mathbb{R}} = \mathrm{id}_{\mathsf{1}} \circ \iota_{\mathbb{R}}$.

Finally, the morphism $\iota_{\mathsf{1}\sqcup\mathsf{1}} : \mathsf{1}\sqcup\mathsf{1} \to \mathsf{1}$ in Cat of Remark 4.6 does not satisfy the hypotheses of Theorem 4.12. By Remark 4.6, since Cat has Eilenberg-Moore objects, higher cokernels and lax descent objects, we have the *lax descent factorization induced by the higher cokernel* of $\iota_{\mathsf{1}\sqcup\mathsf{1}}$ and its semantic factorization. However, in this case, they do not coincide. More precisely, they are respectively given by the commutative triangles below.

$$\begin{tikzcd}
\mathsf{1}\sqcup\mathsf{1} \arrow[rr, "\iota_{\mathsf{1}\sqcup\mathsf{1}}"] \arrow[dr, "\mathrm{id}_{\mathsf{1}\sqcup\mathsf{1}}"'] & & \mathsf{1} \\
& \mathsf{1}\sqcup\mathsf{1} \arrow[ur, "\iota_{\mathsf{1}\sqcup\mathsf{1}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{1}\sqcup\mathsf{1} \arrow[rr, "\iota_{\mathsf{1}\sqcup\mathsf{1}}"] \arrow[dr, "\iota_{\mathsf{1}\sqcup\mathsf{1}}"'] & & \mathsf{1} \\
& \mathsf{1} \arrow[ur, "\mathrm{id}_{\mathsf{1}}"'] &
\end{tikzcd}$$

5. Monadicity and 2-effective monomorphisms

In this section, we show direct consequences of Theorem 4.12 on monadicity. In order to do so, we start by introducing the concept of 2-*effective monomorphism* and *monadicity*.

Henceforth, whenever a 2-category $\mathbb{A}$ has the higher cokernel $\mathcal{H}_p$ of a morphism $p : e \to b$, we use the notation of 2.5 and 2.3. If $\mathbb{A}$ has the higher kernel of a morphism $l : e \to b$, we use the notation of Remark 2.6.

Recall that a morphism $\mathfrak{p} : \mathfrak{e} \to \mathfrak{b}$ of a 2-category $\mathbb{A}$ is an equivalence if there is are a morphism $\mathfrak{l} : \mathfrak{b} \to \mathfrak{e}$ and invertible 2-cells $\mathfrak{l}\mathfrak{p} \Rightarrow \mathrm{id}_{\mathfrak{e}}$, $\mathrm{id}_{\mathfrak{b}} \Rightarrow \mathfrak{p}\mathfrak{l}$. It is a basic coherence result the fact that, whenever we have such a data, we can actually get an adjunction $\mathfrak{l} \dashv \mathfrak{p}$ or an adjunction $\mathfrak{p} \dashv \mathfrak{l}$ with invertible unit and invertible counit (see [34]): these adjunctions are called *adjoint equivalences*.

Definition 5.1. Let $\mathcal{A} : \Delta_{\mathrm{Str}} \to \mathbb{A}$ be a 2-functor. We say that the pair $(p : e \to b, \psi : \mathcal{A}(\mathfrak{d}^1) \cdot p \Rightarrow \mathcal{A}(\mathfrak{d}^0) \cdot p)$, in which p is a morphism and ψ is a 2-cell, is *effective* w.r.t. $\mathsf{lim}(\mathfrak{D}, \mathcal{A})$ if the following statements hold:

- $\mathbb{A}$ has the lax descent object $\mathsf{lim}(\mathfrak{D}, \mathcal{A})$;
- the pair (p, ψ) satisfies the *descent identity* and the *descent associativity* w.r.t. $\mathcal{A}$;
- the induced factorization $p = \mathfrak{d}^{(\mathfrak{D},\mathcal{A})} \circ p^{(\mathcal{A},\psi)}$ is such that $p^{(\mathcal{A},\psi)}$ is an equivalence.

Definition 5.2. [2-effective monomorphism] Let $p : e \to b$ be a morphism of a 2-category $\mathbb{A}$. The morphism p is a 2-*effective monomorphism* of $\mathbb{A}$ if the following statements hold:

- $\mathbb{A}$ has the higher cokernel of p;
- $\mathbb{A}$ has the lax descent object of the higher cokernel $\mathcal{H}_p$;
- the *lax descent factorization induced by the higher cokernel* of p, $p = \mathfrak{d}^p \circ p^{\mathcal{H}}$, is such that $p^{\mathcal{H}}$ is an equivalence, that is to say, (p, α) is effective w.r.t. $\mathcal{H}_p$.

Remark 5.3. The terminology above is motivated by the 1-dimensional case. In a category with suitable pushouts and coequalizers, every morphism p has a factorization induced by the equalizer of the "cokernel pair"

$$b \rightrightarrows b \sqcup_e b$$

of p. If the morphism p is itself the equalizer, p is said to be an *effective monomorphism.*

By Remark 2.6, assuming its existence, the codual of the *lax descent factorization induced by the higher cokernel* of a morphism p gives the same factorization. Hence, we have:

Lemma 5.4 (Self-coduality)**.** *Let p be a morphism of a 2-category $\mathbb{A}$. The morphism p is a 2-effective monomorphism of $\mathbb{A}$ if and only if the morphism corresponding to p is a 2-effective monomorphism in $\mathbb{A}^{\mathrm{co}}$.*

Definition 5.5. [Duality: 2-effective epimorphism] Let $p : e \to b$ be a morphism of a 2-category $\mathbb{A}$. The morphism p is a 2-*effective epimorphism* of $\mathbb{A}$ if the morphism corresponding to p is a 2-effective monomorphism in $\mathbb{A}^{\mathrm{op}}$.

Remark 5.6. [Characterization of 2-effective epimorphisms] As a consequence of proof of Proposition 3.1 of [56], the 2-effective epimorphisms in Cat are precisely the functors that are essentially surjective on objects.

Definition 5.7. [Monadicity, comonadicity, Kleisli morphism] Let $p : e \to b$ be a morphism of a 2-category $\mathbb{A}$. We say that p is *monadic* if the following statements hold:

- p has a codensity monad $\mathtt{t} = (t, m, \eta)$;
- $\mathbb{A}$ has the Eilenberg-Moore object of $\mathtt{t}$;
- the *semantic factorization* $p = \mathtt{u}^{\mathtt{t}} \circ p^{\mathtt{t}}$ is such that $p^{\mathtt{t}}$ is an equivalence.

Dually, $l : b \to e$ is a *Kleisli morphism* if the corresponding morphism in $\mathbb{A}^{\mathrm{op}}$ is monadic, while l is *comonadic* if its corresponding morphism in $\mathbb{A}^{\mathrm{co}}$ is monadic.

By Theorem 4.12 and its dual versions, we get the following characterizations of monadicity, comonadicity and Kleisli morphisms:

Corollary 5.8 (Monadicity theorem)**.** *Assume that $\mathbb{A}$ has the higher cokernel of a morphism $p : e \to b$.*

(1) if $\mathrm{ran}_p p$ exists and is preserved by δ^0, then: p is monadic if and only if p is a 2-effective monomorphism;
(2) if $\mathrm{lan}_p p$ exists and is preserved by δ^1, then: p is comonadic if and only if p is a 2-effective monomorphism.

Proof: The first result follows immediately from the definitions and from Theorem 4.12. The second one is just its codualization (see Lemma 5.4, Theorem 4.13 and Remark 2.6). ∎

Corollary 5.9 (Characterization of Kleisli morphisms). *Assume that $\mathbb{A}$ has the higher kernel of a morphism $l : b \to e$.*

(1) assuming that $\mathrm{rlift}_l l$ *exists and is respected by $\delta_0^{l\downarrow l}$, l is a Kleisli morphism if and only if l is a 2-effective epimorphism;*
(2) assuming that $\mathrm{llift}_p p$ *exists and is respected by $\delta_1^{l\downarrow l}$, l is comonadic if and only if l is a 2-effective epimorphism.*

It is a well known fact that, whenever a morphism is monadic in a 2-category $\mathbb{A}$, it has a left adjoint (see [52, 51]). In our setting, if p is monadic as in Definition 5.7, the existence of a left adjoint follows from (1) since $p^{\mathtt{t}}$ is an equivalence, it has a left adjoint; (2) $\mathtt{u}^{\mathtt{t}}$ has always a left adjoint induced by the underlying morphism of the monad $t : b \to b$, the multiplication $m : t^2 \Rightarrow t$ and the universal property of $b^{\mathtt{t}}$; and (3) composition of right adjoint morphisms is right adjoint [34, 41]. From this fact and Theorem 4.15, we get cleaner versions of our monadicity results:

Corollary 5.10 (Monadicity theorem). *Assume that the 2-category $\mathbb{A}$ has the higher cokernel of a morphism p.*

(1) The morphism p is monadic if and only p has a left adjoint and p is a 2-effective monomorphism.
(2) The morphism p is comonadic if and only if p has a right adjoint and p is a 2-effective monomorphism.

Corollary 5.11 (Characterization of Kleisli morphisms). *Assume that the 2-category $\mathbb{A}$ has the higher kernel of a morphism l.*

(1) The morphism l is a co-Kleisli morphism if and only l has a left adjoint and l is a 2-effective epimorphism.
(2) The morphism l is Kleisli morphism if and only if l has a right adjoint and l is a 2-effective epimorphism.

Remark 5.12. [Monadicity vs comonadicity] It should be noted that, unlike Beck's monadicity theorem in Cat, the condition to get monadicity from a right adjoint morphism is coincides with the condition to get comonadicity from a left adjoint morphism: that is to say, to be a 2-effective monomorphism. Of course, as a consequence, we get that, under the conditions of Corollary 5.10, if the morphism p has a left and a right adjoint morphism, the following statements are equivalent:

i) p is a 2-effective monomorphism;

ii) p is monadic;
iii) p is comonadic.

Remark 5.13. [Beck's monadicity theorem vs formal monadicity theorem] Beck's monadicity theorem [3, 14] states that, in Cat, a functor p is monadic if and only if p has a left adjoint and p creates absolute coequalizers. By our monadicity theorem, we can conclude that, *provided that a functor $p : e \to b$ has a left adjoint, p creates absolute coequalizers if and only if p is a 2-effective monomorphism in* Cat.

However, the 2-effective monomorphisms in Cat are not characterized by the property of creation of absolute coequalizers. For instance, this follows from the fact that, by Lemma 5.4, the concept of 2-effective monomorphism is self codual, while the property of creation of absolute coequalizers is not self dual.

More precisely, one of the fundamental aspects of duality in 1-dimensional category theory is that the usual 2-functor op given by

$$\begin{array}{rcl} \mathrm{op}: & \mathsf{Cat}^{\mathrm{co}} & \to \mathsf{Cat} \\ & e & \mapsto e^{\mathrm{op}} \\ & p : e \to b & \mapsto p^{\mathrm{op}} : e^{\mathrm{op}} \to b^{\mathrm{op}} \\ & \beta & \mapsto \beta^{\mathrm{op}} \end{array}$$

is an involution: in particular, invertible. Therefore a functor $p^{\mathrm{op}} : e^{\mathrm{op}} \to b^{\mathrm{op}}$ is a 2-effective monomorphism in Cat if and only if the morphism $p : e \to b$ is a 2-effective monomorphism in $\mathsf{Cat}^{\mathrm{co}}$. Moreover, by Lemma 5.4, the morphism p is a 2-effective monomorphism in $\mathsf{Cat}^{\mathrm{co}}$ if and only if the corresponding morphism (functor) p is a 2-effective monomorphism in Cat. Hence, by abuse of notation, p is a 2-effective monomorphism in Cat if and only if p^{op} is a 2-effective monomorphism in Cat.

It is clear that a functor $p : e \to b$ creates absolute coequalizers if and only if the corresponding functor of $p^{\mathrm{op}} : e^{\mathrm{op}} \to b^{\mathrm{op}}$ creates absolute equalizers. Since there are functors that create absolute coequalizers but do not create absolute equalizers, the property of creation of absolute equalizers is not self dual. It follows, then, that there are functors that do create absolute coequalizers but are not 2-effective monomorphisms.

For instance, consider the usual forgetful functor between the category of free groups and the category of sets. This functor reflects isomorphisms and has equalizers: hence it creates all equalizers. However, since it has a left adjoint, it does not create absolute coequalizers and it is not a 2-effective monomorphism in Cat (otherwise, it would be monadic). Therefore the image of the morphism corresponding to this functor in $\mathsf{Cat}^{\mathrm{co}}$ by op is a functor that creates absolute coequalizers but it is not a 2-effective monomorphism in Cat.

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