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ON FINITARY FUNCTORS AND FINITELY PRESENTABLE ALGEBRAS

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ABSTRACT: A simple criterion for a functor to be finitary is presented: we call F finitely bounded if for all objects X every finitely generated subobject of FX factorizes through the F-image of a finitely generated subobject of X. This is equivalent to F being finitary for all functors between "reasonable" locally finitely presentable categories, provided that F preserves monomorphisms. We also discuss the question when that last assumption can be dropped.

For finitary regular monads $\mathbb{T}$ on locally finitely presentable categories we characterize the finitely presentable objects in the category of $\mathbb{T}$-algebras in the style known from general algebra: they are precisely the algebras presentable by finitely many generators and finitely many relations.

All this generalizes to locally λ-presentable categories, λ-accessible functors and λ-presentable algebras. As an application we obtain an easy proof that the Hausdorff functor on the category of complete metric spaces is $\aleph_1$-accessible.

1. Introduction

In a number of applications of categorical algebra, *finitary functors*, i.e., functors preserving filtered colimits, play an important role. For example, the classical varieties are precisely the categories of algebras on finitary monads over Set. How does one recognize that a functor F is finitary? For endofunctors of Set there is a simple necessary and sufficient condition: given a set X, every finite subset of FX factorizes through the image by F of a finite subset of X. This condition can be formulated for general functors $F : \mathcal{A} \to \mathcal{B}$: given an object X of $\mathcal{A}$, every finitely generated subobject of FX in $\mathcal{B}$ is required to factorize through the image by F of a finitely generated subobject of X in $\mathcal{A}$. We call such functors *finitely bounded.* For functors $F : \mathcal{A} \to \mathcal{B}$ between locally finitely presentable categories which

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preserve monomorphisms we prove

$$\text{finitary} \Leftrightarrow \text{finitely bounded}$$

whenever finitely generated objects of $\mathcal{A}$ are finitely presentable. (The last condition is, in fact, not only sufficient but also necessary for the above equivalence.)

What about general functors, not necessarily preserving monomorphisms? We prove the above equivalence whenever $\mathcal{A}$ is a strictly locally finitely presentable category, see Definition 3.7. Examples of such categories are sets, vector spaces, group actions of finite groups, and S-sorted sets with S finite. Conversely, if the above equivalence is true for all functors from $\mathcal{A}$ to locally finitely presentable categories, we prove that a weaker form of strictness holds for $\mathcal{A}$.

A closely related topic is the finite presentation of algebras for a monad. If $\mathbb{T}$ is a finitary monad on Set, then the category $\mathsf{Set}^{\mathbb{T}}$ of its algebras is nothing else than the classical concept of a variety of algebras. An algebra A is called *finitely presentable* (in General Algebra) if it can be presented by a finite set of generators and a finite set of equations. If X is a finite set of generators, this means that A can be obtained from the free algebra (TX, μ_X) as a quotient modulo a finitely generated congruence E. Now E is a subalgebra of $(TX, \mu_X)^2$ but it is not "finitely generated" as a subalgebra, but as a congruence. This is explained in Section 4. In case of monads over Set, the above concept coincides with A being a finitely presentable object of $\mathsf{Set}^{\mathbb{T}}$, see [5, Corollary 3.13]. We generalize this result to all locally finitely presentable categories with regular factorizations and all finitary monads preserving regular epimorphisms. We also characterize finitely generated algebras for finitary monads; here no side condition on the monad is required.

All of the above results can be also formulated for locally λ-presentable categories, λ-accessible functors, and algebras that are λ-presentable or λ-generated. We use this to provide a simple proof that the Hausdorff functor on the category of complete metric spaces is countably accessible.

2. Preliminaries

In this section we present properties on finitely presentable and finitely generated objects which will be useful in the subsequent sections.

Recall that an object A in a category $\mathcal{A}$ is called *finitely presentable* if its hom-functor $\mathcal{A}(A, -)$ preserves filtered colimits, and A is called *finitely generated* if $\mathcal{A}(A, -)$ preserves filtered colimits of monomorphisms – more precisely, colimits of filtered diagrams $D : \mathcal{D} \to \mathcal{A}$ for which Dh is a monomorphism in $\mathcal{A}$ for every morphism h of $\mathcal{D}$.

Notation 2.1. For a category $\mathcal{A}$ we denote by

$$\mathcal{A}_{\mathsf{fp}} \qquad \text{and} \qquad \mathcal{A}_{\mathsf{fg}}$$

small full subcategories of $\mathcal{A}$ representing (up to isomorphism) all finitely presentable and finitely generated objects, respectively.

Subobjects $m : M \rightarrowtail A$ with M finitely generated are called *finitely generated subobjects.*

Recall that $\mathcal{A}$ is a *locally finitely presentable* category, shortly *lfp* category, if it is cocomplete, $\mathcal{A}_{\mathsf{fp}}$ is essentially small, and every object is a colimit of a filtered diagram in $\mathcal{A}_{\mathsf{fp}}$.

We now recall [5] a number of standard facts about locally presentable categories.

Remark 2.2. Let $\mathcal{A}$ be an lfp category.

(1) By [5, Proposition 1.61] $\mathcal{A}$ has (strong epi, mono)-factorizations of morphisms.
(2) By [5, Proposition 1.57], every object A of $\mathcal{A}$ is the colimit of its *canonical filtered diagram*

$$D_A : \mathcal{A}_{\mathsf{fp}}/A \to \mathcal{A} \qquad (P \xrightarrow{p} A) \mapsto P,$$

with colimit injections given by the p's.
(3) By [5, Theorem 2.26] $\mathcal{A}$ is a free completion under filtered colimits of $\mathcal{A}_{\mathsf{fp}}$. That is, for every functor $H : \mathcal{A}_{\mathsf{fp}} \to \mathcal{B}$, where $\mathcal{B}$ has filtered colimits, there is an (essentially unique) extension to a finitary functor $\bar{H} : \mathcal{A} \to \mathcal{B}$. Moreover, this extensions can be formed as follows: for every object $A \in \mathcal{A}$ put

$$\bar{H}A = \operatorname{colim} H \cdot D_A.$$

(4) By [5, Proposition 1.62], a colimit of a filtered diagram of monomorphisms has monomorphisms as colimit injections. Moreover, for every compatible cocone formed by monomorphisms, the unique induced morphism from the colimit is a monomorphism too.

(5) By [5, Proposition 1.69], an object A is finitely generated iff it is a strong quotient of a finitely presentable object, i.e., there exists a finitely presentable object A_0 and a strong epimorphism $e : A_0 \twoheadrightarrow A$.
(6) It is easy to verify that every split quotient of a finitely presentable object is finitely presentable again.

Lemma 2.3. *Let $\mathcal{A}$ be an lfp category. A cocone of monomorphisms $c_i : Di \rightarrowtail C$ $(i \in I)$ of a diagram D of monomorphisms is a colimit of D iff it is a* union*; that is, iff id_C is the supremum of the subobjects $c_i : Di \rightarrowtail C$.*

Proof: The 'only if' direction is clear. For the 'if' direction suppose that $c_i : Di \rightarrowtail C$ have the union C, and let $\ell_i : Di \to L$ be the colimit of D. Then, since c_i is a cocone of D, we get a unique morphism $m : L \to C$ with $m \cdot \ell_i = c_i$ for every i. By 2.24, all the ℓ_i and m are monomorphisms, hence m is a subobject of C. Moreover, we have that $c_i \leq m$, for every i. Consequently, since C is the union of all c_i, L must be isomorphic to C via m, because id_C is the largest subobject of C. Thus, the original cocone c_i is a colimit cocone. ∎

Remark 2.4. Colimits of filtered diagrams $D\colon \mathcal{D} \to \mathsf{Set}$ are precisely those cocones $c_i\colon D_i \to C$ $(i \in \mathsf{obj}\,\mathcal{D})$ of D that have the following properties:
(1) (c_i) is jointly surjective, i.e. , $C = \bigcup c_i[D_i]$, and
(2) given i and elements $x, y \in D_i$ merged by c_i, then they are also merged by a connecting morphism $D_i \to D_j$ of D.

This is easy to see: for every cocone $c_i'\colon D_i \to C'$ of D define $f\colon C \to C'$ by choosing for every $x \in C$ some $y \in D_i$ with $x = c_i(y)$ and putting $f(x) = c_i'(y)$. By the two properties, this is well defined and is unique with $f \cdot c_i = c_i'$ for all i.

Recall that an adjunction whose right adjoint is finitary is called a finitary adjunction. The following lemma will be useful along the paper:

Lemma 2.5. *Let $L \dashv R : \mathcal{B} \to \mathcal{A}$ be a finitary adjunction between the lfp categories $\mathcal{B}$ and $\mathcal{A}$. Then we have:*
(1) L preserves both finitely presentable objects and finitely generated ones;
(2) if L is fully faithful, then an object X is finitely presentable in $\mathcal{A}$ iff LX is finitely presentable in $\mathcal{B}$;
(3) if, moreover, L preserves monomorphisms, then X is finitely generated in $\mathcal{A}$ iff LX is finitely generated in $\mathcal{B}$.

Proof: (1) Let X be a finitely presentable object of $\mathcal{A}$ and let $D : \mathcal{D} \to \mathcal{B}$ be a filtered diagram. Then we have the following chain of natural isomorphisms

$$\begin{aligned}\mathcal{B}(LX, \operatorname{colim} D) &\cong \mathcal{A}(X, R(\operatorname{colim} D))\\ &\cong \mathcal{A}(X, \operatorname{colim} RD)\\ &\cong \operatorname{colim}(\mathcal{A}(X, RD(-))\\ &\cong \operatorname{colim}(\mathcal{B}(LX, D(-)).\end{aligned}$$

This shows that LX is finitely presentable in $\mathcal{B}$. Now if X is finitely generated in $\mathcal{A}$ and D is a directed diagram of monos, then RD is also a directed diagram of monos (since the right adjoint R preserves monos). Thus, the same reasoning proves LX to be finitely generated in $\mathcal{B}$.

(2) Suppose that LX is finitely presentable in $\mathcal{B}$ and that $D : \mathcal{D} \to \mathcal{A}$ is a filtered diagram. Then we have the following chain of natural isomorphisms:

$$\begin{aligned}\mathcal{A}(X, \operatorname{colim} D) &\cong \mathcal{B}(LX, L(\operatorname{colim} D))\\ &\cong \mathcal{B}(LX, \operatorname{colim} LD)\\ &\cong \operatorname{colim}(\mathcal{B}(LX, LD(-))\\ &\cong \operatorname{colim}(\mathcal{A}(X, D(-))\end{aligned}$$

Indeed, the first and last step use that L is fully faithful, the second step that L is finitary and the third one that LX is finitely presentable in $\mathcal{B}$.

(3) If LX is finitely generated in $\mathcal{B}$ and $D : \mathcal{D} \to \mathcal{A}$ a directed diagram of monomorphisms, then so is LD since L preserves monomorphisms by assumption. Thus the same reasoning as in (2) shows that X is finitely generated in $\mathcal{A}$. ∎

Lemma 2.6. *Let $\mathcal{A}$ be an lfp category and I a set. An object in the power category $\mathcal{A}^I$ is finitely presentable iff its components*

(1) are finitely presentable in $\mathcal{A}$, and
(2) all but finitely many are initial objects.

Proof: Denote by 0 and 1 the initial and terminal objects, respectively. Note that for every $i \in I$ there are two fully faithful functors $L_i, R_i \colon \mathcal{A} \hookrightarrow \mathcal{A}^I$

defined by:

$$\big(L_i(X)\big)_j = \begin{cases} X & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad \big(R_i(X)\big)_j = \begin{cases} X & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

For every $i \in I$ there is also a canonical projection $\pi_i \colon \mathcal{A}^I \to \mathcal{A}$, $\pi_i\big((X_j)_{j\in I}\big) = X_i$. We have the following adjunctions:

$$L_i \dashv \pi_i \dashv R_i.$$

Sufficiency. Let $A = (A_i)_{i\in I}$ satisfy 1 and 2, then $L_i(A_i)$ is finitely presentable in $\mathcal{A}^I$ by Lemma 2.5 (1). Thus, so is A, since it is the finite coproduct of the $L_i(A_i)$, with $i \in I$, A_i not initial. Obviously, $L_i(A_i)$ is finiely presentable.

Necessity. Let $A = (A_i)_{i\in I}$ be finitely presentable in $\mathcal{A}^I$. Then for every $i \in I$, $\pi_i(A)$ is finitely presentable in $\mathcal{A}$ by Lemma 2.5 (1), proving item 1. To verify 2, for every finite set $J \subseteq I$, let A_J have the components A_j for every $j \in J$ and 0 otherwise. These objects A_J form an obvious directed diagram with a colimit cocone $a_J \colon A_J \to A$. Since A is finitely presentable, there exists J_0 such that id_A factorizes through a_{J_0}, i.e., a_{J_0} is a split epimorphism. Since a split quotient of an initial object is initial, we conclude that 2 holds. ■

Lemma 2.7. *(Finitely presentable objects collectively reflect filtered colimits.) Let $\mathcal{A}$ be an lfp category and $D \colon \mathcal{D} \to \mathcal{A}$ a filtered diagram with objects D_i ($i \in I$). A cocone $c_i \colon D_i \to C$ of D is a colimit of D iff for every $A \in \mathcal{A}_{\mathsf{fp}}$ the cocone*

$$c_i \cdot (-) \colon \mathcal{A}(A, D_i) \longrightarrow \mathcal{A}(A, C)$$

is a colimit of the diagram $\mathcal{A}(A, D-)$ in Set.

Explicitly: for every morphism $f \colon A \to C$, $A \in \mathcal{A}_{\mathsf{fp}}$

(1) a factorization through some c_i exists, and
(2) given two factorizations $f = c_i \cdot q_k$ for $k = 1, 2$, then $q_1, q_2 \colon A \to D_i$ are merged by a connecting morphism of $\mathcal{D}$. This follows from Remark 2.4.

Proof: If (c_i) is a colimit, then since $\mathcal{A}(A, -)$ preserves filtered colimits, the cocone of all $\mathcal{A}(A, c_i) = c_i \cdot (-)$ is a colimit in Set.

Conversely, assume that, for every $A \in \mathcal{A}_{\mathsf{fp}}$, the colimit cocone of the functor $\mathcal{A}(A, D-)$ is $\big(\mathcal{A}(A, c_i)\big)_{i\in\mathcal{D}}$. For every cocone $g_i \colon D_i \to G$ it is our task to prove that there exists a unique $g\colon C \to G$ with $g_i = g \cdot c_i$ for all i. Unicity is clear since (c_i) is a colimit cocone. Now $\big(\mathcal{A}(A, g_i)\big)_{i\in\mathcal{D}}$ forms a cocone of the functor $\mathcal{A}(A, -) \cdot D$. Consequently, there is a unique map $\varphi_A : \mathcal{A}(A, C) \to \mathcal{A}(A, G)$ with $\varphi_A \cdot \mathcal{A}(A, c_i) = \mathcal{A}(A, g_i)$ for all $i \in \mathcal{D}$.

For every morphism $h : A_1 \to A_2$ between objects of $\mathcal{A}_{\mathsf{fp}}$, the square on the right of the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{A}(A_1, D_i) \arrow[r, "\mathcal{A}(A_1,c_i)"] \arrow[rr, bend left, "\mathcal{A}(A_1,g_i)"] & \mathcal{A}(A_1, C) \arrow[r, "\varphi_{A_1}"] & \mathcal{A}(A_1, G) \\
\mathcal{A}(A_2, D_i) \arrow[u, "\mathcal{A}(h,D_i)"] \arrow[r, "\mathcal{A}(A_2,c_i)"] \arrow[rr, bend right, "\mathcal{A}(A_2,g_i)"'] & \mathcal{A}(A_2, C) \arrow[u, "\mathcal{A}(h,C)"'] \arrow[r, "\varphi_{A_2}"] & \mathcal{A}(A_2, G) \arrow[u, "\mathcal{A}(h,G)"']
\end{tikzcd}$$

This follows from the commutativity of the left-hand square and the outward one, which gives the equality $\varphi_{A_1}\cdot\mathcal{A}(h, C)\cdot\mathcal{A}(A_2, c_i) = \mathcal{A}(h, G)\cdot\varphi_{A_2}\cdot\mathcal{A}(A_2, c_i)$, combined with the fact that $\big(\mathcal{A}(A_2, c_i)\big)_{i\in\mathcal{D}}$, being a colimit cocone, is jointly epic.

As a consequence, the morphisms

$$A \xrightarrow{\varphi_A(a)} C \qquad \text{with } a : A \to C \text{ in } \mathcal{A}_{\mathsf{fp}}/C,$$

form a cocone for the canonical filtered diagram $D_C : \mathcal{A}_{\mathsf{fp}}/C \to \mathcal{A}$, of which C is the colimit. Indeed, given a commutative triangle

$$\begin{tikzcd}
A_1 \arrow[rr, "h"] \arrow[dr, "a_1"'] & & A_2 \arrow[dl, "a_2"] \\
& C &
\end{tikzcd}$$

with A_1 and A_2 in $\mathcal{A}_{\mathsf{fp}}$, we have

$$\varphi_{A_1}(a_1) = \varphi_{A_1}(a_2 h) = \varphi_{A_1} \cdot \mathcal{A}(h, C)(a_2) = \mathcal{A}(h, G) \cdot \varphi_{A_2}(a_2) = \varphi_{A_2}(a_2) \cdot h.$$

Thus there is a unique morphism $g\colon C \to G$ making for each $a\colon A \to C$ in $\mathcal{A}_{\mathsf{fp}}/C$ the following triangle commute:

$$\begin{tikzcd}
& A \arrow[dl, "a"'] \arrow[dr, "\varphi_A(a)"] & \\
C \arrow[rr, "g"] & & G
\end{tikzcd}$$

The morphism $g : C \to G$ satisfies $g \cdot c_i = g_i$ for all $i \in \mathcal{D}$. Indeed, fix i; for every $A \in \mathcal{A}_{\mathsf{fp}}$ and $b : A \to D_i$, we have $g_i b = \mathcal{A}(A, g_i)(b) = \varphi_A \cdot \mathcal{A}(A, c_i)(b) = \varphi_A(c_i b) = g c_i b$. And the morphisms $b \in \mathcal{A}_{\mathsf{fp}}/D_i$ are jointly epimorphic, thus $g_i = g \cdot c_i$. ■

Lemma 2.8. *(Finitely generated objects of an lfp category collectively reflect filtered colimits of monomorphisms). Let $\mathcal{A}$ be an lfp category and $D\colon \mathcal{D} \to \mathcal{A}$ a filtered diagram of monomorphisms with ojects $D_i\,(i \in I)$. A cocone $c_i\colon D_i \to C$ of D is a colimit iff for every $A \in \mathcal{A}_{\mathsf{fg}}$ the cocone*

$$c_i \cdot (-)\colon \mathcal{A}(A, D_i) \longrightarrow \mathcal{A}(A, C) \qquad (i \in I)$$

is a colimit of the diagram $\mathcal{A}(A, D-)$ in Set.

Proof: If (c_i) is a colimit, then since $\mathcal{A}(A, -)$ preserves filtered colimits of monomorphisms, the cocone $c_i \cdot (-)\colon \mathcal{A}(A, D_i) \to \mathcal{A}(A, C)$ is a colimit in Set.

Conversely, if $c_i \cdot (-)\colon \mathcal{A}(A, D_i) \to \mathcal{A}(A, C)$ is a colimit of the diagram $\mathcal{A}(A, D-)$ for every $A \in \mathcal{A}_{\mathsf{fg}}$, then it is so for every $A \in \mathcal{A}_{\mathsf{fp}}$. Hence by Lemma 2.7, the cocone (c_i) is a colimit. ■

Corollary 2.9. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ between lfp categories is finitary iff it preserves the canonical colimits: $FA = \operatorname{colim} FD_A$ for every object A of $\mathcal{A}$.*

Indeed, in the notation of Lemma 2.7 we are to verify that $Fc_i\colon FD_i \to FC$ $(i \in I)$ is a colimit of FD. For this, taking into account that lemma and Remark 2.4, we take any $B \in \mathcal{B}_{\mathsf{fp}}$ and prove that every morphism $b\colon B \to FC$ factorizes essentially uniquely through Fc_i for some $i \in \mathcal{D}$. Since $FC = \operatorname{colim} FD_C$ we have a factorization

$$\begin{tikzcd} & FA \arrow[d, "Fa"] \\ B \arrow[ur, dashed, "b_0"] \arrow[r, "b"'] & FC \end{tikzcd} \qquad (A \in \mathcal{A}_{\mathsf{fp}})$$

By Lemma 2.7 there is some $i \in \mathcal{D}$ and $a_0 \in \mathcal{A}(A, D_i)$ with $a = c_i \cdot a_0$ and hence $b = Fc_i \cdot (Fa_0 \cdot b_0)$. The essential uniqueness is clear.

Notation 2.10. Throughout the paper, given a morphism $f : X \to Y$ we denote by $\mathsf{Im}\, f$ the *image of f*, that is, any choice of the intermediate object defined by taking the (strong epi, mono)-factorization of f:

$$f = (\,X \overset{e}{\twoheadrightarrow} \mathsf{Im}\, f \overset{m}{\rightarrowtail} Y\,).$$

We will make use of the next lemma in the proof of Proposition 3.3 and Theorem 4.5.

Lemma 2.11. *In an lfp category, images of filtered colimits are directed unions of images.*

More precisely, suppose we have a filtered diagram $D : \mathcal{D} \to \mathcal{A}$ with objects $D_i\,(i \in I)$ and the colimit cocone $(c_i : D_i \to C)_{i \in I}$. Given a morphism $f : C \to B$, take the factorizations of f and all $f \cdot c_i$ as follows:

$$\text{(1)} \qquad \begin{tikzcd} D_i \arrow[rr, two heads, "e_i"] \arrow[d, "c_i"'] & & \mathsf{Im}(f \cdot c_i) \arrow[d, tail, "m_i"] \arrow[dl, dashed, "d_i"'] \\ C \arrow[r, two heads, "e"] \arrow[rr, bend right, "f"'] & \mathsf{Im}\, f \arrow[r, tail, "m"] & B \end{tikzcd} \qquad (i \in I)$$

Then the subobject m is the union of the subobjects m_i.

Proof: We have the commutative diagram (1), where d_i is the diagonal fill-in. Since $m \cdot d_i = m_i$, we see that d_i is monic. Furthermore, for every morphism $Dg : D_i \to D_j$ we get a monomorphism $\bar{g} : \mathsf{Im}(f \cdot c_i) \rightarrowtail \mathsf{Im}(f \cdot c_j)$ as a diagonal fill-in in the diagram below:

$$\begin{tikzcd} D_i \arrow[r, two heads, "e_i"] \arrow[d, "Dg"'] & \mathsf{Im}(f \cdot c_i) \arrow[d, tail, dashed, "\bar{g}"'] \arrow[dr, tail, "d_i"] & \\ D_j \arrow[r, two heads, "e_j"] & \mathsf{Im}(f \cdot c_j) \arrow[r, tail, "d_j"'] & \mathsf{Im}\, f \end{tikzcd}$$

Since D is a filtered diagram, we see that the objects $\mathsf{Im}(f \cdot c_i)$ form a filtered diagram of monomorphisms; in fact, since d_i and d_j are monic there is at most one connecting morphism $\mathsf{Im}(f \cdot c_i) \to \mathsf{Im}(f \cdot c_j)$.

In order to see that m is the union of the subobjects m_i's, let $d_i' : \mathsf{Im}(f \cdot c_i) \rightarrowtail N$ and $n : N \rightarrowtail \mathsf{Im}\, f$ be monomorphisms such that $n \cdot d_i' = d_i$ for every $i \in I$.

$$\begin{tikzcd} Di \arrow[r, two heads, "e_i"] \arrow[d, "c_i"'] & \mathsf{Im}(f \cdot c_i) \arrow[r, tail, "d_i"] \arrow[d, tail, "d_i'"] & \mathsf{Im}\, f \\ C \arrow[r, dashed, "t"] & N \arrow[ur, tail, "n"'] & \end{tikzcd}$$

Since n is monic, the morphisms $d_i' \cdot e_i$ clearly form a cocone of D, and this induces a unique morphism $t : C \to N$ such that $t \cdot c_i = d_i' \cdot e_i$. Then we obtain the equalities $n \cdot t \cdot c_i = e \cdot c_i$; hence, $n \cdot t = e$. Since n is monic, it follows that it is an isomorphism, i.e., the subobjects $\mathsf{id}_{\mathsf{Im}\, f}$ and n are isomorphic. This shows that m is the desired union. ∎

3. Finitary and Finitely Bounded Functors

In this section we introduce the notion of a finitely bounded functor on a locally presentable category, and we investigate when finitely bounded functors are precisely the finitary ones.

Definition 3.1. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *finitely bounded* provided that, given an object A of $\mathcal{A}$, every finitely generated subobject of FA in $\mathcal{B}$ factorizes through the F-image of a finitely generated subobject of A in $\mathcal{A}$.

In more detail, given a monomorphism $m_0 : M_0 \rightarrowtail FA$ with $M_0 \in \mathcal{B}_{\mathsf{fg}}$ there exists a monomorphism $m : M \rightarrowtail A$ with $M \in \mathcal{A}_{\mathsf{fg}}$ and a factorization as follows:

$$\begin{tikzcd} & FM \arrow[d, "Fm"] \\ M_0 \arrow[ur, dashed] \arrow[r, tail, "m_0"'] & FA \end{tikzcd}$$

Example 3.2. (1) If $\mathcal{B}$ is the category of S-sorted sets, then F is finitely bounded iff for every object A of $\mathcal{A}$ and every element $x \in FA$ there exists a finitely generated subobject $m : X \rightarrowtail A$ (i.e., the coproduct of all sorts of X is finite) such that the image of Fm contains the given element, i.e. $x \in Fm[FX]$.

(2) Let $\mathcal{A}$ be a category with (strong epi, mono)-factorizations. An object of $\mathcal{A}$ is finitely generated iff its hom-functor is finitely bounded. Indeed, by applying 1 we see that $\mathcal{A}(A, -)$ is finitely bounded iff for every morphism $f : A \to B$ there exists a factorization $f = m \cdot g$, where $m : A' \rightarrowtail B$ is monic and A' is finitely generated. This implies that A is finitely generated: for $f = \mathsf{id}_A$ we see that m is invertible. Conversely, if A is finitely generated, then we can take the (strong epi, mono)-factorization of f and use that finitely generated objects are closed under strong quotients [5].

Proposition 3.3. *Let F be a functor between lfp categories preserving monomorphisms. Then F is finitely bounded iff it preserves filtered colimits of monomorphisms.*

Proof: We are given lfp categories $\mathcal{A}$ and $\mathcal{B}$ and a functor $F : \mathcal{A} \to \mathcal{B}$ preserving monomorphisms.

(1) Let F preserve filtered colimits of monomorphisms. Then, for every object A we express it as a canonical filtered colimit of all $p : P \to A$ in $\mathcal{A}_{\mathsf{fp}}/A$ (see Remark 2.2I:canColim). By Lemma 2.11 applied to $f = \mathsf{id}_A$ we see that A is the colimit of its subobjects $\mathsf{Im}\, p$ where p ranges over $\mathcal{A}_{\mathsf{fp}}/A$. Hence, F preserves this colimit:

$$FA = \operatorname*{colim}_{p \in \mathcal{A}_{\mathsf{fp}}/A} F(\mathsf{Im}\, p),$$

and this is a colimit of monomorphisms since F preserves monomorphisms. Given a finitely generated subobject $m_0 : M_0 \rightarrowtail FA$, we thus obtain some p in $\mathcal{A}_{\mathsf{fp}}/A$ such that m_0 factorizes through the F-image of $\mathsf{Im}(p) \rightarrowtail A$. Hence F is finitely bounded.

(2) Let F be finitely bounded. Let $D : \mathcal{D} \to \mathcal{A}$ be a filtered diagram of monomorphisms with a colimit cocone:

$$c_i : D_i \rightarrowtail C \qquad (i \in I).$$

In order to prove that $Fc_i : FD_i \to FC$, $i \in I$, is a colimit cocone, we show that its image under $\mathcal{B}(B, -)$ is a colimit cocone for every finitely generated object B in $\mathcal{B}$ (cf. Lemma 2.8). In other words, given $f : B \to FC$ with $B \in \mathcal{B}_{\mathsf{fg}}$ then

(a) f factorizes through Fc_i for some i in I, and

(b) the factorization is unique.

We do not need to take care of (b): since every c_i is monic by Remark 2.2(4), so is every Fc_i. In order to prove (a), factorize $f : B \to FC$ as a strong epimorphism $q : B \twoheadrightarrow M_0$ followed by a monomorphism $m_0 : M_0 \rightarrowtail FC$. Then M_0 is finitely generated by Remark 2.2(5). Thus, there exists a finitely generated subobject $m : M \rightarrowtail C$ with $m_0 = Fm \cdot u$ for some $u : M_0 \to FM$. Furthermore, since $\mathcal{A}(M, -)$ preserves the colimit of D, m factorizes as $m = c_i \cdot \overline{m}$ for some $i \in I$. Thus $F\overline{m} \cdot u \cdot q$ is the desired factorization:

$$f = m_0 \cdot q = Fm \cdot u \cdot q = Fc_i \cdot F\overline{m} \cdot u \cdot q. \qquad \blacksquare$$

In the following theorem we work with an lfp category whose finitely generated objects are finitely presentable. This holds e.g. for the categories of sets, many-sorted sets, posets, graphs, vector spaces, unary algebras on one operation and nominal sets. Further examples are the categories of commutative monoids (this is known as Redei's theorem [20], see Freyd [11] for a rather short proof), positive convex algebras (i.e. the Eilenberg-Moore algebras for the (sub-)distribution monad on sets [21]), semimodules for Noetherian semirings (see e.g. [9] for a proof). The category of finitary endofunctors of sets also has this property as we verify in Corollary 3.26.

On the other hand, the categories of groups, lattices or monoids do not have that property. A particularly simple counter-example is the slice category $\mathbb{N}/\mathsf{Set}$; equivalently, this is the category of algebras with a set of constants indexed by $\mathbb{N}$. Hence, an object $a : \mathbb{N} \to A$ is finitely generated iff A has a finite set of generators, i.e. $A \setminus a[\mathbb{N}]$ is a finite set. It is finitely presentable iff, moreover, A is presented by finitely many relations, i.e. the kernel of a is a finite subset of $\mathbb{N} \times \mathbb{N}$.

Theorem 3.4. *Let $\mathcal{A}$ be an lfp category in which every finitely generated object is finitely presentable ($\mathcal{A}_{\mathsf{fp}} = \mathcal{A}_{\mathsf{fg}}$). Then for all functors preserving monomorphisms from $\mathcal{A}$ to lfp categories we have the equivalence*

$$\textit{finitary} \iff \textit{finitely bounded.}$$

Proof: Let $F : \mathcal{A} \to \mathcal{B}$ be a finitely bounded functor preserving monomorphisms, where $\mathcal{B}$ is lfp. We prove that F is finitary. The converse follows from Proposition 3.3.

According to Corollary 2.9 it suffices to prove that F preserves the colimits of all canonical filtered diagrams. The proof that FD_A has the colimit cocone given by Fp for all $p : P \to A$ in $\mathcal{A}_{\mathsf{fp}}/A$ uses the fact that this is a filtered diagram in the lfp category $\mathcal{B}$. By Remark 2.4, it is therefore sufficient to prove that for every object $C \in \mathcal{B}_{\mathsf{fp}}$ and every morphism $c : C \to FA$ we have the following two properties:

(1) c factorizes through some of the colimit maps

$$\begin{tikzcd} & FP \arrow[d, "Fp"] \\ C \arrow[ur, dashed, "u"] \arrow[r, "c"'] & FA \end{tikzcd} \qquad (P \in \mathcal{A}_{\mathsf{fp}}),$$

(2) given another such factorization, $c = Fp \cdot v$, then u and v are merged by some connecting morphism; i.e., we have a commutative triangle

$$\begin{tikzcd} P \arrow[rr, "h"] \arrow[dr, "p"'] & & P' \arrow[dl, "p'"] \\ & A & \end{tikzcd} \qquad (P, P' \in \mathcal{A}_{\mathsf{fp}})$$

with $Fh \cdot u = Fh \cdot v$.

Indeed, for every $p : P \to A$ in $\mathcal{A}_{\mathsf{fp}}/A$, by applying Lemma 2.11 to $f = \mathsf{id}_A$, we see that the monomorphisms $m_p : \mathsf{Im}\, p \rightarrowtail A$ form a colimit cocone of a diagram of monomorphisms. Then, by Proposition 3.3, F preserves this colimit, therefore any $c : C \to FA$ factorizes through some $Fm_p : F(\mathsf{Im}\, p) \to FA$. Observe that, since $\mathcal{A}_{\mathsf{fg}} = \mathcal{A}_{\mathsf{fp}}$, we know by Remark 2.2(5) that every $\mathsf{Im}\, p$ is finitely presentable, hence the morphisms m_p are colimit injections and all $e_p : P \twoheadrightarrow \mathsf{Im}\, p$ are connecting morphisms of D_A. Consequently, (1) is clearly satisfied. Moreover, given $u, v : C \to FP$ with $Fp \cdot u = Fp \cdot v$, we have that $Fe_p \cdot u = Fe_p \cdot v$, since Fm_p is monic, thus (2) is satisfied, too. ■

Remark 3.5. Conversely, if every functor from $\mathcal{A}$ to an lfp category fulfils the equivalence in the above theorem, then $\mathcal{A}_{\mathsf{fp}} = \mathcal{A}_{\mathsf{fg}}$. Indeed, for every finitely generated object A, since $F = \mathcal{A}(A, -)$ preserves monomorphisms, we can apply Proposition 3.3 and conclude that F is finitary, i.e., $A \in \mathcal{A}_{\mathsf{fp}}$.

Example 3.6. For Un, the category of algebras with one unary operation, we present a finitely bounded endofunctor that is not finitary. Since in Un finitely generated algebras are finitely presentable, this shows that the condition of preservation of monomorphisms cannot be removed from Theorem 3.4.

Let C_p denote the algebra on p elements whose operation forms a cycle. Define $F\colon \mathsf{Un} \to \mathsf{Un}$ on objects by

$$FX = \begin{cases} C_1 + X & \text{if } \mathsf{Un}(C_p, X) = \emptyset \text{ for some prime } p, \\ C_1 & \text{else.} \end{cases}$$

Given a homomorphism $f : X \to Y$ with $FY = C_1 + Y$, then also $FX = C_1 + X$; indeed, in case $FX = C_1$ we would have $\mathsf{Un}(C_p, X) \neq \emptyset$ for all prime numbers p, and then the same would hold for Y, a contradiction. Thus we can put $Ff = \mathsf{id}_{C_1} + f$. Otherwise Ff is the unique homomorphism to C_1.

We now prove that F is finitely bounded. Suppose we are given a finitely generated subalgebra $m_0 : M_0 \rightarrowtail FX$. If $FX = C_1$ then take $M = \emptyset$ and

$m : \emptyset \rightarrowtail X$ the unique homomorphism. Otherwise we have $FX = C_1 + X$, and we take the preimages of the coproduct injections to see that $m_0 = u{+}m$, where u is the unique homomorphism into the terminal algebra C_1 as shown below:

$$\begin{tikzcd}
M' \arrow[r, "u"] \arrow[d, tail] & C_1 \arrow[d, tail] \\
M_0 \arrow[r, "m_0"] & C_1 + X \\
M \arrow[r, "m"'] \arrow[u, tail] & X \arrow[u, tail]
\end{tikzcd}$$

Then we obtain the desired factorization of m_0:

$$\begin{tikzcd}
& C_1 + M = FM \arrow[d, "\mathsf{id}_{C_1}+m=Fm"] \\
M_0 = M' + M \arrow[ur, "u+M"] \arrow[r, "u+m"'] & C_1 + X = FX
\end{tikzcd}$$

However, F is not finitary; indeed, it does not preserve the colimit of the following chain of inclusions

$$C_2 \hookrightarrow C_2 + C_3 \hookrightarrow C_2 + C_3 + C_5 \hookrightarrow \cdots$$

since every object A in this chain is mapped by F to $C_1 + A$ while its colimit $X = \coprod_{i \text{ prime}} C_i$ is mapped to C_1.

We now turn to the question for which categories $\mathcal{A}$ the equivalence

$$\text{finitary} \iff \text{finitely bounded}$$

holds for *all* functors with domain $\mathcal{A}$.

Definition 3.7. An lfp category $\mathcal{A}$ is called *strictly* or *semi-strictly* lfp provided that every morphism $b : B \to A$ in $\mathcal{A}_{\mathsf{fp}}/A$ factorizes through a morphism $b' : B' \to A$ in $\mathcal{A}_{\mathsf{fp}}/A$ for which some $f : A \to B'$ exists and, in the case of strict lfp, $f \cdot b$ is such a factor, i.e. $b = b' \cdot (f \cdot b)$.

$$\begin{tikzcd}
B \arrow[rr, dashed] \arrow[dr, "b"'] & & B' \arrow[dl, shift left, "b'"] \\
& A \arrow[ur, shift left, "f"] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[rr, "f\cdot b"] \arrow[dr, "b"'] & & B' \arrow[dl, shift left, "b'"] \\
& A \arrow[ur, shift left, "f"] &
\end{tikzcd}$$

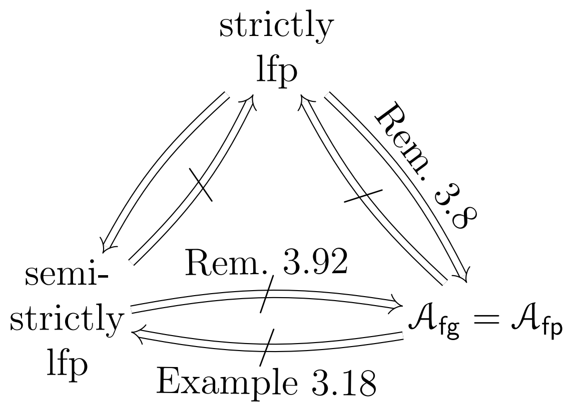
semi-strictly lfp *strictly lfp*

Remark 3.8. In every strictly lfp category we have $\mathcal{A}_{\mathsf{fg}} = \mathcal{A}_{\mathsf{fp}}$: Indeed, given $A \in \mathcal{A}_{\mathsf{fg}}$ express it as a strong quotient $b : B \twoheadrightarrow A$ of some $B \in \mathcal{A}_{\mathsf{fp}}$, see Remark 2.2(5). Then the equality $b = b' \cdot f \cdot b$ implies $b' \cdot f = \mathsf{id}$. Thus, A is a split quotient of a finitely presentable object B', hence, A is finitely presentable by Remark 2.2(6).

Examples 3.9. (1) Set is strictly lfp: given $b : B \to A$ with $B \neq \emptyset$ factorize it as $e : B \twoheadrightarrow \mathsf{Im}\, b$ followed by a split monomorphism $b' : \mathsf{Im}\, b \to A$. Given a splitting, $f \cdot b' = \mathsf{id}$, we have $b = b' \cdot f \cdot b$. The case $B = \emptyset$ is trivial: for $A \neq \emptyset$, b' may be any map from a singleton set to A.

(2) Every lfp category with a zero object $0 \cong 1$ is semi-strictly lfp. Given $b\colon B \to A$, put $b' = b$ and $f = \big(A \xrightarrow{!} 1 \cong 0 \xrightarrow{!} B'\big)$. Examples include the categories of monoids and groups, which are not strictly lfp because in both cases finitely presentable and finitely generated objects differ.

We will see other examples (and non-examples) below. The following figure shows the relationships between the different properties:



Note that from the independence of the lower hand properties we know that neither of them implies ‘strictly lfp’.

Theorem 3.10. *Let $\mathcal{A}$ be a strictly lfp category, and $\mathcal{B}$ an lfp category with $\mathcal{B}_{\mathsf{fg}} = \mathcal{B}_{\mathsf{fp}}$. Then for all functors from $\mathcal{A}$ to $\mathcal{B}$ we have the equivalence*

$$\textit{finitary} \iff \textit{finitely bounded.}$$

Proof: ($\Longrightarrow$) Let $F : \mathcal{A} \to \mathcal{B}$ be finitary. By Remark 3.8 we know that $\mathcal{A}_{\mathsf{fp}} = \mathcal{A}_{\mathsf{fg}}$. Given a finitely generated subobject $m : M \rightarrowtail FA$, write A as the directed colimit of all its finitely generated subobjects $m_i : A_i \rightarrowtail A$.

Since F is finitary, it preserves this colimit, and since M is finitely generated, whence finitely presentable, we obtain some i and some $f : M \to FA_i$ such that $Fm_i \cdot f = m$ as desired.

($\Longleftarrow$) Suppose that $F : \mathcal{A} \to \mathcal{B}$ is finitely bounded. We verify the two properties 1 and 2 in the proof of Theorem 3.4. In order to verify 1, let $c : C \to FA$ be a morphism with C finitely presentable. Then we have the finitely generated subobject $\mathsf{Im}\, c \rightarrowtail FA$, and this factorizes through $Fm : FM \to FA$ for some finitely generated subobject $m : M \rightarrowtail A$ since F is finitely bounded. Then c factorizes through Fm, too, and we are done since M is finitely presentable by Remark 3.8.

To verify (2), suppose that we have $u, v : C \to FB$ and $b : B \to A$ in $\mathcal{A}_{\mathsf{fp}}/A$ such that $Fb \cdot u = Fb \cdot v$. Now choose $f : A \to B'$ with $b = b' \cdot (b \cdot f)$ (see Definition 3.7). Put $h = f \cdot b$ to get $b = b' \cdot h$ as required. Since $Fb \cdot u = Fb \cdot v$, we conclude $Fh \cdot u = Ff \cdot Fb \cdot u = Ff \cdot Fb \cdot u = Fh \cdot v$. ∎

Corollary 3.11. *A functor between strictly lfp categories is finitary iff it is finitely bounded.*

Remark 3.12. Consequently, a set functor F is finitary if and only if it is finitely bounded. The latter means precisely that every element of FX is contained in $Fm[FM]$ for some finite subset $m : M \hookrightarrow X$.

This result was formulated already in [4], but the proof there is unfortunately incorrect.

Open Problem 3.13. Is the above implication an equivalence? That is, given an lfp category $\mathcal{A}$ such that every finitely bounded functor into lfp categories is finitary, does this imply that $\mathcal{A}$ is strictly lfp?

Theorem 3.14. *Let $\mathcal{A}$ be an lfp category such that for functors $F : \mathcal{A} \to \mathsf{Set}$ we have the equivalence*

$$\textit{finitary} \iff \textit{finitely bounded.}$$

Then $\mathcal{A}$ is semi-strictly lfp, and finitely generated objects are finitely presentable ($\mathcal{A}_{\mathsf{fg}} = \mathcal{A}_{\mathsf{fp}}$).

Proof: The second statement easily follows from Example 3.22. Suppose that $\mathcal{A}$ is an lfp category such that the above equivalence holds for all functors from $\mathcal{A}$ to Set. Then the same equivalence holds for all functors $F : \mathcal{A} \to \mathsf{Set}^S$, for S a set of sorts. To see this, denote by $C : \mathsf{Set}^S \to \mathsf{Set}$ the functor forming the coproduct of all sorts. It is easy to see that C creates filtered colimits. Thus, a functor $F : \mathcal{A} \to \mathsf{Set}^S$ is finitary iff $C \cdot F : \mathcal{A} \to \mathsf{Set}$ is. Moreover, F is finitely bounded iff $C \cdot F$ is; indeed, this follows immediately from Example 3.21.

We proceed to prove the semi-strictness of $\mathcal{A}$. Choose a set $S \subseteq \mathcal{A}_{\mathsf{fp}}$ of representatives up to isomorphism. Given a morphism

$$b : B \to A \qquad \text{with } B \in \mathcal{A}_{\mathsf{fp}}$$

we present b' and f as required. Define a functor $F : \mathcal{A} \to Set^S$ on objects Z of $\mathcal{A}$ by

$$FZ = \begin{cases} \mathbb{1} + (\mathcal{A}(s, Z))_{s \in S} & \text{if } \mathcal{A}(A, Z) = \emptyset \\ \mathbb{1} & \text{else,} \end{cases}$$

where $\mathbb{1}$ denotes the terminal S-sorted set. Given a morphism $f : Z \to Z'$ we need to specify Ff in the case where $\mathcal{A}(A, Z') = \emptyset$: this implies $\mathcal{A}(A, Z) = \emptyset$ and we put

$$Ff = \mathsf{id}_{\mathbb{1}} + (\mathcal{A}(s, f))_{s \in S}.$$

Here $\mathcal{A}(s, f) : \mathcal{A}(s, Z) \to \mathcal{A}(s, Z')$ is given by $u \mapsto f \cdot u$, as usual. It is easy to verify that F is a well-defined functor.

(1) Let us prove that F is finitely bounded. The category Set^S is lfp with finitely generated objects $(X)_{s \in S}$ precisely those for which the set $\coprod_{s \in S} X_s$ is finite. Let $m_0 : M_0 \rightarrowtail FZ$ be a finitely generated subobject. We present a finitely generated subobject $m : M \rightarrowtail Z$ such that m_0 factorizes through Fm. This is trivial in the case where $\mathcal{A}(A, Z) \neq \emptyset$: choose any finitely generated subobject $m : M \rightarrowtail Z$ (e.g., the image of the unique morphism from the initial object to Z: cf. Remark 2.2(5)). Then Fm is either $\mathsf{id}_{\mathbb{1}}$ or a split epimorphism, since $FZ = \mathbb{1}$ and in FM each sort is non-empty. Thus, we have t with $Fm \cdot t = \mathsf{id}$ and m_0 factorizes through Fm:

$$\begin{tikzcd} & FM \arrow[d, shift right, two heads, "Fm"'] \\ M_0 \arrow[ur, "t \cdot m_0"] \arrow[r, tail, "m_0"'] & FZ = \mathbb{1} \arrow[u, shift right, tail, "t"'] \end{tikzcd}$$

In the case where $\mathcal{A}(A, Z) = \emptyset$ we have $m_0 = m_1 + m_2$ for subobjects

$$m_1 : M_1 \rightarrowtail \mathbb{1} \qquad \text{and} \qquad m_2 : M_2 \rightarrowtail (\mathcal{A}(s, Z))_{s \in S}.$$

For notational convenience, assume $(M_2)_s \subseteq \mathcal{A}(s, Z)$ and $(m_2)_s$ is the inclusion map for every $s \in S$. Since M_0 is finitely generated, M_2 contains only finitely many elements $u_i : s_i \to Z$, $i = 1, \dots, n$. Factorize $[u_1, \dots, u_n]$ as a strong epimorphism e followed by a monomorphism m in $\mathcal{A}$ (see Remark 2.2.1):

$$\coprod_{i=1}^n s_i \xrightarrow{e} M \xrightarrow{m} Z \, .$$

Then $\mathcal{A}(A, M) = \emptyset$, therefore $Fm = \mathsf{id}_{\mathbb{1}} + (\mathcal{A}(s, m))_{s\in S}$. Since every element $u_i : s_i \to Z$ of M_2 factorizes through m in $\mathcal{A}$, we have

$$u_i = m \cdot u_i' \quad \text{for } u_i' : s_i \to M \text{ with } [u_1', \dots, u_n'] = e.$$

Then the inclusion map $m_2 : M_2 \to (\mathcal{A}(s, Z))_{s\in S}$ has the following form

$$m_2 = \left(M_2 \xrightarrow{v} (\mathcal{A}(s, M))_{s\in S} \xrightarrow{(\mathcal{A}(s,m))_{s\in S}} (\mathcal{A}(s, Z))_{s\in S} \right).$$

The desired factorization of $m_0 = m_1 + m_2$ through $Fm = \mathsf{id}_{\mathbb{1}} + (\mathcal{A}(s, m))_{s\in S}$ is as follows:

$$\begin{tikzcd} & \mathbb{1} + (\mathcal{A}(s, M))_{s\in S} \arrow[d, "\mathsf{id} + (\mathcal{A}(s,m))_{s\in S}"] \\ M_0 = M_1 + M_2 \arrow[ur, "m_1 + v"] \arrow[r, tail, "m_0 = m_1 + m_2"'] & \mathbb{1} + (\mathcal{A}(s, Z))_{s\in S} \end{tikzcd}$$

(2) We thus know that F is finitary, and we will use this to prove that $\mathcal{A}$ is semi-strictly lfp. That is, we find $b' : B' \to A$ in $\mathcal{A}_{\mathsf{fp}}/A$ through which b factorizes and which fulfils $\mathcal{A}(A, B') \neq \emptyset$. Recall from Remark 2.2(2) that $A = \operatorname{colim} D_A$. Our morphism b is an object of the diagram scheme $\mathcal{A}_{\mathsf{fp}}/A$ of D_A. Let D_A' be the full subdiagram of D_A on all objects b' such that b factorizes through b' in $\mathcal{A}$ (that is, such that a connecting morphism $b \to b'$ exists in $\mathcal{A}_{\mathsf{fp}}/A$). Then D_A' is also a filtered diagram and has the same colimit, i.e. $A = \operatorname{colim} D_A'$. Since F preserves this colimit and $FA = \mathbb{1}$, we get

$$\mathbb{1} \cong \operatorname{colim} FD_A'.$$

Assuming that $\mathcal{A}(A, B') = \emptyset$ for all $b' : B' \to A$ in D_A', we obtain a contradiction: the objects of FD_A' are $\mathbb{1} + (\mathcal{A}(s, B'))_{s\in S}$, and since for every $s \in S$ the functor $\mathcal{A}(s, -)$ is finitary, the colimit of all $\mathcal{A}(s, B')$ is $\mathcal{A}(s, A)$. We thus obtain an isomorphism

$$\mathbb{1} \cong \mathbb{1} + (\mathcal{A}(s, A))_{s \in S}.$$

This means $\mathcal{A}(s, A) = \emptyset$ for all $s \in S$, in particular $\mathcal{A}(B, A) = \emptyset$, in contradiction to the existence of the given morphism $b : B \to A$.

Therefore, there exists $b' : B' \to A$ in D'_A, i.e. b' through which b factorizes with $\mathcal{A}(A, B') \neq \emptyset$, as required. ■

Examples 3.15. Here we present some strictly lfp categories.

(1) Set^S is strictly locally finitely presentable iff S is finite. Indeed, if S is finite and $b : B \to A$ is an S-sorted map with B finitely presentable, factorize it as $e : B \twoheadrightarrow \mathsf{Im}\, b = b[B]$ followed by an inclusion $m : \mathsf{Im}\, b \rightarrowtail A$. Choose a finitely presentable S-sorted set M' for which a morphism $m' : M' \to A$ is given such that every nonempty sort of A is also nonempty in M'. Define $f : A \to \mathsf{Im}\, b + M'$ by assigning to every $b(x)$ itself in $\mathsf{Im}\, b$, and to every element $y \in A - b[B]$ of sort s some element of that sort in M'. Then $b' = [m, m'] : \mathsf{Im}\, b + M' \to A$ has the required property, i.e., $b = b' \cdot f \cdot b$ with b' having a finitely presentable domain.

Conversely, if S is infinite, then the morphism b from the initial object to the terminal one does not yield the desired morphisms b' and f.

(2) The category of vector spaces (over a fixed field) is strictly lfp.

Indeed, given $b : B \to A$ with B finite-dimensional, factorize it as $e : B \twoheadrightarrow \mathsf{Im}\, b$ followed by $m : \mathsf{Im}\, b \rightarrowtail A$. There exists a subspace $m' : M' \to A$ with $A = \mathsf{Im}\, b + M'$. The desired triangle is as follows:

$$\begin{tikzcd}
B \arrow[rr, "e = {[\mathsf{id}, 0]} \cdot b"] \arrow[dr, "b = \mathsf{inl} \cdot e"'] & & \mathsf{Im}\, b \arrow[dl, shift left, "m"] \\
& A = \mathsf{Im}\, b + M' \arrow[ur, shift left, "{[\mathsf{id}, 0]}"] &
\end{tikzcd}$$

(3) For every finite group G the category G-Set of actions of G on sets is strictly lfp. This is a special case of the next result. (This does not generalize to finitely presentable groups, see Example 3.17(2) below.)

Recall that a *groupoid* is a category with all morphisms invertible.

Proposition 3.16. *Let $\mathbb{G}$ be a finite groupoid. The category of presheaves on $\mathbb{G}$ is strictly lfp.*

Proof: (1) Put $S = \mathsf{obj}\,\mathbb{G}$. Then the category $\mathsf{Set}^{\mathbb{G}^{\mathrm{op}}}$ of presheaves can be considered as a variety of S-sorted unary algebras. The signature is given by the set of all morphisms of $\mathbb{G}^{\mathrm{op}}$: every morphism $f : X \to Y$ of $\mathbb{G}^{\mathrm{op}}$ corresponds to an operation symbol of arity $X \to Y$ (i.e., variables are of sort X and results of sort Y). This variety is presented by the equations corresponding to the composition in $\mathbb{G}^{\mathrm{op}}$: represent $g \cdot f = h : X \to Y$ in $\mathbb{G}^{\mathrm{op}}$ by $g(fx) = hx$ for a variable x of sort X. Moreover, for every object X, add the equation $\mathsf{id}_X(x) = x$ with x of sort X.

For every algebra A and every element $x \in A$ of sort X the subalgebra which x generates is denoted by A^x. Denote by $\sim_A$ the equivalence on the set of all elements of A defined by $x \sim_A y$ iff $A^x = A^y$. If $I(A)$ is a choice class of this equivalence, then we obtain a representation of A as the following coproduct:

$$A = \coprod_{x \in I(A)} A^x.$$

This follows from $\mathbb{G}$ being a groupoid: whenever $A^x \cap A^y \neq \emptyset$, then $x \sim_A y$.

Moreover, for every homomorphism $h : A \to B$ there exists a function $h_0 : I(A) \to I(B)$ such that on each A^x, $x \in I(A)$, h restricts to a homomorphism $h_0 \colon A^x \to B^{h(x)}$. Indeed, define $h_0(x)$ as the representative of $\sim_B$ with $B^{h(x)} = B^{h_0(x)}$.

(2) Given $x \in A$ of sort X, the algebra A^x is a quotient of the representable algebra $\mathbb{G}(-, X)$. Indeed, the Yoneda transformation corresponding to x, an element of A^x_X of sort X, has surjective components (by the definition of A^x).

Observe that every representable algebra has only finitely many quotients. This follows from the fact that $\mathbb{G}(-, X)$ has finitely many elements, hence, finitely many equivalence relations exist on the set of all elements.

(3) An algebra A is finitely presentable iff $I(A)$ is finite. This follows immediately from $\mathbb{G}$ having only finitely many morphisms.

(4) We are ready to prove that $\mathsf{Set}^{\mathbb{G}^{\mathrm{op}}}$ is strictly lfp. Let a morphism

$$b : B \to A, \qquad B \text{ finitely presentable}$$

be given. Then $\mathsf{Im}\, b$ is, due to (3), also finitely presentable. Due to (1) we know that the complement $\bar{C} = A \setminus \mathsf{Im}\, b$ is also a subalgebra of A. Let

$$e : \bar{C} = \coprod_{x \in I(\bar{C})} \bar{C}^x \to D$$

be the quotient merging two summands iff their domains are isomorphic algebras. Then by (2) the number of summands of $D = \coprod_{y \in I(D)} D^y$ is finite, hence, D is finitely generated. Choose any morphism $g \colon D \to \bar{C}$ by picking, for every D^y, one of the occurrences of D^y in $\bar{C}$. Then $e \cdot g = \mathsf{id}_D$.

The desired triangle with $B' = \mathsf{Im}\, b + D$ (which is finitely presentable) is as follows:

$$\begin{tikzcd} B \arrow[rr, "(\mathsf{id}+e)\cdot b"] \arrow[dr, "b"'] & & \mathsf{Im}\, b + D \arrow[dl, shift left, "\mathsf{id}+g"] \\ & A = \mathsf{Im}\, b + \bar{C} \arrow[ur, shift left, "\mathsf{id}+e"] & \end{tikzcd}$$ ■

Examples 3.17. Here we present lfp categories $\mathcal{A}$ which are not semi-strictly lfp. Moreover, in each case we present a non-finitary *endofunctor* that is finitely bounded.

(1) The category Un. In Example 3.6 we have already presented the promised endofunctor. For the empty algebra B and $A = \coprod_p C_p$, where p ranges over all prime numbers, there exists no finitely presentable algebra B' with $\mathsf{Un}(A, B') \neq \emptyset \neq \mathsf{Un}(B', A)$. Thus Un is not semi-strictly lfp.

(2) The category $\mathbb{Z}$-Set (of actions of the integers on sets). Since this category is equivalent to that of unary algebras with one invertible operation, the argument is as in (1).

(3) The category Gra of graphs and their homomorphisms. Let A denote the graph consisiting of a single infinite path, and let B be the empty graph. There exists no finite graph B' with $\mathsf{Gra}(A, B') \neq \emptyset \neq \mathsf{Gra}(B', A)$. Thus, Gra is not semi-strictly lfp. Analogously to Example 3.6 define an endofunctor F on Gra by

$$FX = \begin{cases} \mathbb{1} + X & \text{if } X \text{ contains no cycle and no infinite path} \\ \mathbb{1} & \text{else} \end{cases}$$

(where $\mathbb{1}$ is the terminal object), and $Ff = \mathsf{id}_{\mathbb{1}} + f$ if the codomain X of f fulfils $FX = \mathbb{1} + X$. This functor is clearly finitely bounded, but for A above, it does not preserve the colimit $A = \operatorname{colim} D_A$ of Remark 2.2(2).

(4) $\mathsf{Set}^{\mathbb{N}}$. If $\mathbb{1}$ is the terminal object, then $\mathsf{Set}^{\mathbb{N}}(\mathbb{1}, B') = \emptyset$ for all finitely presentable objects B. We define F on $\mathsf{Set}^{\mathbb{N}}$ by $FX = \mathbb{1} + X$ if X has only finitely many non-empty components, and $FX = \mathbb{1}$ else.

We next present two examples of rather important categories for which we prove that they are not semi-strictly lfp either.

Example 3.18. Nominal sets are not semi-strictly lfp. Let us first recall the definition of the category Nom of nominal sets (see e.g. [19]). We fix a countably infinite set $\mathbb{A}$ of *atomic names*. Let $\mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ denote the group of all finite permutations on $\mathbb{A}$ (generated by all transpositions). Consider a set X with an action of this group, denoted by $\pi \cdot x$ for a finite permutation π and $x \in X$. A subset $A \subseteq \mathbb{A}$ is called a *support* of an element $x \in X$ provided that every permutation $\pi \in \mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ that fixes all elements of A also fixes x:

$$\pi(a) = a \text{ for all } a \in A \implies \pi \cdot x = x.$$

A *nominal set* is a set with an action of the group $\mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ where every element has a finite support. The category Nom is formed by nominal sets and *equivariant maps*, i.e., maps preserving the given group action. Being a Grothendieck topos, Nom is lfp, and, as shown by Petrişan [18, Proposition 2.3.7], the finitely presentable nominal sets are precisely those with finitely many orbits (where an orbit of x is the set of all $\pi \cdot x$).

It is a standard result that every element x of a nominal set has the least support, denoted by $\mathsf{supp}(x)$. In fact, $\mathsf{supp}\colon X \to \mathcal{P}_{\mathsf{f}}(\mathcal{A})$ is itself an equivariant map, where the nominal structure of $\mathcal{P}_{\mathsf{f}}(\mathcal{A})$ is just element-wise. Consequently, any two elements of the same orbit x_1 and $x_2 = \pi \cdot x_1$ have a support of the same size. In addition, if $f : X \to Y$ is an equivariant map, it is clear that

$$\mathsf{supp}(f(x)) \subseteq \mathsf{supp}(x), \quad \text{for every } x \in X. \tag{2}$$

Now we present a non-finitary endofunctor of Nom which is finitely bounded. Consider for every natural number n the nominal set $P_n = \{Y \subseteq \mathbb{A} \mid |Y| = n\}$ with the action given by $\pi \cdot Y = \{\pi(v) \mid v \in Y\}$. Clearly, $\mathsf{supp}(Y) = Y$ for every $Y \in P_n$. It follows that there is no equivariant map $f : P_m \to X$ whenever $m > |\mathsf{supp}(x)| > 0$ for every $x \in X$, because of property (2).

Thus for $b\colon \emptyset \to \coprod_{0<n<\omega} P_n = A$ the existence of morphisms $A \underset{b'}{\overset{f}{\rightleftarrows}} X$ with X orbit-finite as in Definition 3.7 leads to a contradiction: Indeed suppose first $|\operatorname{\mathsf{supp}}(x)| > 0$ for all $x \in X$, then there is no equivariant map $P_m \to X$ with $m = 1+\max_{x\in X} |\operatorname{\mathsf{supp}}(x)|$, and thus no $f\colon \coprod_{0<n<\omega} P_n \to X$. In the case where $\operatorname{\mathsf{supp}}(x) = \emptyset$ for some $x \in X$, then there is no equivariant map $b'\colon X \to \coprod_{0<n<\omega} P_n$, because $\emptyset \neq \operatorname{\mathsf{supp}}(b'(x)) \subseteq \operatorname{\mathsf{supp}}(x) = \emptyset$ is a contradiction.

Analogously to Example 3.6 we define a functor F on Nom by

$$FX = \begin{cases} \mathbb{1} + X & \text{if } \mathsf{Nom}(P_n, X) = \emptyset \text{ for some } n < \omega \\ \mathbb{1} & \text{else.} \end{cases}$$

For an equivariant map $f : X \to Y$, if $FY = \mathbb{1} + Y$, then also $FX = \mathbb{1} + X$: given $\mathsf{Nom}(P_n, Y) = \emptyset$ for some n, then also $\mathsf{Nom}(P_n, X) = \emptyset$ must hold for the same n. In that case put $Ff = \mathsf{id}_{\mathbb{1}} + f$ and else Ff is the unique equivariant map to $FY = \mathbb{1}$. A very similar argument as in Example 3.6 shows that F is finitely bounded. However, F is not finitary, as it does not preserve the colimit $\coprod_{n<\omega} P_n$ of the chain $P_1 \hookrightarrow P_1+P_2 \hookrightarrow P_1+P_2+P_3 \hookrightarrow \cdots$.

Example 3.19. We prove next that the category $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ of finitary set functors (known to be lfp [5, Theorem 1.46]) has finitely generated objects coincident with the finitely presentable ones, but it fails to be semi-strictly lfp.

Remark 3.20. Recall that a quotient of an object F of $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ is represented by a natural transformation $\varepsilon\colon F \to G$ with epic components. Equivalently, G is isomorphic to F modulo a *congruence* $\sim$. This is a collection of equivalence relations $\sim_X$ on FX ($X \in \mathsf{Set}$) such that for every function $f\colon X \to Y$ given $p_1 \sim_X p_2$ in FX, it follows that $Ff(p_1) \sim_Y Ff(p_2)$.

We are going to characterize finitely presentable objects of $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ as the super-finitay functors introduced in [6]:

Definition 3.21. A set functor F is called *super-finitary* if there exists a natural number n such that Fn is finite and for every set X, the maps $(Ff)_{f\colon n\to X}$ are jointly surjective, i.e. they fulfil $FX = \bigcup_{f\colon n\to X} Ff[Fn]$.

Examples 3.22. (1) The functors $A \times \mathsf{Id}^n$ are super-finitary for all finite sets A and all $n \in \mathbb{N}$.

(2) More generally, let Σ be a finitary signature, i.e., a set of operation symbols σ of finite arities $|\sigma|$. The corresponding *polynomial set functor*

$$H_\Sigma X = \coprod_{\sigma\in\Sigma} X^{|\sigma|}$$

is super-finitary iff the signature has only finitely many symbols. We call such signatures *super-finitary*

(3) Every subfunctor F of $\mathsf{Set}(n,-)$, $n \in \mathbb{N}$, is super-finitary. Indeed, assuming $FX \subseteq \mathsf{Set}(n,X)$ for all X, we are to find, for each $p\colon n \to X$ in FX, a member $q\colon n \to n$ of Fn with $p = Ff(q)$ for some $f\colon n \to X$. That is, with $p = f \cdot q$. Choose a function $g\colon X \to n$ monic on $p[n]$. Then there exists $f\colon n \to X$ with $p = f \cdot g \cdot p$. From $p \in FX$ we deduce $Fg(p) \in Fn$, that is, $g \cdot p \in Fn$. Thus $q = g \cdot p$ is the desired element: we have $p = f \cdot q = Ff(q)$.

(4) Every quotient $\varepsilon\colon F \twoheadrightarrow G$ of a super-finitary functor F is super-finitary. Indeed, given $p \in GX$, find $p' \in FX$ with $p = \varepsilon_X(p')$. There exists $q' \in Fn$ with $p' = Ff(q')$ for some $f\colon n \to X$. We conclude that $q = \varepsilon_n(q')$ fulfils $p = Gf(q)$ from the naturality of ε.

Lemma 3.23. *The following conditions are equivalent for every set functor F:*

(1) F is super-finitary
(2) F is a quotient of the polynomial functor H_Σ for a super-finitary signature Σ, and
(3) F is a quotient of a functor $A \times \mathsf{Id}^n$ (A finite, $n \in \mathbb{N}$).

Proof: $3 \implies 2$ is clear and for $2 \implies 1$ see the Examples 2 and 4 above. To prove $1 \implies 3$, let F be super-finitary and put $A = Fn$ in the above definition. Apply Yoneda Lemma to $\mathsf{Id}^n \cong \mathsf{Set}(n,-)$ and use that $[\mathsf{Set},\mathsf{Set}]_{\mathsf{fin}}$ is cartesian closed:

$$\frac{Fn \xrightarrow{\cong} [\mathsf{Set},\mathsf{Set}]_{\mathsf{fin}}(\mathsf{Set}(n,-),F)}{\varepsilon\colon Fn \times \mathsf{Set}(n,-) \longrightarrow F}$$

The definition of super-finitary shows that ε_X is surjective for every X. ∎

Proposition 3.24. *Super-finitary functors are closed in $[\mathsf{Set},\mathsf{Set}]_{\mathsf{fin}}$ under finite products, finite coproducts, subfunctors, and hence under finite limits.*

Proof: (1) Finite products and coproducts are clear: given quotients $\varepsilon_i\colon A_i\times \mathsf{Id}^{n_i} \twoheadrightarrow F_i$, $i\in\{1,2\}$, then $F_1\times F_2$ is super-finitary due to the quotient

$$\varepsilon_1\times\varepsilon_2\colon (A_1\times A_2)\times \mathsf{Id}^{n_1+n_2}\to F_1\times F_2.$$

Suppose $n_1\geq n_2$, then we can choose a quotient $\varphi\colon A_2\times\mathsf{Id}^{n_1}\twoheadrightarrow A_2\times\mathsf{Id}^{n_2}$. This proves that F_1+F_2 is super finitary due to the quotient

$$\varepsilon_1+(\varepsilon_2\cdot\varphi)\colon (A_1+A_2)\times\mathsf{Id}^{n_1}\cong A_1\times\mathsf{Id}^{n_1}+A_2\times\mathsf{Id}^{n_1}\to F_1+F_2.$$

(2) Let $\mu\colon G\rightarrowtail F$ be a subfunctor of a super-finitary functor F with a quotient $\varepsilon\colon A\times\mathsf{Id}^n\twoheadrightarrow F$. Form a pullback (object-wise in Set) of ε and μ:

$$\begin{tikzcd} H \arrow[r, tail, "\bar\mu"] \arrow[d, two heads, "\bar\varepsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A\times\mathsf{Id}^n \arrow[d, two heads, "\varepsilon"] \\ G \arrow[r, tail, "\mu"] & F \end{tikzcd}$$

For each $a\in A$, the preimage H_a of $\{a\}\times\mathsf{Id}^n\cong\mathsf{Set}(n,-)$ under $\bar\mu$ is super-finitary by Example 3 above. Since $A\times\mathsf{Id}^n=\coprod_{a\in A}\{a\}\times\mathsf{Id}^n$ and preimages under $\bar\mu$ preserve coproducts, we have $H=\coprod_{a\in A}H_a$ and so G is the quotient of the super-finitary functor H.

■

Lemma 3.25. *Let $\mathcal{C}$ be an lfp category with finitely generated objects closed under kernel pairs and in which strong epimorphisms are regular. Then finitely presentable and finitely generated objects coincide.*

Proof: We apply Remark 2.25: Consider a strong epimorphism $c\colon X\twoheadrightarrow Y$ with X finitely presentable. We are to show that Y is finitely presentable. Let $p,q\colon K\rightrightarrows X$ be the kernel pair of c, then K is finitely generated. Hence there is some finitely presentable object K' and a strong epimorphism $e\colon K'\twoheadrightarrow K$:

$$K'\xrightarrow{e}\!\!\!\!\rightarrow K \overset{p}{\underset{q}{\rightrightarrows}} X\xrightarrow{c}\!\!\!\!\rightarrow Y$$

Since the strong epimorphism c is also regular, it is the coequalizer of its kernel pair (p,q); furthermore e is epic, thus c is the coequalizer of $p\cdot e$ and $q\cdot e$. That means that Y is a finite colimit of finitely presentable objects and thus it is finitely presentable. ■

Corollary 3.26. $[\mathsf{Set},\mathsf{Set}]_{\mathsf{fin}}$ *is not semi-strictly lfp.*

Proof: We use the subfunctors

$$\bar{\mathcal{P}} \subseteq \mathcal{P}_0 \subseteq \mathcal{P}$$

of the power-set functor $\mathcal{P}$ given by $\mathcal{P}_0 X = \mathcal{P}X - \{\emptyset\}$ and $\bar{\mathcal{P}}X = \{M \in \mathcal{P}_0 X \mid M \text{ finite}\}$. Then $\bar{\mathcal{P}}$ is an object of $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$. The only endomorphism of $\bar{\mathcal{P}}$ is $\mathsf{id}_{\bar{\mathcal{P}}}$. Indeed for $\mathcal{P}_0$ this has been proven in [6, Proposition 5.4]; the same proof applies to $\bar{\mathcal{P}}$. For a finitely presentable object B of $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$, no natural transformation $\beta\colon B \to \bar{\mathcal{P}}$ is surjective. In fact, given $\varepsilon\colon A \times X^n \twoheadrightarrow B$, everything in the image of $\beta_X \cdot \varepsilon_X$ has cardinality of at most n by naturality of $\beta \cdot \varepsilon$. Furthermore, for such a β, no morphism $\alpha\colon \bar{\mathcal{P}} \to B$ exists – because then β would be a split epimorphism by $\mathsf{id}_{\bar{\mathcal{P}}} = \beta \cdot \alpha$. ∎

Corollary 3.27. *For a finitary set functor, as an object of* $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$*, the following conditions are equivalent:*

(1) finitely presentable,
(2) finitely generated, and
(3) super-finitary.

Proof: To verify $2 \implies 3$, let F be finitely generated. For every finite subset $A \subseteq Fn$, $n \in \mathbb{N}$, we have a subfunctor $F_{n,A} \subseteq F$ given by

$$F_{n,A}X = \bigcup_{f\colon n \to X} Ff[A].$$

Since F is finitary, it is a directed union of all these subfunctors. This implies $F \cong F_{n,A}$ for some n and A, and $F_{n,A}$ is clearly super-finitary.

For $3 \implies 2$, combine Lemma 3.23 and Example 3.221.

$1 \iff 2$ follows by Lemma 3.25. ∎

4. Finitely Presentable Algebras

In the introduction we have recalled the definition of a finitely presentable algebra from General Algebra and the fact that for a finitary monad $\mathbb{T}$ on Set, this is equivalent to A being a finitely presentable object of $\mathsf{Set}^{\mathbb{T}}$. We now generalize this to finitary *regular monads* [16], i.e., those preserving regular epimorphisms, on lfp categories that have regular factorizations.

First, we turn to characterizing finitely generated algebras for *arbitrary* finitary monads.

Remark 4.1. Let $\mathbb{T}$ be a finitary monad on an lfp category $\mathcal{A}$. Then the Eilenberg-Moore category $\mathcal{A}^{\mathbb{T}}$ is also lfp [5, Remark 2.78]. Thus, it has (strong epi, mono)-factorizations. The monomorphisms in $\mathcal{A}^{\mathbb{T}}$, representing *subalgebras*, are precisely the $\mathbb{T}$-algebra morphisms carried by a monomorphism of $\mathcal{A}$ (since the forgetful functor $\mathcal{A}^{T} \to \mathcal{A}$ creates limits). The strong epimorphisms of $\mathcal{A}^{\mathbb{T}}$, representing *strong quotient algebras*, need not coincide with those carried by strong epimorphisms of $\mathcal{A}$ – we do not assume that $\mathbb{T}$ preserves strong epimorphisms.

Recall our terminology that a finitely generated subobject of an object A is a monomorphism $m\colon M \rightarrowtail A$ with M a finitely generated object.

Notation 4.2. Throughout this section given a $\mathbb{T}$-algebra morphism $f\colon X \to Y$ we denote by $\mathsf{Im}\, f$ its image in $\mathcal{A}^{\mathbb{T}}$. That is, we have a strong epimorphism $e\colon X \twoheadrightarrow \mathsf{Im}\, f$ and a monomorphism $m\colon\ \mathsf{Im}\, f \rightarrowtail B$ in $\mathcal{A}^{\mathbb{T}}$ with $f = m \cdot e$.

Definition 4.3. An algebra (A, a) for $\mathbb{T}$ is said to be *generated by a subobject* $m\colon M \rightarrowtail A$ of the base category $\mathcal{A}$ if no proper subalgebra of (A, a) contains m.

The phrase "(A, a) is generated by a finitely generated subobject" may sound strange, but its meaning is clear: there exists a subobject $m\colon M \rightarrowtail A$ with M in $\mathcal{A}_{\mathsf{fg}}$ such that m does not factorize through any proper subalgebra of (A, a).

Example 4.4. The free algebras on finitely presentable objects are shortly called *ffp algebras* below: they are the algebras (TX, μ_X) with X finitely presentable.

(1) Every ffp algebra is generated by a finitely generated object: factorize the unit $\eta_X\colon X \to TX$ in $\mathcal{A}$ as a strong epimorphism $e\colon X \twoheadrightarrow M$ (thus, M is finitely generated by Remark 2.2(5)) followed by a monomorphism $m\colon M \rightarrowtail TX$. Using the universal property, it is easy to see that m generates (TX, μ_X); indeed, suppose we had a subalgebra $s\colon (A, a) \rightarrowtail (TX, \mu_X)$ containing m, via $n\colon M \rightarrowtail A$, say. Then the unique extension of $n \cdot e\colon X \to A$ to a $\mathbb{T}$-algebra morphism $h\colon (TX, \mu_X) \to (A, a)$ satisfies $s \cdot h = \mathsf{id}_{(TX,\mu_X)}$. Thus, s is an isomorphism.

(2) Every ffp algebra is finitely presentable in $\mathcal{A}^{\mathbb{T}}$: apply Lemma 2.5 to the forgetful functor $R\colon \mathcal{A}^{\mathbb{T}} \to \mathcal{A}$ and its left adjoint $LX = (TX, \mu_X)$.

Theorem 4.5. *For every finitary monad $\mathbb{T}$ on an lfp category $\mathcal{A}$ the following conditions on an algebra (A, a) are equivalent:*

(1) (A,a) is generated by a finitely generated subobject,
(2) (A,a) is a strong quotient algebra of an ffp algebra, and
(3) (A,a) is a finitely generated object of $\mathcal{A}^{\mathbb{T}}$.

Proof: (3) $\implies$ (2) First observe that the cocone $Tf\colon TX \to TA$, where (X,f) ranges over $\mathcal{A}_{\mathsf{fp}}/A$, is collectively epimorphic since T preserves the filtered colimit $A = \operatorname{colim} D_A$ of Remark 2.2(2). For every $f\colon X \to A$ consider its unique extension to a $\mathbb{T}$-algebra morphism $a \cdot Tf\colon (TX,\mu_X) \to (A,a)$ and form its factorization in $\mathcal{A}^{\mathbb{T}}$:

$$\begin{tikzcd} TX \arrow[r, two heads, "e_f"] \arrow[d, "Tf"'] & \mathsf{Im}(a \cdot Tf) \arrow[d, tail, "m_f"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

Now observe that $a\colon (TA,\mu_A) \to (A,a)$ is a strong epimorphism in $\mathcal{A}^{\mathbb{T}}$; in fact, the laws of Eilenberg-Moore algebras for $\mathbb{T}$ imply that a is the coequalizer of

$$(TTA,\mu_{TA}) \underset{\mu_A}{\overset{Ta}{\rightrightarrows}} (TA,\mu_A).$$

From Remark 2.22 and the finitarity of the functor T we deduce that Tf : $(TX,\mu_X) \to (TA,\mu_A)$, $f \in \mathcal{A}_{\mathsf{fp}}/A$, is a filtered colimit in $\mathcal{A}^{\mathbb{T}}$. It follows from Lemma 2.11 that (A,a) is a directed union of its subobjects m_f for f in $\mathcal{A}_{\mathsf{fp}}/A$.

Now since (A,a) is finitely generated, id_A factorizes through one of the corresponding colimit injections $m_f\colon \mathsf{Im}(a \cdot Tf) \rightarrowtail A$ for some $f\colon X \to A$ in $\mathcal{A}_{\mathsf{fp}}/A$. Therefore m_f is split epic, whence an isomorphism, and A is a strong quotient of (TX,μ_X) via e_f, as desired.

(2) $\implies$ (1) Let $q\colon (TX,\mu_X) \twoheadrightarrow (A,a)$ be a strong epimorphism in $\mathcal{A}^{\mathbb{T}}$ with X finitely presentable in $\mathcal{A}$. Factorize $q \cdot \eta_X$ as a strong epimorphism followed by a monomorphism in $\mathcal{A}$:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, two heads, "e"'] & TX \arrow[d, two heads, "q"] \\ M \arrow[r, tail, "m"'] & A \end{tikzcd}$$

Then M is finitely generated in $\mathcal{A}$ by Remark 2.2(5). We shall prove that every subalgebra $u\colon (B,b) \rightarrowtail (A,a)$ containing m (i.e., such that there is a morphism $g\colon M \to B$ in $\mathcal{A}$ with $u \cdot g = m$) is isomorphic to (A,a). Let

$e^\sharp\colon (TX, \mu_X) \to (B, b)$ be the unique extension of $g \cdot e$ to a $\mathbb{T}$-algebra morphism:

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[dd, two heads, "e"'] & & TX \arrow[dl, "e^\sharp"'] \arrow[dd, two heads, "q"] \\
& B \arrow[dr, "u"] & \\
M \arrow[ur, "g"] \arrow[rr, tail, "m"'] & & A
\end{tikzcd}$$

Then we see that $u \cdot e^\sharp = q$ because this triangle of $\mathbb{T}$-algebra morphisms commutes when precomposed by the universal morphism η_X. Since q is strong epic, so is u, and therefore u is an isomorphism, as desired.

(1) $\Longrightarrow$ (2) Let $m : M \rightarrowtail A$ be a finitely generated subobject of A that generates (A, a). By Remark 2.2(5), there exists a strong epimorphism $q : X \twoheadrightarrow M$ in $\mathcal{A}$ with X finitely presentable. The unique extension $e = (m \cdot q)^\sharp : (TX, \mu_X) \to (A, a)$ to a $\mathbb{T}$-algebra morphism is an *extremal epimorphism*, i.e., if e factorizes through a subalgebra $u : (B, b) \rightarrowtail (A, a)$, then u is an isomorphism. To prove this, recall that u is also monic in $\mathcal{A}$. Given $e = u \cdot e'$ we use the diagonal fill-in property in $\mathcal{A}$:

$$\begin{tikzcd}
X \arrow[r, two heads, "q"] \arrow[d, "\eta_X"'] & M \arrow[dd, tail, "m"] \arrow[ddl, dashed] \\
TX \arrow[d, "e'"'] & \\
B \arrow[r, tail, "u"'] & A
\end{tikzcd}$$

Since m generates (A, a), this proves that u is an isomorphism. In a complete category every extremal epimorphism is strong, thus we have proven (2).

By Remark 2.2 (5) and the fact that ffp algebras are finitely presentable in $\mathcal{A}^\mathbb{T}$ (see Example 4.4(b)) we have (2) $\Longrightarrow$ (3). ■

As usual, by a *congruence* on a $\mathbb{T}$-algebra (A, a) is meant a subalgebra $(K, k) \rightarrowtail (A, a) \times (A, a)$ forming a kernel pair $\ell, r : (K, k) \rightrightarrows (A, a)$ of some $\mathbb{T}$-algebra morphism. Given a coequalizer $q : (A, a) \twoheadrightarrow (B, b)$ of ℓ, r in $\mathcal{A}^\mathbb{T}$, then (B, b) is called the *quotient algebra* of (A, a) modulo (K, k).

Definition 4.6. A congruence $\ell, r : (K, k) \rightrightarrows (A, a)$ is called a *finitely generated congruence* if there exists a finitely generated subalgebra $m : (K', k') \rightarrowtail$

(K,k) in $\mathcal{A}^{\mathbb{T}}$ such that the quotient of (A,a) modulo (K,k) is also a coequalizer of $\ell \cdot m$ and $r \cdot m$:

$$(K',k') \overset{m}{\rightarrowtail} (K,k) \underset{r}{\overset{\ell}{\rightrightarrows}} (A,a) \overset{q}{\twoheadrightarrow} (B,b).$$

In the next theorem we assume that our base category has regular factorizations, i.e., every strong epimorphism is regular.

Theorem 4.7. *Let $\mathbb{T}$ be a regular, finitary monad on an lfp category $\mathcal{A}$ which has regular factorizations. For every $\mathbb{T}$-algebra (A,a) the following conditions are equaivalent:*

(1) (A,a) is a quotient of an ffp algebra modulo a finitely generated congruence,
(2) (A,a) is a coequalizer of a parallel pair of $\mathbb{T}$-algebra morphisms between ffp algebras:

$$(TY,\mu_Y) \underset{g}{\overset{f}{\rightrightarrows}} (TX,\mu_X) \overset{e}{\twoheadrightarrow} (A,a) \qquad (X,Y \text{ in } \mathcal{A}_{\mathsf{fp}}), \qquad \textit{and}$$

(3) (A,a) is a finitely presentable object of $\mathcal{A}^{\mathbb{T}}$.

Proof: (2) $\Longrightarrow$ (3) Since finitely presentable objects are closed under finite colimits, this follows from Example 4.4(b).

(3) $\Longrightarrow$ (1) First note that the classes of regular and strong epimorphisms in $\mathcal{A}^{\mathbb{T}}$ coincide; indeed, since $\mathbb{T}$ preserves regular epimorphisms, the regular factorizations of $\mathcal{A}$ lift to $\mathcal{A}^{\mathbb{T}}$ (see [16, Proposition 4.17]). Then, by Theorem 4.5, (A,a) is a regular quotient of an ffp algebra via $q : (TX,\mu_X) \twoheadrightarrow (A,a)$, say.

Now take the kernel pair $\ell, r\colon (K,k) \rightrightarrows (TX,\mu_X)$ of q in $\mathcal{A}^{\mathbb{T}}$ and note that q is its coequalizer. Write K in $\mathcal{A}$ as the filtered colimit of its canonical filtered diagram $D_K\colon \mathcal{A}_{\mathsf{fp}}/K \to \mathcal{A}$ (see Remark 2.2(2)) and take for any of the colimit injections $y : Y \to K$ the unique extension $y^\sharp : (TY,\mu_Y) \to (K,k)$ to a $\mathbb{T}$-algebra morphism. Next form for every y in $\mathcal{A}_{\mathsf{fp}}/K$ the following coequalizer in $\mathcal{A}^{\mathbb{T}}$:

$$(TY,\mu_Y) \xrightarrow{y^\sharp} (K,k) \underset{r}{\overset{\ell}{\rightrightarrows}} (TX,\mu_X) \overset{e_y}{\twoheadrightarrow} (A_y,a_y).$$

(a) This defines a filtered diagram $\overline{D} : \mathcal{A}_{\mathsf{fp}}/K \to \mathcal{A}^{\mathbb{T}}$ taking y to e_y. In fact, for every morphism $f : (Y,y) \to (Z,z)$ in $\mathcal{A}_{\mathsf{fp}}/K$ we obtain a $\mathbb{T}$-algebra morphism $a_f : (A_y, a_y) \to (A_z, a_z)$ using the following diagram in $\mathcal{A}^{\mathbb{T}}$ (where we drop the algebra structures):

$$\begin{tikzcd}
TY \arrow[dd, "Tf"'] \arrow[dr, "y^\sharp"] & & & & A_y \arrow[dd, dashed, "a_f"] \\
& K \arrow[r, shift left, "\ell"] \arrow[r, shift right, "r"'] & TX \arrow[urr, two heads, "e_y"] \arrow[drr, two heads, "e_z"'] & & \\
TZ \arrow[ur, "z^\sharp"'] & & & & A_z
\end{tikzcd}$$

Note that a_f is a regular epimorphism in $\mathcal{A}^{\mathbb{T}}$. Furthermore, for every $y : Y \to K$ in $\mathcal{A}_{\mathsf{fp}}/K$ we also obtain a morphism $d_y : A_y \to A$ such that $d_y \cdot e_y = q$:

$$\begin{tikzcd}
TY \arrow[r, "y^\sharp"] & K \arrow[r, shift left, "\ell"] \arrow[r, shift right, "r"'] & TX \arrow[r, two heads, "e_y"] \arrow[d, two heads, "q"'] & A_y \arrow[dl, dashed, "d_y"] \\
& & A &
\end{tikzcd}$$

These morphisms d_y form a cocone on the diagram $\overline{D}$; indeed, we have for every morphism $f : (Y, y) \to (Z, z)$ of $\mathcal{A}_{\mathsf{fp}}/K$ that

$$d_z \cdot a_f \cdot e_y = d_z \cdot e_z = q = d_y \cdot e_y,$$

and we conclude that $d_z \cdot a_f = d_y$ since e_y is epic.

(b) We now show that $(A, a) = \operatorname{colim} \overline{D}$ with colimit injections $d_y : (A_y, a_y) \to (A, a)$. Given any cocone $b_y : (A_y, a_y) \to (B, b)$ of $\overline{D}$, we prove that it factorizes uniquely through (d_y). We first note that all the morphisms $b_y \cdot e_y$ are equal because the diagram is filtered and for every morphism $f : (Y, y) \to (Z, z)$ in $\mathcal{A}_{\mathsf{fp}}/K$ we have the commutative diagram below:

$$\begin{tikzcd}
& A_y \arrow[dd, "a_f"] \arrow[dr, "b_y"] & \\
TX \arrow[ur, two heads, "e_y"] \arrow[dr, two heads, "e_z"'] & & B \\
& A_z \arrow[ur, "b_z"'] &
\end{tikzcd}$$

Let us call the above morphism $q' : TX \to B$, and observe that for every $y : Y \to K$ in $\mathcal{A}_{\mathsf{fp}}/K$ we have

$$q' \cdot \ell \cdot y^\sharp = b_y \cdot e_y \cdot \ell \cdot y^\sharp = b_y \cdot e_y \cdot r \cdot y^\sharp = q' \cdot r \cdot y^\sharp.$$

The cocone of morphisms $y^\sharp : TY \to K$ is collectively epic since so is the colimit cocone $y : Y \to K$, and therefore $q' \cdot \ell = q' \cdot r$. Thus, there exists a unique factorization $h : A \to B$ of q' through $q = \mathsf{coeq}(\ell, r)$, i.e., such that $h \cdot q = q'$. We now have, for every $y : Y \to K$ in $\mathcal{A}_{\mathsf{fp}}/K$,

$$h \cdot d_y \cdot e_y = h \cdot q = q' = b_y \cdot e_y,$$

which implies $h \cdot d_y = b_y$ using that e_y is epic.

Uniqueness of h with the latter property follows immediately: if $k : A \to B$ fulfils $k \cdot d_y = b_y$ for every y in $\mathcal{A}_{\mathsf{fp}}/K$, we have

$$k \cdot q = k \cdot d_y \cdot e_y = b_y \cdot e_y = q' = h \cdot q.$$

(c) Now use that (A, a) is finitely presentable in $\mathcal{A}^{\mathbb{T}}$ to see that there exists some $w : W \to K$ in $\mathcal{A}_{\mathsf{fp}}/K$ and a $\mathbb{T}$-algebra morphism $s : (A, a) \to (A_w, a_w)$ such that $d_w \cdot s = \mathsf{id}_A$. Then $s \cdot d_w$ is an endomorphism of the $\mathbb{T}$-algebra (A_w, a_w) satisfying $d_w \cdot (s \cdot d_w) = d_w$. Since e_w is a coequalizer of a parallel pair of $\mathbb{T}$-algebra morphisms between ffp algebras, (A_w, a_w) is finitely presentable by Example 4.4(b). Since the colimit injection d_w merges $s \cdot d_w$ and id_{A_w}, there exists a morphism $f : (W, w) \to (Y, y)$ in $\mathcal{A}_{\mathsf{fp}}/K$ with a_f merging them too, i.e., such that $a_f \cdot (s \cdot d_w) = a_f$. This implies that $d_y : (A_y, a_y) \to (A, a)$ is an isomorphism with inverse $a_f \cdot s$. Indeed, we have

$$d_y \cdot (a_f \cdot s) = d_w \cdot s = \mathsf{id}_A,$$

and for $(a_f \cdot s) \cdot d_y = \mathsf{id}_{A_y}$ we use that a_f is epic:

$$(a_f \cdot s) \cdot d_y \cdot a_f = a_f \cdot s \cdot d_w = a_f.$$

Thus, $(A, a) \cong (A_y, a_y)$.

(d) It remains to prove that $\ell, r : (K, k) \rightrightarrows (TX, \mu_X)$ is a finitely generated congruence. To see this, take the regular factorization of $y^\sharp : (TY, \mu_Y) \to (K, k)$ for the above y for which $(A, a) \cong (A_y, a_y)$:

$$y^\sharp = \big(\, (TY, \mu_Y) \xrightarrow{e} \mathsf{Im}(y^\sharp) \xrightarrow{m} (K, k)\, \big).$$

Then e_y is also the coequalizer of $\ell \cdot m$ and $r \cdot m$, and $\mathsf{Im}(y^\sharp)$ is a finitely generated $\mathbb{T}$-algebra by Theorem 4.5, as desired.

(1) $\implies$ (2) We are given a regular epimorphism $e : (TX, \mu_X) \twoheadrightarrow (A, a)$ with X finitely presentable in $\mathcal{A}$ and a pair $\ell', r' : (K', k') \rightrightarrows (TX, \mu_X)$ with (K', k') finitely generated, whose coequalizer is e. By Theorem 4.5, there exists a regular quotient $q : (TY, \mu_Y) \twoheadrightarrow (K', k')$ with Y finitely presentable

in $\mathcal{A}$. Since e is a coequalizer of ℓ', r', it is also a coequalizer of the pair $\ell' \cdot q, r' \cdot q$. ■

Open Problem 4.8. Are (1)–(3) above equivalent for all finitary monads (not necessarily regular)?

5. Finitary Monads on Sets

We have seen in Corollary 3.27 that finitely presentable objects of $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ are precisely the finitely generated ones. In contrast, we show that in the category $\mathsf{Mnd_f}(\mathsf{Set})$ of finitary monads on Set the classes of finitely presentable and finitely generated objects do *not* coincide.

Remark 5.1. We apply Lack's result that the category of finitary monads on an lfp category $\mathcal{A}$ is monadic over the category $\mathsf{Sig}(\mathcal{A})$ of signatures (which we recall in Example 5.2), see Corollary 3 of [15]. By using the same proof one can see that the category of finitary endofunctors is also monadic over $\mathsf{Sig}(\mathcal{A})$.

Example 5.2. As an application of Theorem 4.5, we generalize the above fact that $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ has as finitely generated objects precisely the super-finitary functors, see Corollary 3.27, to all lfp categories $\mathcal{A}$. Denote by

$$[\mathcal{A}, \mathcal{A}]_{\mathsf{fin}}$$

the category of all finitary endofunctors of $\mathcal{A}$. An example is the polynomial functor H_Σ for every signature Σ in the sense of Kelly and Power [13]. This means a collection of objects Σ_n of $\mathcal{A}$ indexed by $n \in \mathcal{A}_{\mathsf{fp}}$. Let $|\mathcal{A}_{\mathsf{fp}}|$ be the discrete category of objects of $\mathcal{A}_{\mathsf{fp}}$, then the functor category

$$\mathsf{Sig}(\mathcal{A}) = \mathcal{A}^{|\mathcal{A}_{\mathsf{fp}}|}$$

is the *category of signatures* (whose morphisms from $\Sigma \to \Sigma'$ are collections of morphisms $e_n \colon \Sigma_n \to \Sigma'_n$ for $n \in |\mathcal{A}_{\mathsf{fp}}|$). The *polynomial functor* H_Σ is the coproduct of the endofunctors $\mathcal{A}(n, -) \bullet \Sigma_n$, where $\bullet$ denotes copowers of Σ_n, shortly:

$$H_\Sigma X = \coprod_{n \in \mathcal{A}_{\mathsf{fp}}} \mathcal{A}(n, X) \bullet \Sigma_n.$$

We obtain an adjoint situation

$$\begin{tikzcd} {[\mathcal{A}, \mathcal{A}]_{\mathsf{fin}}} \arrow[r, shift left, "U"] \arrow[r, phantom, "\top"] & \mathsf{Sig}(\mathcal{A}) \arrow[l, shift left, "\Phi"] \end{tikzcd}$$

where the forgetful functor U takes a finitary endofunctor F to the signature

$$U(F)_n = Fn \qquad (n \in \mathcal{A}_{\mathsf{fp}})$$

and Φ takes a signature Σ to the polynomial endofunctor $\Phi\Sigma = H_\Sigma$. The resulting monad $\mathbb{T}$ is given by

$$T(\Sigma)_n = \coprod_{m \in \mathcal{A}_{\mathsf{fp}}} \mathcal{A}(m,n) \bullet \Sigma_m.$$

As mentioned above, [15, Corollary 3] implies that the forgetful functor U is monadic. Thus, the category $[\mathcal{A},\mathcal{A}]_{\mathsf{fin}}$ is equivalent to the Eilenberg-Moore category of the monad $\mathbb{T}$. By Theorem 4.5, finitely generated objects of $[\mathcal{A},\mathcal{A}]_{\mathsf{fin}}$ are precisely the strong quotients of ffp algebras for $\mathbb{T}$. Now by Lemma 2.6, a signature Σ is finitely presentable in $\mathcal{A}^{|\mathcal{A}_{\mathsf{fp}}|}$ iff for the initial object 0 of $\mathcal{A}$ we have

$$\Sigma_n = 0 \text{ for all but finitely many } n \in \mathcal{A}_{\mathsf{fp}}$$

and

$$\Sigma_n \text{ is finitely presentable for every } n \in \mathcal{A}_{\mathsf{fp}}.$$

Let us call such signatures *super-finitary.* We thus obtain the following result.

Proposition 5.3. *For an lfp category $\mathcal{A}$, a finitary endofunctor is finitely generated in $[\mathcal{A},\mathcal{A}]_{\mathsf{fin}}$ iff it is a strong quotient of a polynomial functor H_Σ with Σ super-finitary.*

Example 5.4. Another application of results of Section 4: the category

$$\mathsf{Mnd_f}(\mathcal{A})$$

of all finitary monads on an lfp category $\mathcal{A}$. Lack proved in [15] that this category is also monadic over the category of signatures. More precisely, for the forgetful functor $V \colon \mathsf{Mnd_f}(\mathcal{A}) \to [\mathcal{A},\mathcal{A}]_{\mathsf{fin}}$ the composite

$$UV \colon \mathsf{Mnd_f}(\mathcal{A}) \to \mathsf{Sig}(\mathcal{A})$$

is a monadic functor. Recall from Barr [8] that every finitary endofunctor H generates a free monoid; let us denote it by H^*. The corresponding free monad $\mathbb{T}$ for UV assigns to every signature Σ the signature derived from the free monad on Σ (w.r.t. UV), or, equivalently, from the free monad H_Σ^* on the polynomial endofunctor H_Σ. Thus the monad $\mathbb{T}$ is given by the following rule for Σ in $\mathsf{Sig}(\mathcal{A})$:

$$(T\Sigma)_n = H_\Sigma^* n \qquad \text{for all } n \in \mathcal{A}_{\mathsf{fp}}.$$

(Example: if $\mathcal{A} = \mathsf{Set}$ then H_Σ^* assigns to every set X the set $H_\Sigma^* X$ of all Σ-terms with variables in X.) In general, it follows from [?] that the underlying functor of H_Σ^* is the colimit of the following ω-chain in $[\mathcal{A}, \mathcal{A}]_{\mathsf{fin}}$:

$$\mathsf{Id} \xrightarrow{w_0} H_\Sigma + \mathsf{Id} \xrightarrow{H_\Sigma w_0 + \mathsf{id}} H_\Sigma(H_\Sigma + \mathsf{Id}) + \mathsf{Id} \longrightarrow \cdots W_n \xrightarrow{w_n} W_{n+1} \longrightarrow \cdots$$

Here, $W_0 = \mathsf{Id}$ and $W_{n+1} = H_\Sigma W_n + \mathsf{Id}$, and w_0 is the coproduct injection and $w_{n+1} = H_\Sigma w_n + \mathsf{id}$. The monad H_Σ^* is thus the free $\mathbb{T}$-algebra on Σ and the ffp algebras are precisely H_Σ^* for Σ super-finitary.

Definition 5.5. Let Σ be a signature in an lfp category $\mathcal{A}$.

(1) By a Σ-*equation* is meant a parallel pair

$$f, f' \colon k \longrightarrow H_\Sigma^* n \qquad \text{with } k, n \in \mathcal{A}_{\mathsf{fp}}$$

of morphisms in $\mathcal{A}$.

(2) A quotient of H_Σ^* in $\mathsf{Mnd}_{\mathsf{f}}(\mathcal{A})$ is said to *satisfy* the equation if its n-component merges f and f'.

(3) By a *presentation* of a monad $\mathbb{M}$ in $\mathsf{Mnd}_{\mathsf{f}}(\mathcal{A})$ is meant a signature Σ and a collection of Σ-equations such that the least quotient of H_Σ^* satisfying all of the given equations has the form $c \colon H_\Sigma^* \twoheadrightarrow \mathbb{M}$.

If $\mathcal{A} = \mathsf{Set}$, this is the classical concept of a presentation of a variety by equations. Indeed, given a pair $f, f' \colon 1 \to H_\Sigma^* n$, which is a pair of Σ-terms in n variables, satisfaction of the equation $f = f'$ in the sense of general algebra means that precisely $c_n \cdot f = c_n \cdot f'$. And a general pair $f, f' \colon k \to H_\Sigma^* n$ can be substituted by k pairs of terms in n variables.

Remark 5.6. (1) Every finitary monad $\mathbb{M}$ has a presentation. Indeed, since this is an algebra for the monad $\mathbb{T}$, it is a coequalizer of a parallel pair of monad morphisms between free algebras for $\mathbb{T}$:

$$H_\Gamma^* \overset{\ell}{\underset{r}{\rightrightarrows}} H_\Sigma^* \xrightarrow{c} \mathbb{M}$$

To give a monad morphism ℓ is equivalent to giving a signature morphism

$$\ell_n \colon \Gamma_n \longrightarrow H_\Sigma^* n \qquad (n \in |\mathcal{A}_{\mathsf{fp}}|).$$

Analogously for $r \mapsto (r_n)$. Thus, to say that c merges ℓ and r is the same as to say that it satisfies the equations $\ell_n, r_n \colon \Gamma_n \to H_\Sigma^* n$ for all n. And the above coequalizer c is the least such quotient.

(2) Every equation $f, f' \colon k \to H^*_\Sigma n = \operatorname{colim}_{r\in\mathbb{N}} W_r n$ can be substituted, for some number r (the "depth" of the terms), by an equation $g, g' \colon k \to W_r n$. This follows from k being finitely presentable.

Theorem 5.7. *Let $\mathcal{A}$ be an lfp category with regular factorizations. A finitary monad is, as an object of $\mathsf{Mnd_f}(\mathcal{A})$,*

(1) finitely generated iff it has a presentation by Σ-equations with Σ super-finitary, and

(2) finitely presentable iff it has a presentation by finitely many Σ-equations with Σ super-finitary.

Proof: (1) Let $\mathbb{M}$ have a presentation with Σ super-finitary. Then $\mathbb{M}$ is a (regular) quotient of an ffp-algebra H^*_Σ for $\mathbb{T}$, thus, it is finitely generated by Theorem 4.5.

Conversely, if $\mathbb{M}$ is finitely generated, it is a (strong) quotient $c \colon H^*_\Sigma \twoheadrightarrow \mathbb{M}$ for Σ super-finitary. It is sufficient to show that c is a regular epimorphism in $\mathsf{Mnd_f}(\mathcal{A})$, then the argument that $\mathbb{M}$ has a presentation using Σ is as in Remark 5.6.

Since $\mathcal{A}$ has regular factorizations, so does $\mathsf{Sig}(\mathcal{A}) = \mathcal{A}^{|\mathcal{A}_{\mathsf{fp}}|}$. And the monad $\mathbb{T}$ on $\mathsf{Sig}(\mathcal{A})$ given by

$$(T\Sigma)_n = H^*_\Sigma n \qquad (n \in \mathcal{A}_{\mathsf{fp}})$$

is regular. Indeed, for every regular epimorphism $e \colon \Sigma \twoheadrightarrow \Gamma$ in $\mathsf{Sig}(\mathcal{A})$ we have regular epimorphisms $e_n \colon \Sigma_n \twoheadrightarrow \Gamma_n$ in $\mathcal{A}$ ($n \in \mathcal{A}_{\mathsf{fp}}$), and the components of Te are the morphisms

$$(Te)_m = \coprod_{n\in|\mathcal{A}_{\mathsf{fp}}|} \mathcal{A}(n,m) \bullet e_n \qquad (m \in \mathcal{A}_{\mathsf{fp}}).$$

Since coproducts of regular epimorphisms in $\mathcal{A}$ are regular epimorphisms, we conclude that each $(Te)_m$ is regularly epic in $\mathcal{A}$. Thus, Te is regularly epic in $\mathsf{Sig}(\mathcal{A})$.

Consequently, the category of $\mathbb{T}$-algebras has regular factorizations. Since c is a strong epimorphism, it is regular.

(2) We can apply Theorem 4.7: an algebra $\mathbb{M}$ for $\mathbb{T}$ is finitely presentable iff it is a coequalizer in $\mathsf{Mnd_f}(\mathcal{A})$ as follows:

$$H^*_\Gamma \underset{r}{\overset{\ell}{\rightrightarrows}} H^*_\Sigma \xrightarrow{c} \mathbb{M}$$

for some super-finitary signatures Γ and Σ. By the preceding remark, we can substitute ℓ and r by a collection of equations $\Gamma_n \rightrightarrows H^*_\Sigma n$, and since Γ is super-finitary, this collection is finite. Therefore, every finitely presentable object of $\mathsf{Mnd_f}(\mathcal{A})$ has a super-finitary presentation.

Conversely, let $\mathbb{M}$ be presented by a super-finitary signature Σ and equations

$$f_i, f_i' \colon k_i \longrightarrow H^*_\Sigma n_i \qquad (i = 1, \dots, r).$$

Let Γ be the super-finitary signature with

$$\Gamma_k = \coprod_{\substack{i \in I \\ k_i = k}} k_i.$$

Then we have signature morphisms

$$f, f' \colon \Gamma \longrightarrow T(\Sigma)$$

derived from the given pairs in an obvious way. For the corresponding monad morphisms

$$\bar{f}, \bar{f}' \colon H^*_\Gamma \longrightarrow H^*_\Sigma$$

we see that the coequalizer of this pair is the smallest quotient $c \colon H^*_\Sigma \twoheadrightarrow \mathbb{M}$ with $c_{n_i} \cdot f_i = c_{n_i} \cdot f_i'$ for all $i = 1, \dots, n$. This follows immediately from the fact that c is a regular epimorphism in $\mathsf{Mnd_f}(\mathcal{A})$. Indeed, since $\mathcal{A}$ has regular factorizations, so does $\mathsf{Sig}(\mathcal{A})$, a power of $\mathcal{A}$. Since, moreover, $\mathbb{T}$ is a regular monad, the category $\mathsf{Mnd_f}(\mathcal{A})$ of its algebras has regular factorizations, thus, every strong epimorphism is regular. ■

Corollary 5.8. *A finitary monad on* Set *is a finitely presentable object of* $\mathsf{Mnd_f}(\mathsf{Set})$ *iff the corresponding variety of algebras has a presentation (in the classical sense) by finitely many operations and finitely many equations.*

Most of "everyday" varieties (groups, lattices, boolean algebras, etc.) yield finitely presentable monads. Vector spaces over a field K yield a finitely presentable monad iff K is finite – equivalently, that monad is finitely generated. However, there are finitely generated monads in $\mathsf{Mnd_f}(\mathsf{Set})$ that fail to be finitely presentable. We prove that the classes of finitely presentable and finitely generated objects differ in $\mathsf{Mnd_f}(\mathsf{Set})$ by relating monads to monoids via an adjunction.

Remark 5.9. Recall that every set functor has a unique strength. This follows from the result by Kock [14] that a strength of an endofunctor on a closed monoidal category bijectively corresponds to a way of making that functor enriched (see also Moggi [17, Proposition 3.4]). For every monad (T, η, μ) on Set we have a canonical strength, i.e. a family of morphisms

$$s_{X,Y} : TX \times Y \to T(X \times Y)$$

natural in X and Y and such that the following axioms hold

$$\begin{tikzcd} TX \times 1 \arrow[r, "s_{1,X}"] \arrow[dr, "\cong"'] & T(X \times 1) \arrow[d, "\cong"] \\ & TX \end{tikzcd} \qquad \begin{tikzcd} TX \times Y \times Z \arrow[r, "s_{X,Y} \times Z"] \arrow[dr, "s_{X,Y\times Z}"'] & T(X \times Y) \times Z \arrow[d, "s_{X\times Y,Z}"] \\ & T(X \times Y \times Z) \end{tikzcd}$$

$$\begin{tikzcd} TX \times Y \arrow[r, "s_{X,Y}"] & T(X \times Y) \\ X \times Y \arrow[u, "\eta_X \times Y"] \arrow[ur, "\eta_{X\times Y}"'] \end{tikzcd} \qquad \begin{tikzcd} TTX \times Y \arrow[r, "s_{TX,Y}"] \arrow[d, "\mu_X \times Y"'] & T(TX \times Y) \arrow[r, "Ts_{X,Y}"] & TT(X \times Y) \arrow[d, "\mu_{X\times Y}"] \\ TX \times Y \arrow[rr, "s_{X,Y}"] & & T(X \times Y) \end{tikzcd}$$

In fact, for a given monad T on Set one defines the canonical strength by the commutativity of the following diagrams

$$\begin{tikzcd} TX \times Y \arrow[rr, "s_{X,Y}"] & & T(X \times Y) \\ & TX \times 1 \cong TX \cong T(X \times 1) \arrow[ul, bend left, "TX\times y"] \arrow[ur, bend right, "T(X\times y)"'] & \end{tikzcd} \quad \text{for every } y\colon 1 \to Y.$$

Notation 5.10. (1) We have a functor $R : \mathsf{Mnd_f}(\mathsf{Set}) \to \mathsf{Mon}$ sending a monad (T, η, μ) with strength s to the monoid $T1$ with unit $\eta_1 : 1 \to T1$ and the following multiplication:

$$m : T1 \times T1 \xrightarrow{s_{1,T1}} T(1 \times T1) \xrightarrow{\cong} TT1 \xrightarrow{\mu_1} T1.$$

(2) We define a functor $L : \mathsf{Mon} \to \mathsf{Mnd_f}(\mathsf{Set})$ as follows. For every monoid $(M, *, 1_M)$ we have the free M-set monad LM with the following object assignment, unit and multiplication:

$$LM(X) = M \times X, \quad \eta_X : x \mapsto (1_M, x), \quad \mu_X : (n, (m, x)) \mapsto (n * m, x).$$

This extends to a functor $L : \mathsf{Mon} \to \mathsf{Mnd_f}(\mathsf{Set})$.

For example, the finite powerset monad $\mathcal{P}_\mathsf{f}$ induces the monoid $(\{0, 1\}, \wedge, 1)$ of boolean values with conjunction as multiplication.

Proposition 5.11. *We have an adjoint situation $L \dashv R$ with the following unit ν and counit ϵ:*

$$\nu_M \colon M \xrightarrow{\cong} M \times 1 = RLM$$
$$\epsilon_T \colon LRT = T1 \times (-) \xrightarrow{s_{1,-}} T(1 \times (-)) \xrightarrow{T\cong} T,$$

where s is the strength of T.

Proof: It is not hard to see that ν_M is a monoid morphism, because the monoid structure in $M \times 1 = RLM$ boils down to the monoid structure of M. Furthermore, ν_M is clearly natural in M.

For every monad T, ϵ_T is a natural transformation $T1 \times (-) \to T$ because of the naturality of the strength s. The axioms for the strength imply that $s_{1,-} : T1 \times (-) \to T(1 \times (-))$ is a monad morphism by straightforward diagram chasing. To see ν and ϵ establish an adjunction, it remains to check the triangle identities:

- The identity $\epsilon_{LM} \cdot L\nu_M = \mathsf{id}_{LM}$ is just the associativity of the product:

$$LM \xrightarrow{L\nu_M} LRLM \xrightarrow{\qquad\qquad\epsilon_{LM}\qquad\qquad} LM$$
$$M \times (-) \xrightarrow{\nu_M \times (-)} (M \times 1) \times (-) \xrightarrow{\cong} M \times (1 \times (-)) \xrightarrow{M \times \cong} M \times (-)$$

  The composite is obviously just the identity on $M \times (-)$.
- The identity $R\epsilon_T \cdot \nu_{RT} = \mathsf{id}_{RT}$ follows directly from the first axiom of strength:

$$RT \xrightarrow{\nu_{RT}} RLRT \xrightarrow{\qquad\quad R\epsilon_T \quad\qquad} RT$$
$$\begin{tikzcd} T1 \arrow[r, "\nu_{T1}"] \arrow[rr, bend right, "T\nu_1"'] & T1 \times 1 \arrow[r, "s_{1,1}"] & T(1 \times 1) \arrow[r, "\cong"] & T1 \end{tikzcd}$$

■

Corollary 5.12. *In the category of finitary monads on* Set *the classes of finitely presentable and finitely generated objects do not coincide.*

Proof: Note that from the fact that the unit of the adjunction $L \dashv R$ is an isomorphism we see that L is fully faithful. Thus, we may regard Mon as a full coreflective subcategory of $\mathsf{Mnd_f}(\mathsf{Set})$. Furthermore, the right-adjoint R preserves filtered colimits; this follows from the fact that filtered colimits in $\mathsf{Mnd_f}(\mathsf{Set})$ are created by the forgetful functor into $[\mathsf{Set}, \mathsf{Set}]_{\mathsf{fin}}$ where they are formed objectwise. In addition, L preserves monomorphisms; in fact, for an injective monoid morphism $m : M \rightarrowtail M'$ the monad morphism Lm :

$LM \to LM'$ is monic since all its components $m \times \mathsf{id}_X : M \times X \to M' \times X$ are clearly injective. By Lemma 2.5, we therefore have that a monoid M is finitely presentable (resp. finitely generated) if and only if the monad LM is finitely presentable (resp. finitely generated).

Now it is well-known that in the category Mon of monoids finitely presentable and finitely generated objects do not coincide; see Campbell et al. [10, Example 4.5] for an example of a finitely generated monoid which is not finitely presentable. ■

6. λ-Accessible Functors and λ-Presentable Algebras

Almost everything we have proved above generalizes to locally λ-presentable categories for every infinite regular cardinal λ. Recall that an object A of a category $\mathcal{A}$ is *λ-presentable* (*λ-generated*) if its hom-functor $\mathcal{A}(A, -)$ preserves λ-filtered colimits (of monomorphisms). A category $\mathcal{A}$ is *locally λ-presentable* if it is cocomplete and has a set of λ-presentable objects whose closure under λ-filtered colimits is all of $\mathcal{A}$. Functors preserving λ-filtered colimits are called *λ-accessible*.

All of Remark 2.2 holds for λ in lieu of $\aleph_0$, with the same references in [5].

If $\lambda = \aleph_1$ we speak about *locally countably presentable categories*, *countably presentable objects*, etc.

Examples 6.1. (1) Complete metric spaces. We denote by

$$\mathsf{CMS}$$

the category of complete metric spaces of diameter ≤ 1 and non-expanding functions, i.e. functions $f : X \to Y$ such that for all $x, y \in X$ we have $d_Y(f(x), f(y)) \leq d_X(x, y)$. This category is locally countably presentable. The classes of countably presentable and countably generated objects coincide and these are precisely the compact spaces.

Indeed, every compact (= separable) complete metric space is countably presentable, see [2, Corollaries 2.9]. And every countably generated space A in CMS is separable: consider the countably filtered diagram of all spaces $\bar{X} \subseteq A$ where X ranges over countable subsets of A and $\bar{X}$ is the closure in A. Since A is the colimit of this diagram, id_A factorizes through one of the embeddings $\bar{X} \hookrightarrow A$, i.e., $A = \bar{X}$ is separable.

(2) Complete partial orders. Denote by

$$\omega\mathsf{CPO}$$

the category of ω-cpos, i.e., of posets with joins of ω-chains and monotone functions preserving joins of ω-chains. This is also a locally countably presentable category. An ω-cpo is countable presentable (equivalently, countably generated) iff it has a countable subset which is dense w.r.t. joins of ω-chains.

Following our convention in Section 3 we speak about a *λ-generated subobject* $m : M \rightarrowtail A$ of A if M is a λ-generated object of $\mathcal{A}$. This leads to a generalization of the notion of finitely bounded functors to λ-bounded ones. The latter terminology stems from Kawahara and Mori [12], where endofunctors on sets were considered. Our terminology is slightly different in that λ-generated subobjects in Set have cardinality less than λ, whereas subsets of cardinality less than or equal to λ were considered in loc. cit.

Definition 6.2. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *λ-bounded* provided that given an object A of $\mathcal{A}$, every λ-generated subobject $m_0 : M_0 \rightarrowtail FA$ in $\mathcal{B}$ factorizes through the F-image of a λ-generated subobject $m : M \rightarrowtail A$ in $\mathcal{A}$:

$$\begin{tikzcd} & FM \arrow[d, "Fm"] \\ M_0 \arrow[ur, dashed] \arrow[r, tail, "m_0"'] & FA \end{tikzcd}$$

Theorem 6.3. *Let $\mathcal{A}$ be a locally λ-presentable category in which every λ-generated object is λ-presentable. Then for all functors from $\mathcal{A}$ to locally λ-presentable categories preserving monomorphisms we have the equivalence*

$$\lambda\text{-accessible} \iff \lambda\text{-bounded.}$$

The proof is completely analogous to that of Theorem 3.4.

Example 6.4. The Hausdorff endofunctor $\mathcal{H}$ on CMS was proved to be accessible (for some λ) by van Breugel et al. [22]. Later it was shown to be even finitary [2]. However, these proofs are a bit involved. Using Theorem 6.3 we provide an easy argument why the Hausdorff functor is countably accessible. (Which, since CMS is not lfp but is locally countable presentable, seems to be the "natural" property.)

Recall that for a given metric space (X, d) the distance of a point $x \in X$ to a subset $M \subseteq X$ is defined by $d(x, M) = \inf_{y \in M} d(x, y)$. The *Hausdorff distance* of subsets $M, N \subseteq X$ is defined as the maximum of $\sup_{x \in M} d(x, N)$ and $\sup_{y \in N} d(y, M)$. The *Hausdorff functor* assigns to every complete metric space X the space $\mathcal{H}X$ of all non-empty compact subsets of X equipped with the Hausdorff metric. It is defined on non-expanding maps by taking the direct images. We now easily see that $\mathcal{H}$ is countably accessible:

(1) $\mathcal{H}$ preserves monomorphisms. Indeed, given $f : X \rightarrowtail Y$ monic, then $f[M] \neq f[N]$ for every pair M, N of distinct elements of $\mathcal{H}X$, thus $\mathcal{H}f$ is monic, too.

(2) $\mathcal{H}$ is countably bounded. In order to see this, let $m_0 : M_0 \hookrightarrow \mathcal{H}X$ be a subspace with M_0 compact, and choose a countable dense subset $S \subseteq M_0$. For every element $s \in S$ the set $m_0(s) \subseteq X$ is compact, hence, separable; choose a countable dense set $T_s \subseteq m_0(s)$. For the countable set $T = \bigcup_{s \in S} T_s$ form the closure in X and denote it by $m : M \hookrightarrow X$. Then M is countably generated, and $M_0 \subseteq \mathcal{H}m[\mathcal{H}M]$; indeed, for every $x \in M_0$ we have $m_0(x) \subseteq M$ because M is closed, and this holds whenever $x \in S$ (due to $m_0(x) = \overline{T_x}$).

Definition 6.5. A locally λ-presentable category $\mathcal{A}$ is called *strictly* or *semi-strictly* locally λ-presentable provided that every morphism $b : B \to A$ in $\mathcal{A}_\lambda/A$ factorizes through a morphism $b' : B' \to A$ in $\mathcal{A}_\lambda/A$ for which some $f : A \to B'$ exists and, in the case of strict locally λ-presentable, $f \cdot b$ is such a factor, i.e. $b = b' \cdot (f \cdot b)$.

$$\begin{tikzcd} B \arrow[rr, dashed] \arrow[dr, "b"'] & & B' \arrow[dl, shift left, "b'"] \\ & A \arrow[ur, shift left, "f"] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "f \cdot b"] \arrow[dr, "b"'] & & B' \arrow[dl, shift left, "b'"] \\ & A \arrow[ur, shift left, "f"] & \end{tikzcd}$$

semi-strictly locally λ-presentable   *strictly locally λ-presentable*

Examples 6.6. (1) Set^S is strictly locally λ-presentable iff $\mathsf{card}\, S < \lambda$. This is analogous to Example 3.15(1).

(2) The category $\mathsf{Set}^{\mathbb{G}^{\mathrm{op}}}$ of presheaves on a groupoid $\mathbb{G}$ of α elements is strictly locally λ-presentable whenever $\lambda > 2^\alpha$. The proof is analogous to that of Proposition 3.16. For the set S of objects of $\mathbb{G}$ we work with the corresponding S-sorted unary algebras. Every representable algebra has at most α elements, hence, at most 2^α equivalence relations

on the set of its elements. Therefore, representable algebras have less than λ quotients. Thus, the algebra D in part (4) of that proof is λ-generated. The rest of the proof is unchanged.

(3) The category of groups is not strictly locally λ-presentable for any infinite cardinal λ.

Indeed, let A be a simple group of cardinality at least λ^λ. (Recall that for every set X of cardinality ≥ 5 the group of even permutations on X is simple.) Since groups form an lfp category, there exists a non-zero homomorphism $b : B \to A$ with B finitely presentable. Given a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "f \cdot b"] \arrow[dr, "b"'] & & B' \arrow[dl, "b'"] \\ & A & \end{tikzcd} \qquad \text{for some } f : A \to B'$$

we will prove that B' is not λ-presentable. Indeed, since b is non-zero, we see that so is $f : A \to B'$. Since A is simple, f is monic, hence $\mathsf{card}\, B' \geq \lambda^\lambda$. However, every λ-presentable group has cardinality at most λ.

(4) The category Nom of nominal sets is strictly locally countably presentable. In order to prove this we first verify that countably presentable objects are precisely the countable nominal sets. Let X be a countably presentable nominal set. Then every countable choice of orbits of X yields a countable subobject of X in Nom. Thus X is a countably directed union of countable subobjects. Since X is countably presentable, it follows that X is isomorphic to one of these subobjects. Thus, X is countable.

Conversely, every countable nominal set is countably presentable since countably filtered colimits of nominal sets are formed on the level of sets (i.e. these colimits are preserved and reflected by the forgetful functor $\mathsf{Nom} \to \mathsf{Set}$).

Now let $b\colon B \to A$ be a morphism in Nom with B countable. Clearly, we have $A = \mathsf{Im}(b) + C$ for some subobject C of A. Indeed, every nominal set is a coproduct of its orbits, and equivariance of b implies that is must be a coproduct of some of the orbits of A. Furthermore, let $m\colon C_1 \rightarrowtail C$ be a subobject obtained by choosing one orbit from each isomorphism class of orbits of C. We obtain a surjective equivariant map $e\colon C \twoheadrightarrow C_1$ by choosing, for every orbit in $C \setminus C_1$, a concrete isomorphism to an orbit of C_1 and for every $x \in C_1 \subseteq C$, $e(x) = x$. Then we have $e \cdot m = \mathsf{id}_{C_1}$, i.e. m is a split monomorphism of Nom. In the appendix we prove that there are (up to isomorphism) only countably many single-orbit nominal sets. Hence, C_1 is countable, and thus so is $B' = \mathsf{Im}(b) + C_1$. Moreover, the morphisms $b' = \mathsf{id} + m\colon B' \to A$ and $f\colon \mathsf{id} + e\colon A \to B'$ clearly satisfy the desired property $b = b' \cdot f \cdot b$.

Theorem 6.7. *Let $\mathcal{A}$ be a locally λ-presentable category.*

(1) If $\mathcal{A}$ is strictly locally λ-presentable, then for all functors from $\mathcal{A}$ to locally λ-presentable categories we have

$$\lambda\text{-accessible} \iff \lambda\text{-bounded.}$$

(2) Conversely, if this equivalence holds for all functors to locally λ-presentable categories, then $\mathcal{A}$ is semi-strictly locally λ-presentable.

The proof is completely analogous to those of Theorems 3.10 and 3.14.

Remark 6.8. For λ-accessible monads on locally λ-presentable categories all the results of Section 4 have an appropriate statement and completely analogous proofs. We leave the explicit formulation to the reader.

Remark 6.9. Assume that we work in a set theory distinguishing between sets and classes (e.g. Zermelo-Fraenkel theory) or distinguishing universes, so that by a "a class" we take a member of the next higher universe of that of all small sets. Then we form a super-large category

$$\mathsf{Class}$$

of classes and class functions. It plays a central role in the paper of Aczel and Mendler [1] on terminal coalgebras. An endofunctor F of Class in that paper is called *set-based* if for every class X and every element $x \in FX$ there exists a subset $i : Y \rightarrowtail X$ such that x lies in $Fi[FX]$. This corresponds to ∞-bounded where ∞ stands for "being large". The corresponding concept of ∞-accessibility is evident:

Definition 6.10. A diagram $D : \mathcal{D} \to \mathsf{Class}$, with $\mathcal{D}$ not necessarily small, is called *∞-filtered* if every smalll subcategory of $\mathcal{D}$ has a cocone in $\mathcal{D}$. An endofunctor of Class is *∞-accessible* if it preserves colimits of ∞-filtered diagrams.

Proposition 6.11. *An endofunctor of* Class *is set-based iff it is ∞-accessible.*

Proof: (1) For every morphism $b : B \to A$ in Class with B small factorizes in Set/A through a morphism $b' : B' \to A$ in Set/A where the factorization f fulfils $b = b' \cdot (f \cdot b)$. (Shortly: Class is strictly locally ∞-presentable.) The proof is the same as that of Example 3.9(2).

(2) The rest is completely analogous to part (1) of the proof of Theorem 3.10 ∎

Remark 6.12. Assuming, moreover, that all proper classes are mutually bijective, it follows that *every* endofunctor on Class is ∞-accessible, see [3].

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Appendix A. Details on Single-Orbit Nominal Sets

In this appendix we prove that in the category Nom of nominal sets there are (up to isomorphism) only countably many nominal sets having only one orbit. To this end we consider the nominal sets $\mathbb{A}^{\#n}$ of injective maps from $n = \{0, 1, \dots, n-1\}$ to $\mathbb{A}$. The group action on $\mathbb{A}^{\#n}$ is component-wise, in other words, it is given by postcomposition: for $t\colon n \rightarrowtail \mathbb{A}$ and $\pi \in \mathfrak{S}_\mathsf{f}(\mathbb{A})$ (i.e. a bijective map $\pi\colon \mathbb{A} \to \mathbb{A}$) the group action is the composed map $\pi \cdot t\colon n \rightarrowtail \mathbb{A}$. Thus, for every $t : n \rightarrowtail \mathbb{A}$ of $\mathbb{A}^{\#n}$, $\mathsf{supp}(t) = \{t(i) \mid i < n\}$.

Lemma A.1. *Up to isomorphism, there are only countably many single-orbit sets.*

Proof: Every single-orbit nominal set Q whose elements have supports of cardinality n is a quotient of the (single-orbit) nominal set $\mathbb{A}^{\#n}$ (see [19, Exercise 5.1]). Indeed, if $Q = \{\pi \cdot x \mid \pi \in \mathfrak{S}_\mathsf{f}(\mathbb{A})\}$ with $\mathsf{supp}(x) = \{a_0, \dots, a_{n-1}\}$, let $t\colon n \rightarrowtail \mathbb{A}$ be the element of $\mathbb{A}^{\#n}$ with $t(i) = a_i$ and define $q\colon \mathbb{A}^{\#n} \twoheadrightarrow Q$ as follows: for every $u \in \mathbb{A}^{\#n}$ it is clear that there is some $\pi \in \mathfrak{S}_\mathsf{f}(\mathbb{A})$ with $u = \pi \cdot t$; put $q(u) = \pi \cdot x$. This way, q is well-defined (since $\mathsf{supp}(x) = \{t(i) \mid i < n\}$) and equivariant.

The quotients of $\mathbb{A}^{\#n}$ are given by equivariant equivalence relations on $\mathbb{A}^{\#n}$. We prove that, for every $n \in \mathbb{N}$, we have a bijective correspondence between the set of all quotients with $|\mathsf{supp}([t]_\sim)| = n$ for all $t \in \mathbb{A}^{\#n}$ and the set of all subgroups of $\mathfrak{S}_\mathsf{f}(n)$.

(1) Given an equivariant equivalence $\sim$ on $\mathbb{A}^{\#n}$ put

$$S = \{\sigma \in \mathfrak{S}_\mathsf{f}(n) \mid \forall (t\colon n \rightarrowtail \mathbb{A})\colon t \cdot \sigma \sim t\}.$$

Note that since $\sim$ is equivariant (and composition of maps is associative), $\forall$ can equivalently be replaced by $\exists$:

$$S = \{\sigma \in \mathfrak{S}_\mathsf{f}(n) \mid \exists (t\colon n \rightarrowtail \mathbb{A})\colon t \cdot \sigma \sim t\}.$$

It is easy to verify that S is a subgroup of $\mathfrak{S}_\mathsf{f}(n)$. Moreover, we have that, for every $t, u \in \mathbb{A}^{\#n}$,

$$t \sim u \quad \iff \quad u = t \cdot \sigma \quad \text{for some } \sigma \in S. \tag{3}$$

Indeed, "$\Longleftarrow$" is obvious. For "$\Longrightarrow$" suppose that $t \sim u$. Since $|\mathsf{supp}([t]_\sim)| = n$, we have that $\mathsf{supp}(t) = \mathsf{supp}([t]_\sim) = \mathsf{supp}([u]_\sim) = \mathsf{supp}(u)$; thus, there is some $\sigma \in \mathfrak{S}_\mathsf{f}(n)$ such that $u = t \cdot \sigma$. Consequently, $t \sim t \cdot \sigma$, showing that $\sigma \in S$.

(2) For every subgroup S of $\mathfrak{S}_{\mathsf{f}}(n)$, it is clear that the relation $\sim$ defined by (3) is an equivariant equivalence. We show that, moreover, $|\,\mathsf{supp}([t]_\sim)| = n$ for every $t \in \mathbb{A}^{\#n}$. We have $|\,\mathsf{supp}([t]_\sim)| \leq n$ because the canonical quotient map $[-]_\sim$ is equivariant. In order to see that $|\,\mathsf{supp}([t]_\sim)|$ is not smaller than n, assume $a \in \mathsf{supp}(t) \setminus \mathsf{supp}([t]_\sim)$ and take any element $b \notin \mathsf{supp}(t)$. Then $(a\,b) \cdot [t]_\sim = [t]_\sim$, i.e. there is some $\sigma \in \mathfrak{S}_{\mathsf{f}}(n)$ with $(a\,b) \cdot t \cdot \sigma = t$, which is a contradiction to $b \notin \mathsf{supp}(t) = \mathsf{supp}(t \cdot \sigma) = \{t(i) \mid i < n\}$.

(3) It remains to show that, given two subgroups S and S' which determine the same equivariant equivalence relations $\sim$ via (3), then $S = S'$. Indeed, given $\sigma \in S$, we have $t = (t \cdot \sigma) \cdot \sigma^{-1}$ and therefore $t \cdot \sigma \sim t$ for every $t \in \mathbb{A}^{\#n}$. By (3) applied to S', this implies that $t = t \cdot \sigma \cdot \sigma'$ for some $\sigma' \in S'$. Since t is monic, we obtain $\sigma \cdot \sigma' = \mathsf{id}_n$, i.e., $\sigma = (\sigma')^{-1} \in S'$. This proves $S \subseteq S'$, and the reverse inclusion holds by symmetry. ■

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