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VARIATIONAL SOLUTIONS TO THE ABSTRACT EULER EQUATION

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Abstract: We study a class of nonlinear evolutionary equations of a certain structure reminiscent of the incompressible Euler equations. This includes, in particular, the ideal MHD, multidimensional Camassa-Holm, EPDiff, Euler-α and Korteweg-de Vries equations, and two models of incompressible elastodynamics. We interpret the "abstract Euler equation" as a concave maximization problem in the spirit of *Y. Brenier. Comm. Math. Phys. (2018) 364(2) 579-605*. An optimizer determines a "time-noisy" version of the original unknown function, and the latter one may be retrieved by time-averaging. Assuming a certain "trace condition", which holds for the above-mentioned examples, we prove the existence of the generalized solutions determined by the maximizers.

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1. Introduction

The Euler equations of motion of a homogeneous incompressible inviscid fluid [1] are

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$(u \cdot \nu)|_{\partial\Omega} = 0, \tag{1.3}$$

$$u(0) = u_0. \tag{1.4}$$

The unknowns are $u : [0,T] \times \Omega \to \mathbb{R}^d$ and $p : [0,T] \times \Omega \to \mathbb{R}$. Here Ω is the periodic box $\mathbb{T}^d$ or an open domain in $\mathbb{R}^d$ with sufficiently regular boundary. The Euler equations may be rewritten in the form

$$\partial_t u = \mathcal{P}\mathcal{L}(u \otimes u), \quad u(t,\cdot) \in \mathcal{P}(X^d), \quad u(0,\cdot) = u_0 \in \mathcal{P}(X^d), \tag{1.5}$$

where $X := L^2(\Omega)$ and $\mathcal{P} : X^d \to X^d$ is the Leray-Helmholtz projector [40], whereas

$$\mathcal{L} = -\operatorname{div}, \quad \mathcal{L} : D(\mathcal{L}) \subset X_s^{d\times d} \to X^d$$

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(we refer to the Notation and conventions subsection at the end of the Introduction for the meaning of the symbol $X_s^{d\times d}$). The kinetic energy

$$\mathcal{K}_t := \frac{1}{2}\int_\Omega |u|^2(t,x)\,dx$$

is formally conserved due to

$$(\mathcal{L}(u\otimes u),u) = 0, \quad u \in \mathcal{P}(X^d) \tag{1.6}$$

for any sufficiently smooth vector field u.

In this paper, we study the following abstract generalization of (1.5): find

$$v : [0,T] \to X^n$$

solving

$$\partial_t v = PL(v\otimes v), \quad v(t,\cdot) \in P(X^n), \quad v(0,\cdot) = v_0 \in P(X^n). \tag{1.7}$$

Here $(\Omega, \mathcal{A}, \mu)$ is a measure space, $X := L^2(\Omega)$, $n \in \mathbb{N}$,

$$P : X^n \to X^n$$

is any orthogonal projector (i.e., a self-adjoint idempotent linear operator), and

$$L : D(L) \subset X_s^{n\times n} \to X^n$$

is a closed densely defined linear operator, satisfying

$$(L(v\otimes v),v) = 0, \quad v \in P(X^n), \tag{1.8}$$

provided v is a *sufficiently smooth* vector field (see the Notation and conventions subsection for the meaning of this expression).

This setting can be further generalized, see Remarks 2.7 and 2.8. As we will see, the examples of (1.7) include the ideal MHD, multidimensional Camassa-Holm, EPDiff, Euler-α and Korteweg-de Vries equations, and two models of incompressible elastodynamics.

Brenier [5] recently suggested to regard the incompressible Euler system (1.1)–(1.4) as a concave maximization problem. He also discussed the relation of his approach with the theory of convex integration [11, 10]. In this paper, we adapt his ideas to suit the general equation (1.7). We will see that the concave maximization problem generates the "time-noisy" function $V := v + (t-T)\partial_t v$, and hence the unknown v can be retrieved by time averaging.

In Section 2, we discuss the abstract theory and prove an abstract existence theorem, and in Section 3 we examine the above-mentioned examples.

Notation and conventions. We use the notations $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times n}_s$ for the spaces of $n \times n$ matrices and symmetric matrices, resp., with the scalar product generated by the Frobenius norm. The symbol $\mathbb{R}^{(n\times n)\times(n\times n)}$ denotes the space of matrices with matricial entries. For a tensor $\Xi \in \mathbb{R}^{(n\times n)\times(n\times n)}$, define the matrices $\widehat{\Xi}, \breve{\Xi} \in \mathbb{R}^{n\times n}$ by

$$\widehat{\Xi}_{ij} = \sum_k \Xi_{ik,jk}, \quad \breve{\Xi}_{ij} = \sum_k \Xi_{ki,kj}.$$

For a matrix $M \in \mathbb{R}^{n\times n}$, define the tensors $\widehat{M}^*, \breve{M}^* \in \mathbb{R}^{(n\times n)\times(n\times n)}$ by

$$\widehat{M}^*_{ik,jl} = M_{ij}\delta_{kl}, \quad \breve{M}^*_{ik,jl} = M_{kl}\delta_{ij}.$$

For a tensor $\Upsilon \in \mathbb{R}^{n\times n\times n}$, denote

$$\widetilde{\Upsilon}_{ijk} := \Upsilon_{ikj}.$$

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Denote for brevity $X = L^2(\Omega)$. Let $X^{n\times n}_s$ be the subspace of $X^{n\times n}$ consisting of symmetric-matrix-valued functions. The parentheses $(\cdot,\cdot)$ will stand for the scalar products in X^n and $X^{n\times n}_s$. For $A, B \in X^{n\times n}_s$, we write $A \geq B$ and $A > B$ when $A - B$ is a nonnegative-definite-matrix-function and is a strictly-positive-definite-matrix-function, resp. The action of a matrix-function A from $X^{n\times n}_s$ on a vector-function ξ from X^n is denoted $A.\xi$ or simply $A\xi$.

Fix $n \in \mathbb{N}$ and the operators P, L as above. Let $L^* : D(L^*) \subset X^n \to X^{n\times n}_s$ be the adjoint of L. Fix some linear dense subspace $\mathcal{R} \subset X$. Assume that

$$\mathcal{R} \subset L^\infty(\Omega)$$

and

$$\mathcal{R}^n \subset D(L) \subset X^n,\ \mathcal{R}^{n\times n}_s \subset D(L) \subset X^{n\times n}_s,$$
$$L(\mathcal{R}^{n\times n}_s) \subset \mathcal{R}^n,\ L^*(\mathcal{R}^n) \subset \mathcal{R}^{n\times n}_s,\ P(\mathcal{R}^n) \subset \mathcal{R}^n.$$

We will abuse the language and call the elements of $\mathcal{R}$ *sufficiently smooth* functions. For example, if Ω is a Riemannian manifold, we can take the set of conventional smooth functions as our $\mathcal{R}$.

Fix also a linear dense subspace $\widehat{\mathcal{R}} \subset L^2((-\epsilon, T+\epsilon)\times\Omega)$. Assume that

$$\widehat{\mathcal{R}} \subset L^\infty((-\epsilon, T+\epsilon)\times\Omega), \quad \partial_t\widehat{\mathcal{R}} \subset \widehat{\mathcal{R}}, \quad \widehat{\mathcal{R}}(t) = \mathcal{R},\ t \in [0, T],$$

and

$$L(\widehat{\mathcal{R}}_s^{n\times n}) \subset \widehat{\mathcal{R}}^n,\ L^*(\widehat{\mathcal{R}}^n) \subset \widehat{\mathcal{R}}_s^{n\times n},\ P(\widehat{\mathcal{R}}^n) \subset \widehat{\mathcal{R}}^n.$$

A time-dependent function $v : [0,T] \to X$ is called sufficiently smooth if $v \in \widehat{\mathcal{R}}\big|_{[0,T]}$.

2. The abstract results

The abstract Euler equation (1.7) admits the following natural weak formulation:

$$\int_0^T [(v,w) + (v,\partial_t a) + (v\otimes v, L^* a)]\,dt + (v_0, a(0)) = 0 \tag{2.1}$$

for all sufficiently smooth vector fields $a : [0,T] \to P(X^n)$, $a(T) = 0$, $w : [0,T] \to (I-P)(X^n)$.

We now observe that (1.8) implies

$$((u+r)\otimes(u+r), L^*(u+r)) - ((u-r)\otimes(u-r), L^*(u-r)) - 2(r\otimes r, L^* r) = 0$$

for $u, r \in P(X^n)$ sufficiently smooth, whence

$$(u\otimes u, L^* r) + 2\,(r\otimes u, L^* u) = 0. \tag{2.2}$$

Consequently, (2.1) can be formally recast as

$$\int_0^T [(v,w) + (v,\partial_t a) - 2(a\otimes v, L^* v)]\,dt + (v_0, a(0)) = 0. \tag{2.3}$$

This implies the following strong reformulation of (1.7):

$$\partial_t v + 2P[L^* v.v] = 0, \quad v(t,\cdot) \in P(X^n), \quad v(0,\cdot) = v_0. \tag{2.4}$$

Let us now rewrite problem (2.1) in terms of the test functions $B := L^* a$ and $E := \partial_t a + w$. We first observe that

$$(v_0, a(0)) = -\int_0^T (v_0, \partial_t a) = -\int_0^T (v_0, E) \tag{2.5}$$

since w is orthogonal to v_0. The link between B and E can alternatively be described by the conditions

$$\partial_t B = (L^* \circ P)E, \quad B(T) = 0. \tag{2.6}$$

Indeed, any pair (B, E) satisfying (2.6) generates a pair (a, w) such that $B = L^* a$, $E = \partial_t a + w$, and vice versa. It suffices to take $a(t) = \int_T^t PE$, $w = E - PE$. Hence, (2.1) becomes

$$\int_0^T [(v - v_0, E) + (v \otimes v, B)]\, dt = 0 \tag{2.7}$$

for all sufficiently smooth vector fields $B : [0, T] \to X_s^{n\times n}$, $E : [0, T] \to X^n$ satisfying the constraints (2.6).

For a technical reason, we now need to extend the class of test functions in (2.7) (this may make the problem more difficult but definitely not simpler). Observe that (2.6) can be rewritten in the following weak form

$$\int_0^T [(B, \partial_t \Psi) + (E, PL\Psi)]\, dt = 0 \tag{2.8}$$

for all sufficiently smooth vector fields $\Psi : [0, T] \to X_s^{n\times n}$, $\Psi(0) = 0$. Accordingly, the new weak formulation of (1.7) is to look for functions $v \in L^2((0, T) \times \Omega; \mathbb{R}^n)$ which satisfy (2.7) for all vector fields $B \in L^\infty((0, T) \times \Omega; \mathbb{R}_s^{n\times n})$, $E \in L^2((0, T) \times \Omega; \mathbb{R}^n)$ meeting the constraint (2.8).

Formally, (1.8) implies that the energy

$$K_t := \frac{1}{2}(v(t), v(t))$$

is conserved, which yields

$$\int_0^T K_t = T K_0.$$

Both of these properties may however fail for the weak solutions. The idea of Brenier [5], which we reemploy here, is to look for a solution that minimizes $\int_0^T K_t$. This can be recast as a saddle-point problem:

$$\mathcal{I}(v_0) = \inf_v \sup_{E,B} \int_0^T \left[(v - v_0, E) + \frac{1}{2}(v \otimes v, I + 2B)\right] dt \tag{2.9}$$

where the supremum is taken along all pairs (E, B) satisfying the linear constraint (2.8). The dual problem is

$$\mathcal{J}(v_0) = \sup_{E,B:(2.8)} \inf_v \int_0^T \left[(v - v_0, E) + \frac{1}{2}(v \otimes v, I + 2B)\right] dt. \tag{2.10}$$

Since inf sup ≥ sup inf, one has $\mathcal{I}(v_0) \geq \mathcal{J}(v_0)$.

It is easy to see that any solution to (2.10) necessarily satisfies

$$I + 2B \geq 0. \tag{2.11}$$

Assume for a while that

$$I + 2B > 0. \tag{2.12}$$

Then

$$\inf_v \left[(v, E) + \frac{1}{2}(v \otimes v, I + 2B) \right] = -\frac{1}{2}((I + 2B)^{-1} E, E)$$
$$= \inf_{z \otimes z \leq M} \left[(z, E) + \frac{1}{2}(M, I + 2B) \right] =: K_-(E, B), \tag{2.13}$$

and the first infimum is achieved at $v = -(I + 2B)^{-1} E$. Consequently, (2.10) becomes

$$\mathcal{J}(v_0) = \sup_{E,B:\,(2.8),(2.11)} -\int_0^T (v_0, E)\, dt + \int_0^T K_-(E, B)\, dt. \tag{2.14}$$

As mentioned in [5], this is reminiscent of the Benamou-Brenier formula from the optimal transport theory [2, 41].

If $I + 2B$ is non-negative definite but not invertible at some $(t, x) \in [0, T] \times \Omega$, (2.14) still makes sense with

$$\int_0^T K_-(E, B)\, dt := \inf_{z \otimes z \leq M} \int_0^T \left[(z, E) + \frac{1}{2}(M, I + 2B) \right] dt, \tag{2.15}$$

cf. the last equality in (2.13), where $(z, M) : [0, T] \to X^n \times X_s^{n \times n}$ are sufficiently smooth.

The following theorem shows that a sufficiently smooth solution to (1.7) on a small time interval $[0, T]$ determines a solution to the optimization problem (2.14), and vice versa. This advocates the possibility to view the maximizers of (2.14) as generalized variational solutions to (1.7), see also Remark 2.6 below.

Theorem 2.1. *Let v be a sufficiently smooth solution to* (1.7) *(or, equivalently, to* (2.4)*), satisfying*

$$I \geq 2(t - T) L^* v(t),\ t \in [0, T]. \tag{2.16}$$

Then there exists a pair (B_+, E_+) that maximizes (2.14)*. Namely, one has*

$$B_+ = L^* a,\ E_+ = \partial_t a + w,$$

where

$$a = (T-t)v,\ w = 2(t-T)(I-P)[L^* v.v]. \tag{2.17}$$

The original variable v can be retrieved by means of the formula

$$v(t) = \frac{1}{T-t}\int_t^T (-PE_+)(s)\,ds, \quad t < T. \tag{2.18}$$

Proof: By construction, (E_+, B_+) verify (2.6) and thus (2.8). Moreover, (2.16) implies (2.11) for B_+. Let us observe that

$$v + 2B_+.v + E_+ = 0. \tag{2.19}$$

Indeed, using (2.4) we compute

$$\begin{aligned} v + 2B_+.v + E_+ &= v + 2(T-t)L^* v.v + (-v + (T-t)\partial_t v) + 2(t-T)(I-P)[L^* v.v] \\ &= (T-t)\partial_t v + 2(T-t)P[L^* v.v] = 0. \end{aligned} \tag{2.20}$$

On the other hand, since v satisfies (2.7), we have

$$\int_0^T [(v - v_0, E_+) + (v \otimes v, B_+)]\,dt = 0. \tag{2.21}$$

Hence, by (2.19),

$$\int_0^T [-(v_0, E_+) + (v \otimes v, B_+)]\,dt = \int_0^T (v \otimes v, (I + 2B_+))\,dt, \tag{2.22}$$

whence

$$\int_0^T [(v_0, E_+) + (v \otimes v, B_+)]\,dt = -\int_0^T (v \otimes v, I)\,dt. \tag{2.23}$$

Since v solves (2.7), we have $\mathcal{I}(v_0) = \frac{1}{2}\int_0^T (v \otimes v, I)\,dt = TK_0$. Thus, we need to show that

$$\int_0^T -(v_0, E_+) + K_-(E_+, B_+)\,dt = TK_0, \tag{2.24}$$

so that there is no duality gap. Indeed, (2.19) and (2.23) yield

$$\int_0^T -(v_0, E_+) + K_-(E_+, B_+)\, dt$$
$$= \int_0^T -(v_0, E_+)\, dt + \inf_{z\otimes z\le M} \int_0^T \left[-(z, (I+2B_+)v) + \frac{1}{2}(M, I+2B_+) \right] dt$$
$$= \int_0^T -\left[(v_0, E_+) + \frac{1}{2}((I+2B_+)v, v) \right] dt$$
$$= \int_0^T \left[(v\otimes v, I) - \frac{1}{2}(v, v) \right] dt = TK_0$$

because the energy conservation holds for the strong solutions.

Finally,

$$-PE_+ = -\partial_t a = v + (t-T)\partial_t v, \tag{2.25}$$

so

$$\int_t^T -PE_+(s)\, ds = \int_t^T [v(s) + (s-T)\partial_s v]\, ds = (T-t)v(t), \tag{2.26}$$

providing (2.18). ■

Corollary 2.2. *If in Theorem 2.1 one has*

$$I > 2(t-T)L^* v(t),\ t\in[0,T], \tag{2.27}$$

then the solution can be also retrieved by the formula

$$v(t) = \left(-(I+2B_+)^{-1}E_+\right)(t), \quad t\in[0,T]. \tag{2.28}$$

Indeed, it suffices to observe that (2.27) means that $I + 2B_+ > 0$, and if this holds, (2.28) is equivalent to (2.19).

Definition 2.3. The operator L is said to satisfy the trace condition if a uniform (w.r.t. to a.e. $x\in\Omega$ or any extra parameter) lower bound on the eigenvalues of the matrix $L^*\zeta(x)$, $\zeta\in D(L^*)\cap P(X^n)$, implies a uniform upper bound on its eigenvalues (a.e. in Ω).

Remark 2.4. The trace condition is particularly satisfied provided

$$PL(qI) = 0 \tag{2.29}$$

for any $q \in X$ sufficiently smooth. where $I \in \mathbb{R}_s^{n\times n}$ is the identity matrix. It suffices to observe that the trace of $L^*\zeta$ vanishes almost everywhere. Indeed,

$$(\mathrm{Tr}(L^*\zeta), q) = (L^*\zeta, qI) = (\zeta, PL(qI))$$

since $\zeta \in P(X^n)$. This applies to the Euler equation (1.5) because

$$\mathcal{P}[-\operatorname{div}(qI)] = \mathcal{P}(-\nabla q) = 0.$$

The next theorem shows existence of variational solutions.

Theorem 2.5. *Assume that L satisfies the trace condition. Then for any $v_0 \in P(X^n)$ there exists a maximizer*

$$(E, B) \in L^2((0,T)\times\Omega; \mathbb{R}^n) \times L^\infty((0,T)\times\Omega; \mathbb{R}_s^{n\times n})$$

of (2.14), *and* $\mathcal{J}(v_0) \geq 0$.

Proof: It suffices to consider to the pairs (E, B) that meet the restrictions (2.8), (2.11). Testing (2.14) with $E = 0$, $B = 0$, we see that $\mathcal{J}(v_0) \geq 0$. Let (E_m, B_m) be a maximizing sequence. Without loss of generality, it satisfies

$$0 \leq J(v_0) \leq \frac{1}{n} - \int_0^T (v_0, E_m)\, dt + \int_0^T K_-(E_m, B_m)\, dt. \tag{2.30}$$

Since $I + 2B_m \geq 0$, the eigenvalues of B_m are uniformly bounded from below, and the trace condition implies a uniform L^∞ bound on B_m. Hence, $I + 2B_m \leq kI$ with some constant $k > 0$. By the definition of K_- in (2.13), we have

$$K_-(E_m, B_m) \leq \inf_{z\otimes z \leq M} \left[(z, E_m) + \frac{k}{2}(M, I) \right] = -\frac{1}{2k}(E_m, E_m). \tag{2.31}$$

We infer that

$$\frac{1}{2k}\int_0^T (E_m, E_m) \leq \frac{1}{n} - \int_0^T (v_0, E_m)\, dt \leq \frac{1}{n} + 2kTK_0 + \frac{1}{4k}\int_0^T (E_m, E_m), \tag{2.32}$$

which gives a uniform $L^2((0,T)\times\Omega;\mathbb{R}^n)$-bound on E_m. The functional (2.15) is concave and upper semicontinuous on $L^2((0,T)\times\Omega;\mathbb{R}^n)\times L^\infty((0,T)\times\Omega;\mathbb{R}_s^{n\times n})$ as an infimum of affine continuous functionals. The functional $\int_0^T (v_0, \cdot)\, dt$ is a linear bounded functional on $L^2((0,T)\times\Omega;\mathbb{R}^n)$. Consequently, every weak-* accumulation point of (E_m, B_m) is a maximizer of (2.14). Note that the constraints (2.8), (2.11) are preserved by the limit. ∎

Remark 2.6. Let (E,B) be any maximizer of (2.14). Set $V := -PE$. Formula (2.18), in contrast to (2.28), does not rely on strict positive-definiteness of $I+2B$. We thus can define a generalized solution to (1.7) by setting

$$v := \frac{1}{T-t}\int_t^T V(s)\,ds \in H^1_{loc} \cap C([0,T);X^n), \tag{2.33}$$

cf. (2.18).

Remark 2.7. Assume that $\mu(\Omega)$ is finite. Then the theory above can be adapted to the setting

$$\partial_t v + PAv = PL(v\otimes v), \quad v(t,\cdot)\in P(X^n), \quad v(0,\cdot) = v_0 \in P(X^n) \tag{2.34}$$

where

$$P : X^n \to X^n$$

is any orthogonal projector, and

$$L : D(L) \subset X_s^{n\times n} \to X^n, \quad A : D(A)\subset X^n \to X^n$$

are linear operators, satisfying

$$(L(v\otimes v), v) = (Av, v) = 0, \quad v \in P(X^n), \tag{2.35}$$

for any sufficiently smooth vector field v. Set

$$\tilde v_0 = (v_0, 1) \in X^{n+1} \simeq X^n \times X,$$

$$\tilde v = (v,1) : [0,T] \to X^{n+1},$$

$$\tilde P : X^{n+1} \to X^{n+1},\ \tilde P(v,q) = \left(Pv, \int_\Omega q\,d\mu\right),$$

$$\tilde L : D(\tilde L) \subset X_s^{(n+1)\times(n+1)} \to X^{n+1},\ D(\tilde L) = \left(\begin{array}{c|c} D(L) & D(A) \\ \hline D(A)^\top & X \end{array}\right),$$

$$\tilde L\left(\begin{array}{c|c} M & v \\ \hline v^\top & q \end{array}\right) = \left(\begin{array}{c} LM - Av \\ \hline 0 \end{array}\right).$$

Tautologically,

$$\partial_t 1 = 0. \tag{2.36}$$

The "system" (2.34), (2.36) can be recast as

$$\partial_t \tilde v = \tilde P \tilde L(\tilde v\otimes\tilde v), \quad \tilde v(t,\cdot) \in \tilde P(X^{n+1}), \quad \tilde v(0,\cdot) = \tilde v_0 \in \tilde P(X^{n+1}), \tag{2.37}$$

which has the structure of (1.7). Moreover, any $\tilde{v} \in \tilde{P}(X^{n+1})$ can be expressed as

$$(v, a) \simeq \begin{pmatrix} v \\ \hline a \end{pmatrix}, \quad v \in P(X^n), \quad a = cst.$$

Hence, due to (2.35),

$$\left(\tilde{L}(\tilde{v} \otimes \tilde{v}), \tilde{v}\right) = \left(\tilde{L}\left(\begin{pmatrix} v \\ \hline a \end{pmatrix} \otimes \begin{pmatrix} v \\ \hline a \end{pmatrix}\right), \begin{pmatrix} v \\ \hline a \end{pmatrix}\right) = \left(\begin{pmatrix} L(v \otimes v) - aAv \\ \hline 0 \end{pmatrix}, \begin{pmatrix} v \\ \hline a \end{pmatrix}\right) = 0, \tag{2.38}$$

i.e., condition (1.8) is met. If (2.29) holds for L, it is valid for $\tilde{L}$ as well. Indeed, in this situation we have

$$\tilde{P}\tilde{L}\begin{pmatrix} qI & 0 \\ \hline 0 & q \end{pmatrix} = \tilde{P}\begin{pmatrix} L(qI) \\ \hline 0 \end{pmatrix} = 0$$

for any $q \in X$ sufficiently smooth.

Remark 2.8. We reckon that with some effort the theory above can be generalized to the situation when X^n is replaced with the space of L^2 vector fields on a Riemannian manifold.

3. Applications

To fix the ideas, in this section we restrict ourselves to the case of the periodic box $\Omega = \mathbb{T}^d$. The symbol $\mathcal{P}$ denotes the Leray-Helmholtz projector in X^d, and I in most cases stands for the $d \times d$ identity matrix. Many of the examples below are known to be the geodesic equations on infinite-dimensional Lie groups, cf. [1, 23]. To the best of our knowledge, for $d > 2$ the existence of global weak solutions akin to (2.1) for arbitrary initial data has never been established for any of these examples excluding the last one.

Incompressible ideal MHD. The incompressible ideal MHD equations [1] read

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = \operatorname{div}(b \otimes b), \tag{3.1}$$

$$\partial_t b + \operatorname{div}(b \otimes u) = \operatorname{div}(u \otimes b), \tag{3.2}$$

$$\operatorname{div} u = 0, \tag{3.3}$$

$$\operatorname{div} b = 0, \tag{3.4}$$

$$u(0) = u_0, \quad b(0) = b_0. \tag{3.5}$$

The unknowns are $u, b : [0,T]\times\Omega \to \mathbb{R}^d$ and $p : [0,T]\times\Omega \to \mathbb{R}$. The ideal MHD equations are the geodesic equations on the semidirect product of the Lie group of volume-preserving diffeomorphisms with the dual of its Lie algebra [1]. We refer to [7, 13] for some recent results concerning existence and non-existence of weak solutions. Since

$$\operatorname{div}\operatorname{div}(b\otimes u) = \operatorname{div}\operatorname{div}(u\otimes b),$$

we can rewrite (3.1), (3.2) in the equivalent form

$$\partial_t u = \mathcal{P}(\operatorname{div}(b\otimes b) - \operatorname{div}(u\otimes u)), \tag{3.6}$$

$$\partial_t b = \mathcal{P}(\operatorname{div}(u\otimes b) - \operatorname{div}(b\otimes u)). \tag{3.7}$$

Set

$$n = 2d,\ v = (u,b) : [0,T] \to X^n \simeq X^d \times X^d,$$
$$P : X^n \to X^n, \quad P(v,\beta) = (\mathcal{P}v, \mathcal{P}\beta),$$
$$L : D(L) \subset X_s^{n\times n} \to X^n, \quad L\left(\begin{array}{c|c} M & N \\ \hline N^\top & S \end{array}\right) = \left(\frac{\operatorname{div} S - \operatorname{div} M}{\operatorname{div} N - \operatorname{div}(N^\top)}\right).$$

Then (3.1)–(3.5) becomes the abstract Euler equation (1.7). It is straightforward to check that (1.8) holds for $v = (u,b)$ sufficiently smooth. Let $q \in X$ be a sufficiently smooth function. Then

$$PL\left(\begin{array}{c|c} qI & 0 \\ \hline 0 & qI \end{array}\right) = P\left(\frac{\nabla q - \nabla q}{0}\right) = 0.$$

In view of Remark 2.4, Theorem 2.5 and Remark 2.6 are applicable, and we get

Corollary 3.1. *For any* $(u_0, b_0) \in X^d \times X^d$ *with* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$, *there exists a generalized solution* (2.33) *to* (3.1)–(3.5).

Multidimensional Camassa-Holm. The multidimensional Camassa-Holm system [25, 15] looks like

$$\partial_t m + (\nabla u)^\top . m + \operatorname{div}(m\otimes u) = 0, \tag{3.8}$$

$$m = u - \nabla \operatorname{div} u, \tag{3.9}$$

$$u(0) = u_0. \tag{3.10}$$

The unknown is $u : [0,T]\times\Omega \to \mathbb{R}^d$. It describes the geodesics of the diffeomorphism group with H^1_{div} metric, see, e.g., [22]. A distinct geodesic interpretation was discussed in [15]. Relaxed solutions in the spirit of the generalized flows of Brenier [3] were recently constructed in [14]. We recall (cf. [22, 4]) that, loosely speaking, there is a "fiber-base" duality between the Monge-Kantorovich transport [41] and Euler's equations (1.1)-(1.4). In a similar way, one can think, cf. [15], about a "fiber-base" duality between (3.8)–(3.10) and the unbalanced optimal transport [24, 8, 31].

We now define the relevant projector. Namely, for each $(v,\sigma) \in X^{d+1} \simeq X^d \times X$, we consider its orthogonal projection over the vector fields of the form $(u, \operatorname{div} u)$. This is related to the "duality" above and to the unbalanced version of Brenier's polar factorization theorem [4] that was discussed in preliminary preprint versions of [15]. The explicit expression of the projector is

$$P : X^{d+1} \to X^{d+1},$$

$$P\begin{pmatrix} v \\ \hline \sigma \end{pmatrix} = \mathcal{P}_{\operatorname{div}}\begin{pmatrix} v \\ \hline \sigma \end{pmatrix} := \begin{pmatrix} v - \nabla(I-\Delta)^{-1}(\sigma - \operatorname{div} v) \\ \hline \sigma - (I-\Delta)^{-1}(\sigma - \operatorname{div} v) \end{pmatrix}. \tag{3.11}$$

Set

$$n = d+1,\ v = (u, \operatorname{div} u) : [0,T] \to X^n \simeq X^d \times X,$$

$$L : D(L) \subset X_s^{n\times n} \to X^n, \quad L\left(\begin{array}{c|c} M & v \\ \hline v^\top & q \end{array}\right) = \begin{pmatrix} -\operatorname{div} M \\ \hline -\operatorname{div} v + \frac{1}{2}\operatorname{Tr} M + \frac{1}{2} q \end{pmatrix}.$$

We claim that the Camassa-Holm system (3.8)–(3.10) is tantamount to the abstract Euler equation (1.7) with P and L just defined. Indeed, denote $p := \operatorname{div} u$, $p_0 := \operatorname{div} u_0$ in (3.8)–(3.10). After some calculations, one finds that (3.8)–(3.10) is equivalent to

$$\partial_t u = -\operatorname{div}(u \otimes u) + \nabla\left[\partial_t p + \operatorname{div}(up) - \frac{1}{2}|u|^2 - \frac{1}{2}p^2\right], \tag{3.12}$$

$$p = \operatorname{div} u, \tag{3.13}$$

$$u(0) = u_0, p(0) = p_0. \tag{3.14}$$

Tautologically,

$$\partial_t p = -\operatorname{div}(up) + \frac{1}{2}|u|^2 + \frac{1}{2}p^2 + \left[\partial_t p + \operatorname{div}(up) - \frac{1}{2}|u|^2 - \frac{1}{2}p^2\right]. \tag{3.15}$$

The system (3.12)–(3.15) can be rewritten as

$$\partial_t v = L(v \otimes v) + \begin{pmatrix} \nabla \xi \\ \hline \xi \end{pmatrix}, \quad v(0) = v_0, \tag{3.16}$$

where

$$v(t) = \left(\frac{u(t)}{p(t)}\right) \in P(X^n)$$

and

$$\xi = \partial_t p + \operatorname{div}(up) - \frac{1}{2}|u|^2 - \frac{1}{2}p^2. \tag{3.17}$$

Applying the projector P to both sides of (3.16), we get (1.7). Reciprocally, (1.7) implies (3.16) where ξ necessarily satisfies (3.17) due to (3.15).

A not very tedious calculation verifies (1.8) for $v = (u, \operatorname{div} u)$ sufficiently smooth. However,

$$PL\left(\begin{array}{c|c} qI & 0 \\ \hline 0 & q \end{array}\right) = P\left(\begin{array}{c} -\nabla q \\ \hline \frac{d+1}{2}q \end{array}\right),$$

which yields that the requirement (2.29) is not met, and we need to find another way to secure the trace condition. It will be based on the following simple multidimensional variant of the Grönwall-Bellman lemma.

Lemma 3.2. *Consider a function $\psi \in W^{1,1}(\mathbb{T}^d)$ such that a.e. in $\mathbb{T}^d$ one has*

$$|\nabla \psi(x)| \le c\psi(x) \tag{3.18}$$

with a constant c. Then $\psi \in C(\mathbb{T}^d)$, and

$$|\psi(x)| \le e^{\frac{c\sqrt{d}}{2}} \int_{\mathbb{T}^d} \psi(y)\, dy, \quad x \in \mathbb{T}^d. \tag{3.19}$$

Proof: By Sobolev embedding, $\psi \in L^p(\mathbb{T}^d)$, $1 - \frac{1}{n} = \frac{1}{p}$, whence $\psi \in W^{1,p}(\mathbb{T}^d)$. Bootstrapping, we derive that $\psi \in W^{1,\infty}(\mathbb{T}^d) \subset C(\mathbb{T}^d)$. Consequently, $\log \psi \in W^{1,\infty}(\mathbb{T}^d)$ because

$$|\nabla \log \psi(x)| \le c \tag{3.20}$$

due to (3.18). Since $|\mathbb{T}^d| = 1$, there is $x^0 \in \mathbb{T}^d$ such that $\psi(x^0) = \int_{\mathbb{T}^d} \psi(y)\, dy$. By (3.20),

$$|\log \psi(x) - \log \psi(x^0)| \le c|x - x^0| \le \frac{c\sqrt{d}}{2}, \quad x \in \mathbb{T}^d, \tag{3.21}$$

which implies (3.19). ∎

We return to the Camassa-Holm system. The adjoint operator is

$$L^* : D(L^*) \subset X^n \to X_s^{n\times n}, \quad L^*\left(\frac{\phi}{\chi}\right) = \frac{1}{2}\left(\begin{array}{c|c} \nabla\phi + (\nabla\phi)^\top + \chi I & \nabla\chi \\ \hline (\nabla\chi)^\top & \chi \end{array}\right).$$

If $(\phi,\chi) \in P(X^n)$, then $\chi = \operatorname{div}\phi$. If the eigenvalues of

$$\left(\begin{array}{c|c} \nabla\phi + (\nabla\phi)^\top + \chi I & \nabla\chi \\ \hline (\nabla\chi)^\top & \chi \end{array}\right)(x), \quad \chi = \operatorname{div}\phi, \tag{3.22}$$

are bounded from below, there is $k \geq 0$ such that

$$\left(\begin{array}{c|c} \nabla\phi + (\nabla\phi)^\top + (\chi+k) I & \nabla\chi \\ \hline (\nabla\chi)^\top & \chi+k \end{array}\right)(x) \geq 0.$$

In particular, $\chi + k \geq 0$. Moreover, considering the principal minors of order 2, we see that

$$(\chi + k + 2\partial_{x_i}\phi_i)(\chi+k) \geq (\partial_{x_i}\chi)^2.$$

Thus,

$$3(\chi+k)^2 = (3\chi+3k)(\chi+k) \geq (3\chi+k)(\chi+k) \geq |\nabla\chi|^2.$$

Since $\int_{\mathbb{T}^d} \chi(y)\,dy = 0$, Lemma 3.2 implies that

$$\chi(x) + k \leq k e^{\frac{\sqrt{3d}}{2}}, \quad x \in \mathbb{T}^d. \tag{3.23}$$

This provides a uniform bound on the trace of the matrix in (3.22). Hence, the eigenvalues of this matrix are bounded from above, and the trace condition holds. We infer

Corollary 3.3. *For every $u_0 \in X^d$, there exists a generalized solution* (2.33) *to* (3.8)–(3.10).

EPDiff. The EPDiff equations [16, 19, 44, 20, 35, 27] are

$$\partial_t m + (\nabla u)^\top . m + \operatorname{div}(m \otimes u) = 0, \tag{3.24}$$

$$m = u - \Delta u, \tag{3.25}$$

$$u(0) = u_0. \tag{3.26}$$

The unknown is $u : [0,T]\times\Omega \to \mathbb{R}^d$. The EPDiff equations are the geodesic equations on the diffeomorphism group with H^1 metric, see, e.g., [22].

For each $(v, M) \in X^{d(1+d)} \simeq X^d \times X^{d\times d}$, we consider its orthogonal projection over the fields of the form $(u, \nabla u)$. This is related to the matricial optimal transport [6]. More profoundly, we reckon that there is a "fiber-base" duality between the matricial transport as considered in [6] and the EPDiff equations, cf. the discussion of the Camassa-Holm example. The explicit expression of the projector is

$$P : X^{d(1+d)} \to X^{d(1+d)},$$

$$P\begin{pmatrix} v \\ \hline M \end{pmatrix} = \mathcal{P}_\nabla \begin{pmatrix} v \\ \hline M \end{pmatrix} := \begin{pmatrix} v - \operatorname{div}(I - \nabla \operatorname{div})^{-1}(M - \nabla v) \\ \hline M - (I - \nabla \operatorname{div})^{-1}(M - \nabla v) \end{pmatrix}. \tag{3.27}$$

Remark 3.4. The operator $(I - \nabla \operatorname{div})^{-1}$ can be viewed as the Riesz isomorphism between the Hilbert spaces E^* and E, where $E := \{M \in X^{d\times d} | \operatorname{div} M \in X^d\}$ is equipped with the scalar product $(M, N)_E = (M, N) + (\operatorname{div} M, \operatorname{div} N)$, cf. [40]. Consequently, (3.27) defines a bounded linear operator on $X^{d(1+d)}$.

Set

$$n = d(1+d),\ v = (u, \nabla u) : [0, T] \to X^n \simeq X^d \times X^{d\times d},$$

$$L : D(L) \subset X_s^{n\times n} \to X^n,$$

$$L\left(\begin{array}{c|c} M & \Upsilon \\ \hline \Upsilon^\top & \Xi \end{array}\right) = \begin{pmatrix} 0 \\ \hline -\operatorname{div}(\Upsilon^\top) + M + \widehat{\Xi} - \widecheck{\Xi} + \frac{1}{2} I \operatorname{Tr} M + \frac{1}{2} I \operatorname{Tr} \widehat{\Xi} \end{pmatrix}.$$

Let us now interpret the EPDiff equations as an abstract Euler equation. Denote $G := \nabla u$, $G_0 := \nabla u_0$ in (3.24)–(3.26). A tedious calculation shows that (3.24)–(3.26) is equivalent to

$$\partial_t u = \operatorname{div}\Big[\partial_t G + \operatorname{div}(G \otimes u) - u \otimes u - (\widehat{G \otimes G}) + (\widecheck{G \otimes G}) \\ - \frac{1}{2} I \operatorname{Tr} u \otimes u - \frac{1}{2} I \operatorname{Tr}(\widehat{G \otimes G})\Big], \tag{3.28}$$

$$G = \nabla u, \tag{3.29}$$

$$u(0) = u_0,\ G(0) = G_0. \tag{3.30}$$

Tautologically,

$$\partial_t G = -\operatorname{div}(G \otimes u) + u \otimes u + (\widehat{G \otimes G}) - (\widecheck{G \otimes G}) + \frac{1}{2} I \operatorname{Tr} u \otimes u \\ + \frac{1}{2} I \operatorname{Tr}(\widehat{G \otimes G}) + \Big[\partial_t G + \operatorname{div}(G \otimes u) - u \otimes u - (\widehat{G \otimes G}) + (\widecheck{G \otimes G}) \\ - \frac{1}{2} I \operatorname{Tr} u \otimes u - \frac{1}{2} I \operatorname{Tr}(\widehat{G \otimes G})\Big]. \tag{3.31}$$

The system (3.28)–(3.31) can be rewritten as

$$\partial_t v = L(v \otimes v) + \begin{pmatrix} \operatorname{div} \xi \\ \hline \xi \end{pmatrix}, \quad v(0) = v_0, \tag{3.32}$$

where

$$v(t) = \begin{pmatrix} u(t) \\ \hline G(t) \end{pmatrix} \in P(X^n)$$

and

$$\xi = \partial_t G + \operatorname{div}(G \otimes u) - u \otimes u - (\widehat{G \otimes G}) + (\widetilde{G \otimes G}) - \frac{1}{2} I \operatorname{Tr} u \otimes u - \frac{1}{2} I \operatorname{Tr}(\widehat{G \otimes G}). \tag{3.33}$$

Applying the projector P to both sides of (3.32), we get (1.7). Reciprocally, (1.7) implies (3.32) where ξ necessarily satisfies (3.33) due to (3.31).

A direct calculation shows that (1.8) holds for $v = (u, \nabla u)$ sufficiently smooth, but the requirement (2.29) is not met.

The adjoint operator is

$$L^* : D(L^*) \subset X^n \to X_s^{n \times n},$$

$$L^* \begin{pmatrix} \phi \\ \hline \Phi \end{pmatrix} = \frac{1}{2} \left(\begin{array}{c|c} \Phi + \Phi^\top + I \operatorname{Tr} \Phi & \nabla \Phi \\ \hline (\nabla \Phi)^\top & \widehat{\Phi}^* + \widehat{(\Phi^\top)}^* - \widetilde{\Phi}^* - \widetilde{(\Phi^\top)}^* + (\operatorname{Tr} \Phi) \widetilde{I}^* \end{array} \right).$$

If $(\phi, \Phi) \in P(X^n)$, then $\Phi = \nabla \phi$. If the eigenvalues of

$$\left(\begin{array}{c|c} \Phi + \Phi^\top + I \operatorname{Tr} \Phi & \nabla \Phi \\ \hline (\nabla \Phi)^\top & \widehat{\Phi}^* + \widehat{(\Phi^\top)}^* - \widetilde{\Phi}^* - \widetilde{(\Phi^\top)}^* + (\operatorname{Tr} \Phi) \widetilde{I}^* \end{array} \right)(x), \quad \Phi = \nabla \phi, \tag{3.34}$$

are bounded from below, there is $k \geq 0$ such that

$$\left(\begin{array}{c|c} \Phi + \Phi^\top + (k + \operatorname{Tr} \Phi) I & \nabla \Phi \\ \hline (\nabla \Phi)^\top & \widehat{\Phi}^* + \widehat{(\Phi^\top)}^* - \widetilde{\Phi}^* - \widetilde{(\Phi^\top)}^* + (k + \operatorname{Tr} \Phi) \widetilde{I}^* \end{array} \right)(x) \geq 0.$$

Taking the trace of the last block, we deduce that $k + \operatorname{Tr} \Phi \geq 0$. Moreover, the non-negativity of the principal minors of order 2 yields

$$(k + \operatorname{Tr} \Phi + 2\Phi_{ii})(2\Phi_{jj} - 2\Phi_{ll} + k + \operatorname{Tr} \Phi) \geq (\partial_{x_i} \Phi_{jl})^2.$$

Letting $j = l$ and performing the summation w.r.t. to the remaining indices, we arrive at

$$3(k + \operatorname{Tr} \Phi)^2 \geq (k + 3 \operatorname{Tr} \Phi)(k + \operatorname{Tr} \Phi) \geq |\nabla \operatorname{Tr} \Phi|^2.$$

But $\Phi = \nabla \phi$, so $\int_{\mathbb{T}^d} \operatorname{Tr} \Phi(y)\, dy = 0$. As in the Camassa-Holm case above, Lemma 3.2 implies a uniform bound on $\operatorname{Tr} \Phi$ and thus on the trace of the matrix in (3.34). This yields the trace condition, and leads to

Corollary 3.5. *For every $u_0 \in X^d$, there exists a generalized solution* (2.33) *to* (3.24)–(3.26).

Euler-α. The Euler-α equations [17, 18, 34, 37] (with $\alpha = 1$ for definiteness) may be written as

$$\partial_t m + (\nabla u)^\top . m + \operatorname{div}(m \otimes u) + \nabla p = 0, \tag{3.35}$$

$$m = u - \Delta u, \tag{3.36}$$

$$\operatorname{div} u = 0, \tag{3.37}$$

$$u(0) = u_0. \tag{3.38}$$

The unknowns are $u : [0,T] \times \Omega \to \mathbb{R}^d$ and $p : [0,T] \times \Omega \to \mathbb{R}$. These equations are the geodesic equations on the group of volume-preserving diffeomorphisms with H^1 metric, see, e.g., [22]. This example is quite similar to the previous one. We first recast (3.35)–(3.38) in the form

$$\partial_t u + \nabla p = \operatorname{div}\Big[\partial_t G + \operatorname{div}(G \otimes u) - u \otimes u - (\widehat{G \otimes G}) + (\widecheck{G \otimes G}) - \frac{1}{2} I \operatorname{Tr} u \otimes u - \frac{1}{2} I \operatorname{Tr}(\widehat{G \otimes G})\Big], \tag{3.39}$$

$$\partial_t G = -\operatorname{div}(G \otimes u) + u \otimes u + (\widehat{G \otimes G}) - (\widecheck{G \otimes G}) + \frac{1}{2} I \operatorname{Tr} u \otimes u + \frac{1}{2} I \operatorname{Tr}(\widehat{G \otimes G}) + \Big[\partial_t G + \operatorname{div}(G \otimes u) - u \otimes u - (\widehat{G \otimes G}) + (\widecheck{G \otimes G}) - \frac{1}{2} I \operatorname{Tr} u \otimes u - \frac{1}{2} I \operatorname{Tr}(\widehat{G \otimes G})\Big]. \tag{3.40}$$

$$G = \nabla u, \tag{3.41}$$

$$\operatorname{Tr} G = 0, \tag{3.42}$$

$$u(0) = u_0,\ G(0) = G_0, \tag{3.43}$$

cf. (3.28)–(3.31). Set

$$n = d(1+d),\ v = (u, G) : [0,T] \to X^n \simeq X^d \times X^{d \times d},$$

$$L : D(L) \subset X_s^{n \times n} \to X^n,$$

$$L\left(\begin{array}{c|c} M & \Upsilon \\ \hline \Upsilon^\top & \Xi \end{array}\right) = \left(\begin{array}{c} 0 \\ \hline -\operatorname{div}(\Upsilon^\top) + M + \widehat{\Xi} - \widecheck{\Xi} + \frac{1}{2} I \operatorname{Tr} M + \frac{1}{2} I \operatorname{Tr} \widehat{\Xi} \end{array}\right).$$

Consider the set

$$Y := \mathcal{P}_\nabla X^n \cap \{(u, G) | \operatorname{Tr} G = 0\},$$

where $\mathcal{P}_\nabla$ was defined in (3.27). It is clear that Y is a closed linear subspace of X^n. Let

$$P : X^n \to Y$$

be the corresponding orthogonal projector. The orthogonal complement of Y consists of the elements of the form $(\operatorname{div}\xi, \xi + qI)$, $\xi \in X^{d\times d}$, $q \in X$. Rewrite the system (3.39)–(3.43) as

$$\partial_t v = L(v \otimes v) + \begin{pmatrix} \operatorname{div}\xi \\ \hline \xi + pI \end{pmatrix}, \quad v(0) = v_0, \tag{3.44}$$

where

$$v(t) = \begin{pmatrix} u(t) \\ \hline G(t) \end{pmatrix} \in P(X^n)$$

and

$$\xi = \partial_t G + \operatorname{div}(G\otimes u) - u\otimes u - (\widehat{G\otimes G}) + (\widecheck{G\otimes G}) - \frac{1}{2} I \operatorname{Tr} u\otimes u - \frac{1}{2} I \operatorname{Tr}(\widehat{G\otimes G}) - pI. \tag{3.45}$$

Applying the projector P to both sides of (3.44), we get the abstract Euler equation (1.7). Reciprocally, (1.7) implies (3.44) where ξ necessarily satisfies (3.45) due to the trivial equality (3.40).

As in the previous example, (1.8) holds for $v = (u, \nabla u)$ sufficiently smooth. In contrast to EPDiff, (2.29) is now valid since

$$PL\left(\begin{array}{c|c} qI & 0 \\ \hline 0 & \widecheck{qI}^{*} \end{array}\right) = P\begin{pmatrix} 0 \\ \hline (d+1)qI \end{pmatrix} = 0.$$

Thus we have

Corollary 3.6. *For every* $u_0 \in X^d$, $\operatorname{div} u_0 = 0$, *there exists a generalized solution* (2.33) *to* (3.35)–(3.38).

Incompressible isotropic Hookean elastodynamics. The "neo-Hookean" model of motion of incompressible isotropic elastic fluid [32, 38, 39, 29, 30, 28] reads

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = \operatorname{div}(FF^\top), \tag{3.46}$$

$$\partial_t F + \operatorname{div}(F \otimes u) = (\nabla u)F, \tag{3.47}$$

$$\operatorname{div} u = 0, \tag{3.48}$$

$$\operatorname{div} F^\top = 0, \tag{3.49}$$

$$u(0) = u_0, \quad F(0) = F_0. \tag{3.50}$$

The unknowns are $u : [0,T] \times \Omega \to \mathbb{R}^d$, $F : [0,T] \times \Omega \to \mathbb{R}^{d\times d}$ and $p : [0,T] \times \Omega \to \mathbb{R}$. Consider the projector

$$P : X^{d\times d} \to X^{d\times d}, \qquad P(M) = \mathcal{P}_d(M) := \Big(\mathcal{P}M_1 \,\big|\, \mathcal{P}M_2 \,\big|\, \cdots \,\big|\, \mathcal{P}M_d\Big), \quad (3.51)$$

where $M_1, \dots, M_d \in X^d$ are the columns of the matrix M. Obviously,

$$\operatorname{div}(\mathcal{P}_d M)^\top = 0, \quad M \in X^{d\times d}.$$

It is straigtforward to check that

$$\operatorname{div}(\operatorname{div}(F \otimes u))^\top = \operatorname{div}((\nabla u)F)^\top,$$

which allows us to project (3.47) onto $\mathcal{P}_d X^{n\times n}$. Set

$$n = d(1+d),\ v = (u,F) : [0,T] \to X^n \simeq X^d \times X^{d\times d},$$

$$P : X^n \to X^n, \quad P(v,\Phi) = (\mathcal{P}v, \mathcal{P}_d\Phi),$$

$$L : D(L) \subset X_s^{n\times n} \to X^n, \quad L\left(\begin{array}{c|c} M & \Upsilon \\ \hline \Upsilon^\top & \Xi \end{array}\right) = \left(\frac{\operatorname{div}\widehat{\Xi} - \operatorname{div} M}{\operatorname{div}\widetilde{\Upsilon} - \operatorname{div}(\Upsilon^\top)}\right).$$

Then (3.46)–(3.50) can be recast in the form of the abstract Euler equation (1.7). The structure of equations (3.46)–(3.50) (in particular, their similarity with the ideal MHD equations) allows us to conjecture that they determine the geodesics on some Lie group. The conservativity condition (1.8) holds for $v = (u,F)$ sufficiently smooth (this can be verified straightforwardly). Let us check (2.29). Let $q \in X$ be a sufficiently smooth function. Then

$$PL\left(\begin{array}{c|c} qI & 0 \\ \hline 0 & \widetilde{qI} \end{array}\right) = P\left(\frac{d\nabla q - \nabla q}{0}\right) = 0.$$

As a result, we have

Corollary 3.7. *For any* $(u_0, F_0) \in X^d \times X^{d\times d}$ *with* $\operatorname{div} u_0 = 0$, $\operatorname{div} F_0^\top = 0$*, there exists a generalized solution* (2.33) *to* (3.46)–(3.50).

Conservative incompressible elastic fluid. The motion of the incompressible Oldroyd-B viscoelastic material (also known as Jeffreys' fluid) is described [36, 26, 46] by the problem

$$\partial_t u + \operatorname{div}(u \otimes u) - \mu \Delta u + \nabla p = \operatorname{div} \tau, \tag{3.52}$$

$$\partial_t \tau + \operatorname{div}(\tau \otimes u) + Q(\nabla u, \tau) + a\tau = \frac{1}{2}(\nabla u + (\nabla u)^\top), \tag{3.53}$$

$$\operatorname{div} u = 0, \tag{3.54}$$

$$u(0) = u_0, \quad \tau(0) = \tau_0. \tag{3.55}$$

The unknowns are $u : [0,T] \times \Omega \to \mathbb{R}^d$, $\tau : [0,T] \times \Omega \to \mathbb{R}_s^{d\times d}$ and $p : [0,T] \times \Omega \to \mathbb{R}$. When the retardation time vanishes, we get Maxwell's fluid (this corresponds to $\mu = 0$). The choice $a = 0$ (cf. [32, 45]) tallies with the damping-free case when the relaxation time blows up. We restrict ourselves to the purely hyperbolic case $a = \mu = 0$, which coheres with a purely elastic fluid. Note that (cf. [32, 21]) the purely hyperbolic system with $Q = -\nabla u \tau - \tau(\nabla u)^\top$(the upper-convective case) can be made equivalent to (3.46)-(3.50) if one assumes the ansätze

$$\tau = FF^\top, \quad \operatorname{div} F^\top = 0. \tag{3.56}$$

This makes sense because the constraints (3.56) are preserved along the flow. Here we do not assume neither (3.56) nor even positive-definiteness of τ. The term Q is related to frame-invariance and is known to create mathematical difficulties. We consider the simplified model with $Q \equiv 0$, cf. [12, 32, 45, 46, 26]. This model, unlike (3.46)–(3.50), is not frame-indifferent, but it is invariant to the transformations which keep the frame inertial (e.g., to the Galilean transformation). We arrive at the following conservative problem:

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = \operatorname{div} \tau, \tag{3.57}$$

$$\partial_t \tau + \operatorname{div}(\tau \otimes u) = \frac{1}{2}(\nabla u + (\nabla u)^\top), \tag{3.58}$$

$$\operatorname{div} u = 0, \tag{3.59}$$

$$u(0) = u_0, \quad \tau(0) = \tau_0. \tag{3.60}$$

Set

$$n = d + d(d+1)/2, \ v = (u, \tau) : [0,T] \to X^n \simeq X^d \times X_s^{d\times d},$$
$$P : X^n \to X^n, \quad P(v, \varsigma) = (\mathcal{P} v, \varsigma),$$

$$A : D(A) \subset X^n \to X^n, \quad A(v,\varsigma) = -\frac{1}{2}(2\,\mathrm{div}\,\varsigma, \nabla v + (\nabla v)^\top),$$
$$L : D(L) \subset X_s^{n\times n} \to X^n, \quad L\left(\begin{array}{c|c} M & \Upsilon \\ \hline \Upsilon^\top & \Xi \end{array}\right) = \left(\begin{array}{c} -\mathrm{div}\,M \\ \hline -\mathrm{div}(\Upsilon^\top) \end{array}\right).$$

Then (3.57)–(3.60) can be written in the abstract form (2.34). Condition (2.35) follows by integration by parts. Moreover, (2.29) is satisfied since

$$PL\left(\begin{array}{c|c} qI & 0 \\ \hline 0 & \widetilde{qI}^* \end{array}\right) = P\left(\begin{array}{c} -\mathrm{div}(qI) \\ \hline 0 \end{array}\right) = \left(\begin{array}{c} -\mathcal{P}\nabla q \\ \hline 0 \end{array}\right) = 0$$

for each $q \in X$ sufficiently smooth. In light of Remark 2.7, we have the following corollary:

Corollary 3.8. *For any* $(u_0, \tau_0) \in X^d \times X_s^{d\times d}$ *with* $\mathrm{div}\, u_0 = 0$, *there exists a generalized solution* (2.33) *of the extended system* (2.37) *tantamount to* (3.57)–(3.60).

Korteweg-de Vries. Let $\Omega = \mathbb{T}^1$. The Korteweg-de Vries equation is

$$\partial_t v + v_{xxx} = 6vv_x, \quad v(0) = v_0. \tag{3.61}$$

The unknown is $v : [0,T]\times\Omega \to \mathbb{R}$. It is the geodesic equation for the Virasoro group [23]. The Korteweg-de Vries equation is known to be globally well-posed [9] but we still consider this example for the sake of curiosity.

Set

$$n = 1, \quad P = I, \quad A : D(A) \subset X \to X, \quad A(v) = -v_{xxx},$$
$$L : D(L) \subset X \to X, \quad L(\sigma) = -3\sigma_x.$$

Then (3.61) can be written in the abstract form (2.34). Condition (2.35) can be easily verified via integration by parts. However, (2.29) is not satisfied.

As in Remark 2.7, consider the extended problem (2.37) with

$$\tilde{P}(v,a) = \left(v, \int_\Omega a\,d\mu\right),$$

$$\tilde{L}\begin{pmatrix} \sigma & z \\ z & a \end{pmatrix} = \begin{pmatrix} -3\sigma_x + z_{xxx} \\ 0 \end{pmatrix}.$$

The adjoint operator is

$$L^* : D(L^*) \subset X^2 \to X_s^{2\times 2}, \quad L^*\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6\phi_x & -\phi_{xxx} \\ -\phi_{xxx} & 0 \end{pmatrix}.$$

If there is $k \geq 0$ such that

$$\begin{pmatrix} 6\phi_x + k & -\phi_{xxx} \\ -\phi_{xxx} & k \end{pmatrix} \geq 0,$$

then

$$6k\phi_x + k^2 - \phi_{xxx}^2 \geq 0.$$

Consequently,

$$\int_{\mathbb{T}^1} \phi_{xxx}^2 \leq k^2.$$

By Wirtinger inequality, ϕ_x is uniformly bounded in $W^{2,2}(\mathbb{T}^1)$ and thus in $L^\infty(\mathbb{T}^1)$. Accordingly, the trace of $L^*(\phi, \psi)$ is uniformly bounded, which implies the trace condition.

Corollary 3.9. *For any $v_0 \in X$, there exists a generalized solution* (2.33) *of the extended system* (2.37) *tantamount to* (3.61).

Remark 3.10. Some of the examples above (namely, the Euler-α and the ideal MHD) as well as the incompressible Euler itself are known to have dissipative solutions in the spirit of Lions [33] (see [42], [43]). The quadratic conservative structure of the abstract Euler equation (1.7) complies nicely with Lions' concept (see [42, Appendix] for a related discussion). We have little doubt that all the examples of Section 3 admit dissipative solutions (this should not be difficult to prove but lies beyond the scope of this article). It would be interesting to find a link between the variational solutions (2.33) and the dissipative solutions.

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