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MATRIX TODA AND VOLTERRA LATTICES

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Abstract: We consider matrix Toda and Volterra lattice equations and their relation with matrix biorthogonal polynomials. From that relation, we give a method for constructing a new solution of these systems from another given one. An illustrative example is presented.

Keywords: Matrix biorthogonal polynomials, matrix Toda lattice, matrix Volterra lattice, Symmetrized process, block Jacobi Matrices.
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1. Introduction

The Toda lattice

$$\begin{cases} \dot{b}_n(t) = a_n(t) - a_{n+1}(t) \\ \dot{a}_{n+1}(t) = a_{n+1}(t)(b_n(t) - b_{n+1}(t)) \end{cases}, \qquad a_0 = 0, \qquad n = 0, 1 \ldots, \tag{1}$$

when both $(a_n(t))_{n\in\mathbb{N}}$ and $(b_n(t))_{n\in\mathbb{N}}$ are real or complex functions has been well studied in the literature from different points of view (*cf.* [2, 3, 6, 13, 24, 26]). In particular, if we assume that $a_n(t) \neq 0$, $n > 0$, and we define a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ recursively by

$$p_{n+1}(x,t) = (x - b_n(t))p_n(x,t) - a_n(t)p_{n-1}(x,t), \quad p_{-1} = 0, \quad p_0 = 1,$$

then using Favard's theorem [9], for each $t \in \mathbb{R}$, there exists a linear functional $u(t)$ such that the sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ is orthogonal with respect to $u(t)$. This connection with orthogonal polynomials was used in [7, 13, 26] to give sufficient conditions for the construction of a new solution $(\widetilde{a}_n(t))_{n\in\mathbb{N}}$

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and $(\widetilde{b}_n(t))_{n\in\mathbb{N}}$ of (1) from another given one. Both solutions are linked by a Bäcklund transformation, given by

$$\begin{aligned} a_n &= \gamma_{2n-1}\gamma_{2n}, & b_n &= \gamma_{2n}+\gamma_{2n+1}+C, & n&\in\mathbb{N},\\ \widetilde{a}_n &= \gamma_{2n}\gamma_{2n+1}, & \widetilde{b}_n &= \gamma_{2n+1}+\gamma_{2n+2}+C, & n&\in\mathbb{N}, \end{aligned}$$

where $(\gamma_n)_{n\in\mathbb{N}}$ is solution of the Volterra lattice

$$\dot{\gamma}_{n+1} = \gamma_{n+1}(\gamma_n - \gamma_{n+2}), \qquad \gamma_0 = 0, \qquad n = 0, 1 \dots. \tag{2}$$

Here we emphasize that the dot '˙' means differentiation with respect to $t \in \mathbb{R}$. Both the Volterra and the new solution of the Toda lattices are strongly related with Darboux transformations of orthogonal polynomials (or equivalently, to the LU and UL factorization of its associated Jacobi matrix [8]). In [5] the above analysis is generalized to high-order Toda and Volterra lattices.

The Darboux transformations has also been used by Spiridonov and Zhedanov to study the discrete-time Toda [28] and Volterra lattices [29] and their connection with Askey-Wilson polynomials.

More recently, a matrix interpretation of high-order Toda lattices is given in [4] to consider the following semi-infinite system of differential equations

$$\begin{cases} \dot{a}_n = c_n - c_{n-2}, \\ \dot{b}_n = c_n a_{n+1} - c_{n-1} a_n + d_n - d_{n-2}, \\ \dot{c}_n = c_n(b_{n+1} - b_n) + d_n a_{n+2} - d_{n-1} a_n, \\ \dot{d}_n = d_n(b_{n+2} - b_n), \end{cases} \qquad n \in \mathbb{N}, \tag{3}$$

with initial conditions $a_0 = b_0 = c_0 = d_0 = c_1 = 0$, where a_n, b_n, c_n and d_n are complex functions depending on $t \in \mathbb{R}$. This system can be characterized in terms of matrix orthogonal polynomials satisfying the following three term recurrence relation

$$xV_m(x) = A_{m+1}V_{m+1} + B_n V_m(x) + C_m V_{m-1}(x), \qquad n \in \mathbb{N},$$

where

$$A_m = \begin{bmatrix} 1 & 0 \\ a_{2m+3} & 1 \end{bmatrix}, \qquad B_m = \begin{bmatrix} b_{2m+1} & a_{2m+2} \\ c_{2m+1} & b_{2m+2} \end{bmatrix}, \qquad C_m = \begin{bmatrix} d_{2m-1} & c_{2m} \\ 0 & d_{2m} \end{bmatrix}.$$

Notice that from the above differential system, (3) can be written in matrix notation as

$$\begin{cases} \dot{A}_m &= A_m D_{m+1} - D_m A_m \\ \dot{B}_m &= A_m C_{m+1} - C_m A_{m-1} + B_m D_m - D_m B_m \, , \qquad\qquad n \in \mathbb{N}, \\ \dot{C}_m &= B_m C_m - C_m B_{m-1} + C_m D_{m-1} - D_m C_m \end{cases}$$

where $D_m = \begin{bmatrix} 0 & 0 \\ c_{2m+1} & 0 \end{bmatrix}$. With this new interpretation, the authors find, under some conditions, a representation of the vector of linear functional associated with the polynomials $(V_n)_{n\in\mathbb{N}}$, and show that the orthogonality governs the high-order Toda lattice.

Noncommutative extension of Toda and Volterra lattices is not just a generalization. We can see that, in recent years, the interest on non-commutative analogue of solition equations and their integrability have been increased [16], such is the case of Hirota-Miwa equation [15, 21], KdV equation [10, 19, 25], KP equation [14, 18, 27], mKP equation [16, 18] and the Toda and Volterra lattices [18], among others. In particular, the non-commutative Toda equation, have been extensively studied in the literature (*cf.* [11, 18, 17, 22, 23]). For example, in [27] the author introduces the non-commutative Toda lattice from Sato-Wilson equations and shows that with an appropriate change of variables, the Toda equation can be written as in (1), but now with $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ belonging to a ring. Moreover, the author derives the bilinear identities for the Baker-Akhiezer functions and calculates the N-soliton solutions. More recently, in [20] the Darboux and the binary Darboux are used for the construction of the solution (in terms of quasi-determinants) of non-commutative Toda Lattices.

The propose of this manuscript is to deal with a very special type of noncommutative Toda and Volterra lattices. In fact, we will consider Toda and Volterra lattices as in (1) and (2), but now with $N \times N$ matrix complex functions, a_n, b_n and γ_n. In particular, we are interested in relating the matrix Toda and Volterra equations with sequences of matrix bi-orthogonal polynomials which are associated with a matrix sesquilinear forms. The matrix bi-orthogonal polynomials allow us to find a new solution of Toda and Volterra lattices from a given one using the symmetrization process (*cf.* Section 3).

This work is organized as follows: In Section 2, we present the basic theory about matrix biorthogonal polynomials and a Favard's matrix theorem.

Following the ideas given in [9] we expose the symmetrization process for matrix sesquilinear forms associated with a matrix of the linear functionals, this process involves the LU block factorization (*cf.* [8] in the scalar case) and the matrix Christoffel transformation [1]. In Section 4, we study a matrix Toda system and use this symmetrization process (the analogous to the Bäcklund transformation) to construct a new solution from another given one. Both solutions are liked to each other by a matrix Volterra lattice. We also give a very instructive example which motivates the study in Section 5 of the matrix Volterra lattice (or equivalently matrix 2-Toda lattice), where we give characterizations of the solution of a Volterra lattice, its corresponding matrix of linear functionals associated with the block Jacobi matrix and its sequence of matrix biorthogonal polynomials.

2. Matrix biorthogonal polynomials

First of all we will fix some notation. Let $\mathbb{C}$ be the set of complex numbers, and denote by $\mathbb{C}^{N\times N}$ the linear space of $N\times N$ matrices with complex entries. For an arbitrary finite or semi-infinite matrix A, the matrix $A^\dagger$ is its transpose conjugate. We will denote by I and $\mathbf{0}$ the identity and zero $N\times N$ matrices, respectively.

A *sesquilinear form* on the bimodule of matrix polynomials $\mathbb{C}^{N\times N}[x]$, with *real variable*, is a map

$$\langle\cdot,\cdot\rangle : \mathbb{C}^{N\times N}[x]\times\mathbb{C}^{N\times N}[x]\to\mathbb{C}^{N\times N},$$

such that for any triple $P,Q,R\in\mathbb{C}^{N\times N}[x]$ and for all $A,B\in\mathbb{C}^{N\times N}$:

1. $\langle A\,P(x)+B\,Q(x),R(x)\rangle = A\,\langle P(x),R(x)\rangle + B\,\langle Q(x),R(x)\rangle$;
2. $\langle P(x),A\,Q(x)+B\,R(x)\rangle = \langle P(x),Q(x)\rangle\,A^\dagger + \langle P(x),R(x)\rangle\,B^\dagger$.

If $\langle P(x),Q(x)\rangle = \langle Q(x),P(x)\rangle^\dagger$, then $\langle\cdot,\cdot\rangle$ is called a *symmetric sesquilinear form.*

Given a matrix of linear functionals, *i.e.*

$$u = \begin{bmatrix} u_{0,0} & \cdots & u_{0,N-1} \\ \vdots & \ddots & \vdots \\ u_{N-1,0} & \cdots & u_{N-1,N-1}\end{bmatrix},$$

where $u_{i,j}$ belongs to the dual space of $\mathbb{C}[x]$, we define its associated sesquilinear form $\langle P, Q\rangle_u$ as follows

$$\big(\langle P, Q\rangle_u\big)_{i,j} = \sum_{k,l=0}^{N-1} [u_{k,l}, P_{i,k}(x)\,\overline{Q}_{j,l}(x)], \qquad i,j = 0,1\cdots,N-1.$$

Here, $[u_{i,j}, p(x)]$ is the action of the linear functional $u_{i,j}$ on the scalar polynomial $p(x)$.

An important property of the sesquilinear form defined in terms of a matrix of linear functionals is that $\langle x\,P(x), Q(x)\rangle_u = \langle P(x), x\,Q(x)\rangle_u$. Hereinafter we only work with matrix sesquilinear forms satisfying the above property and, unless otherwise stated, we will assume that $\langle \mathrm{I}, \mathrm{I}\rangle_u = \mathrm{I}$.

Definition 1. *The n-th moment of a sesquilinear form associated with a matrix of linear functionals is defined as*

$$u_n = \begin{bmatrix} \langle u_{0,0}, x^n\rangle & \dots & \langle u_{0,N-1}, x^n\rangle \\ \vdots & \ddots & \vdots \\ \langle u_{N-1,0}, x^n\rangle & \dots & \langle u_{N-1,N-1}, x^n\rangle \end{bmatrix}, \qquad n \in \mathbb{N}.$$

We also define the block moment matrix M and its n-th truncation as

$$M = \begin{bmatrix} u_0 & u_1 & \cdots \\ u_1 & u_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad M_n = \begin{bmatrix} u_0 & \cdots & u_{n-1} \\ \vdots & \ddots & \vdots \\ u_{n-1} & \cdots & u_{2n-2} \end{bmatrix}, \qquad n \in \mathbb{N}, \tag{4}$$

Let $\chi(x) = \begin{bmatrix} \mathrm{I} & \mathrm{I}x & \mathrm{I}x^2 & \cdots \end{bmatrix}^\dagger$. Observe that with this notation, the block moment matrix can be expressed as $M = \langle \chi(x), \chi(x)\rangle_u$.

Proposition 1. *The block moment matrix M satisfies*

$$M\Lambda^\dagger = \Lambda M, \qquad \textit{where} \qquad \Lambda = \begin{bmatrix} \mathbf{0} & \mathrm{I} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \mathrm{I} & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5}$$

Proof: The proof follows from the identity $\Lambda\chi(x) = x\,\chi(x)$. ■

Definition 2. *A sesquilinear form $\langle\cdot,\cdot\rangle_u$ is said to be quasi-definite if all block leading sub-matrices of the corresponding block moment matrix are nonsingular.*

Proposition 2 (*cf.* [1])**.** *If the sesquilinear form* $\langle\cdot,\cdot\rangle_u$ *is quasi-definite, then its block moment matrix* M *has an unique Gauss-Borel factorization,*

$$M = S_1^{-1} H (S_2)^{-\dagger}, \tag{6}$$

where S_1, S_2 *are lower unitriangular block matrices and* H *is a block diagonal matrix. Moreover, if* $M = M^\dagger$*, then* $S_1 = S_2$*.*

Definition 3. *Let* $\langle\cdot,\cdot\rangle_u$ *be a quasi-definite sesquilinear form, such that the associated block moment matrix has a Gauss-Borel factorization as in* (6)*. The first and second families of matrix biorthogonal polynomials with respect to* $\langle\cdot,\cdot\rangle_u$ *are defined by*

$$P^{[1]}(x) = \begin{bmatrix} P_0^{[1]}(x) \\ P_1^{[1]}(x) \\ \vdots \end{bmatrix} = S_1 \chi(x), \qquad P^{[2]}(x) = \begin{bmatrix} P_0^{[2]}(x) \\ P_1^{[2]}(x) \\ \vdots \end{bmatrix} = S_2 \chi(x).$$

Proposition 3 (Biorthogonality)**.** *The first and second families of monic matrix polynomials* $(P_n^{[1]})_{n\in\mathbb{N}}$ *and* $(P_n^{[2]})_{n\in\mathbb{N}}$ *are biorthogonal,* i.e.,

$$\left\langle P_n^{[1]}(x), P_m^{[2]}(x) \right\rangle_u = \delta_{n,m} H_n, \qquad n, m \in \mathbb{N},$$

where H_n *is the* (n,n)*-block element of the block semi-infinite matrix* H *obtained in the Gauss-Borel factorization* (6)*. These biorthogonal relations yields,*

$$\begin{aligned} &\left\langle P_n^{[1]}(x), x^m \mathrm{I} \right\rangle_u = \mathbf{0}, && \left\langle x^m \mathrm{I}, P_n^{[2]}(x) \right\rangle_u = \mathbf{0}, && m = 0, \dots, n-1, \\ &\left\langle P_n^{[1]}(x), x^n \mathrm{I} \right\rangle_u = H_n, && \left\langle x^n \mathrm{I}, P_n^{[2]}(x) \right\rangle_u = H_n, && n \in \mathbb{N}. \end{aligned}$$

Definition 4. *The matrices*

$$J_1 = S_1 \Lambda S_1^{-1}, \qquad J_2 = S_2 \Lambda S_2^{-1}, \tag{7}$$

are said to be the Jacobi matrices associated with the moment matrix M*.*

Proposition 4. *The two block tridiagonal Jacobi matrices in* (7) *are related by*

$$H^{-1} J_1 = J_2^\dagger H^{-1},$$

Moreover, we have that

$$J_1 P^{[1]}(x) = x P^{[1]}(x), \qquad J_2 P^{[2]}(x) = x P^{[2]}(x). \tag{8}$$

Proof: The relation between the above two Jacobi matrices follows from the Gauss-Borel factorization and by (5). The relation (8) follows from the definitions of the Jacobi matrices (7). ■

Observe that as a consequence of the above proposition we get that J_1 and J_2 have a three diagonal block shape with the block I on the superdiagonal,

$$J_1 = \begin{bmatrix} b_0^{[1]} & \mathrm{I} & & \\ a_1^{[1]} & b_1^{[1]} & \mathrm{I} & \\ & \ddots & \ddots & \ddots \end{bmatrix}, \qquad J_2 = \begin{bmatrix} b_0^{[2]} & \mathrm{I} & & \\ a_1^{[2]} & b_1^{[2]} & \mathrm{I} & \\ & \ddots & \ddots & \ddots \end{bmatrix}. \tag{9}$$

The equations in (8) means that $(P_n^{[1]})_{n\in\mathbb{N}}$ and $(P_n^{[2]})_{n\in\mathbb{N}}$, respectively satisfies a three term recurrence relation, for $n \in \mathbb{N}$,

$$\begin{aligned} xP_n^{[1]}(x) &= P_{n+1}^{[1]}(x) + b_n^{[1]}P_n^{[1]}(x) + a_n^{[1]}P_{n-1}^{[1]}(x), \quad P_0^{[1]}(x) = \mathrm{I}, \quad P_{-1}^{[1]}(x) = \mathbf{0}, \\ xP_n^{[2]}(x) &= P_{n+1}^{[2]}(x) + b_n^{[2]}P_n^{[2]}(x) + a_n^{[2]}P_{n-1}^{[2]}(x), \quad P_0^{[2]}(x) = \mathrm{I}, \quad P_{-1}^{[2]}(x) = \mathbf{0}, \end{aligned}$$

where for $i = 1, 2$, $a_n^{[i]}$, and $b_n^{[i]}$ are $N \times N$ matrices. Moreover, $a_n^{[1]} = H_n H_{n-1}^{-1}$.

Proposition 5. *Given a matrix of linear functionals u and its sequence of moments $(u_n)_{n\in\mathbb{N}}$. Then $u_n = (J_1^n)_{0,0}$, $n \in \mathbb{N}$.*

Proof: We know from (8) that $J_1^n P^{[1]}(x) = x^n P^{[1]}(x)$, $n \in \mathbb{N}$, and the first block line gives us

$$x^n P_0^{[1]}(x) = \sum_{k=0}^{n} (J_1^n)_{0,k} P_k^{[1]}(x).$$

Now, applying the sesquilinear form and making use of the biorthogonality condition we get the desired result. ■

Definition 5 (*cf.* [12])**.** *Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times N}$, $C \in \mathbb{C}^{N\times m}$, and $D \in \mathbb{C}^{N\times N}$, with A a nonsingular matrix. The last quasi-determinant of the block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, is given by*

$$\Theta_* \begin{bmatrix} A & B \\ C & D \end{bmatrix} = D - CA^{-1}B.$$

Remark 1. *The sequences of matrix polynomials* $(P_n^{[1]})_{n\in\mathbb{N}}$ *and* $(H_n)_{n\in\mathbb{N}}$ *can be written in term of the moments* $(u_n)_{n\in\mathbb{N}}$ *as follows (*cf. [1]*)*

$$P_n^{[1]}(x) = x^n I_p - \begin{bmatrix} u_n & \cdots & u_{2n-1} \end{bmatrix} M_n^{-1} \begin{bmatrix} I_p \\ xI_p \\ \vdots \\ x^{n-1} I_p \end{bmatrix}, \qquad \textit{and}$$

$$H_n = \Theta_* \left[\begin{array}{c|c} \multirow{3}{*}{M_n} & u_n \\ & \vdots \\ & u_{2n-1} \\ \hline u_n \cdots u_{2n-1} & u_{2n} \end{array}\right],$$

where M_n *is given in* (4)*. From these representations and using the three term recurrence relation, we have*

$$a_n^{[1]} = H_n H_{n-1}^{-1}, \quad b_n^{[1]} = D_n H_n, \quad \textit{where} \quad D_n = \Theta_* \left[\begin{array}{c|c} \multirow{3}{*}{M_n} & u_{n+1} \\ & \vdots \\ & u_{2n} \\ \hline u_n \cdots u_{2n-1} & u_{2n+1} \end{array}\right].$$

There exist similar formulas for $(P_n^{[2]})_{n\in\mathbb{N}}$*,* $a_n^{[2]}$ *and* $b_n^{[2]}$*.*

Definition 6. *Let* u *be a matrix of linear functionals with sequence of moments* $(u_n)_{n\in\mathbb{N}}$*. Let* $r = \sup\{|x| : x \in \operatorname{supp} u\} < \infty$ *and consider the disks about infinity,* $\mathbb{D} = \{z \in \mathbb{C} : |z| > r\}$*. For* $z \in \mathbb{D}$ *we define the Stieltjes matrix function as*

$$F(z) = \left\langle \frac{1}{z-x}\mathrm{I}, \mathrm{I} \right\rangle_u .$$

Observe that in this definition we have that the geometric series $\displaystyle\sum_{k=0}^{\infty} \left(\frac{x}{z}\right)^k$ is uniform convergent in any compact subset of $\mathbb{D}$. With this in mind, $F(z)$ can also be written as

$$F(z) = \sum_{k=0}^{\infty} \frac{u_k}{z^{k+1}}.$$

Proposition 6 (Matrix Favard's theorem). *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be two sequences of matrices with* a_n *nonsingular for every* $n \in \mathbb{N}$. *If we define the matrix polynomials* $(P_n^{[1]})_{n\in\mathbb{N}}$ *by:* $P_0(x) = \mathrm{I}$, $P_{-1}(x) = \mathbf{0}$, *and*

$$P_{n+1}^{[1]}(x) = (x\mathrm{I} - b_n)P_n^{[1]}(x) - a_n P_{n-1}^{[1]}(x), \qquad n \geq 0, \tag{10}$$

then we can find a matrix of linear functionals u *such that its sesquilinear form satisfies*

$$\langle P_n^{[1]}(x), x^m I\rangle_u = a_n \cdots a_1\, \delta_{n,m},$$

where $\delta_{n,m}$ *is the Kronecker delta function.*

Proof: To prove the above result we inductively define the moments $(u_n)_{n\in\mathbb{N}}$ of the matrix of the linear functional u by

$$u_0 = \langle \mathrm{I}, \mathrm{I}\rangle_u = \mathrm{I}, \qquad \left\langle P_n^{[1]}(x), \mathrm{I}\right\rangle_u = \mathbf{0}, \qquad n = 1, 2 \ldots.$$

So, we can define u_1 using the fact that

$$\mathbf{0} = \left\langle P_1^{[1]}(x), \mathrm{I}\right\rangle_u = \langle (x\mathrm{I} - b_0), \mathrm{I}\rangle = u_1 - b_0 u_0.$$

In the same way u_2 can be defined from

$$\mathbf{0} = \left\langle P_2^{[1]}(x), \mathrm{I}\right\rangle_u = \left\langle x^2\mathrm{I} - xb_1 - xb_0 - b_1 b_0, \mathrm{I}\right\rangle_u = u_2 - (b_0 + b_1)u_1 - b_1 b_0 u_0.$$

Following this process we can find all the moments of u. From the above definition and the recurrence relation (10), we arrive to the orthogonality conditions

$$\left\langle P_n^{[1]}(x), x^m\mathrm{I}\right\rangle_u = \mathbf{0}, \ \ m < n, \ \text{ as well as } \ H_n = \left\langle P_n^{[1]}(x), x^n\mathrm{I}\right\rangle_u = a_n \cdots a_1.$$

As a last comment, it is important to point out that from the Proposition 4, we can construct the second family of matrix biorthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_u$, *i.e.* $(P_n^{[2]})_{n\in\mathbb{N}}$. ■

3. Symmetrized sesquilinear forms

Let $(P_n^{[1]})_{n\in\mathbb{N}}$ and $(P_n^{[2]})_{n\in\mathbb{N}}$ be the sequences of matrix biorthogonal polynomials with respect to a quasi-definite sesquilinear form $\langle \cdot, \cdot \rangle_u$. We are interested in finding conditions such that the Jacobi matrices J_1 and J_2, have

LU block factorization, *i.e*,

$$J_i = L_iU_i = \begin{bmatrix} \mathrm{I} & & & \\ \gamma_2^{[i]} & \mathrm{I} & & \\ & \gamma_4^{[i]} & \mathrm{I} & \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \gamma_1^{[i]} & \mathrm{I} & & \\ & \gamma_3^{[i]} & \mathrm{I} & \\ & & \gamma_5^{[i]} & \mathrm{I} \\ & & & \ddots & \ddots \end{bmatrix}, \qquad i = 1, 2. \tag{11}$$

where $(\gamma_{2n+1}^{[i]})_{n\in\mathbb{N}}$ are nonsingular matrices.

The above factorization will be our principal tool to construct a new solution of a Matrix Toda system (as in (18), Section 4) from another given one.

Lemma 1. *Let J_i, $i = 1, 2$, be the Jacobi matrix associated with $(P_n^{[i]})_{n\in\mathbb{N}}$. Then, the $P_n^{[i]}(0)$ is, for each $n \in \mathbb{N}$, a non-singular matrix, if and only if there exist block matrices L_i and U_i as in* (11) *such that $J_i = L_iU_i$. Moreover,*

$$\gamma_{2n+1}^{[i]} = -P_{n+1}^{[i]}(0)\big(P_n^{[i]}(0)\big)^{-1}, \qquad n \in \mathbb{N}. \tag{12}$$

Proof: If the factorization exists, then necessarily

$$b_n^{[i]} = \gamma_{2n+1}^{[i]} + \gamma_{2n}^{[i]}, \qquad n \geq 0, \qquad a_n^{[i]} = \gamma_{2n}^{[i]}\gamma_{2n-1}^{[i]}, \qquad n \geq 1,$$

with $\gamma_0 = \mathbf{0}$. Now, we are going to use the induction process to prove (12). Observe that for $n = 0$, $b_0^{[i]} = -P_1^{[i]}(0)(P_0^{[i]}(0))^{-1} = \gamma_1^{[i]}$. Suppose now that for $k \leq n$, $\gamma_{2k-1}^{[i]} = -P_k^{[i]}(0)(P_{k-1}^{[i]}(0))^{-1}$. Using the three term recurrence relation (9),

$$-P_{n+1}^{[i]}(0) = b_n^{[i]}P_n^{[i]}(0) + a_n^{[i]}P_{n-1}^{[i]}(0),$$

then

$$\begin{aligned} -P_{n+1}^{[i]}(0)(P_n^{[i]}(0))^{-1} &= b_n^{[i]} + a_n^{[i]}P_{n-1}^{[i]}(0)(P_n^{[i]}(0))^{-1} \\ &= b_n^{[i]} + a_n^{[i]}\left[P_n^{[i]}(0)(P_{n-1}^{[i]}(0))^{-1}\right]^{-1} \\ &= b_n^{[i]} - \gamma_{2n}^{[i]}\gamma_{2n-1}^{[i]}(\gamma_{2n-1}^{[i]})^{-1} \\ &= b_n^{[i]} - \gamma_{2n}^{[i]} = \gamma_{2n+1}^{[i]}. \end{aligned}$$

Thus, if $P_n^{[i]}(0)$, is a nonsingular matrix, then the block elements of the matrix U_i are well defined. Defining the block elements of the matrix L_i as $\gamma_{2n}^{[i]} = b_n^{[i]} - \gamma_{2n+1}^{[i]}$ we obtain the result.

Conversely, assume that there exist block matrices L_i and U_i as in (11) such that $J_i = L_iU_i$. Using induction, we are going to prove that $P_n^{[i]}(0)$

is a non-singular matrix for every $n \in \mathbb{N}$. Notice that $P_0(0) = \mathrm{I}$. Suppose now that $(P_k^{[i]}(0))_{k=0}^{n-1}$ are non-singular matrices and that the matrix $P_n^{[i]}(0)$ is singular. Since J_i has LU block factorization as in (11), then is not difficult to check that

$$(J_i)_n = (L_i)_n (U_i)_n,$$

where $(A)_n$ is the block n-th truncation of the matrix A. From here

$$x \begin{bmatrix} P_0^{[i]}(x) \\ \vdots \\ P_{n-2}^{[i]}(x) \\ P_{n-1}(x) \end{bmatrix} = (J_i)_n \begin{bmatrix} P_0^{[i]}(x) \\ \vdots \\ P_{n-2}^{[i]}(x) \\ P_{n-1}^{[i]}(x) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_n^{[i]}(x) \end{bmatrix} = (L_i)_n (U_i)_n \begin{bmatrix} P_0^{[i]}(x) \\ \vdots \\ P_{n-2}^{[i]}(x) \\ P_{n-1}^{[i]}(x) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_n^{[i]}(x) \end{bmatrix}.$$

Evaluating the above in zero, we get

$$(L_i)_n (U_i)_n \begin{bmatrix} P_0^{[i]}(0) \\ \vdots \\ P_{n-2}^{[i]}(0) \\ P_{n-1}^{[i]}(0) \end{bmatrix} = - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_n^{[i]}(0) \end{bmatrix}.$$

Taking into account that $(L_i)_n$ is a non-singular matrix and its inverse is also a block triangular matrices with I's in the main diagonal, then $P_n^{[i]}(0) = -\gamma_{2n-1}^{[i]} P_{n-1}^{[i]}(0)$. Now, as from the hypothesis $\gamma_{2n-1}^{[i]}$ and $P_{n-1}^{[i]}(0)$ are non singular matrices, we get a contradiction. Thus $P_n^{[i]}(0)$ is a non-singular matrix. ■

Now, let consider $\widetilde{J}_i = U_i L_i$, $i = 1, 2$. It is not difficult to check that $\widetilde{J}_i$ is also a block tridiagonal matrix

$$\widetilde{J}_i = \begin{bmatrix} \widetilde{b}_0^{[i]} & \mathrm{I} & & & \\ \widetilde{a}_1^{[i]} & \widetilde{b}_1^{[i]} & \mathrm{I} & & \\ & \widetilde{a}_2^{[i]} & \widetilde{b}_2^{[i]} & \mathrm{I} & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{13}$$

thus the sequences $(\widetilde{P}_n^{[i]})_{n \in \mathbb{N}}$, $i = 1, 2$, of matrix polynomials defined by the recurrence formula: $\widetilde{P}_{-1}^{[i]}(x) = \mathbf{0}$, $\widetilde{P}_0^{[i]}(x) = \mathrm{I}$, and

$$x \widetilde{P}_n^{[i]}(x) = \widetilde{P}_{n+1}^{[i]}(x) + \widetilde{b}_n^{[i]} \widetilde{P}_n^{[i]}(x) + \widetilde{a}_n^{[i]} \widetilde{P}_{n-1}^{[i]}(x), \qquad n \geq 0,$$

are also biorthogonal with respect to a matrix sesquilinear form. But what is the corresponding matrix of the functionals? The answer is given in the next result.

Proposition 7. *Let u be a matrix of linear functionals and J_i, $i = 1, 2$ its respectively block Jacobi matrices with* LU *block factorization as in* (11)*, (or equivalently, for every $n \in \mathbb{N}$, $P_n^{[i]}(0)$ is a nonsingular matrix). Then the matrix polynomials $(\widetilde{P}_n^{[1]})_{n\in\mathbb{N}}$ and $(\widetilde{P}_n^{[2]})_{n\in\mathbb{N}}$ associated with $\widetilde{J}_1 = U_1L_1$ and $\widetilde{J}_2 = U_2L_2$ are the sequences of matrix biorthogonal polynomials with respect to the sesquilinear form generated by the matrix of linear functionals xu.*

Proof: Define $Q^{[i]} = U_iP^{[i]}$ with $i = 1, 2$. The hypothesis implies that $Q_n^{[i]}(x) = P_{n+1}^{[i]}(x) + \gamma_{2n+1}^{[i]}P_n^{[i]}(x)$ (observe that $Q_n^{[i]}(x)$ is a polynomial of degree $n + 1$). From the LU block factorization we also have that $xP^{[i]} = (L_iU_i)P^{[i]} = L_iQ^{[i]}$, thus

$$xP_n^{[i]}(x) = Q_n^{[i]}(x) + \gamma_{2n}^{[i]}Q_{n-1}^{[i]}(x), \qquad n \in \mathbb{N},$$

but this implies that $Q_n^{[i]}(0) = -\gamma_{2n}^{[i]}Q_{n-1}^{[i]}(0)$ and taking into account that by definition $\gamma_0^{[i]} = \mathbf{0}$, then $Q_n^{[i]}(0) = \mathbf{0}$ for every $n = 0, 1, \ldots$. From here, $Q_n^{[i]}(x)$ can be written as $x\widetilde{P}_n^{[i]}(x)$ where $\widetilde{P}_n^{[i]}(x)$ is a matrix polynomial of degree n. As a consequence

$$x\widetilde{P}_n^{[i]}(x) = P_{n+1}^{[i]}(x) + \gamma_{2n+1}^{[i]}P_n^{[i]}(x).$$

with $\gamma_{2n+1}^{[i]} = -P_{n+1}^{[i]}(0)\big(P_n^{[i]}(0)\big)^{-1}$. In [1] was shown that $(\widetilde{P}_n^{[1]})_{n\in\mathbb{N}}$ and $(\widetilde{P}_n^{[2]})_{n\in\mathbb{N}}$ are precisely the families of matrix biorthogonal polynomials associated with the sesquilinear form $\langle\cdot,\cdot\rangle_{xu}$. ■

Definition 7 (*cf.* [1])**.** *The matrix of linear functionals xu is said to be the Christoffel transformation of u.*

Now, we define the sequences of matrix polynomials $(S_n^{[1]})_{n\in\mathbb{N}}$, $(S_n^{[2]})_{n\in\mathbb{N}}$ by

$$S_{2n}^{[i]}(x) = P_n^{[i]}(x^2), \qquad S_{2n+1}^{[i]}(x) = x\widetilde{P}_n^{[i]}(x^2), \qquad i = 1, 2,$$

it is not difficult to check that $(S_n^{[i]})_{n\in\mathbb{N}}$ for $i = 1, 2$, satisfies the following three term recurrence relation

$$xS_n^{[i]}(x) = S_{n+1}^{[i]}(x) + \gamma_n^{[i]}S_{n-1}^{[i]}(x), \qquad n \in \mathbb{N},$$

or equivalently, its corresponding block Jacobi matrix has the shape

$$\Gamma_1^{[i]} = \begin{bmatrix} \mathbf{0} & \mathrm{I} & & & \\ \gamma_1^{[i]} & \mathbf{0} & \mathrm{I} & & \\ \mathbf{0} & \gamma_2^{[i]} & \mathbf{0} & \mathrm{I} & \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \qquad i = 1, 2.$$

Let $\mathfrak{s}$ be the matrix of linear functionals for which the sequences $(S_n^{[1]})_{n\in\mathbb{N}}$ and $(S_n^{[2]})_{n\in\mathbb{N}}$ are biorthogonal and $(w_n)_{n\in\mathbb{N}}$ its corresponding sequence of matrix moments, then

$$w_{2n} = u_n \qquad \text{and} \qquad w_{2n+1} = \mathbf{0}, \qquad n \in \mathbb{N}.$$

Observe also that from the LU block factorization, we have the following relations for $i = 1, 2$,

$$b_n^{[i]} = \gamma_{2n+1}^{[i]} + \gamma_{2n}^{[i]}, \quad n \geq 0, \qquad a_n^{[i]} = \gamma_{2n}^{[i]}\gamma_{2n-1}^{[i]}, \qquad n \geq 1, \tag{14}$$

$$\widetilde{b}_n^{[i]} = \gamma_{2n+2}^{[i]} + \gamma_{2n+1}^{[i]}, \quad n \geq 0, \qquad \widetilde{a}_n^{[i]} = \gamma_{2n+1}^{[i]}\gamma_{2n}^{[i]} \qquad n \geq 1, \tag{15}$$

with the convention $\gamma_0^{[i]} = \mathbf{0}$. From here and Proposition 7,

$$\begin{aligned} \gamma_{2n+1}^{[i]} &= -P_{n+1}^{[i]}(0)\big(P_n^{[i]}(0)\big)^{-1}, \qquad & \gamma_{2n}^{[i]} &= -a_n^{[i]}P_{n-1}^{[i]}(0)\big(P_n^{[i]}(0)\big)^{-1}, \\ \gamma_{2n+1}^{[2]} &= (H_n^{-1}\gamma_{2n+1}^{[1]}H_n)^{\dagger}, & \gamma_{2n}^{[2]} &= (H_n^{-1}\gamma_{2n}^{[1]}H_n)^{\dagger}. \end{aligned} \tag{16}$$

The above, give us the following representation for $\widetilde{a}_n^{[i]}$ and $\widetilde{b}_n^{[i]}$

$$\begin{aligned} \widetilde{b}_n^{[i]} &= P_{n+2}^{[i]}(0)\big(P_{n+1}^{[i]}(0)\big)^{-1} + b_{n+1}^{[i]} - P_{n+1}^{[i]}(0)\big(P_n^{[i]}(0)\big)^{-1}, \\ \widetilde{a}_n^{[i]} &= P_{n+1}^{[i]}(0)\big(P_n^{[i]}(0)\big)^{-1}a_n^{[i]}P_{n-1}^{[i]}(0)\big(P_n^{[i]}(0)\big)^{-1}. \end{aligned} \tag{17}$$

4. Matrix Toda Lattice

In the scalar case, if we have a moment functional $\mu(t) : \mathbb{C}[x] \to \mathbb{C}$ depending on a parameter $t \in \mathbb{R}$, then it is clear that its moments also depend on t, *i.e* $\langle \mu(t), x^n \rangle = \mu_n(t)$. We can define the derivative of a moment functional with respect to t as follows,

$$\left\langle \frac{d}{dt}\mu(t), p(x) \right\rangle = \lim_{h\to 0} \frac{\langle \mu(t+h), p(x)\rangle - \langle \mu(t), p(x)\rangle}{h}, \qquad p(x) \in \mathbb{C}[x].$$

An important property that can be proven from the above definition is the following. Suppose that we have a polynomial depending on t, *i.e.* $p(x,t) =$

$\sum_{k=0}^{m} c_k(t)x^k$ then

$$\frac{d}{dt}\left\langle \mu(t), p(x,t)\right\rangle = \left\langle \frac{d}{dt}\mu(t), p(x,t)\right\rangle + \left\langle \mu(t), \frac{d}{dt}p(x,t)\right\rangle .$$

Extrapolating the above argument, we can define the derivative of a sesquilinear form associated with a matrix of linear functionals that depends on a parameter t as follows

$$\begin{aligned}\frac{d}{dt}\left\langle P(x), Q(x)\right\rangle_{u(t)} \\ &= \langle \frac{d}{dt}P(x), Q(x)\rangle_{u(t)} + \langle P(x), Q(x)\rangle_{\frac{d}{dt}u(t)} + \langle P(x), \frac{d}{dt}Q(x)\rangle_{u(t)}.\end{aligned}$$

Consider now the following semi-infinite system of matrix differential equations

$$\begin{cases}\dot{b}_n(t) = a_n(t) - a_{n+1}(t), \\ \dot{a}_{n+1}(t) = a_{n+1}(t)b_n(t) - b_{n+1}(t)a_{n+1}(t)\end{cases}, \quad a_0(t) = \mathbf{0}, \quad n \in \mathbb{N}. \tag{18}$$

To give an interpretation of the above system, we start by considering the following sequence of matrix orthogonal polynomials $(P_n^{[1]})_{n\in\mathbb{N}}$ satisfying the three term recurrence relation: $P_{-1}^{[1]}(x,t) = \mathbf{0}$, $P_0^{[1]}(x,t) = \mathrm{I}$, and

$$xP_n^{[1]}(x,t) = P_{n+1}^{[1]}(x,t) + b_n(t)P_n^{[1]}(x,t) + a_n(t)P_{n-1}^{[1]}(x,t) \quad n \geq 0. \tag{19}$$

From Proposition 6, we know that there exists a sesquilinear form $\langle\cdot,\cdot\rangle_u$ (depending on t) such that the sequence $(P_n^{[1]})_{n\in\mathbb{N}}$ is the first family of matrix biorthogonal polynomials, as well as that the sequences $(a_n(t))_{n\in\mathbb{N}}$ and $(b_n(t))_{n\in\mathbb{N}}$ can be written in term of the quasi-determinants (*cf.* Remark 1). In this Section we are interested in showing the relationship between the sequence of matrix polynomials $(P_n^{[1]})_{n\in\mathbb{N}}$ (as well as, its associated matrix of the linear functionals) and the Toda matrix system (18).

If we assume that J_1 is as in (9), we have that the system (18) can be described in terms of a Lax pair (J_1, J_{1-}), *i.e.*

$$\dot{J}_1(t) = J_{1-}J_1 - J_1J_{1-}, \tag{20}$$

where J_{1-} is the following block matrix

$$J_{1-}=\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \\ a_1(t) & \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & a_2(t) & \mathbf{0} & \\ & \ddots & \ddots & \ddots\end{bmatrix}. \tag{21}$$

Lemma 2. *Let J_1 be a solution of a Toda lattice system as in* (18)*. Then, the following relation holds,*

$$\dot{H}_n=-b_nH_n+H_nb_0, \qquad n\in\mathbb{N}.$$

Proof: Taking into account that $a_n(t)=H_nH_{n-1}^{-1}$ then from (18) we get

$$\dot{H}_nH_{n-1}^{-1}-H_nH_{n-1}^{-1}\dot{H}_{n-1}H_{n-1}^{-1}=H_nH_{n-1}^{-1}b_{n-1}-b_nH_nH_{n-1}^{-1}.$$

After some manipulations, we arrive to

$$H_n^{-1}\dot{H}_n-H_{n-1}^{-1}\dot{H}_{n-1}=H_{n-1}^{-1}b_{n-1}H_{n-1}-H_n^{-1}b_nH_n.$$

Thus, for every $n\in\mathbb{N}$ we see that

$$H_n^{-1}(\dot{H}_n+b_nH_n)=H_0^{-1}(\dot{H}_0+b_0H_0),$$

and remembering that $H_0=\mathrm{I}$, we complete the proof. ∎

Proposition 8. *If J_1 is a solution of a Toda lattice as in* (18)*, then J_2 is a solution of the semi-infinite matrix differential system*

$$\begin{cases}\dot{b}_n^{[2]}=(a_n^{[2]}-a_{n+1}^{[2]})+(b_0^{[2]}b_n^{[2]}-b_n^{[2]}b_0^{[2]})\\ \dot{a}_{n+1}^{[2]}=a_{n+1}^{[2]}b_n^{[2]}-b_{n+1}^{[2]}a_{n+1}^{[2]}+(b_0^{[2]}a_{n+1}^{[2]}-a_{n+1}^{[2]}b_0^{[2]})\end{cases}, \quad a_0^{[2]}=\mathbf{0}, \quad n\in\mathbb{N}.$$

Proof: From Proposition 4 we have that

$$\begin{aligned}\dot{J}_2^{\dagger}&=-\dot{H}^{-1}J_1H+H^{-1}\dot{J}_1H+H^{-1}J_1\dot{H}\\ &=-H^{-1}\dot{H}(H^{-1}J_1H)+H^{-1}(J_{1-}J_1-J_1J_{1-})H+H^{-1}J_1\dot{H}\\ &=(H^{-1}J_{1-}H-H^{-1}\dot{H})J_2^{\dagger}-J_2^{\dagger}(H^{-1}J_{1-}H-H^{-1}\dot{H}).\end{aligned} \tag{22}$$

On the other hand, from Lemma 2 we get

$$H^{-1}J_{1-}H-H^{-1}\dot{H}=\begin{bmatrix}\mathbf{0} & & & \\ \mathrm{I} & (b_1^{[2]}-b_0^{[2]})^{\dagger} & & \\ \mathbf{0} & \mathrm{I} & (b_2^{[2]}-b_0^{[2]})^{\dagger} & \\ & \ddots & \ddots & \ddots\end{bmatrix}.$$

Thus, replacing the above in (22) we obtain

$$(\dot{J}_2^{\dagger})_{n,m} = \begin{cases} (a_n^{[2]\dagger} - a_{n+1}^{[2]\dagger}) + (b_n^{[2]\dagger} b_0^{[2]\dagger} - b_0^{[2]\dagger} b_n^{[2]\dagger}), & \text{if} \quad m = n, \\ b_n^{[2]\dagger} a_{n+1}^{[2]\dagger} - a_{n+1}^{[2]\dagger} b_{n+1}^{[2]\dagger} + (a_{n+1}^{[2]\dagger} b_0^{[2\dagger]} - b_0^{[2]\dagger} a_{n+1}^{[2]\dagger}), & \text{if} \quad m = n+1, \end{cases}$$

and the result follows. ∎

From Proposition 8 we note that J_2 is not a solution of the matrix Toda lattice. However this holds true, if we apply a transformation as we will see in the next result.

Proposition 9. *Let J_1 be a solution of a Toda lattice system as in* (18)*. If for every H_n, we can find two matrix functions V_n and W_n satisfying*

$$\dot{V}_n = -b_n V_n, \qquad \dot{W}_n = -b_0 W_n,$$

and such that $H_n = V_n W_n^{-1}$, then

$$\begin{cases} \frac{d}{dt}(W_0^{\dagger} b_n^{[2]} W_0^{-\dagger}) = W_0^{\dagger} a_n^{[2]} W_0^{-\dagger} - W_0^{\dagger} a_{n+1}^{[2]} W_0^{-\dagger} & n = 0, 1 \ldots, \\ \frac{d}{dt}(W_0^{\dagger} a_{n+1}^{[2]} W_0^{-\dagger}) = W_0^{\dagger} a_{n+1}^{[2]} b_n^{[2]} W_0^{-\dagger} - W_0^{\dagger} b_{n+1}^{[2]} a_{n+1}^{[2]} W_0^{-\dagger}, & n = 0, 1, \ldots. \end{cases}$$

Proof: First of all observe that if we take derivative on $H_n = V_n W_n^{-1}$ then, we obtain the result given in Lemma 2. From the fact that $\frac{d}{dt}(H_n W_0) = -b_n H_n W_0$ and taking into account that $b_n^{[2]\dagger} = H_n^{-1} b_n H_n$ and $a_n^{[2]\dagger} = H_{n-1}^{-1} H_n$ we get the desired result. ∎

Proposition 10. *Let J_1 be a solution of a Toda lattice as in* (18) *and $(P_n^{[1]})_{n\in\mathbb{N}}$ its first family of associated matrix biorthogonal polynomials. Then, for every $n \in \mathbb{N}$,*

$$\dot{P}_n^{[1]}(x,t) = a_n(t) P_{n-1}^{[1]}(x,t).$$

Proof: From the relation $xP^{[1]} = J_1 P^{[1]}$ (recall that we are taking the derivate on the variable t), we have

$$x\dot{P}^{[1]} = \dot{J}_1 P^{[1]} + J_1 \dot{P}^{[1]} = (J_{1-} J_1 - J_1 J_{1-}) P^{[1]} + J_1 \dot{P}^{[1]}.$$

From here

$$(xI - J_1)(\dot{P}^{[1]} - J_{1-} P^{[1]}) = \mathbf{0}.$$

So we get the result. ∎

Proposition 11. *If J_1 is a solution of a Toda lattice, then for all $n = 1, 2 \dots$ the following identity is satisfied*

$$(\dot{J_1^n}) = J_{1-}J_1^n - J_1^n J_{1-}.$$

Proof: The proof will be by induction. For $n = 1$, it follows from (20). For $n = 2$ we have

$$\begin{aligned}\dot{J_1^2} &= \dot{J}_1 J_1 + J_1 \dot{J}_1 = (J_{1-}J_1 - J_1J_{1-})J_1 + J_1(J_{1-}J_1 - J_1J_{1-})\\ &= J_{1-}J_1^2 - J_1^2 J_{1-}.\end{aligned}$$

Suppose that the property is satisfied by $n = m - 1$ to prove by $n = m$

$$\begin{aligned}\dot{J_1^m} &= \dot{J_1^{m-1}}\, J_1 + J_1^{m-1}\, \dot{J}_1\\ &= (J_{1-}J_1^{m-1} - J_1^{m-1}J_{1-})J_1 + J_1^{m-1}(J_{1-}J_1 - J_1J_{1-})\\ &= J_{1-}J_1^m - J_1^m J_{1-}.\end{aligned}$$

thus for every $n \in \mathbb{N}$ we get the result. ■

Corollary 1. *Let u be a matrix of linear functionals such that the corresponding block Jacobi matrix is a solution of* (18)*. Then the sequence of matrix moments $(u_n)_{n\in\mathbb{N}}$ satisfies the following matrix differential equation*

$$\dot{u}_n(t) = u_n(t)u_1(t) - u_{n+1}(t), \qquad n \in \mathbb{N}.$$

Proof: The result follows from the fact that $u_n(t) = (J_1^n)_{0,0}$. ■

Corollary 2. *Let u be a matrix of linear functionals such that its corresponding block Jacobi matrix is a solution of* (18)*. Then its associated Stieltjes function satisfies the following matrix differential equation*

$$\dot{F}(z) = F(z)(u_1(t) - z\mathrm{I}) + \mathrm{I}. \tag{23}$$

Proof: Using Corollary 1 we have that

$$\dot{F}(z) = \sum_{k=0}^{\infty} \frac{\dot{u}_k(t)}{z^{k+1}} = F(z)u_1(t) - z\sum_{k=0}^{\infty} \frac{u_{n+1}(t)}{z^{n+2}},$$

and the result follows from the definition of $F(z)$. ■

Proposition 12. *If J_1 is a solution of a Toda lattice as in* (18) *and u is the corresponding matrix of linear functionals, then the following equation holds*

$$\dot{u}(t) = u(t)u_1(t) - xu(t). \tag{24}$$

Proof: Using the original representation of the Stieltjes matrix function $F(z)$ and (23) we have

$$\begin{aligned}\dot{F}(z) &= \left\langle \frac{\mathrm{I}}{z-x}, u_1(t)^\dagger - \bar{z}\mathrm{I} \right\rangle_{u(t)} + \langle \mathrm{I}, \mathrm{I} \rangle_{u(t)}, \\ &= \left\langle \frac{\mathrm{I}}{z-x}, u_1(t)^\dagger \right\rangle_{u(t)} + \left\langle \frac{z}{z-x}, \mathrm{I} \right\rangle_{u(t)} + \langle \mathrm{I}, \mathrm{I} \rangle_{u(t)},\end{aligned}$$

and so

$$\dot{F}(z) = \left\langle \frac{\mathrm{I}}{z-x}, \mathrm{I} \right\rangle_{u(t)u_1(t)} - \left\langle \frac{\mathrm{I}}{z-x}, \mathrm{I} \right\rangle_{xu(t)}.$$

Thus, if we denote by $\widehat{u}$ the matrix of linear functionals $\widehat{u} = -\frac{d}{dt}u + uu_1(t) - ux$, then the corresponding Stieltjes function is the zero matrix; but this implies that every matrix moment of $\widehat{u}$ is equal to $\mathbf{0}$, and from here $\widehat{u}$ is equal to the zero matrix of linear functionals. ■

Corollary 3. *If in particular* $\langle P, Q \rangle_u = \int PW(x,t)Q^\dagger dx$ *and its corresponding Jacobi matrix* J_1 *satisfies* (18), *then the matrix weight function* $W(x,t)$ *has the structure*

$$W(x,t) = e^{-xt}W(x,0)K(t), \tag{25}$$

where $K(t)$ *satisfies the following matrix differential equation*

$$\dot{K}(t) = K(t)u_1(t).$$

Proof: Recall that if J_1 satisfies (18), then u satisfies (24), but this implies that

$$\dot{W}(x,t) = W(x,t)(u_1(t) - x\mathrm{I}).$$

On the other hand, if we take derivative in (25),

$$\dot{W}(x,t) = W(x,t)(K^{-1}(t)\dot{K}(t) - x\mathrm{I}),$$

and taking into account our hypothesis, we get the result. ■

Suppose now that we have a Jacobi matrix $J_1(t)$ with LU block factorization as in (11). If the sequence of matrix $(\gamma_n)_{n\in\mathbb{N}}$ is a solution of a Volterra lattice, *i.e.*

$$\dot{\gamma}_{n+1}(t) = \gamma_{n+1}(t)\gamma_n(t) - \gamma_{n+2}(t)\gamma_{n+1}(t), \qquad n \geq 0, \tag{26}$$

then using the representation (14) it is easy to prove that J_1 and $\widetilde{J}_1$ are solutions of the Toda lattice (18). Here we are interested in the reciprocal result.

Theorem 1. *Let* $(b_n)_{n\in\mathbb{N}}$ *and* $(a_n)_{n\in\mathbb{N}}$ *be a solution of the Toda lattice* (18) *satisfying the hypothesis of Lemma 1. If* γ_n *is defined by* (14)*, then* (15) *defines a new solution of* (18).

Proof: First of all, notice that

$$\frac{d}{dt}\Big(P_n^{[1]}(x,t)^{-1}\Big) = -P_n^{[1]}(x,t)^{-1}\Big(\frac{d}{dt}P_n^{[1]}(x,t)\Big)P_n^{[1]}(x,t)^{-1}.$$

Let $\widetilde{J}_1$ be as in (13). If we take derivative in (17), we get that $\widetilde{J}_1$ is also a solution of the Toda lattice. Now, defining the sequence of matrices $(\gamma_n)_{n\in\mathbb{N}}$ as in (16), and taking derivative we obtain

$$\begin{aligned}\dot{\gamma}_{2n+1} &= -a_{n+1}P_{n+1}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1} \\ &\qquad\qquad + P_{n+1}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1}a_n(t)P_{n-1}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1} \\ &= -\gamma_{2n+2}\gamma_{2n+1} + \gamma_{2n+1}\gamma_{2n}, \\ \dot{\gamma}_{2n} &= -(a_nb_{n-1} - b_na_n)P_{n-1}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1} \\ &\qquad - a_na_{n-1}P_{n-2}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1} - \big(a_nP_{n-1}^{[1]}(0,t)(P_n^{[1]}(0,t))^{-1}\big)^2;\end{aligned}$$

and, as $b_n = -P_{n+1}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1} - a_nP_{n-1}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1}$ we get that

$$\begin{aligned}\dot{\gamma}_{2n} &= a_n(t) - P_{n+1}^{[1]}(0,t)(P_n^{[1]}(0,t))^{-1}a_n(t)P_{n-1}^{[1]}(0,t)(P_n^{[1]}(0,t))^{-1} \\ &= \gamma_{2n}\gamma_{2n-1} - \gamma_{2n+1}\gamma_{2n}.\end{aligned}$$

Observe that from the above, if $(b_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ are solutions of the Toda lattice (18) and we know the sequence of matrix polynomials $(P_n^{[1]}(x,t))_{n\in\mathbb{N}}$, that satisfies the recurrence relation (19) and $P_n^{[1]}(0,t)$, $n\geq 0$, is a nonsingular matrix, then the matrix sequences $(\widetilde{a}_n(x))_{n\in\mathbb{N}}$ and $(\widetilde{b}_n(x))_{n\in\mathbb{N}}$ defined by

$$\begin{aligned}\widetilde{b}_n(t) &= P_{n+2}^{[1]}(0,t)\big(P_{n+1}^{[1]}(0,t)\big)^{-1} + b_{n+1}(t) - P_{n+1}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1}, \\ \widetilde{a}_n(t) &= P_{n+1}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1}a_n(t)P_{n-1}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1},\end{aligned}$$

are also solution of the matrix Toda lattice (18). Moreover, in this case

$$\gamma_{2n+1}(t) = -P_{n+1}^{[1]}(0,t)\big(P_n^{[1]}(0,t)\big)^{-1},\ \ \gamma_{2n}(t) = -a_n(t)P_{n-1}^{[i]}(0,t)\big(P_n^{[i]}(0,t)\big)^{-1},$$

is solution of a matrix Volterra lattice. ∎

Next, we are going to show an illustrative example.

Example 1. *Suppose that we have the following matrix weight*

$$W(x,t)=e^{(1-x)t}\begin{bmatrix}x & -1\\ 0 & x\end{bmatrix}e^{-x}x^{\alpha}, \qquad x\in]0,\infty[, \qquad \alpha>-1.$$

Observe that in this case $\int W(x,t)dx$ *is an invertible matrix different from the identity. However, this hypothesis is not necessary in the symmetrization process. If* $(P_n^{[1]})_{n\in\mathbb{N}}$ *is the sequence of left matrix orthogonal polynomials with respect to* $W(x,t)$*, then making a suitable change of the variable, we obtain that*

$$P_n^{[1]}(x,t)=\frac{1}{(1+t)^n}Q_n^{[1]}((t+1)x), \tag{27}$$

where

$$Q_n^{[1]}(x)=\begin{bmatrix}1 & 1\\ 0 & \frac{1}{(t+1)}\end{bmatrix}\begin{bmatrix}L_n^{\alpha+1}(x) & -\frac{n}{\alpha+1}L_{n-1}^{\alpha+2}(x)\\ 0 & L_n^{\alpha+1}(x)\end{bmatrix}\begin{bmatrix}1 & -(t+1)\\ 0 & (t+1)\end{bmatrix}, \tag{28}$$

and $(L_n^{\alpha})_{n\in\mathbb{N}}$ *is the sequence of scalar monic Laguerre polynomials of parameter* α *which are orthogonal with respect to the measure* $d\mu=e^{-x}x^{\alpha}dx$ *and has the following monomial representation (*cf. [9] *for other characterizations),*

$$L_n^{\alpha}(x)=\frac{(-1)^n}{n!}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}.$$

On the other hand, we know that $P_n^{[1]}(x,t)$ *satisfies a three term recurrence relation*

$$P_{n+1}^{[1]}(x,t)=(x\mathrm{I}-b_n(t))P_n^{[1]}(x,t)-a_n(t)P_{n-1}^{[1]}(x,t).$$

Using the representation given in (27) *and the properties of the Laguerre polynomials, we find that*

$$b_n(t)=\begin{bmatrix}\frac{2+\alpha+2n}{t+1} & -\frac{2}{1+\alpha}\\ 0 & \frac{2+\alpha+2n}{t+1}\end{bmatrix}, \qquad a_n(t)=\begin{bmatrix}\frac{n(\alpha+n+1)}{(t+1)^2} & 0\\ 0 & \frac{n(\alpha+n+1)}{(t+1)^2}\end{bmatrix},$$

and

$$H_n=\frac{e^t n!\Gamma(\alpha+n+2)}{(t+1)^{\alpha+2n+2}}\begin{bmatrix}1 & -\frac{(t+1)}{\alpha+1}\\ 0 & 1\end{bmatrix}.$$

Besides, observe that $(b_n)_{n\in\mathbb{N}}$ *and* $(a_n)_{n\in\mathbb{N}}$ *satisfy the Toda lattice* (18). *Moreover, from the representation* (27) *and* (28)

$$P_n^{[1]}(0) = \frac{1}{(t+1)^n}\begin{bmatrix}1 & 1\\ 0 & \frac{1}{(t+1)}\end{bmatrix}\begin{bmatrix}L_n^{\alpha+1}(0) & -\frac{n}{\alpha+1}L_{n-1}^{\alpha+2}(0)\\ 0 & L_n^{\alpha+1}(0)\end{bmatrix}\begin{bmatrix}1 & -(t+1)\\ 0 & (t+1)\end{bmatrix},$$

where $L_n^{\alpha}(0) = (-1)^n\dfrac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)}$. *Thus, if we use the symmetrization process we obtain that*

$$\gamma_{2n+1}(t) = \frac{\alpha+n+2}{t+1}\begin{bmatrix}1 & \frac{(t+1)}{(\alpha+1)(\alpha+2)}\\ 0 & 1\end{bmatrix}, \qquad \gamma_{2n}(t) = \frac{n}{t+1}\begin{bmatrix}1 & -\frac{(t+1)}{(\alpha+1)(\alpha+2)}\\ 0 & 1\end{bmatrix}.$$

It is not difficult to check that the sequence of matrices $(\gamma_n)_{n\in\mathbb{N}}$ *satisfies* (26) *and consequently we can construct a second solution of the matrix Toda lattice. Moreover, defining the following sequence of matrix polynomials,* $(S_n^{[1]})_{n\in\mathbb{N}}$, *by*

$$S_{2n}^{[1]}(x,t) = \begin{bmatrix}\frac{1}{(t+1)^n} & \frac{1}{(t+1)^n}\\ 0 & \frac{1}{(t+1)^{n+1}}\end{bmatrix}\begin{bmatrix}L_n^{\alpha+1}((t+1)x^2) & -\frac{n}{\alpha+1}L_{n-1}^{\alpha+2}((t+1)x^2)\\ 0 & L_n^{\alpha+1}((t+1)x^2)\end{bmatrix}\begin{bmatrix}1 & -(t+1)\\ 0 & (t+1)\end{bmatrix},$$

$$S_{2n+1}^{[1]}(x,t) = \begin{bmatrix}\frac{x}{(t+1)^n} & \frac{x}{(t+1)^n}\\ 0 & \frac{x}{(t+1)^{n+1}}\end{bmatrix}\begin{bmatrix}L_n^{\alpha+2}((t+1)x^2) & -\frac{n}{\alpha+3}L_{n-1}^{\alpha+2}((t+1)x^2)\\ 0 & L_n^{\alpha+2}((t+1)x^2)\end{bmatrix}\begin{bmatrix}1 & -(t+1)\\ 0 & (t+1)\end{bmatrix},$$

then, we have that

$$xS_n^{[1]}(x,t) = S_{n+1}^{[1]}(x,t) + \gamma_n(t)S_{n-1}^{[1]}(x,t).$$

We can also see that $(S_n^{[1]})_{n\in\mathbb{N}}$ *is the first family of matrix biorthogonal polynomials with respect to the matrix weight*

$$W(x,t) = e^{(1-x^2)t\mathrm{I}}\begin{bmatrix}x^2 & -1\\ 0 & x^2\end{bmatrix}e^{-x^2}|x|^{2\alpha+1}, \qquad x\in\mathbb{R}.$$

Due to the close relation between the matrix lattices of Toda and Volterra we are going to study the properties of the last one.

5. Matrix Volterra System

Suppose that we have the following matrix Volterra system

$$\dot{\gamma}_{n+1}(t) = \gamma_{n+1}(t)\gamma_n(t) - \gamma_{n+2}(t)\gamma_{n+1}(t), \quad n\ge 0. \tag{29}$$

This system can be described in terms of a Lax pair $(\Gamma_1, \Gamma_{1-}^2(t))$, *i.e.*

$$\dot{\Gamma}_1(t) = \Gamma_{1-}^2\Gamma_1 - \Gamma_1\Gamma_{1-}^2, \tag{30}$$

where $\Gamma_1(t)$ and $\Gamma^2_{1-}(t)$ are the following block matrices

$$\Gamma_1(t) = \begin{bmatrix} \mathbf{0} & \mathrm{I} & & & \\ \gamma_1(t) & \mathbf{0} & \mathrm{I} & & \\ \mathbf{0} & \gamma_2(t) & \mathbf{0} & \mathrm{I} & \\ & \mathbf{0} & \gamma_3(t) & \mathbf{0} & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \quad \Gamma^2_{1-}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots \\ \gamma_2(t)\gamma_1(t) & \mathbf{0} & \cdots \\ \mathbf{0} & \gamma_3(t)\gamma_2(t) & \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Observe that for each $t \in \mathbb{R}$, we have associated to the matrix $\Gamma_1(t)$ a sequence of matrix polynomials $(S^{[1]}_n)_{n\in\mathbb{N}}$ defined by

$$S^{[1]}_{n+1}(x,t) = xS^{[1]}_n(x,t) - \gamma_n(t)S^{[1]}_{n-1}(x,t),$$

or, in more compact form,

$$S^{[1]}\Gamma_1(t) = xS^{[1]}, \quad \text{where} \quad S^{[1]} = \begin{bmatrix} S^{[1]\dagger}_0(x,t) & S^{[1]\dagger}_1(x,t) & \cdots \end{bmatrix}^\dagger. \tag{31}$$

We will denote by $\mathfrak{s}$ the matrix of linear functionals associated with $(S^{[1]}_n)_{n\in\mathbb{N}}$, and $(w_n)_{n\in\mathbb{N}}$ will be the corresponding sequence of moment matrices.

Theorem 2. *The following conditions are equivalent*

(a) $(\gamma_n)_{n\in\mathbb{N}}$ *is a solution of* (29), *i.e.* $\dot{\Gamma}_1(t) = \Gamma^2_{1-}\Gamma_1 - \Gamma_1\Gamma^2_{1-}$.

(b) *For* $n \in \mathbb{N}$,

$$\frac{d}{dt}(\Gamma^n_1)_{0,0} = (\Gamma^2_1)_{0,0}(\Gamma^n_1)_{0,0} + (\Gamma^n_1)_{0,2}(\Gamma^2_1)_{2,0}. \tag{32}$$

(c) *For* $n \in \mathbb{N}$, *the moments satisfy that* $\dot{w}_n = w_n w_2 - w_{n+2}$.

(d) *The Stieltjes function satisfies the following differential equation*

$$\dot{F}(z) = F(z)(w_2(t) - z^2\mathrm{I}) + zw_0(t). \tag{33}$$

(e) *The following equations for the matrix of the moment functionals hold*

$$\dot{\mathfrak{s}}(t) = \mathfrak{s}(t)w_2(t) - x^2\mathfrak{s}(t). \tag{34}$$

(f) *If* $(S^{[1]}_n)_{n\in\mathbb{N}}$ *is the first sequence of matrix orthogonal polynomials associated with* Γ_1, *then*

$$\frac{d}{dt}S^{[1]}_n(x,t) = \gamma_n(t)\gamma_{n-1}(t)S^{[1]}_{n-2}(x,t). \tag{35}$$

Proof: We will prove this theorem according to the following scheme

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (a).$$

We begin proving that $(a) \Rightarrow (b)$. First of all we establish that (a) implies that

$$\dot{\Gamma}_1^n = \Gamma_{1-}^2 \Gamma_1^n - \Gamma_1^n \Gamma_{1-}^2, \qquad n \in \mathbb{N}. \tag{36}$$

The proof will be by induction. For $n = 2$ we have

$$\begin{aligned}(\dot{\Gamma_1^2}) &= \dot{\Gamma}_1 \Gamma_1 + \Gamma_1 \dot{\Gamma}_1 = (\Gamma_{1-}^2 \Gamma_1 - \Gamma_1 \Gamma_{1-}^2)\Gamma_1 + \Gamma_1(\Gamma_{1-}^2 \Gamma_1 - \Gamma_1 \Gamma_{1-}^2) \\ &= \Gamma_{1-}^2 \Gamma_1^2 - \Gamma_1^2 \Gamma_{1-}^2.\end{aligned}$$

Suppose that (36) is satisfied by $n = m - 1$, and analyze when $n = m$.

$$\begin{aligned}(\dot{\Gamma}_1^m) &= (\dot{\Gamma}_1^{m-1})\Gamma_1 + \Gamma_1^{m-1}\dot{\Gamma}_1 \\ &= (\Gamma_{1-}^2 \Gamma_1^{m-1} - \Gamma_1^{m-1}\Gamma_{1-}^2)\Gamma_1 + \Gamma_1^{m-1}(\Gamma_{1-}^2 \Gamma_1 - \Gamma_1 \Gamma_{1-}^2) \\ &= \Gamma_{1-}^2 \Gamma_1^m - \Gamma_1^m \Gamma_{1-}^2.\end{aligned}$$

To prove (32), we note that

$$(\dot{\Gamma}_1^n)_{0,0} = \sum_{k=0}^{\infty} (\Gamma_{1-}^2)_{0,k}(\Gamma_1^n)_{k,0} - (\Gamma_1^n)_{0,k}(\Gamma_{1-}^2)_{k,0} = -(\Gamma_1^n)_{0,2}(\Gamma_{1-}^2)_{2,0}.$$

On the other hand, is not difficult to check that $(\Gamma_{1-}^2)_{2,0} = (\Gamma_1^2)_{2,0}$ and

$$(\Gamma_1^{n+2})_{0,0} = (\Gamma_1^n)_{0,0}(\Gamma_1^2)_{0,0} + (\Gamma_1^n)_{0,2}(\Gamma_1^2)_{2,0}.$$

From here

$$\begin{aligned}(\dot{\Gamma}_1^n)_{0,0} &= -(\Gamma_1^n)_{0,2}(\Gamma_{1-}^2)_{2,0} - (\Gamma_1^n)_{0,0}(\Gamma_{1-}^2)_{0,0} + (\Gamma_1^n)_{0,0}(\Gamma_{1-}^2)_{0,0} \\ &= (\Gamma_1^n)_{0,0}(\Gamma_{1-}^2)_{0,0} - (\Gamma_1^{n+2})_{0,0}.\end{aligned}$$

To prove that $(b) \Rightarrow (c)$, we use (32) and Proposition 5 to obtain

$$\dot{w}_n = w_n w_2 - w_{n+2}.$$

To prove that $(c) \Rightarrow (d)$, we use the definition of the Stieltjes function, and the fact that $w_1(t) = 0$.

$$\begin{aligned}\dot{F}(z) &= \sum_{k=0}^{\infty} \frac{\dot{w}_k(t)}{z^{k+1}} = \sum_{k=0}^{\infty} \frac{w_k(t)}{z^{k+1}} w_2(t) - \frac{w_{k+2}(t)}{z^{k+1}} \\ &= F(z)(w_2(t) - z^2 \mathrm{I}) + z w_0(t).\end{aligned}$$

To prove that $(c) \Rightarrow (d)$, we use the original representation of the Stieltjes function $F(z)$. Notice that from (33), we have that

$$\begin{aligned}\dot{F}(z) &= \left\langle \frac{\mathrm{I}}{z-x}, w_2(t)^\dagger - \bar{z}^2\mathrm{I} \right\rangle_{\mathfrak{s}(t)} + z\,\langle \mathrm{I},\mathrm{I}\rangle_{\mathfrak{s}(t)} + \langle x\mathrm{I},\mathrm{I}\rangle_{\mathfrak{s}(t)} \\ &= \left\langle \frac{\mathrm{I}}{z-x}, w_2(t)^\dagger - \bar{z}^2\mathrm{I} \right\rangle_{\mathfrak{s}(t)} + z\left\langle \frac{z-x}{z-x}\mathrm{I},\mathrm{I}\right\rangle_{\mathfrak{s}(t)} + \left\langle x\frac{z-x}{z-x}\mathrm{I},\mathrm{I}\right\rangle_{\mathfrak{s}(t)} \\ &= \left\langle \frac{\mathrm{I}}{z-x}, w_2(t)^\dagger \right\rangle_{\mathfrak{s}(t)} - \left\langle x^2\frac{\mathrm{I}}{z-x},\mathrm{I}\right\rangle_{\mathfrak{s}(t)} \\ &= \left\langle \frac{\mathrm{I}}{z-x},\mathrm{I}\right\rangle_{\mathfrak{s}(t)w_2(t)} - \left\langle \frac{\mathrm{I}}{z-x},\mathrm{I}\right\rangle_{x^2\mathfrak{s}(t)}.\end{aligned}$$

Thus, denoting by $\widehat{\mathfrak{s}}$ the matrix of linear functionals

$$\widehat{\mathfrak{s}} = \frac{d}{dt}\mathfrak{s} - \mathfrak{s}(t)w_2(t) - \mathfrak{s}(t)x^2,$$

then its corresponding Stieltjes function is equal to the zero matrix, but this implies that every moment matrix of $\widehat{\mathfrak{s}}$ is equal to $\mathbf{0}$, and from here $\widehat{\mathfrak{s}}$ is equal to the zero matrix of linear functionals.

To prove that $(d) \Rightarrow (e)$, we use the fact that $\dot{S}_n^{[1]}(x,t) - \gamma_n\gamma_{n-1}S_{n-2}^{[1]}(x,t)$ is a matrix polynomial of degree less or equal to $n-1$. Let $m = 0,\dots,n-1$. From the hypothesis

$$\begin{aligned}0_N &= \frac{d}{dt}\left\langle S_n^{[1]}(x,t), x^m\mathrm{I}\right\rangle_{\mathfrak{s}} = \left\langle S_n^{[1]}(x,t), x^m\mathrm{I}\right\rangle_{\dot{\mathfrak{s}}} + \left\langle \dot{S}_n^{[1]}(x,t), x^m\mathrm{I}\right\rangle_{\mathfrak{s}} \\ &= \left\langle S_n^{[1]}(x,t), x^m\mathrm{I}\right\rangle_{\mathfrak{s}} w_2(t) - \left\langle x^2S_n^{[1]}(x,t), x^m\mathrm{I}\right\rangle_{\mathfrak{s}} + \left\langle \dot{S}_n^{[1]}(x,t), x^m\mathrm{I}\right\rangle_{\mathfrak{s}} \\ &= \left\langle \dot{S}_n^{[1]}(x,t) - \gamma_n\gamma_{n-1}S_{n-2}^{[1]}(x,t), x^m\mathrm{I}\right\rangle_{\mathfrak{s}},\end{aligned}$$

and it implies that $\dot{S}_n(x,t) = \gamma_n\gamma_{n-1}S_{n-2}(x,t)$.

Before proving that $(e) \Rightarrow (a)$ we observe that using the notation in (31) we can rewrite (35) as $\frac{d}{dt}S^{[1]} = {\Gamma_1}^2_-(t)S^{[1]}$. With this in mind and recall that $S^{[1]}\Gamma_1(t) = xS^{[1]}$ we get

$$\dot{\Gamma}_1(t)S^{[1]}(x,t) + \Gamma_1(t)\dot{S}^{[1]}(x,t) = x\dot{S}^{[1]}(x,t).$$

Thus,

$$\dot{\Gamma}_1(t)S^{[1]}(x,t) + (\Gamma_1(t) - xI){\Gamma_1}^2_-(t)S^{[1]}(x,t) = \mathbf{0},$$
$$(\dot{\Gamma}_1(t) - {\Gamma_1}^2_-\Gamma_1 + \Gamma_1{\Gamma_1}^2_-)S^{[1]}(x,t) = \mathbf{0}.$$

And since for each t, $(P_n(x,t))_{n\in\mathbb{N}}$ is a basis of left module $\mathbb{C}^{N\times N}[x]$ we conclude that

$$\dot{\Gamma}_1(t) - {\Gamma_1}^2_-\Gamma_1 + \Gamma_1{\Gamma_1}^2_- = \mathbf{0}. \qquad \blacksquare$$

Corollary 4. *If in particular* $\langle P, Q\rangle_{\mathfrak{s}} = \int PW(x,t)Q^{\dagger}dx$ *and its associated Jacobi matrix* Γ_1 *satisfies* (29)*, then the matrix weight function* $W(x,t)$ *has the structure*

$$W(x,t) = e^{-x^2t}W(x,0)K(t), \tag{37}$$

where $K(t)$ *satisfies the matrix differential equation* $\dot{K}(t) = K(t)w_2(t)$*.*

Proof: Recall that from Theorem 2, Γ_1 satisfies (34), which implies that

$$\dot{W}(x,t) = W(x,t)(w_2(t) - x^2\mathrm{I}).$$

On the other hand, taking derivative in (37),

$$\dot{W}(x,t) = W(x,t)(K^{-1}(t)\dot{K}(t) - x^2\mathrm{I}),$$

and making use of our hypothesis, we get the result. $\blacksquare$

Corollary 5 (Lax-type Theorem)**.** *Let* $S^{[1]}$ *be the block column vector of matrix monic polynomials* $S^{[1]}_n(x,t)$ *and let* $\lambda(t)$ *be a spectral point of the Jacobi matrix* $\Gamma_1(t)$*, i.e.*

$$\Gamma_1 S^{[1]}(\lambda(t)) = \lambda(t)S^{[1]}(\lambda(t)). \tag{38}$$

If $\Gamma_1(t)$ *satisfies* (30)*, then* $\lambda(t)$ *does not depend on* t*.*

Proof: Taking derivative in (38)

$$\dot{\Gamma}_1 S^{[1]}(\lambda(t)) + \Gamma_1\dot{S}^{[1]}(\lambda(t)) = \dot{\lambda}(t)S^{[1]}(\lambda(t)) + \lambda(t)\dot{S}^{[1]}(\lambda(t)),$$

and using (30) we get

$$(\lambda(t)I - \Gamma_1)(\Gamma^2_{1-}S^{[1]}(\lambda) - \dot{S}^{[1]}(\lambda)) = \dot{\lambda}(t)S^{[1]}(\lambda(t)).$$

Again using that $\dot{S}^{[1]}(x,t) = \Gamma^2_{1-}S^{[1]}(x,t)$ we get $\dot{\lambda}(t)S^{[1]}(\lambda(t)) = \mathbf{0}$ but this implies that $\dot{\lambda}(t) = \mathbf{0}$. $\blacksquare$

Remark 2. *We emphasize that we do not have the reciprocal of the last result.*

In fact, if the spectral points of the Jacobi matrix $\Gamma_1(t)$ do not depend on t, then there exists a semi-infinite block matrix $C = [c_{j,k}]_{j,k=0}^{\infty}$ such that $\dot{\Gamma}_1 = C\Gamma_1 - \Gamma_1 C$, where C has the shape

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ * & \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & * & \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \cdots \\ * & \mathbf{0} & * & \mathbf{0} & * & \mathbf{0} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The sequence $(\gamma_n)_{n\in\mathbb{N}}$ must satisfy that

$$\dot{\gamma}_n = c_{n,n-2} - c_{n+1,n-1}, \qquad n \in \mathbb{N};$$

moreover, for every $n > 2$ and $m = 2, 3, \ldots$

$$\mathbf{0} = c_{n,n-2m} + c_{n,n-2(m-1)}\gamma_{n-2(m-1)} - \gamma_n c_{n-1,n-(2m-1)} - c_{n+1,n-(2m-1)}.$$

Observe then that in this case we cannot assure that C is equal to Γ_-^2. This is due among other things to the fact that any matrix high-order Volterra lattice also satisfies the Corollary 5 (*cf.* [5]), where we said that Γ_1 is a solution of a matrix high-order Volterra lattice if it satisfies that

$$\dot{\Gamma}_1(t) = (\Gamma_{1-})^{2m}\Gamma_1 - \Gamma_1(\Gamma_{1-})^{2m},$$

for some $m \in \mathbb{N}$.

We remark that all the results given here can be generalized, using similar techniques, to study this type of high order Volterra systems.

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